INSERVICE TRAINING PROGRAMME IN MATHEMATICS FOR THE PGTs OF NVS

02-06-2003 to 22-06-2003

REPORT

Academic Coordinator

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(NATIONAL COUNCIL OF EDUCATIONAL RESEARCH AND TRAINING, NEW DELHI)

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PREFACE

The 21-day training programme in Mathematics for the PGTs of NVS, New Delhi was held in RIE, Mysore from 2^{nd} to 22^{nd} June 2003.

The programme was arranged at the request of NVS, New Delhi. The main objective of the programme was to enrich the content level of the teachers as per the revised curriculum.

The present volume contains a detailed report as well as some enrichment material other than the ones given in the training programme to be used in the classroom transactions. I am extremely happy to place on the record that all the participating teachers took great interest in learning new ideas and that they were very punctual in their schedule of training.

I am grateful to Prof. J.S. Rajput, Director, NCERT, for having selected RIE, Mysore as the venue for the programme. My thanks are also to authorises NVS, New Delhi for not only providing funds for the programme but also deputing Mr. Palaniappan, Principal, NVS, Mandya as a liaison officer from NVS.

I am indeed thankful to Prof. G. Ravindra, Principal, RIE, Mysore, for giving full cooperation and guidance to conduct the programme. I also wish to thank all the resource persons and guest lecturers who have greatly contributed and shared their valuable experiences with the participants.

My thanks are also to my colleagues in Mathematics Department for their support, guidance and participation, both during planning and conduct of the programme. I wish to thank my colleagues in other sections and departments for their cooperation.

Lastly, I express my thanks to the administrative and accounts staff for their help in making the programme a grand success.

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ABOUT THE TRAINING PROGRAMME

Need to upgrade periodically the professional competence of teachers at all levels in general and senior secondary teachers in particular cannot be overemphasised. In order to improve the capabilities of the teachers in content and pedagogy, the NVS arranges inservice training of teachers at various levels in the form of orientation and refresher courses. In recent times, introduction of career advancement schemes have made it obligatory for the plus two level teachers to undergo refreshers courses of three weeks duration. Hence there is a felt need for a training or enrichment package designed to cater to the special needs of plus two level teachers. The present programme was held at RIE, Mysore from 2nd to 22nd June 2003 for PGTs in Mathematics of NVS. The programme was planned and implemented by the Mathematics section of DESM of RIE. In addition to the Mathematics faculty, faculty members from the Department of Education also worked as resource persons. Guest lectures and popular talks were arranged using the expertise of external resource persons of eminence.

The main objectives of the training programme was to

(i) *enrich the content competency of teachers so that they can execute the revised curriculum with greater confidence.*

(ii) *make the teachers aware of recent thrust areas in the field of education so as to improve their professional competence and*

(iii) *make them familiar with certain skills and strategies required for effective teaching in the present day classrooms.*

The programme consisted of four lecture sessions per day and compulsory reference work in the library at the end of each day. The topics for the lecture sessions were included after identification of difficult areas, identified in a special session on the very first day. The topics covered were as mentioned below:

(i) *Calculus (Differential and Integral)*

(ii) *Differential Equations*

(iii) *Statics and Dynamics*

(iv) *3D Geometry*

(v) *Probability and Statistics*

(vi) *Computers (with hands on experience)*

(vii) *Mathematical Logic*

(viii) *Boolean Algebra*

(ix) *Teaching of Concepts in Mathematics*

(x) *Evaluation in Mathematics*

(xi) *Conic Sections and Advanced Level Problem Solving*

(xii) *Value Education*

(xiii) *Action Research*

(xiv) *Creativity in Teaching and Learning*

(xv) *Commercial Mathematics*

(xvi) *Linear Programming*

(xvii) *Mathematical Modelling*

(xviii) *Mathematics Laboratory*

Pre-test and Post-test were conducted for the teachers to study the impact of the training programme.

To sumup varieties of experiences were provided to the participants in order to enhance content enrichment and professional competence. It is hoped that the programme has sufficiently motivated the teachers which is also revealed by the Pre-test and Post-test conducted during the programme.

B.C. BASTI
Academic Coordinator

LIST OF RESOURCE PERSONS

I. RIE Faculty

Prof. G. Ravindra

Prof. K. Dorasami

Prof. N.M. Rao

Prof. D. Basavayya

Dr. B.S.P. Raju

Dr. B.S. Upadhyaya

Mr. B.C. Basti

Dr. N.N. Prahallada

Dr. C.G.V. Murthy

Dr. A.S.N. Rao Sindhe

Dr. (Mrs) Kalpana Venugopal

II. External Resource Persons

Dr. N.B. Badarinarayana

Mr. M.V. Gopalakrishna

Dr. G.T. Narayana Rao

Dr. Shamanna

(Popular Talks)

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SETS, RELATIONS AND FUNCTIONS

BY

B.C. BASTI

Introduction : an overview.

In this write up (Instructional material) an attempt has been made to discuss the important concepts of 'sets relations and functions. Although these concepts are as old as man's history of civilization, formal introduction of these concepts into mathematics has been very recent. Through the use of these concepts one gains an understanding of the structures and patterns that occur in Mathematics. Pedagogically it has been widely accepted that the concept of sets greatly helps unification of several branches of Mathematics at the school level.

The topic of 'sets' invariably finds a place in the school curriculum all over the world. As an introduction to modern or the so called new Mathematics and as a "language", its importance is accepted.

Sets — Preliminaries

The words, class 'collection' 'assemblage' are synonemous because they convey the idea of a 'set'. Intuitively a set is a 'collection' of objects. The objects may be physical objects, numbers, any kind of symbols or even ideas.

In Mathematics, the term 'set' is used to mean a 'well-defined' collection of objects. Why do we insert the adjective 'well defined' in the description of the term 'set'? Let us study a few examples of 'collections' of objects.

Ex. 1. All states in the Indian Union

Ex. 2. All rivers of Karnataka

Ex. 3. All multiples of the nnmber 7

Ex. 4. Some interesting books

Ex. 5. The students of that class

Ex. 6. The collection of all circles having a given point as their centre

Ex. 7. The good films produced in Bombay in the year 1981

A scrutiny of the 'collections' given in the above list reveals that in the case of examples 1, 2, 3, 6, there is no difficulty in identifying the objects present in each collection. Whether an object is in the given set or not can be clearly judged in these cases. But the collections in examples 4, 5 and 7 have been described by the words like 'interesting' 'that class' and 'good films'. These descriptions render the sets 'ambiguous'. We are unable to identify clearly the objects of these collections.

Hence the collections in examples 1, 2, 3, 6 are 'well defined' $\therefore$ they are examples of sets. Where as those in examples 4, 5, 7 are *Not well defined* collections.

A set is therefore a well defined collection. The following collections are well defined.

1. The set of all lines passing through a given point.
2. The set of all two legged animals.
3. The set of all primes less than 14.
4. The set of all employees of the NCERT

Each of the above is an example of a *well defined collection* because in these cases the basic requirement that "*given any object what so ever and a set, it must be possible to determine whether or not the object is in the set in question*" is satisfied.

Exercise :

which of the following collections are sets ?

a) Rational numbers.
b) The students studying this book.
c) The paintings in Salarjung Museum
d) The contents of little boys' pockets
e) The ripe oranges

Notation and representation of sets

It is customary to denote sets by capital letters, A, B, C etc. The objects in a set are called 'members of the set' or 'elements of the set'. The 'elements' of any set are usually denoted by small letters a, b, c etc.

If 'a' is an element of A then we write $a \in A$ read as (a belongs to A). The notation $a \notin A$ indicates that 'a' *does not* belong to A.

There are two ways of representing a set.

I. In the first method we make a list of all the members of the set, separating them by commas, and we enclose them within 'braces' or flower brackets. This method is called roster method or tabular form.

Ex. 1. The set A of all numerals on the dial of a clock can be represented by roster method as

$$A=(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12)$$

Ex. 2. The solutions of the equation $x^2-5x+4=0$ are listed as the set (1, 4) using roster method.

Ex. 3. The set of all days of the week by roster method of representation becomes

(Sunday, Monday. Tuesday, Wednesday, Thursday, Friday, Saturday).

II. The second method of designating a set is called 'rule method' or 'set builder form'. In this method, a rule or a common property of all the elements is stated.

For example, to represent a *set* B of *all even numbers,* we use the letter x (usually) to represent an arbitrary element and write

$$B=(x/x \text{ is even}).$$

Which reads:

"B is a set of all x such that x is even".

If S is the set of all elements x with the property p by set builder method we write.

$$S=\{x/x \text{ has the property } p\}$$

Here the property p is called the defining property.

Ex. 1. $N=(1, 2, 3, 4, 5......)$

in the set builder from $N=(x/x$ is a natural number).

Ex. 2. C=set of all capital cities in Europe in the set builder form

$C=[x/x$ is a capital city in Europe]

Exercise : Express the following sets in (1) roster form (2) set builder form.

a) All integers between —5 and +5

b) Solution of $x^2-3x+2=0$.

c) All equilateral triangles in a plane.

d) All plays written by Shakespeare.

e) All noble laureates from India.

At this stage we mention two important rules in the representation of sets.

1) The order in which the elements are listed in a set is *immaterial*, since we are interested in the set as a whole.

for ex : A= [2, 5, 3, 6, 4]

this set can as well be written as

A= [2, 3, 4, 5, 6].

Similarly the set [2, 0, 1] is the same as the set [0, 1, 2].

2) Each element of a set is listed *once* only.

Example 1. Let the scores of five students in an examination be given by

57, 81, 81, 75, 44.

The set representing these scores is

S= [57, 81, 75, 44]

Note that in S the score 81 is listed *once* only, even though if appears twice in the original list.

Ex. 2 : [1, 1, 2. 2, 5, 7] is the same set as [1, 2, 5, 7].

Q : Why should duplication or reptition be avoided while listing the elements in a set ? give reason.

Finite sets, infinite set, Empty set.

Consider (1) the set [5]=A

(2) B= [a, e, i, o, u]

(3) C= [x/x is an integer]

(4) D= The set of all stars of first magnitude

Let us examine each of these sets as to the *number of elements* it has.

The set A has a single element in it. We call this set a 'singleton'. This set has the least number of elements.

The set B has *five* elements in it. It's elemets can be counted as 5.

The set though has a large number of elements has only a *finite number* of elements.

The set C has infinitely many elements in it meaning that, the process of listing its elements wil never end. Another example of a set with infinitely many elements is the 'set of all points in a line segment'.

It is clear that sets may be of any size in so far as the number of elements are concerned.

In the above examples, A, B, and D are *finite sets* while C is an *infinite set.*

Frequently we even find it convenient to consider a set containing *no* elements, such as, the *set of all points at which two parallel lines intersect.*

Ex. 2 The set of all common factors of 3 and 7,
Ex. 3. The set of even primes greater than 5 and less than 20.

These set are all sets with *no* elements, in them. Such a set is referred to as *empty or null set.* Null set is denoted by the symbol ϕ or $\{\ \}$.

Subsets and Equal sets ; Equivalent sets :

If every element of set A is also an element of the set B then A is called the *subset* of B. A is *a* subset of B if and only if.

For all x, $x \in A \leftrightarrow x \in B$. We denote this relationship by $A \subset B$. We write '*A is contained in B* or $B \supset A$ (B contains A).

$$\boxed{\therefore\ A \subset B \rightarrow \text{for all } x\ x \in A \leftarrow\!\!\rightarrow x \in B}$$

Ex. 1. $A = [x/x$ is *a*. Counting number]
ie $A = \{1, 2, 3, 4 \ldots\ 5 \ldots\}$

$B = \{3, 5, 7\ \ldots\}$ $C = [5, 10, 15, 20 \ldots\ldots]$
$D = [2, 4, 6, 8 \ldots\ \ldots]$

Observe that B, C & D are subsets of A. The sets B, C, D are constructed by selecting the elements from A.

For all x ie $\forall x\ x \in B \rightarrow x \in$ ie $B \subset A$

$\forall x . x \in C \rightarrow x \in A$ ie $C \subset A$

$\forall x\ x \in D \rightarrow x \in A$ ie $D \subset A$

Ex. 2 : $A = [1, 2, 3, 10, 11]$

$B = [10, 11]$. Here $B \subset A$ $\because$ all the elements of B are elements of A. But A has some elements that are not found in B.

In this example B is called a *proper* subset of A.

Ex. 3. Let $A = [5, 6, 7, 8, 10]$, $B = [5, 6, 7, 8]$
$C = [7, 8, 9]$, $D = [5]$, $E = [10]$.

Here B, C, D, E are all *proper* subsets of A $\because$ *Not all* elements of A are in B or C or D or E. It is important to note that

1) Every set is a subset of itself
2) Null set is a subset of every set

Though these statements surprise us, they are the direct consequences of the *definition of a subset* for,

1) $A \subset A$ $\rightarrow$ for all $x\ x \in A \rightarrow x \in A$ which is always true. Hence $A \subset A$, A is an *Uniproper* subset of A.

2) $\phi \subset A \rightarrow$. Every element of ϕ is also an element of A. Trouble would arise if there is some element in ϕ which fails to be in A. Since ϕ has no

elements at all the requirement of a subset is trivially satisfied for ϕ. Hense $\phi \subset A$. In fact ϕ *is a subset of every set.* ϕ is an *improper subset* of any set <Note that ϕ and the given set itself are the two improper subsets of any given set>.

Ex. $A = [0, 1, 2]$. Let us list all its subsets.

$[0]$ $[1]$ $[2]$ $[0, 1]$ $[0, 2]$ $[1, 2]$ are the proper subsets of A and

$[0, 1, 2]$ and ϕ are its improper subsets.

Equal sets : Two sets are said to be equal if they have idenitcally same elements-

$A = [1, 2, 4]$ and $B = [1, 2, 2^2]$ are equal sets

$A = B$.

Ex. 2, $A = [1, 2, 2, 1]$ and $B = [1, 2]$ are identical sets or epual sets, $A = B$.

Ex. 3. Let $[1, 5, 6] = \mathbf{A}$

Here $A = B$

$[6, 1, 5] = B$

These are equal sets. Here *every element of* A *is an element of* B. ie. $A \subset B$

and *every element of* B *is also an element of* A ie. $B \subset A$.

From this example we note that $A = B$ if $A \subset B$, and $B \subset A$

$$\boxed{A = B \text{ if } A \subset B \text{ and } B \subset A}$$

$$\begin{array}{lccccc} A: & 1 & 2 & 3 & 4 \ldots\ldots\ldots\ldots & n \ldots\ldots\ldots \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ B: & 3 & 6 & 9 & 12 \ldots\ldots\ldots\ldots & 3n \ldots\ldots\ldots \end{array}$$

Here A and B are equivalent, because their elements can be matched or *a* one-one correspondence is possible between their elements.

$$\begin{array}{lccccl} \text{Ex. 4}: & [1^2 & 2^2 & 3^2 & 4^2 \ldots\ldots\ldots\ldots &]=A \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ & [1^3, & 1^3, & 1^3. & 1^3 \ldots\ldots\ldots\ldots &]=B \end{array}$$

As in ex. 3 here also, A and B are eqivalent. In Ex. 3 & 4, A and B being infinite sets, we donot ask "Do they have same number of elements ? instead we set up a one-one Correspondence for the elements of A and B and decide that these sets are *equivalent.* Are all equal sets equivalent ? the answer is obviously 'Yes'.

Cardinal numbers and Infinite sets :

The concept of 'counting numbers' or 'natural numbers' as a set, was developed because of man's desire to compare sets of various objects. Consider a set of *ten* books, *a* basket of *ten* apples, *a* pack of *ten* wolves-all these sets have *10* objects in them. This fact as we know, is arrived at, by the counting process.

What is the principle underlying the process of counting ? Recall, that counting involves a 'matching process' or setting up a 'one-to-one correspondence'

Equivalent sets :

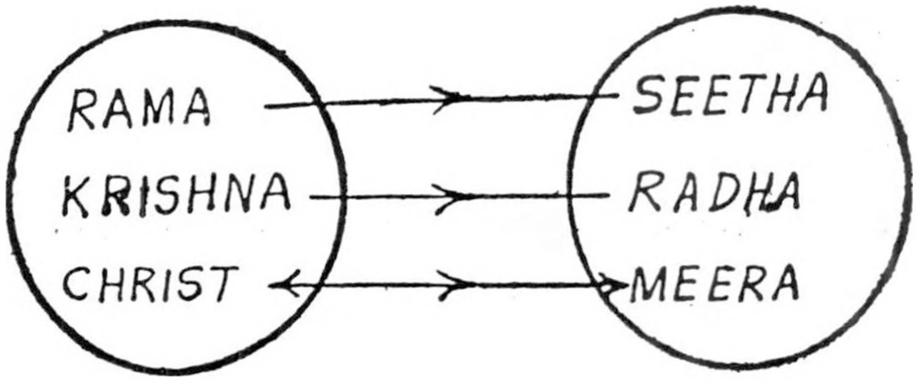


Fig. 1

Ex. 1. A=[Rama, Krishna, Christ]

B=[Seetha, Radha, Meera]

Here, A and B are not equal sets but,
A and B have both three elements in them. We say A and B are equivalent sets.
ie. the elements of these sets can be *matched*

Ex. 2. A=The cricket team from England

B=The Cricket team from India.

Note that the players are different in each team but A and B have same number of players

∴ A and B are equivalent sets.

Ex. 3. A=[1, 2, 3, 4........]

B=[3, 6, 9, 12........]

A and B are infiinte sets.

We can match the elements of A and B or set up a one-one Correspondence between the elements of and A B as shown here.

between elements of the given set and the elements of *some* standard counting set. The set of natural numbers 'N' is the standard counting set.

The set A= [a, b, c, d, e], has 5 elements in it because A is equivalent to the set.

N_5=[1, 2, 3, 4, 5] which is the subset of N. We say *5* is the cardinal number of this finite set A. Note that A is equivalent to N_5.

If A= [$3, 3^2, 3^3 \ldots\ldots\ldots\ldots 3^n$.]

We know that here A is equivalent to the finite subset N_n= [1,2,3........n] of N. So the cardial number of this finite set A is 'n'. A is equivalent to n_n. If two finite sets are equivalent they have the same cordinal number.

It is now clear that the cardinal *Number* of any *finite set* is a *specific natural number.*

Infinite sets have a special property which makes them interesting to study.

We have seen that two finite sets are equivalent iff they contain same number of elements.

Ex. 1: Now let N=[1, 2, 3........ ..]=[$x/x \in N$]

E= [2, 4, 6............]=[$x/x=2n$, $n \in N$]

sets

N and E are equivalent sets. Note that E, here is a proper subset of N. ie. *not all* elements of N are elements of E. We can still set up a one- One correspondance between N & E

$$\begin{array}{lllll} N: & 1, & 2, & 3 \ldots\ldots & n \ldots\ldots \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ E: & 2 & 4 & 6 \ldots\ldots & 2n \ldots\ldots \end{array}$$

$\therefore$ N is equivalent to E

Clearly, the infinite set N is equivalent to *its proper* subset E of even numbers.

Ex. 2: $I = [0, \pm 1, \pm 2, \pm 3 \ldots\ldots]$
$P = [-1, -2, -3 \ldots\ldots]$
$Q = [0, 1, 3, 5, 7 \ldots\ldots]$

Note that I is an infinite set and
$P \subset I$ P is a proper subset of I
$Q \subset I$ Q ,, I

$\therefore$ I is equivalent to P (its proper subset)
I is equivalent to Q (its proper subset)

This property of "*a* set being equivalent to a proper subset of itself" is characteristic of '*Infinite sets*'. So we state.

≪*A set is infinite, if it is equivalent to a proper subset of itself, otherwise it is finite*≫.

The cardinal number of the standard counting set N *does not* correspond to any finite natural number, as N is an *infinite set.* The cardinal number of N is sometimes denoted by α (alpha null). All infinite sets which are equivalent to the set N have the same cardinal number α (alpha null).

The idea of cardial numbers was first developed by Georg Cantor in a remarkable series of articles published in 1872. Prior to Cantor's study of infinite Msets, athematicians used the symbol ∞ indiscriminately to indicate the 'number' of elements in all kinds of infinite sets. Cantor's work revolutionised the concept of 'infinity' in mathematics.

Exercise :

(1) How many elements are in [a, a, a, a, a]

(2) Can there be unequal empty sets ? Explain.

(3) Extend the definitions of union and intersetion to n sets. n—finite+ve integer).

(4) Find all subsets of [0, 1, 2].

State whether each statement is correct ?

(a) [1, 4, 3]=[4, 3, 1].

(b) [4]$\in$ [(4)]

(c) [4] $\subset$ [(4)]

(d) [ϕ] a subset of every set

(e) [1, 2, 3, 1, 3, 2] $\subset$ [1, 2, 3].

(5) State whether following sets are finite or infinite

(a) Set all lines parallel to X-ax is

(b) Set of all circles through the origin (.00)

(c) The set of all animals living on Earth

Venn diagrams and Universal set.

To understand the relationships among sets, as also properties of sets, we often use simple diagrams called Venn diagrams. These are strictly schematic representations. Although they cannot be used to prove statements, they are excellent visual aids to verify important set relationships. In these diagrams sets are represented by circular areas.

Ex : The concept of $A \subset B$ $A \neq B$ is shown in the venu diagram as

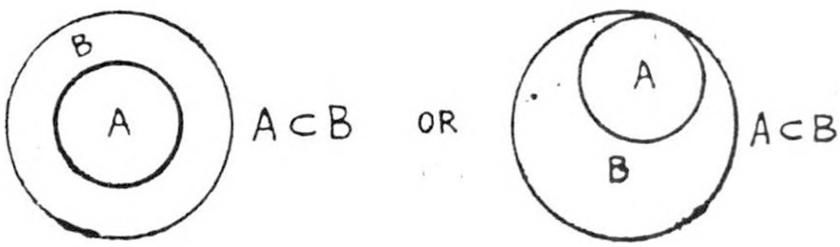


Fig 2

Ex. 2 :

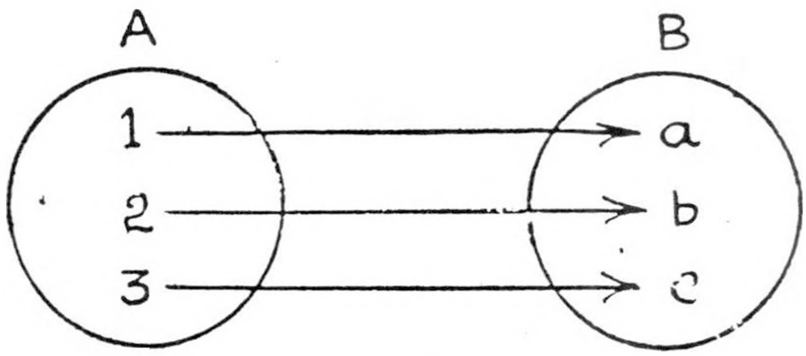


Fig 3

This diagram illustrate the 1—1 correspondence between the sets

$A = [1, 2, 3]$ and $B = [a, b, c]$

A is equivalent to B.

Universal set : In any discussion on sets, all sets under investigation will very likely be subsets of a fixed set. We call this set 'Universal set' or 'Universe of discourse'. We denote this by set U.

ex : 1 Any study about population of human beings, will have the set of all human beings in this world as the Universal set.

Ex : 2 : If A= the set of rectangles.
B= the set of all circles
C= The set of all triangles

the Universal set for these sets is the set of all plane figures.

In a Venn diagram, Universal set U is usualy represented by a rectangle. All the subsets of the Universal set are shown as circles in this rectangle.

For Ex. 3 the venn diagram is shown here.

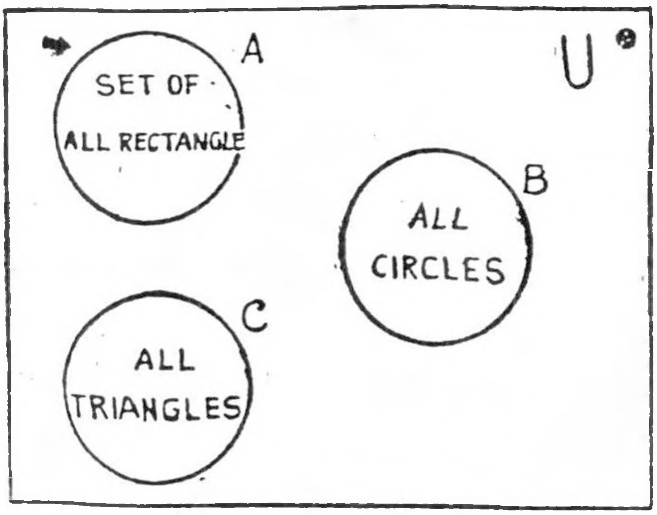


Fig 4.

Exercise : Let $Q=[x/x$ is a quadrilateral]

$H=[x/x$ is a rhombus]

$R=[x/x$ is a rectangle]

$S=[x/x$ is a square]

Decide which sets are the proper subsets of others. Draw Venn diagrams to illustrate their relationships.

Exercise 2 : Draw Venu diagram to illustrate the sets,

A = The set of all boys in your state

B = Set of all boys in your school

C = The set of all boys in your mathematics class

OPERATIONS ON SETS

In arithmetic we are taught how to add, subtract and multiply numbers. What exactly in done in each of these processes ? Recall that for each pair of numbers x and y, We assign a number $x+y$ called the **sum** of x and y, a number $x-y$ called the **difference** of x and y and a number xy called the **product** of x and y, This process of assigning (associating) a number with a pair of numbers is nothing but 'binary operation' on numbers. In fact the fundamental operations on numbers are all 'binary operations'.

Let us extend the idea of 'operation to sets. We wish to construct new sets from the given sets, while there are various ways of assigning a 'new set' to a given pair of sets, we in this section, discuss three important ways of constructing new sets by devising binary operations called

1) Union 2) Intersection 3) difference of sets. We later see that these operations have certain properties similar to the usual operations of arithmetic.

Union of sets

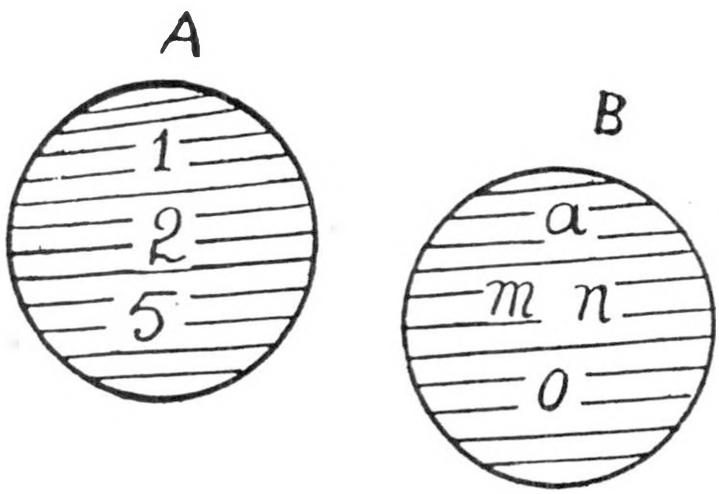


Fig 5.

Consider $A = [1, 2, 5]$

$B = [a, n, m, o]$

Let us form the set $C = [1, 2, 5, a, n, m, o]$
Note that C is the set of all those elements which are either in A or B or both, In other words.

$$C = [x / x \in A \text{ or } x \notin B]$$

Note that we have used 'or' in the inclusive sense.

We refer to C as the 'Union' of A and B, and we write

$$C = A \cup B \quad (A \text{ union } B)$$

Refer to the Venn diagram Fig 6

Shaded portion represents $A \cup B$

Ex. 2 : Let $A = [1, 2, 3, 4, 5]$

$B = [1, 2, 3, 8, 6]$

easily, $C = A \cup B = [1, 2, 3, 4, 5, 8, 6]$

$\because A \cup B = C$ satisfies the property

$C = [x / x \in A \text{ or } x \in B]$

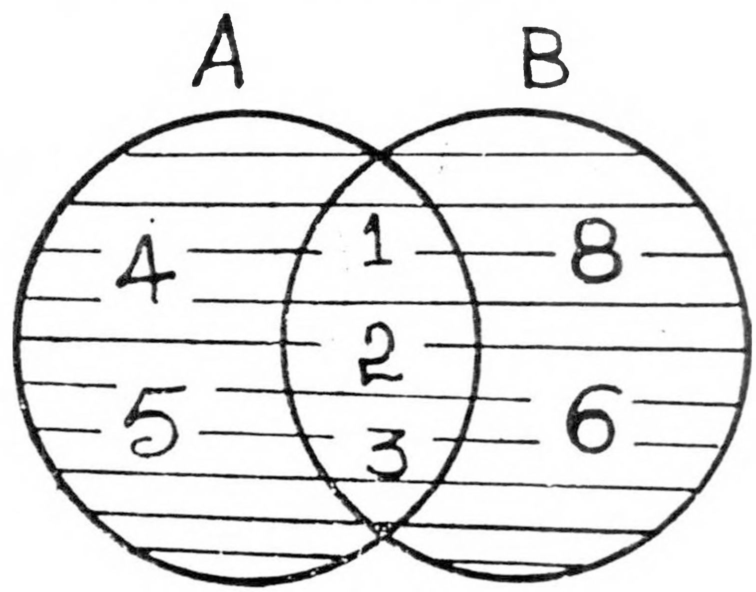


Fig 6.

Shaded portion in fig 6

represents $A \cup B$.

Ex. 3 : Let $A = [a, b, c]$

$B = [a]$

Here. $A \cup B = [a, b, c] = A$ itself

The shaded postion in AUB is fig 2.1 (c)

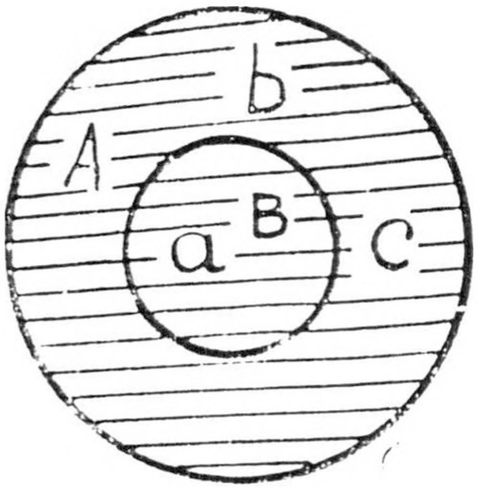


Fig 7.

We now define AUB the union of any sets A and B

$$\boxed{AUB = \{x/x \in A \text{ or } x \in B\}}$$

If $A = N$ (set of nat. numbers), $B = \{x/x = x^2, x \in N\}$ then $AUB = A$.

Union of sets as a binary operation on sets. To each pair of sets A and B, a set called **AUB** is assigned such that

$$AUB = [x/x \in A \text{ or } x \in B]$$

Remarks : By the definition of 'union' the following properties directly follow.

1) The set A U B is identical with the set BUA
2) The set AUB contains the set A as well as B

ie. $A \subset AUB$
$B \subset AUB$

3) If $A \subset B$ as we have already seen $A \cup B = B$ itself

4) The union of any set with itself is the given set itself ie. $A \cup A = A$

Exercise

1) What is $A \cup B$ if either A or B is an empty set ?

2) For what choice of sets do we get $A \cup B = \phi$?

3) Give an example of a set A U B such that A U B is equivalent to A or B

4) Write the venn diagram to represent the set A U B U C for sets A, B, C

Intersection of sets.

Intersection of the sets is another binary operation on sets. To each pair of sets A and B, we assign a set called '*Intersection of the sets A and B*, (denoted by $A \cap B$), according to the requirement that

$$A \cap B = \{ x / x \in A \text{ and } x \in B \}$$

ie $A \cap B$ Consists of all the elements *that are of common to the sets* A *and* B. The shaded portion in the following Venn diagram represents $A \cap B$ in each case.

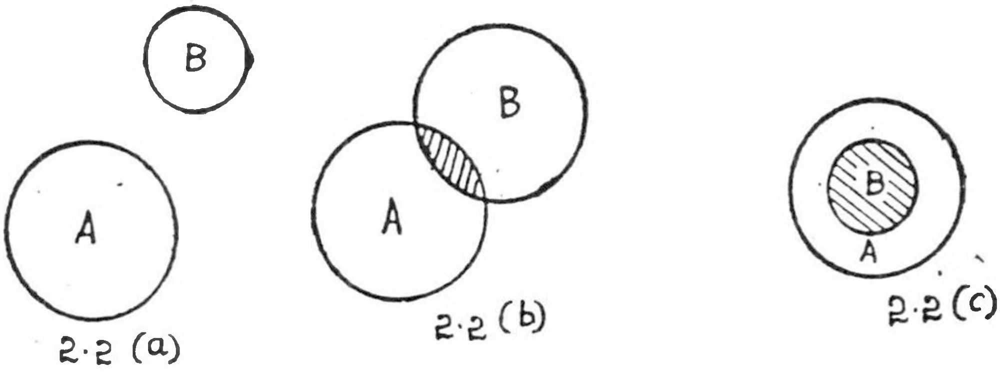


Fig 8.

In fiig 2.2 (a) absence of common elements in A and B explains why there is no shaded portion to represent A ∩ B. The sets A and B in this case do not interscet.........A ∩ B is a null set. We call such sets A and B as disjoint sets. Note that the sets in 2. 2 (b) and 2.2 (c) are not disjoint sets.

Ex. 1 : Let A=[1, 2, 3, 4..... ..]

B=[0, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$........]

these are both infinite sets. They have no common elements between them.

Obviously A ∩ B={x/x∈A and x∈B}=ϕ(empty set)

∴ A and B are **disjoint sets.**

Ex. 2, A={ x/x is a perfect square }

B={ x/x is an even number }

A ∩ B=[x/x is a perfect square **and** x is an even number] Observe that A ∩ B is not empty ∵ some perpect squares like 4, 16 etc. ∈A U B

Note the following consequences of the definition of the set A ∩ B

1) The set A ∩ B is equal to the set B ∩ A

2) A ∩ A is always=A itself

3) $A \cap B$ is a subset of A, ie $A \cap B \subset A$

$A \cap B$ is a subset of B, ie $A \cap B \subset B$

4) If $A \subset B$ then $A \cap B = A$ it self

5) If A and B are disjoint, $A \cap B = \phi$, ϕ is a subset of every set and ∴ of A and B.

Exercise : 1) For any sets A, B and C Find using Venn diagrams the set $(A \cap B) \cap C$ and the set $A \cap (B \cup C)$

2) Let $A = [1, 2, 3, 4]$

$B = [2, 4, 6, 8]$

$C = [3, 4, 5, 6]$

Find (a) $A \cap B$, (b) $A \cap C$ (c) $B \cap C$ (d) $B \cap B$.

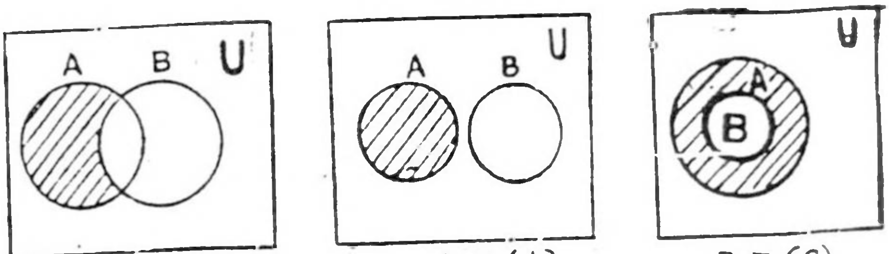
The difference of sets A and B :

The difference $A - B$ of the set A and B is a set such that

$R = A - B = [\ x / x \in A$ but $x \in B\ ]$ we could also say

(read A minus B) $A - B = [\ x \in A$ and $x \in B\ ]$

In the Venn diagrams that follow $A - B$ is given by the shaded are, a



2·3 (a) 2·3 (b) 2·3 (c)

Fig 9.

Ex. 1. $A = [5, 6, 9, 16]$, $B = [5, 6]$

$A - B = [9. 16]$

Ex. 2 $A = \{x/x = 2x, \quad n \in N\} \equiv \{2, 4, 6, 8........\}$

$B = \{x/x = 3n \quad n \in N\} \equiv \{3, 6, 9, 12........\}$

$A - B = [2, 4, 8, 10, 14........] \equiv [x/x = 2n$ but $x \neq 3n, \quad n \in N]$

For any set A the difference set U—A is the difference of the universal set and A is called the complement of the set A in U ie complement of A=U—A.

We write $A^1 = U - A$

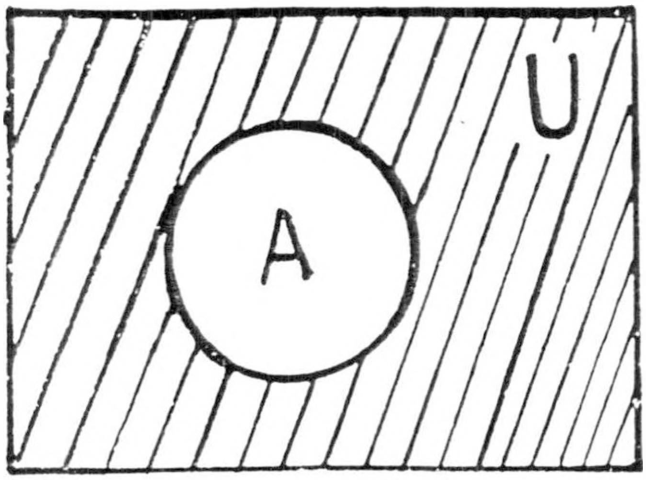


Fig 11.

The shaded postion of the diagram (Fig. 11) represents A^1 the complement of A

$\therefore \quad A^1 + [x/x \epsilon U$ and $x \in A]$

Example : Let $U = [1, 2, 3, 4........]$

$A = [2, 4, 6, 8............]$

then $A^1 = [1, 3, 5, 7............] = U - A$

Remark : 1) For any set A

$A \cup A^1 = U$ the universal set

2) U^1 the complement of the universal set $= \phi$

3) ϕ^1 the complement of the empty set is the universal set U.

4) The set A and its complement A^1 are always disjoint in $A \cap A^1 = \phi$

We can use Venn diagrams to understand some simple relationships among the operations of union, intersection, difference of sets and complements.

1) **$A-B$, $A \cap B$ and $B-A$ are mutually disjoint.**

the corresponding diagram are

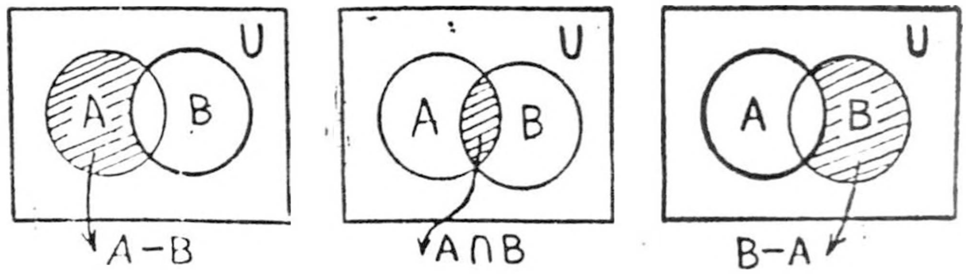


Fig 12.

from these diagrams we have

$A-B$, $A \cap B$ and $B-A$ are mutually disjoint sets.

2) $A-B = A \cap B^1$ Let us draw Venn diagrams

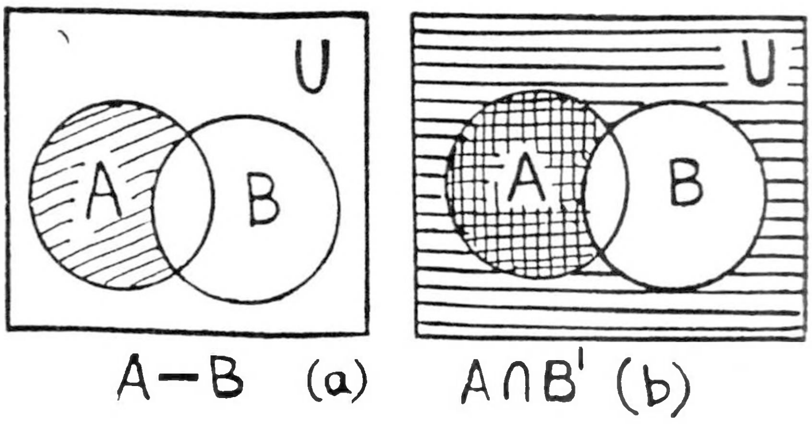


Fig 13.

∴ Shaded area in the diagram (a). is $A-B$ Horizontally shaded area in the diagram (b) is the set B^1. The double hatched (shaded) area in the figure (b) is the set $A \cap B^1$ From figs (a) and (b), we have

$$A-B=A\cap B^1$$

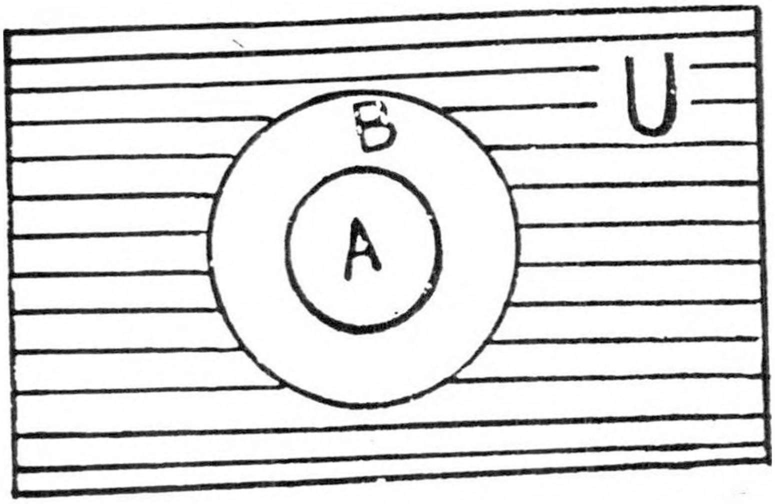


Fig (c)

Fig 14.

3) If $A \subset B$ then $B^1 \subset A^1$

horizontally shaded area in diagram (c) is the set B^1

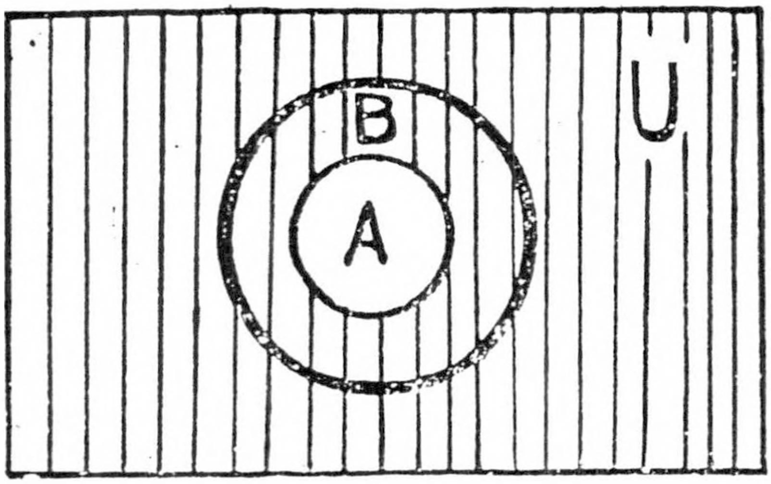


Fig (d)

Fig 15.

Vertically shaded postion of fig (d) represents the the set A^1.

From the two figures it is clear that B^1 is contained in A^1ie the region of B^1 is included in the region of A^1

Exercises : Verify by the drawing the Venn diagrams the following set theoretic relations.

1) $(A \cap B) \cup (A-B)=A$

2) $(A-B) \cup B=A \cup B$

3) $(A-B) \cup B=\phi$

4) $A-B=A-(A \cap B)$

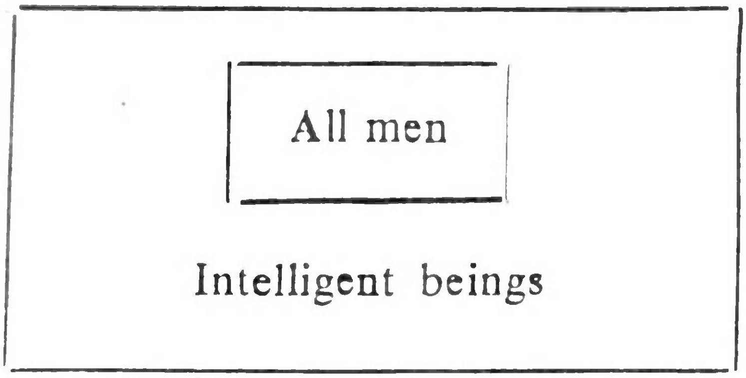
Use of Venn diagrams and knowledge of sets in solving some problems :

Venn diagrams illustrate the relationships that exist among given sets. Many verbal statements can be conveniently translated into statements about sets and represented in Venn diagrams.

Ex : This statement,

"All men are intelligent" can be rewritten using language of sets as

'The set of all men is a subset of the set of all intelligent beings' we can now use Venn diagrams to represent this idea as in fig 2.4 (a)



There are some problems which can be solved using the language of sets and Venn diagrams. In these problems, we restate the problem as a statement about sets, and study these sets using Venn diagrams.

Example :

Problem : In a group of 40 students who drink tea or coffee or both, 26 drink tea of whom 16 drink tea but not coffee. How many drink coffee but not tea ?

We recognize the different sets of students as

A= set of students who drink coffee

B= set of students who drink tea.

(B has 26 elements).

then A U B the set of all student drink coffee or tea or both.

From the data A U B has 40 elements in it.

B—A is the set of students who drink tea but not coffee.

It is given that B—A has 16 elements. We have to find number of elements in A—B. In fig (a) (AUB) is horizontally hatched area)

$\therefore$ (A U B)—B is the set of all those who are strictly coffee drinkers they are 40—26=14 in number. (A U B)—B is the double hatched area of fig (b) The set (A U B)—B is the same as the set A—B ; the horizontally hatched area in fig c.

$\therefore$ A—B, the set of all who drink coffee but not tea contains 14 elements.

From the diagrams (A U B)—B=A—B

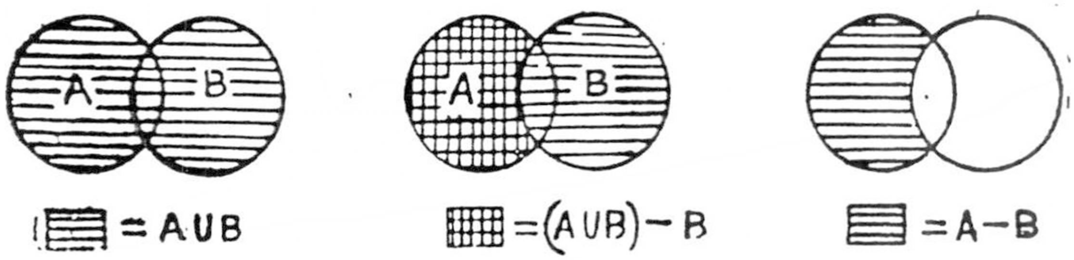


Fig 16·

In the same problem if we want to find how many students drink both tea and coffee ie. we want the number of elements in the set A U B What is the answer ?

Exercises : Solve problems given in the exercise 3.4 chapter 3, of the Text book of Maths. VIII standard

x x x

Use of Venn diagrams & sets in testing the validity of arguments in Logic :

What is an argument ? An argument is an assertive statement. An argument, therefore, is true or false but not both. Argument occurs in a reasoning process. Every argument contains two parts, First part is called *premises*. Permises is made up of a number of statements. Second part of argument is called *conclusion*. Conclusion is a single statement.

An argument is of the form

$$\text{Premises} \begin{cases} S_1 \\ S_2 \\ S_3 \\ \vdots \\ S_n \end{cases}$$

$$\overline{\text{Conclusion} \therefore S.}$$

Which means that the statements S_1, $S_2 \cdots\cdots S_n$ of the premises lead to the conclusion S. If the conclusion S is arrived at *logically* from the premises, then the argument is said to be *valid*. If the conclusion does not logically follow from the premises, we say the argument is *invalid*.

Consider the example of an argument :

S_1 : Some animals are clever

S_2 : Man is an animal

$\therefore$ S Man is clever

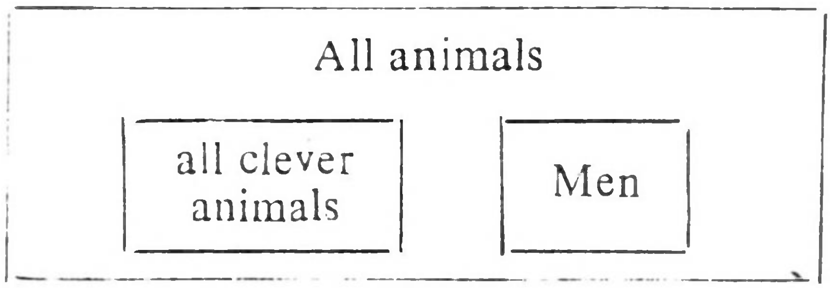
Here the statements S_1 and S_2 are both true but the conclusion S does not *logically follow* from the premises (S_1 and S_2). Therefore the *argument is invalid*.

Let us use Venn diagrams to test this argument.

Let A : The set of all animals

B : The set of all clever animals we know that A and B are related by the statement S_1 of the argument.

By S_1, B is a *proper* subsect of A. By S_2 it is clear that the set of all men is a subset of the set of all animals. Refer to the diagram.



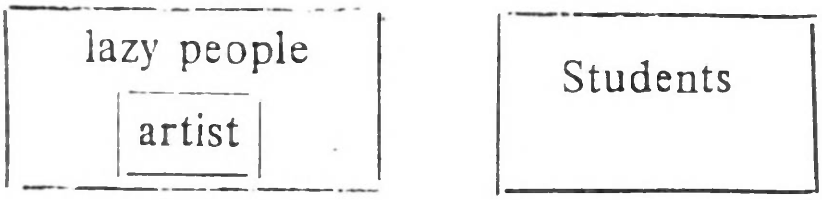
The conclusion of the argument is valid only when the "set of all men" *interesects* the "set of all clever animals." But the diagram shows that the set of Men is *disjoint* with "the set of all clever animals" ∴ the argument is tested by Venn diagrams and it is found to be *invalid.*

*E*x. 2 : Consider the argument

Premises
S_1 : No student
S_2 : John is an artist
S_3 All artists are lazy

Conclusion ∴ John is not a student

When we display in a diagram the relationships that occure among the sets available in this argument we get the following Venn diagram.



From the diagram it is clear that 'no artist is a student'. ∴ the conclusion of the argument is justified. ∴ Argument is valid

Exercises : Test the validity of the argument ;

1. S_1 : All lawyers are wealthy

   S_2 : Poets are temperamental

   S_3 : Raghava is a lawyer.

   S_4 : No temperamental person is wealthy

   ∴ Raghavan is not a poet.

2 S_1 : All students are lazy

   S_2 : No body who is wealthy is a student

   ∴ Lazy people are not wealthy.

3. S_1 : No college professor is wealthy

   S_3 : Some poets are wealthy

   Some poets are colloge professors

A comparison of set operations with number operations :

We recall the usefulness of Venn diagrams in visualising set relationships.

It will be instructive to verify the following properties of the binary operations 'Union' and 'Insterse-ction' of sets by Venn diagrams. In the following, '*addition*' of numbers is compared with '*Union*' of sets. '*Multiplication*' and '*Intersection*' are compared as operations.

Like number operations, 'Union' and 'Intersection' are both commutative and associative operavions, since

1. $A \cap B = B \cup A$ 1. $A \cap B = B \cap A$

2. $(A \cup B) \cup C = A \cup (B \cup C)$ 2. $A \cap B) \cap C = A \cap (B \cup C$

For universal set U and the null set ϕ we have.

3. $A \cup \phi = \phi \cup A = A$ 3^1 $U \cap A = A \cap U = A$

Compare the roles of ϕ and U here with those of corresponding numbers O and 1. w.r.t. 'addition' and multiplication respectively. Recall ;

$a + o = o + a = a \forall$ numbers a, $a \times 1 = 1 \times a, \forall a \in R$

4) $A \cup A = A \ \forall$ Sets A, 4^1) $A \cap A = A$ A sets A

We have no anologous property in number operations for, $a+a=a$ *need not be ture* except when $a=o$

$a \times a=a$ also is not generally true

This additional property in set operations is called 'idempotence'.

5) $\phi \cap A=A \cap \phi=\phi$ for all sets

This property of the set ϕ is comparable with that of the number O w.r.t. multiplication in our number-system.

$a \times o=o \times a$ for numbers a.

6) The operations of union and intersections distribute over each other, since

$$A \cup (B \cap C) = (A \cup B) \cap (A \cap C) \text{ and} \qquad (1)$$

$$A \cap (B \cap C) = (A \cap B) \cup (A \cap C) \qquad (2)$$

Whereas we have single distributive law for numbers. We know multiplication *alone* distributes over addition. The distributive law of number system compares with the *distributive* law (2) of sets ie.

$a(b+c) = ab+ac \; \forall$ numbers $a.b.c$

The reason for attempting to compare set operations with number operations is that all our new operations and new Mathematical systems find their inspiration or motivation from the properties of number operations. Sets form a 'mathematical system' and hence we can study sets as a 'Mathematical system in its own right.

Relations and Functions

Every one is familiar with the idea of relations as a form of 'connection' between two or more things. Relation is a concept which permeates everyday life. We commonly hear of such relations as 'the husband of', 'friend of' 'is to the left of', 'is taller than', 'is the same size as' 'is between'. A quick scrutiny of elementary Mathematics makes it evident to us that after all Mathematics is a study of a veriety of spatial and quantitative relationships. Elementary Geometry studies such relations as 'is parallel to', 'is collinear with', is 'congruent to', 'is similar to'. Where as Arithmetic is dominated by the relations like 'is equal to' 'is a factor of' 'is greater than' 'is a product of'.

Mathematicians study the concept of 'relation' and its properties in an abstract way.

In this section let us study 'relation' and the related concept of 'function' as Mathematical concepts.

Ordered pairs, Product sets, Graphs.

In the study of operations on sets, we were interested in construction of new sets and their properties. A more elaborate and useful way of constructing a new set from a given pair of sets is by the 'ordered pairs' of elements.

The concepts of 'ordered pair' and 'ordered triple' are not new to us. Recall that points in a plane are represented by ordered pairs of numbers. The ordered pair (5, 4) and the ordered pair (4, 5) represent two different points in the plane. Idea of 'ordered pairs' is basic to Analytical Geometry. 'Ordered pairs' play vital role in the construction of Mathematical systems.

Intuitively, an *orderd pair* consists of two elements a and b where a is called first component and b is called the second component. We denote this ordered pair as (a, b).

Orderd pairs (a, b) and (b, a) are different.

Example : The rational number $\frac{2}{3}$ is represented by the ordered pair (2, 3).

(3, 2) represents the rational number $\frac{3}{2}$.

The ordered pairs $(a, b)=(c, d)$ if

$$a=c \text{ and } b=d$$

We know that ordered pairs can be plotted graphically.

Ex. 1. Sketch the graph of all ordered pairs (x, y) x, $y \in Z$ (Integers) & satisfying the property $0 < x < 5$ and $y \leq 4$.

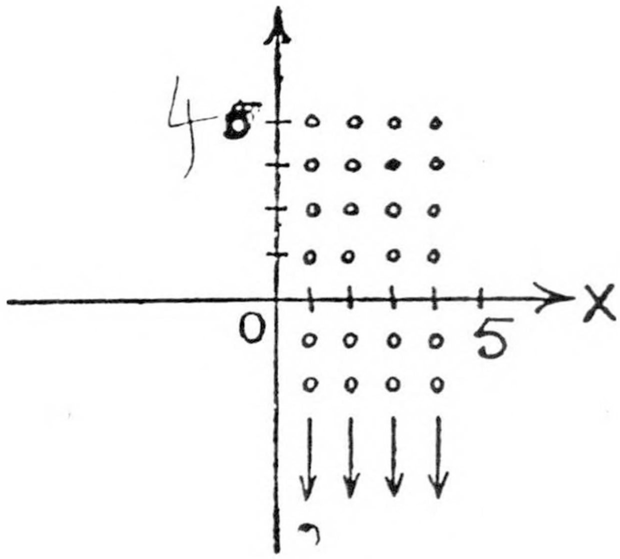


Fig 18

This graph shows all the orderd pairs (x, y) of integers subject to the condition that $0 < x < 5$ and $y \leq 4$

The graph is made up of infinitely many discrete points.

Product Sets : (Cartesian Products).

Ex : Let $A=\{1, 2, 3, 4,\}$

$B=\{8, 6\}$.

The cartesian product $A \times B$ of these two sets is the set of ordered pairs

$A \times B = \{(1,8), (2, 6), (3, 8), (3, 6), (1, 6), (2, 8), (4, 8) (4, 6).\}$

We read $A \times B$ as 'A cross B'

Ex. 2. The product set $A \times B$ of the sets $A=\{1, 4\}$, $B=\{3, 2.\}$ is

$A \times B=[(1, 3), (1, 2), (4, 3), (4, 2)]$

$A \times B$ is graphically shown as thes et of 4 (dots) discrete points.

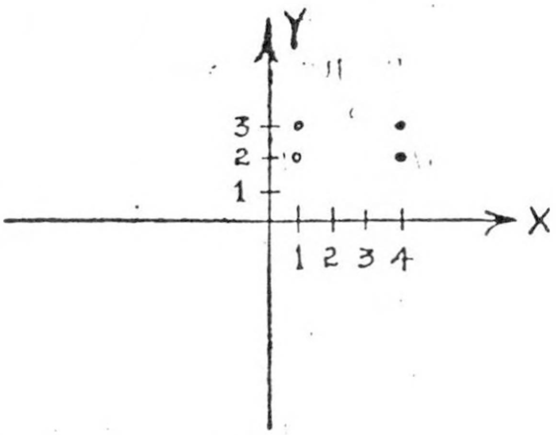


Fig 20

Let $U=[1, 2, 3]$

Draw the graph of $U \times U$. $U \times U$ is a discrete set of 9 (dots) points in the carteisan plane.

Ex. 3 :

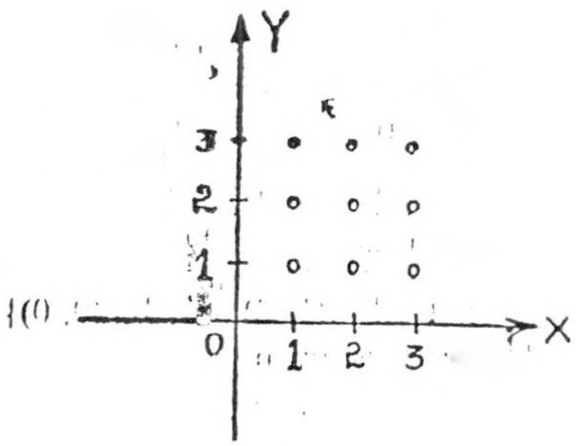


Fig 21

We define the cartesian product of any two sets **A** and **B** :

$$\boxed{A \times B = [(a, b)/a \in A \text{ ahd } b \in B].}$$

In general $A \times B \neq B \times B \times A$

for

$$B \times A = \{(b, a)/b \in B \text{ and } a \in A\},$$

Every ordered pair of numbers $(x \cdot y)$ belongs to the set $R \times R$, where R is the set of reals. To any ordered pair (x, y) of numbers, there corresponds a point in the cartesian plane. Therefore cartesian plane is nothing but the set of all ordered pairs of $R \times R$.

Any set of ordered pairs of numbers or cartesian product of any two sets is always a subset of $R \times R$.

Exercise :

1. If $A = [6, 8]$, $B = [5, 3]$ $C = [3, 4]$.

   Find (a) $A \times (B \cup C)$. (b) $A \times (B \cap C)$,

   (c) $(A \times B) \cup (A \times C)$

2. Graph $S = [(x, y)/2x - 3y = 6]$. $x, y \in Z$.

3. If the universal set

   $U = \{(5, 5), (4, 1), (1, 2), (8, 3), (0, 0)\}$

   Using U as given

   a) Find $[(x, y)/x > y]$

   b) Find $[(x, y)/(x + y) = 5]$

c) Find $[(x, y)/x \text{ is even}]$

d) Find $[(x, y)/x=4y]$

Relation as a set of Ordered Pairs :

Ex. 1 : Consider the sets of names of men and their respective native places.

A= [Rama, Rahim, Govinda, Srihari]

B= [Mysore, Gulbarga, Hassan, Melkote]

A and B are connected by the relation 'is a native of' this can be illustrated in a diagram.

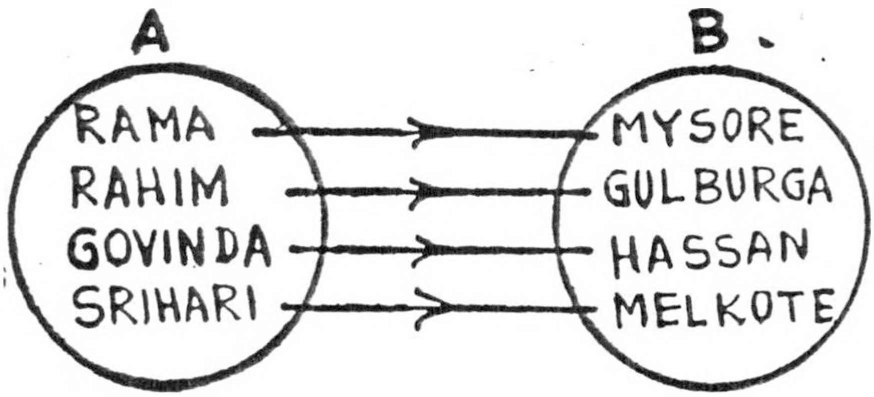


Fig 22

We can express the relation between the elements of A and B as a set R of ordered pairs.

R_1= (Rama, Mysore), (Rahim, Gulbarga) (Govinda, Hassan), (Srihari, Melkote)

This set of ordered pairs R completely expresses the *relation* in question.

Ex. 2 : Consider $A=[1, 2, 3]$
$B=[2, 4, 6]$

Let the relation be "is the double of". The related elements in these two sets are thls relation 1 & 2, 2 & 4, 3 & 6 is given by the set of ordered pairs

$$R_2 = [(1, 2), (2, 4), (3, 6)].$$

Ex. 3 : Write the set of all ordered pairs of elements
$A = [3, 9, 12,]$
$B = [4, 10, 2, 12]$

Using the condition 'is greater than'. Obviouslye
$R_3 = [(3, 2), (9, 4), (9, 2), (12, 4), (12, 10), (12, 2)]$

R the set of ordered pairs describes the relation in each example given above. We ∴ generalize

≪A relation from a set A to a set B is a set of ordered pairs≫.

Refer to the examples 1, 2, 3 in above. Relation R from a set A to set B in each of the above examples eonsists of the ordered pairs of *only* related elements. In each case if we compute $A \times B$ then we realize that R has only *some* elements of $A \times B$.

In Ex. 1 $A \times B$ has 16 elements in it.

$R_1 = [(R, M), (R. G), (G, H), (S, M)]$ is on ly a subset ie. $R_1 \subset A \times B$.

In Ex. 2 $A \times B = \begin{Bmatrix} (1, 2), (1, 4), (1, 6) (2, 2) (2, 4) \\ (2, 6) (3, 1) (3, 4) (3, 6) \end{Bmatrix}$

Where as

$R_2 = [(1, 2), (2, 4), (3, 6)]$
$\therefore\ R_2 \subset A \times B$

In Ex. 3 $A \times B$ consists of 12 ordered pairs,

$$A \times B = \begin{bmatrix} (3, 4) & (3, 10) & (3, 2) & (3, 12) & (9, 4) & (9, 10) \\ (9, 2) & (9, 12) & (12, 4) & (12, 10) & (12, 2) & (12, 12) \end{bmatrix}$$

where as the relation $R_3 = \begin{bmatrix} (3, 2) & (9, 4) & (9, 2) & (12, 4) \\ (12, 10) & (12, 2) & & \end{bmatrix}$

$R_3 \subset A \times B$

Example : Graph the relation $R = [(x, y)/x \in R, y \in R, x = 2)]$

The Vertical line consists of all point whose x coordinate $= 2$ and y coordinate varies over the set R

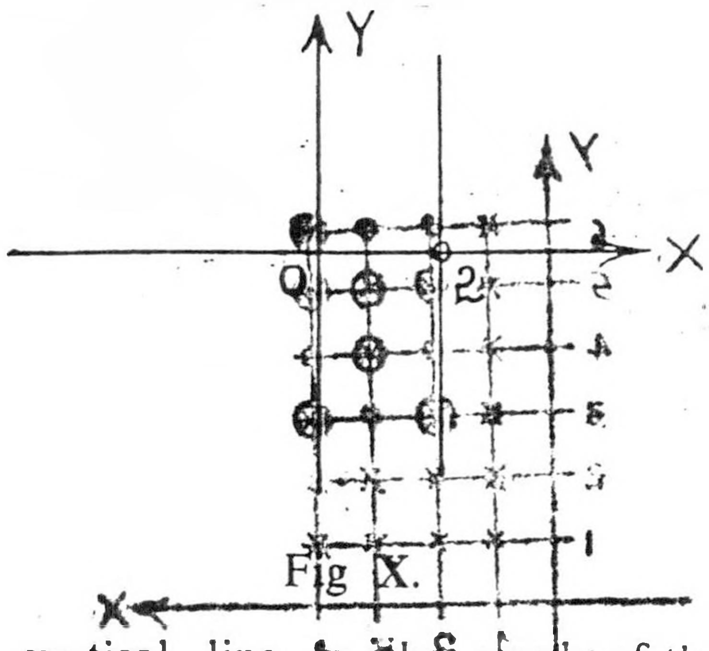


Fig X.

This vertical line is the graph of the relation $R = [(x, y)/x = 2]$

Note that relation R is a subset of $R \times R$

we have the generalization.

Any relation involves two sets, say A and B. hence it is called binary relation.

2) Relation is from set A to set B.

3) Relation is a set of ordered pairs of related elements.

4) Relation is a subset of the Cartesian product $A \times B$

5) All binary relations (it involues two sets A & B) determine subsets of $R \times R$

Graphs of relations :

1) Let $A = [2, 3, 4]$ $B = [3, 4. 5, 6]$

Sketch the graph of the relation from A to B given by $R = [(x, y)/ x \text{ divides } y]$

We know $R \subset A \times B$ and $R = [(2, 4)\ (2, 6)\ (3, 3)\ (3, 6)\ (4, 4)]$

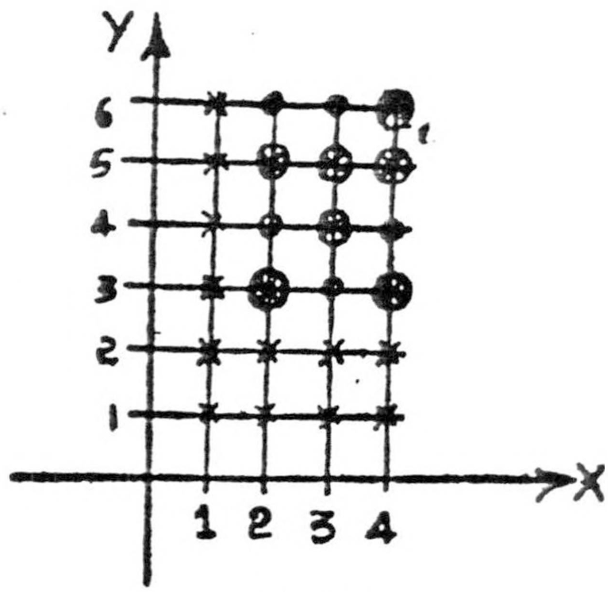


Fig 23.

The five (dots) points constitute the graph of the relation R.

The set of all points indicated by $(\times)$ along with the dots constitute the points in the set $A \times B$.

Let R be the relation in the set of real numbers defined by $y < x+1$. Graph of the relation R **gives** the set of all points in the shaded areas.

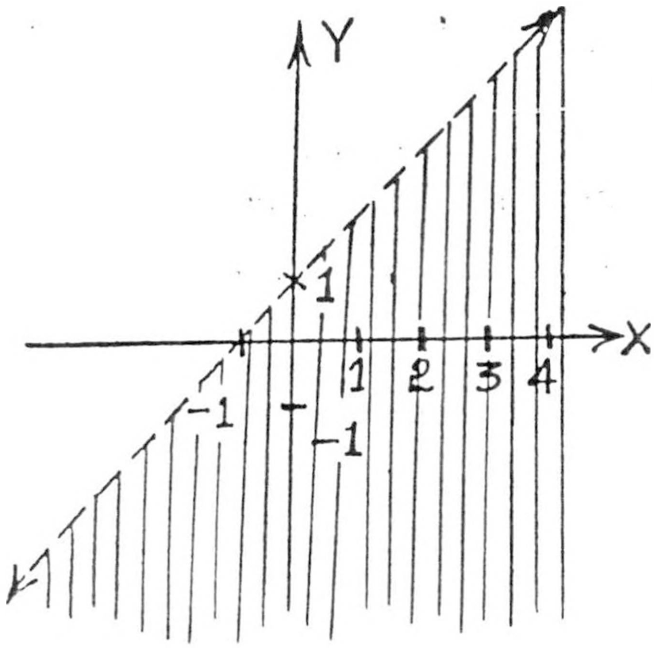


Fig 24.

The line $Y=x+1$ is shown by dotted line to show that it does not belong to the graph. The graph of the relation R. ie.

The shaded area consists of the point below the line $Y=x+1$

Domain Range and Inverse relation :

Let R be a relation from the set A to set B. R is a binary relation (Why ?. Recall that R is a set of ordered pairs and that $R \subset A \times B$.

The inverse relotion R^{-1} is also a set of ordered pairs.

$$R^{-1} = \{(b,a)/(a,b) \in R\},$$

The inverse relation R^{-1} Consists of those ordered pairs obtained by reversing the elements of (components of) ordered pairs of R.

Ex : If $A=[1, 2, 3]$ $B=[a, b]$ and $R=[(1, a)\ (2, b)\ (3, a)]$ is a relation from [to] then $R^{-1}=[(a,1), (b,2), (a,3)]$ is the inverse relation of R.

Exercise :

1. Write the cartesian product of [1, 2, 3] and [3, 4, 5]. Display it graphically.
2. Let $R=[x,y)/x\in R,\ y\in R,\ x^2+y^2=16]$. Sketch this relation in a Graph.
3. Let R be a relation in the set of N drfined by $2x+4y=15$. Describe this set in the set builder form. Find R^{-1} sketch the relrtion graphically.

Let R be a relation from A to B. The domain of the relation R is defined as the set of all first elements (first components) of the ordered pairs that belong to R.

We know that $R<A\times B$

D = Domain of $R=[a/(a, b)\in R]$.

The range E of the relation R is the set of all second components (second elements) of the ordered pairs that belong to R.

Range $R=E=[b/(a, b)\in R]$.

Example : Let relation

$$R=[(3,1),\ (4,5,)\ (6,7,)\ (10,11,)\ (8,13)]$$

The domain D of R has all the first components of the ordered pairs in R

$\therefore$ $D=[3, 4, 6. 10. 8]$

Range $E=[1, 5, 7, 11, 13]$

The inverse relation $R^{-1}=[(1, 3)\ (5, 4)\ (7, 6)\ (11, 10)\ (13, 8)]$

Assignment :

Sketch the following product set in a diagram by shading the appropriate once.

1) $[-3, 3] \times [-1, 2]$
2) $[-3, 1] \times [-2, 2]$
3) $[2, 3] \times [-3, 4]$

Suppose A B C have 3, 4 and 5 elements respectively, how many elements are there in

(i) $A \times B \times C$ (ii) $B \times A \times C$ (iii) $B \times C \times A$?

4) Let $A = B \cap C$ which, if any of the following is ture ?

1) $A \times A = (B \times B) \cap (C \times C)$
2) $A \times A = (B \times C) \cap (C \times B)$

5) Verify whether the relation

$(S \times W) \cap (S \times V) = S \times (W \cap V)$ holds by taking $S = [a, b]$ $W = [1, 2, 3, 4, 5]$ $V = [3, 5, 7, 9]$

6) If $S_1 = [(x, y) \ / \ x \in R, \ y \in R, \ y > -x + 1]$
and $S_2 = [(x, y) \ / \ x \in R, \ y \in R, \ x^2 + y^2 \leq 25]$

Graph the relations S_1 and S_2 the set $S_1 \cap S_2$.

7) If the Universal set $U = [1, 2, 3, 4, 5]$ list the order pairs of the following relations in $U \times U$.

$R_1 = [(x, y) / x \times y \text{ is even}]$
$R_2 = [(x, y) / x - y = 6]$
$R_3 = [(x, y) / x > 2 \text{ and } y = 3]$

8) Find the domain, range and the inverse relation for R_1 R_2 R_3 in problem7.

Functions :

The word function was first introduced by Descartes in 1637. He used this word to mean the positive integral power x^n of a variable x. Leibnitz associated this term with curves. Bernoulli (1667-1748) regarded a function as made up of a varible and constants. Euler (1707-1783) regarded a function as an equation involving variables and constants. The Eulerian concept of a function was used until Fourier studied this concept in connection with Trigonometric series.

Function concept is refined by the use of set theory. Function is a special kind of relation between two sets of elements.

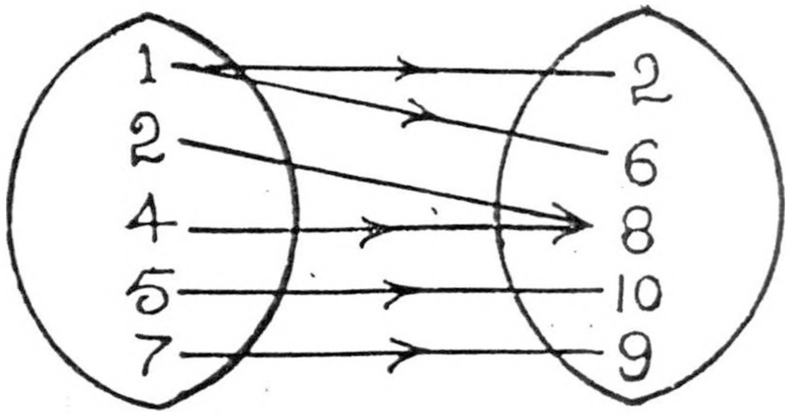


Fig 25.

Consider the two sets

Ex. 1 : A= [1, 2, 3, 4, 5, 6]

B=[2, 4, 6, 8, 10. 12, 1, 3, 5, 7, 9, 11]

Let us define a relation R for A to B the relation

R=[(1, 2) (1, 6) (2, 8) (4, 8) (5, 10) (7, 9)]

See fig 25

In this set R The are two ordered pairs [1,2] and [1, 6] in which the first component is thes ame number 1

Ex. 2 : Let R= [(1, 3) (2, 3) (3, 8) (4, 6) (5, 6) (6, 6)]

In this relation R, no first component of any ordered pair repeats more than once, eventhough there are three ordered pairs

(4, 6), (5, 6) (6, 6) with same second component 6.

Refer to fig 26

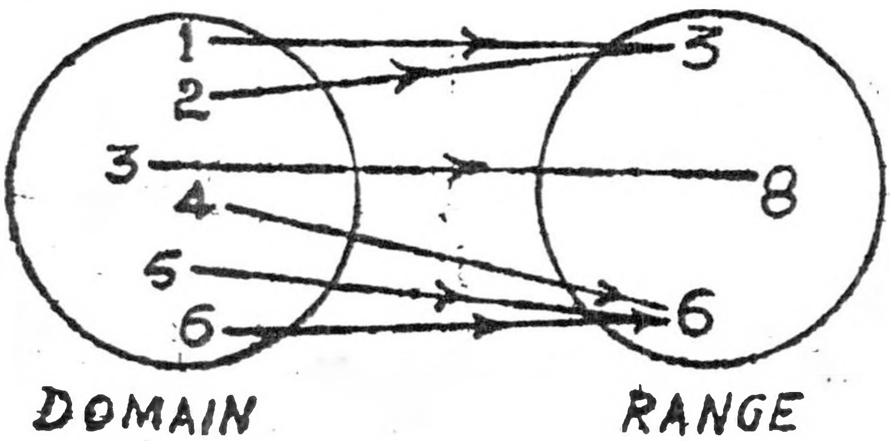


Fig 26.

Ex. 3 : Consider the relation R

R = [(1,1), (2,9), (2,3), (2,3), (4,3), (4,9), (5,7), (5,4)]

In this example of R

Domain of R = [1, 2, 4, 5]

Range of R = [1, 9, 3, 7, 4]

Note that the element 4 of the domain (first component) is related to 3 and 9 ie. we have (4,3), (4,9) in R similarly, we have

(2, 9), (2, 3) in R

and (5, 7), (5, 4) in R.

The elements 4, 2, 5 of the domain are *repeated* in R. This is clear by diagram (fig 27).

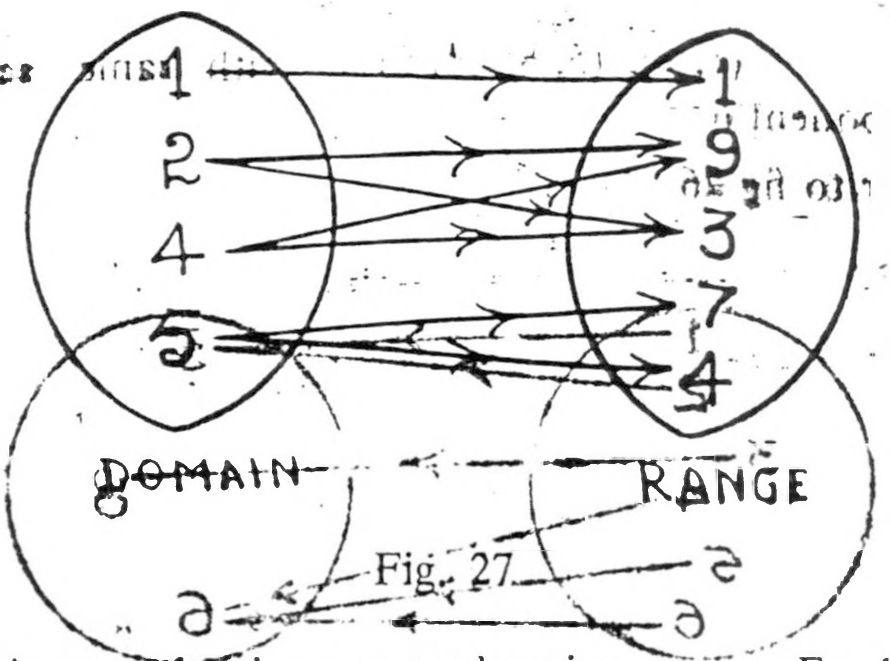


Fig. 27

Among the three examples given now. Ex. 1 and Ex. 3 have ordered pairs whose *first component repeats* Where as in Ex. 2 no two order pairs have the same number for their *first component.*

Observe that the sets R in each of these example. forms a *relation*. But *not* all these relations are qualified to be called *functions.*

Definition : Function F is a set of ordered pairs. The set of all *first components* of the ordered pairs forms the domain D of F. The set of all *second compnents* is called the range E of F or every $a \in D$ there exists $b \in E$ such that

$$(a, b) \in F$$

Each element of D appears *exactly once as the first element* in the ordered pairs of F. this condition rewritten becomes ;

$$\text{if } (a\ b) \in F \text{ and if } (a\ c) \in F \text{ then } b = c,$$

Let us apply this definition to the examples given in the beginning of this section Ex. 1 and Ex. 3 fail to satisfy the condition of the above definition. Hence they are *not functions.* Ex. 2 satisfies all the conditions of the definition ∴ this set represents a function.

Exercise :

Which of the relations below are functions ?

a. $R_1 = [(4,3), (4,15), (6,3), (8,9)]$

$R_2 = [(4,9), (6,11), (8,3)]$

$R_3 = [(4,1), (4,2), (4,3)]$,

b. Find those relations below wich are not functions.

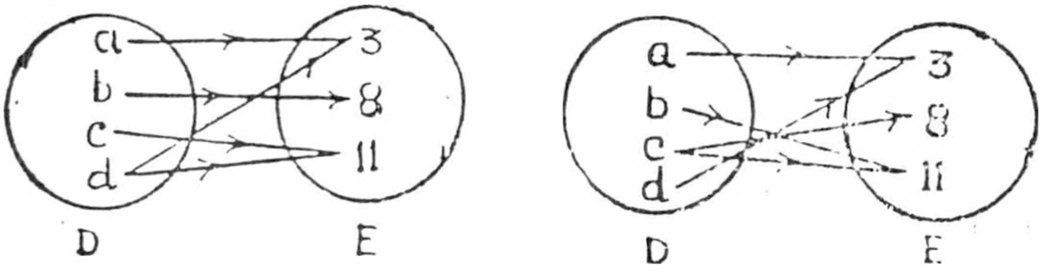


Fig. 28

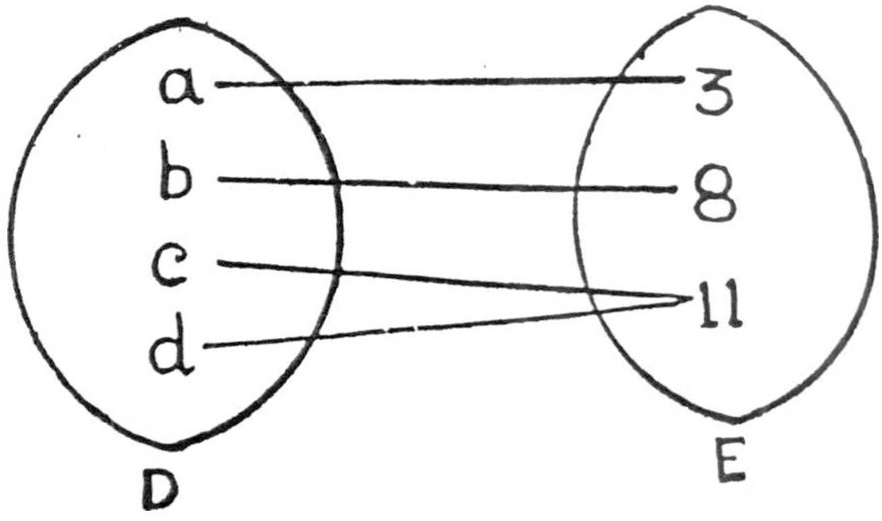


Fig 29

Recall the differeuce betwe:n a function and a relation.

If $R=[(x, y) \in N \times N ;\ y=2x]$

is a relation. Is it a function ? Yes it is a function because no two ordered pairs in R have the *same first* component. Every ordered pair in R is of the type $(n, 2n)$ where $n \in N$. If $R=[(x, y) \in N \times N : x > y$

Is this relation a function ?

It cannot be because R has ordered pairs like (5, 1), (5, 2), (5, 3), (5, 4) in it. This cannot happen in a function.

Function as rule, Correspondence-Notations

The function of f given by

$f=\{(x.y) \in N \times N :\ y=x+5\}$ can also be denoted by

$f : N \rightarrow N$ defined by $f(x)=x+5$

If we just write $f : x \rightarrow x+5$ we know how the function is defined but we donot know how the domain is chosen.

Recall that the function f associates (assigns) with each element x of its domain *exactly one* element y of its range. $f: x \rightarrow y$ or $f: x \rightarrow f(x)$ tells us that f Consists of all ordered pairs $(x, f(x).)$ $f(x)$ is called the image of x under f

Assignment ; 1) Given $S=[3, 4, 5]$ $T=[x, y, z]$

a) Is $A=(3, x), (4, y) (5, x)$ a relation in $S \times T$

b) Is A a function from S to T ?

2) List the elements of the relation R which is the inverse relation of

$A=[(0, -4), (1, 4), (2, 2), (3, 4), (4, 4)]$

a) Is A a function ? b) Is R a function ?

3) Graph the function $f: x \rightarrow y$ defined by $y=x^2$ with its domain the set $[x : -2 \leq x \leq 2 \quad x \in R]$

4) Given that the domain of the function f defined by $f(x)=\sqrt{x}$ is the set of reals ; $0 \leq x \leq 2$, *find* f^{-1} and graph it.

Mathematical Induction

By B.C. Basti

> **SYLLABUS :** *Introduction, Principle of Mathematical induction proving different tyupes of problems of equality, inequality and divisibility by the method of principle of mathematical induction.*

POINTS TO REMEMBER

1. Introduction : The word 'Induction' means the method of inferring a general statement from the validity of particular cases. We must be cautious here that in mathematics this kind of inference is not allowed, even when a huge list of particular cases have been verified. Mathematical induction is a principle by which one can conclude a statement for all positive integers, after providing certain related propositions.

Let us see an example to explain the need for our caution.

We know that the numbers 13, 23, 43, 53, 73 etc. are prime numbers. And the numbers 33, 63, 93 etc. are composite. From these particular cases we formulate a general statement. A number of the form $10n + 3$ is prime. If n is not divisible by 3. Is this a true statement ?

Even if there are hundreds of particular cases where this is known to be true, we can not conclude that this general statement is true.

If fact this statement is not true in general when the number 143 is of the form $10n + 3$ with $n = 14$, but it is not a prime.

We say that 143 is a counter example to the statement.

Even when we do not have a counter example, we can not conclude that a general statement is true simply because it has been found to be true in all its particular cases that have been verified. We can at the best say that it is a reasonable conjecture.

2. Preparation for Induction : A notation : consider the statements of the form

(*i*) n is divisible by 3.

(*ii*) The number $10n + 3$ is prime.

(*iii*) $2^n > n$.

All these are statements concerning the natural numbers $n = 1, 2, 3, \dots$. We use the notations $P(n)$ or $P_1(n)$ or $P_2(n)$ etc. to denote such statements. When we give values for $n = 1, 2, \dots$. We obtain particular statements. If in the statement $P(n)$, we substitute $n = 3$, the particular statement so obtained, is denoted by $P(3)$.

3. Peano's Axioms : Let N be the set of natural numbers. Then the properties satisfied by N, known as the Peano's axioms are :

Axiom 1. $1 \in N$, *i.e.*, 1 is a natural number.

Axiom 2. For each $n \in N$; there exists a unique natural number $n^* \in N$ called the successor of n.

Axiom 3. $1 \neq n^*, \forall n \in N$, *i.e.*, 1 is not the successor of any natural number.

Axiom 4. $\forall m, n \in N, m^* = n^* \Rightarrow m = n$, *i.e.*, each natural number, if it is a successor, it is the successor of a unique natural number.

Axiom 5. Principle of finite induction (P.F.I.). If $S \subset N$ be such that

(*i*) $1 \in S$ and

(*ii*) $m \in S \Rightarrow m^* \in S$, then $S = N$.

Note : Axiom 1 assures us that N is not a null set, *i.e.*, $N \neq \phi$. Axiom S is commonly known as the induction axiom or principle of mathematical induction.

4. Mathematical Induction : This principle of mathematical induction.

The principle of mathematical induction states :

Let $P(n)$ be a statement involving the natural number n.

(*a*) If $P(1)$ is true and

(*b*) If $P(k + 1)$ is true whenever $P(k)$ is true.

Then, we conclude that $P(n)$ is true for $\forall n \in N$.

5. Working Rule : In order to prove that a statement P(n) is true for all natural numbers, we should verify

Step 1. P(1) is true.

Step 2. Verify that P(k + 1) is true, whenever P(k) is true.

The method of induction is a powerful tool for proving theorems in mathematics, first we prove the result for $n = 1$. After that assuming the result to be true for $n = K$, we prove it to be true for $n = K + 1$.

It should be kept in mind that both parts are absolutely necessary for the proof.

DEFINITION AND IMPORTANT RESULT ON CHAPTER 2

1. **Mathematical Induction.** The principle of mathematical induction states :

Let P(n) be a statement involving the natural number n.

(a) If P(1) is true and
(b) If P(k + 1) is true whenever P(k) is true,
then, we conclude that P(n) is true for $\forall\, n \in N$.

2. **Working Rule.** In order to prove that a statement P(n) is true for all natural numbers, we should verify

Step I. P(1) is true.

Step II. Verify that P(k + 1) is true, whenever P(k) is true.

The method of induction is a powerful tool for proving theorems in mathematics.

TEXT BOOK EXERCISE 2.1
TYPE—I
(*SOLVED EXAMPLES*)

Example 1. If P(n) is the statement
"$n(n + 1)(n + 2)$ is divisible by 12"
Prove that P(3) and P(4) are true but P(5) is not true. [*T.B.Q. 1*]

Sol. P(n) is $n(n + 1)(n + 2)$ is divisible by 12.
P(3) is $3(3 + 1)(3 + 2)$ is divisible by 12.
i.e., 60 is divisible by 12
It is true.
P(4) is $4(4 + 1)(4 + 2)$ is divisible by 12
i.e., 120 is divisible by 12
It is true.
P(5) is $5(4 + 1)(5 + 2)$ is divisible by 12
i.e., 210 is divisible by 12
It is not true.

PRACTICE EXERCISE 2.1 (*i*)

1. If P(n) is the statement "$n(n + 1)(n + 2)$ is a multiple of 6" is it true ? [*A.I.C.B.S.E. 1978 ; D.B. 1984*]
2. If P(n) is the statement "$n(n + 1)(2n + 1)$ is divisible by 6" is it true ? [*C.B.S.E. 1980*]
3. If P(n) is the statement "$n^3 + 2$ is a multiple of 5", then show that P(1) is not true.
4. Prove that $n(n + 1)(n + 5)$ is divisible by 6 for all $n \in N$.
5. If P(n) is the statement "$n(n + 1)$ is even" then what is P(4) ?

TEXT BOOK EXERCISE 2.1
TYPE—II
(*SOLVED EXAMPLES*)

Example 1. If P(n) is the statement "$n^2 > 100$". Prove that whenever P(r) is true, P(r + 1) is also true. [*T.B.Q. 2*]

Sol. P(n) : $n^2 > 100$
$\therefore$ P(n) : $r^2 > 100$
Now P(r + 1) : $(r + 1)^2 > 100$
We know that
$$r^2 > 100 \qquad \text{[From P}(r)\text{]}$$
Adding both sides $2r + 1$
$$r^2 + 2r + 1 > 100 + 2r + 1$$
$$(r + 1)^2 > (100 + 2r + 1) \qquad ...(i)$$
Also $100 + 2r + 1 > 100$ as $2r + 1$ is positive ...(ii)
$$\therefore \quad (r + 1)^2 > 100 \qquad \text{[From } (i) \text{ and } (ii)\text{]}$$
Hence P(r + 1) is true.

Example 2. If P(n) is the statement "$2^n \geq 3n$" and if P(r) is true, prove that P(r + 1) is true. [*T.B.Q. 3*]

Sol. P(n) ; $2^n \geq 3n$
$\because$ P(r) is true. [Given]
$$2^r \geq 3r$$
$$P(r + 1) ;\ 2^{r+1} \geq 3(r + 1)$$
Since $2^r \geq 3r$ [From P(r)]
Multiplying both sides by 2, we get
$$2^{r+1} \geq 6 \qquad ...(i)$$
$$r > 1$$
$$\therefore \quad 3r > 3$$
$$\Rightarrow \quad (3r + 3r) > (3 + 3r)$$
$$\Rightarrow \quad 6r > (3r + 3)$$
$$\Rightarrow \quad 6r > 3(r + 1) \qquad ...(ii)$$

From (*i*) and (*ii*) $2^{r+1} > 3(r+1)$

Hence $P(r+1)$ is true.

Example 3. If $P(n)$ is the statement "$2^{3n} - 1$ is an integral multiple of 7", prove that $P(1), P(2)$ and $P(3)$ are true. [*T.B.Q. 4*]

Sol. $P(n)$; $2^{3n} - 1$ is an integral multiple of 7.

$P(1)$; $2^{3\times 1} - 1 = 7$, is an integral multiple of 7. [It is true]

$P(2)$; $2^{3\times 2} - 1 = 63$, is an integral multiple of 7. [It is true]

$P(3)$; $2^{3\times 3} - 1 = 511$, is an integral multiple of 7. [It is true.]

PRACTICE EXERCISE 2.1 (*ii*)

1. Use the principle of mathematical induction to prove the following statements for all $n \in N$.
   (*i*) $3^{2n} - 1$ is divisible by 8.
   (*ii*) $10^{2n-1} + 1$ is divisible by 11 for $n \in N$.
   (*iii*) $3^n > 2^n$, for all $n \in N$
   (*iv*) $2^n > n$.
   (*v*) $7^{2n} + (2^{3n-3})\, 3^{n-1}$ is divisible by 25, $n \in N$
2. Prove by method of induction the following statements for all $n \in N$.
   (*i*) For each natural number n, $6^{n+2} + 7^{2n+1}$ is divisible by 43.
   (*ii*) Prove that $n^2 > 2n \;\forall\; n \geq 3$, by using the principle of mathematical induction.
   (*iii*) Prove by method of induction that $7^{2n} - 1$ is divisible by 48, where n is a positive integer.
3. Use the principle of mathematical induction to prove each of the following statements :
   (*i*) $10^n + 3.4^{n+2} + 5$ divisible by 9.
   (*ii*) $5^{2n} - 1$ is divisible by 24 for every natural number n.
   (*iii*) $n^4 < 10^n$, where n is positive integer.
4. Use the principle of mathematical induction to prove each of the following statements :
   (*i*) $2^n < 3^n$, $n \in N$.
   (*ii*) $(1 + x)^n > 1 + nx$ for $n \geq 2$ and $x > -1$.
   (*iii*) Let $P(n)$ be the statement "$3^n > n$". Is $P(1)$ true ? What is $P(n + 1)$?

TEXT BOOK EXERCISE 2.1
TYPE—III
(*SOLVED EXAMPLES*)

Example 1. If $P(n)$ is the statement "$2^{3n} - 1$ is an integral multiple of 7", and if $P(r)$ is true, prove that $P(r + 1)$ is true. [*T.B.Q. 5*]

Sol. $P(n)$: $2^{3n} - 1$ is an integral multiple of 7

$\because$ $P(r)$ is true. [Given]

$\therefore$ "$2^{3r} - 1$ is an integral multiple of 7".

$P(r + 1)$: $2^{3(r+1)} - 1$ is an integral multiple of 7.

Consider
$$\begin{aligned} 2^{3(r+1)} - 1 &= 2^{3r+3} - 1 \\ &= 2^{3r}\,.\,2^3 - 1 \\ &= 8(2^{3r}) - 1 \\ &= 8(2^{3r}) - 8 + 7 \\ &= 8(2^{3r} - 1) + 7. \end{aligned}$$

$\because$ $2^{3r} - 1$ is an integral multiple of 7, so $8(2^{3r} - 1)$ is an integral multiple of 7. Also 7 is a multiple of 7. Since sum of the two number which are integral multiple of 7 is also an integral multiple of 7.

So $2^{3(r+1)} - 1$ is an integral multiple of 7.

Hence $P(r + 1)$ is true.

Example 2. If $P(n)$ is the statement that sum of the first n natural numbers is divisible by $(n + 1)$, prove that if $P(n)$ is true, then $P(r + 2)$ is true. [*T.B.Q. 6*]

Sol. $P(n)$: $1 + 2 + 3 + 4 + \ldots + n$ is divisible by $(n + 1)$.

$\Rightarrow$ $\dfrac{n(n+1)}{2}$ is divisible by $(n + 1)$

$\because$ $P(r)$ is true.

$\therefore$ $1 + 2 + 3 + \ldots + r$ is divisible by $(r + 1)$

i.e., $\dfrac{r(r+1)}{2}$ is divisible by $(r + 1)$.

Now $P(r + 2)$: $1 + 2 + 3 + \ldots + (r + 2)$ is divisible by $(r + 3)$

Consider
$$\begin{aligned} & 1 + 2 + 3 \ldots + r + (r + 1) + (r + 2) \\ &= (1 + 2 + 3 + \ldots + r) + (2r + 3) \\ &= \frac{r(r+1)}{2} + (2r + 3) \quad \text{[Using } P(r)] \\ &= \frac{r^2 + r + 4r + 6}{2} \\ &= \frac{(r+2)(r+3)}{2} \end{aligned}$$

which is divisible by $(r + 3)$

Hence $P(r + 2)$ is true.

PRACTICE EXERCISE 2.1 (*ii*)

1. Use the principle of mathematical induction to prove that the following statements for all $n \in N$.
   (*i*) $1 + 2 + 3 + \ldots + n = \dfrac{n(n+1)}{2}$

(*ii*) $1^2 + 2^2 + 3^2 + \dots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

(*iii*) $1^3 + 2^3 + 3^3 + \dots + n^3 = \left[\dfrac{n(n+1)}{2}\right]^2$.

2. Show that if the statement

$$P(n) : 2 + 4 + 6 + \dots + 2n = n(n+1) + 2$$

is true for $n = K$, then it is true for $n = K + 1$ can we conclude that $P(n)$ is true for every natural number n ?

3. Use the principle of mathematical induction to prove that the following statements for all $n \in N$.

(*i*) Prove that : $1^2 + 3^2 + 5^2 + \dots + (2n-1)^2 = \dfrac{n(2n-1)(2n+1)}{3}$, $n \in N$

(*ii*) Prove that : $1 + 4 + 7 + \dots + (3n-2) = \dfrac{n(3n-1)}{2}$, $n \in N$

(*iii*) $1 + 3 + 5 + \dots + (2n-1) = n^2 \; \forall \; n \in N$.

4. Use the principle of mathematical induction to prove that the following statements for all $n \in N$.

(*i*) $a + (a+d) + (a+2d) + \dots + a + (n-1)d = \dfrac{n}{2}[2a + (n-1)d]$

(*ii*) $1.2 + 2.3 + 3.4 + \dots n(n+1) = \dfrac{n(n+1)(n+2)}{2}$

(*iii*) $2 + 2^2 + 2^3 + \dots + 2^n = 2(2^n - 1)$

(*iv*) $2 + 6 + 10 + \dots + (4n-2) = 2n^2$.

TEXT BOOK EXERCISE 2.1
TYPE—IV
(*SOLVED EXAMPLES*)

Example 1. Given an example of a statement $P(n)$ such that it is true for all n. [*T.B.Q. 7*]

Sol. Consider

$$P(n) : 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

i.e., sum of first n natural number is $\dfrac{n(n+1)}{2}$

$P(1)$: L.H.S. $= 1$

R.H.S. $= \dfrac{1(1+1)}{2} = 1 \Rightarrow P(1)$ is true

$P(2)$: L.H.S. $= 1 + 2 = 3$

R.H.S. $= \dfrac{2(2+1)}{2} = 3 \Rightarrow P(2)$ is true

$P(3)$: L.H.S. $= 1 + 2 + 3 = 6$

R.H.S. $= \dfrac{3(3+1)}{2} = 6 \Rightarrow P(3)$ is true

$P(4)$: L.H.S. $= 1 + 2 + 3 + 4 = 10$

R.H.S. $= \dfrac{4(4+1)}{2} = 10$

Similarly for any value of $n \in N$, $P(n)$ is true.

Example 2. Given an example of a statement $P(n)$ such that $P(3)$ is true, but $P(4)$ is not true. [*T.B.Q. 8*]

Sol. Consider $P(n)$: "$3n^2 + n$ is divisible by 3"

$P(3) : 3 \times 3^2 + 3 = 3 \times 9 + 3$
$= 27 + 3 = 30$ is divisible by 3

$\therefore$ It is true.

Again $P(4) : 3 \times (4)^2 + 4 = 48 + 4 = 52$ is divisible by 3. It is not true.

PRACTICE EXERCISE 2.1 (*iv*)

1. Verify that if $(2.1 + 1) + (2.2 + 1) + \dots (2n + 1) = n^2 + 2n + 11$, $n \in N$ is true for $n = m$ then it is also true for $n = m + 1$ can we conclude that it is true for every $n \in N$?
2. Apply the principal of mathematical induction to prove that for all $n \in N$
$$1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n-1)}{6}$$
3. If $P(n)$ is the statement "$n(n+1)$ is even" then what is $P(7)$?
4. If $P(n)$ is the statement "$n^3 + n$ is divisible by 3"
(*i*) Is the statement $P(5)$ is true ?
(*ii*) Is the statement $P(6)$ is true ? [*Imp.*]
5. If $P(n)$ is the statement that the sum of first n natural numbers is divisible by $(n+1)$, prove that if $P(r)$ is true, then $P(r+2)$ is true.
6. If $P(n)$ be the statement "$C(n, r) \le n$, for all $1 \le r \le n$" is $P(3)$ is true.

TEXT BOOK EXERCISE 2.2
TYPE—I
(*SOLVED EXAMPLES*)

Example 1. Prove that the following by the principle of induction : the sum of the first n natural number is $\dfrac{n(n+1)}{2}$ [*T.B.Q. 1*]

Sol. Let the given statement is $P(n)$

i.e., $P(n) : 1 + 2 + 3 + \dots + n = \dfrac{1}{2}n(n+1)$

when $n = 1$, L.H.S. $= 1$

$$\text{R.H.S.} = \frac{1}{2}(1+1) = \frac{1}{2}\times 2 = 1$$

$$\text{L.H.S.} = \text{R.H.S.} \Rightarrow \text{P(1) is true.}$$

Now assume that P(k) is true

$$P(k) : 1+2+3+\ldots+k = \frac{k(k+1)}{2}$$

$$P(k+1) : 1+2+3+\ldots+k+(k+1) = \frac{1}{2}(k+1)(k+2)$$

We wish to prove P(k + 1) is true whenever P(k) is true. Let us examine its L.H.S.

$$\text{L.H.S.} = 1+2+3+\ldots+k+(k+1)$$

$$= \frac{k(k+1)}{2} + (k+1), \text{ since P}(k) \text{ is true}$$

$$= (k+1)\left(\frac{1}{2}k+1\right) = \frac{1}{2}(k+1)(k+2) = \text{R.H.S.}$$

Thus P(k + 1) is true whenever P(k) is true.

By the principle of mathematical induction, P(n) is true for $n \in$ N.

☞ **Example 2. Prove the following by the principle of mathematical induction**

$$\mathbf{1+4+7+\ldots+(3n-2) = \frac{n(3n-1)}{2}}.$$

[T.B.Q. 3]

Sol. Let the given statement be P(n)

$$\text{Now} \quad P(n) : 1+4+7+\ldots+(3n-2) = \frac{n(3n-1)}{2}$$

when n = 1, $\quad$ L.H.S. = 1

$$\text{R.H.S.} = \frac{1(3\times 1-1)}{2} = 1\times\frac{2}{2} = 1$$

$$\text{L.H.S.} = \text{R.H.S.} \Rightarrow \text{P(1) is true.}$$

Now assume that P(k) is true

$$P(k) : 1+4+7+\ldots+(3k-2) = \frac{k(3k-1)}{2} \quad \ldots(i)$$

Now we shall show that P(k + 1) is true

$$1+4+7+\ldots+(3k-2)+[3(k+1)-2] = (k+1)\left[\frac{3k+3-1}{2}\right]$$

$$1+4+7+\ldots+(3k-2)+[3(k+1)-2]$$
$$= \frac{k(3k-1)}{2} + [3(k+1)-2] \quad \text{[From } (i)]$$

$$= \frac{k(3k-1)}{2} + \frac{(3k+1)}{1}$$

$$= \frac{3k^2-k+6k+2}{2} = \frac{3k^2+5k+2}{2}$$

$$= \frac{3k^2+3k+2k+2}{2} = \frac{3k(k+1)+2(k+1)}{2}$$

$$= \frac{(3k+2)(k+1)}{2}$$

$$= \frac{(k+1)[3(k+1)-1]}{2}$$

Clearly, P(k + 1) is true.

Hence P(n) is true for all positive integers.

Example 3. Prove that the following by principle of induction $4+8+12+\ldots+4n = 2n(n+1)$.

[T.B.Q. 5]

Sol. Let P(n) : $4+8+12+\ldots+4n = 2n(n+1)$

For $n = 1$ $\quad$ L.H.S. = 4,

R.H.S. = $2\times 1(1+1) = 2\times 2 = 4$

L.H.S. = R.H.S. $\Rightarrow$ P(1) is true.

Let P(k) be true, then

P(k) : $4+8+12+\ldots+4k = 2k(k+1)$

Now P(k + 1) : $4+8+12+\ldots+4k+4(k+1)$

$= 2(k+1)(k+2)$

L.H.S. of P(k + 1) = $4+8+12+\ldots+4k+4(k+1)$

$= 2k(k+1)+4(k+1)$

[Using P(k)]

$= 2(k+1)\times(k+2) = $ R.H.S.

$\therefore$ P(k + 1) is true.

Hence by principle of mathematics induction P(n) is true $\forall\, n \in$ N.

PRACTICE EXERCISE 2.2 (*i*)

1. Let P(n) be the statement "$n^2 + n$ is even".
   Then (*a*) P(1) is the statement "2 is even". It is true.
   (*b*) If P(r) is true for some r, then to prove that P(r + 1) is true.
2. Using the principle of mathematical induction prove that each of the following statements for every natural number n.
   (*i*) $1+3+5+7+\ldots+(2n-1) = n^2$
   (*ii*) $1+2+3+\ldots+n = \frac{1}{2}n(n+1)$
   (*iii*) $2+2^2+2^3+\ldots+2^n = 2(2^n-1)$
   (*iv*) $1.3+3.5+5.7+\ldots+(2n-1)(2n+1) = \frac{n(4n^2+6n-1)}{3}$ *[Imp.]*

[N.M.O.C. 1993 (Set B)] *[A.I.S.S.E. 1985]*

(*v*) $3.6 + 6.9 + 9.12 + ... + 3n(3n + 3) = 3n(n + 1)(n + 2)$

(*vi*) $x + 4x + 7x + 7x + ... + (3n - 2)x = \frac{1}{2}n(3n - 1)x$ [*A.I.S.S.E. 1980*]

(*vii*) $(2.1 + 1) + (2.2 + 1) + (2.3 + 1) ... + (2n + 1) = (n + 1)^2 - 1.$

3. Using the principle of mathematical induction prove that for every natural number n
$2.3^2 + 2^2.3^3 + 2^3.3^4 + 2^4.3^5 + ... + 2^n . 3^{n+1} = \frac{18}{5}(6^n - 1).$ [*Imp.*]

TEXT BOOK EXERCISE 2.2
TYPE—II
(*SOLVED EXAMPLES*)

Example 1. Prove the following by the principle of induction : $n(n + 1)(n + 2)$ is divisible by 6, where n is a natural number. [*T.B.Q. 2*]
[*D.S.S.E. 1984, 1980*] [*A.I.S.S.E. 1978*]

Sol. Let the given statement be $P(n)$

$P(n) : n(n + 1)(2n + 1)$ is divisible by 6

Step 1. $P(1) : 1(1 + 1)(2 \times 1 + 1) = 1 \times 2 \times 3 = 6$ which is divisible by 6.

$\therefore$ $P(1)$ is true.

Step 2. Let $P(k)$ be true.

$\therefore$ $P(k) : k(k + 1)(2k + 1)$ is divisible by 6

$$\begin{aligned} P(k + 1) : \ & (k + 1)(k + 1 + 1)(2k + 2 + 1) \\ &= (k + 1)(k + 2)(2k + 1 + 2) \\ &= (k + 1)(k + 2)(2k + 1) + 2(k + 1)(k + 2) \\ &= (k + 1)(2k + 1)k + 2(k + 1)(2k + 1) + (k + 1)(k + 2)2 \\ &= k(k + 1)(2k + 1) + 2[(k + 1)(2k + 1) + (k + 1)(k + 2)] \\ &= k(k + 1)(2k + 1) + 2[(k + 1)(2k + 1 + k + 2)] \\ &= k(k + 1)(2k + 1) + 2[(k + 1)(3k + 3)] \\ &= k(k + 1)(2k + 1) + 6(k + 1)(k + 1) \\ &= k(k + 1)(2k + 1) + 6(k + 1)^2. \end{aligned}$$

From (*i*), $k(k + 1)(2k + 1)$ is divisible by 6 and $6(k + 1)^2$ is divisible by 6 because 6 is one of its factor.

Hence $P(n)$ is divisible by 6 for all natural number n.

Example 2. Prove the following by the principle of mathematical induction

If 3^{2n}, where n is a natural number, is divided by 8, the remainder is always 1. [*T.B.Q. 4*]

Using principle of mathematical induction prove that $3^{2n} - 1$ divisible by 8 for every natural number n. [*Annual Exam. 1994*]

Sol. Let the given statement be $P(n)$

$$P(n) : 3^{2n} = M(8) + 1$$

or $$P(n) : 3^{2n} - 1 = M(8)$$

Step 1. When $n = 1$, then

$$\text{L.H.S.} = 3^{2\times1} - 1 = 3^2 - 1 = 9 - 1 = 8 = M(8)$$

$\Rightarrow$ $P(1)$ is true.

Step 2. Let $P(k)$ be true

$$P(k) : 3^{2k} - 1 = \text{Multiple of } 8 \quad ...(i)$$

Now it is to be proved that $P(k + 1)$ is true.

$$P(k + 1) : 3^{2(k+1)} - 1 = M(8)$$

$$\begin{aligned} \text{L.H.S.} &= 3^{2(k+1)} - 1 \\ &= 3^{2k} . 3^2 - 1 \\ &= (9)3^{2k} - 1 \\ &= 9(3^{2k}) - 9 + 8 \\ &= 9(3^{2k} - 1) + 8 \\ &= M(8) + 8 \quad [\text{Using } (i) \\ &= M(8) + M(8) = M(8) \quad ...(ii) \end{aligned}$$

Hence, $P(k + 1)$ is true.

Combining (*i*) and (*ii*) by P.M.I. $P(n)$ is true for every natural number n.

Prove the following by the principle of mathematical induction.

Example 3. The sum $S_n = n^3 + 3n^2 + 5n + 3$ is divisible by 3 for any positive integer n. [*T.B.Q. 8*]

Sol. Let the given statement be $P(n)$

$$P(n) : S_n = n^3 + 3n^2 + 5n + 3 = M(3)$$

Step 1. When $n = 1$, then

$$n^3 + 3n^2 + 5n + 3 = (1)^3 + 3(1)^2 + 5(1) + 3 = 12 = M(3)$$

$\therefore$ $P(1)$ is true.

Step 2. Let $P(k)$ be true.

i.e., $$P(k) : k^3 + 3k^2 + 5k + 3 = M(3) \quad ...(i)$$

Now it is to be prove that $P(k + 1)$ is true.

$$\begin{aligned} P(k + 1) : \ & (k + 1)^3 + 3(k + 1)^2 + 5(k + 1) + 3 \\ &= (k + 1)[(k + 1)^2 + 5] + 3[(k + 1)^2 + 1] \\ &= (k + 1)[(k^2 + 2k + 1 + 5] + 3[k^2 + 2k + 1 + 1] \\ &= (k + 1)(k^2 + 2k + 6) + 3[(k + 1)^2 + 1] \\ &= (k^3 + 3k^2 + 8k + 6) + 3[(k + 1)^2 + 1] \\ &= (k^3 + 3k^2 + 5k + 3) + (3k + 3) + 3[(k + 1)^2 + 1] \\ &= M(3) + 3(k + 1) + 3[(k + 1)^2 + 1] \quad [\text{Using } (i) \\ &= M(3) + M(3) + M(3) = M(3) \end{aligned}$$

$\therefore$ $P(k + 1)$ is true. ...(*ii*)

Combining (*i*) and (*ii*) by principle of mathematical induction, we get $P(n)$ is true for all positive integers.

PRACTICE EXERCISE 2.2 (*ii*)

Prove the following by principle of mathematical induction :

1. Prove that for $n \in N$ $10^n + 3.4^{n+2} + 5$ is 5 is divisible by 9.
2. Prove the following by principle of mathematical induction.

   Prove that $3^{2n+2} - 8n - 9$ is divisible by 64 for every natural number n.
3. Use the principle of mathematical induction to prove that $n(n + 1)(n + 2)$ is a multiple of 6 for all natural number n.
4. Use the principle of mathematical induction to prove that $3^{2n} + 1$ is divisible by 8 for all $n \in N$. [*N.M.O.C. 1994 (Set B)*] [*Imp.*]
5. Use principle of mathematical induction prove that $10^{2n-1} + 1$ is divisible by 11 for all $n \in N$.
6. Use principle of mathematical induction,
   (*i*) Prove that $8^n - 3^n$ is divisible by 5 for all $n \in N$.
   (*ii*) Prove that $4^n - 3n - 1$ is multiple of 9 for all $n \in N$.
   (*iii*) Prove that $9^n - 8n - 1$ is a multiple of 64 for all $n \in N$.
   (*iv*) Prove that $n^3 + (n + 1)^3 + (n + 2)^3$ is divisible by 9 for every natural number.
   (*v*) Prove that $n(n + 1)(n + 5)$ is divisible by 6 for all $n \in N$.

TEXT-BOOK EXERCISE 2.2
TYPE—III
(*SOLVED EXAMPLES*)

☞ **Example 1. If x and y are any two distinct integers, then $x^n - y^n$ is an integral multiple of $(x - y)$.** [*T.B.Q. 6*]

Sol. Let the given statement be P(n)

i.e., $P(n): x^n - y^n = M(x - y), x - y \neq 0$

Step 1. When $n = 1$

$$x^n - y^n = x - y = M(x - y)$$

$\Rightarrow$ P(1) is true.

Step 2. Assume that P(k) is true

i.e., Let $x^k - y^k = M(x - y),\ x - y \neq 0$ ...(*i*)

We shall show that P(k + 1) is true

i.e., $x^{k+1} - y^{k+1} = M(x - y)$

Now $x^{k+1} - x^k y + x^k y - y^{k+1}$

$= x^k(x - y) + y(x^k - y^k)$

$= M(x - y) + yM(x - y)$ [Using (*i*)]

$= M(x - y)$

$\Rightarrow$ P(k + 1) is true

$\therefore$ By principle of mathematical induction.

P(n) is true for all $n \in N$.

PRACTICE EXERCISE 2.2 (*iii*)

1. By the principle of mathematical induction prove that
$$a + (a + d) + (a + 2d) + ... + [a + (n - 1)d] = \frac{n}{2}[2a + (n - 1)d]$$
2. Using principle of mathematical induction prove that
$$a + ar + ar^2 + ... + ar^{n-1} = \frac{a(1 - r^n)}{(1 - r)},\ r \neq 1.$$
3. Using principle of mathematical induction prove that $x^n - a^n$ is divisible by $(x - a)$, $\forall\ n \in N$. [*Imp.*]
4. Prove by the principle of induction that $x^{2n} - y^{2n}$ is divisible by $(x - y)$, where n is a positive integer.

TEXT-BOOK EXERCISE 2.2
TYPE—IV
(*SOLVED EXAMPLES*)

☞ **Example 1. Prove the following by the principle of induction**

$$1^2 + 2^2 + 3^2 + ... + n^2 = \frac{1}{6}n(n + 1)(2n + 1)$$

for every positive integer n. [*T.B.Q. 8*]

Sol. Let $P(n): 1^2 + 2^2 + 3^2 + ... + n^2 = \dfrac{n(n + 1)(2n + 1)}{6}$

When $n = 1$

L.H.S. $= (1)^2 = 1$

R.H.S. $= \frac{1}{6} . 1(1 + 1)(2 + 1)$

$= \frac{1}{6} \times 2 \times 3 = 1$

$\therefore$ L.H.S. = R.H.S. $\Rightarrow$ P(1) is true.

Assume that P(k) is true

i.e. $1^2 + 2^2 + 3^2 + ... + k^2 = \frac{1}{6}k(k + 1)k(2k + 1)$ ...(*i*)

Now $P(k + 1): 1^2 + 2^2 + 3^2 + ... + k^2 + (k + 1)^2 = \frac{1}{6}(k + 1)(k + 2)(2k + 3)$

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$$\text{L.H.S.} = 1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2$$

$$= \frac{1}{6} k(k+1)(2k+1) + (k+1)^2$$

[Using (*i*)]

$$= \frac{1}{6}(k+1)[k(2k+1) + 6(k+1)]$$

$$= \frac{1}{6}(k+1)(2k^2 + 7k + 6)$$

$$= \frac{1}{6}(k+1)[2k^2 + 4k + 3k + 6]$$

$$= \frac{1}{6}(k+1)[2k(k+2) + 3(k+2)]$$

$$= \frac{1}{6}(k+1)(k+2)(3k+3)$$

= R.H.S. of P$(k+1)$

$\Rightarrow$ P$(k+1)$ is true.

$\therefore$ By principle of mathematical induction, P(n) is true for $n \in$ N.

Example 2. Prove the following by the principle of induction

$$\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{(2n+1)}.$$

[*T.B.Q. 9*]

Sol. Let P(n) denote the given statement

$$P(n): \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$$

For $n = 1$, $\quad \text{L.H.S.} = \frac{1}{1.3} = \frac{1}{3}$

$$\text{R.H.S.} = \frac{1}{2 \times 1 + 1} = \frac{1}{3}$$

L.H.S. = R.H.S. $\Rightarrow$ P(1) is true

Let P(k) is true then

$$\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots + \frac{1}{(2k-1)(2k+1)} = \frac{k}{(2k+1)}$$

Now $\quad P(k+1): \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots + \frac{1}{(2k-1)(2k+1)}$

$$+ \frac{1}{(2k+1)(2k+3)} = \frac{(k+1)}{(2k+3)}$$

L.H.S. of P$(k+1)$

$$= \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots + \frac{1}{(2k-1)(2k+1)}$$

$$+ \frac{1}{(2k+1)(2k+3)}$$

$$= \frac{k}{(2k+1)} + \frac{1}{(2k+1)(2k+3)} \qquad \text{[Using P}(k)\text{]}$$

$$= \frac{k(2k+1)+1}{(2k+1)(2k+3)} = \frac{2k^2+3k+1}{(2k+1)(2k+3)}$$

$$= \frac{(2k+1)(k+1)}{(2k+1)(2k+3)} = \frac{k+1}{2k+3} = \text{R.H.S.}$$

$\Rightarrow$ P$(k+1)$ is true.

Hence, by principle of mathematical induction, P(n) is true for all natural number n.

PRACTICE EXERCISE 2.2 (*iv*)

1. Use the principle of mathematical induction to prove the following statement for all $n \in$ N.

   (*i*) $1.2 + 2.3 + 3.4 + \ldots + n(n+1) = \frac{n(n+1)(n+2)}{3}$

   (*ii*) $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots + \frac{1}{n(n+1)} = \frac{n}{(n+1)}$.

   (*iii*) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$.

   (*iv*) $1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$.

   (*v*) $1.3 + 3.5 + 5.7 + \ldots + (2n-1)(2n+1) = \frac{n(4n^2+6n-1)}{3}$.

2. Using principle of mathematical induction prove each of the following statements

   (*i*) $1 + 2 + 3 + \ldots + k < \frac{1}{8}(2k+1)^2 \; \forall \; k \in \text{N}$

   (*ii*) $1^2 + (1^2 + 2^2) + \ldots + (1^2 + 2^2 + \ldots + n^2) = \frac{n(n+1)^2(n+2)}{12}$

   (*iii*) $\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \ldots + \frac{1}{(3m-2)(3m+1)} = \frac{m}{3m+1}$.

   (*iv*) $1.3.5 + 3.5.7 + \ldots + (2n-1)(2n+1)(2n+3) = n(n+2)(2n^2+4n-1)$

TEXT-BOOK EXERCISE 2.2
TYPE—V
(*SOLVED EXAMPLES*)

Example 1. If a set has n elements, prove that it has 2^n subsets. [*T.B.Q. 10*]

Sol. Let $P(n): 1 + 2 + 3 + 4 + \ldots = 2^n$

For $n = 1$ $\quad$ L.H.S. $= 2^1 = 2$,

R.H.S. $= 2^1 = 2$

$\therefore$ L.H.S. $=$ R.H.S. $\Rightarrow$ P(1) is true.

Let $P(k)$ be true, then

$P(k): 2^0 + 2^1 + 2^2 + \ldots + 2^k = 2^k$

Now $P(k + 1): 2^0 + 2^1 + 2^2 + \ldots + 2^{k+1} = 2^k.2$ which is true also.

Hence by the principle of mathematical induction $P(n)$ is true for all values of n.

PRACTICE EXERCISE 2.2 (*v*)

1. Prove by using principle of mathematical induction

$$\underbrace{7 + 77 + 777 + \ldots + 777\ldots7}_{n \text{ digits}} = \frac{7}{81}(10^{n+1} - 9n - 10)$$

2. Prove by using principle of mathematical induction
$1.4.7 + 2.5.8 + 3.6.9 + \ldots n(n + 3)(n + 6)$
$$\frac{n}{4}(n + 1)(n + 6)(n + 7).$$

MISCELLANEOUS EXERCISE
(*SOLVED EXAMPLES*)

Example 1. Prove by induction that the sum of the first n odd natural numbers is n^2. [*T.B.Q. 1*]

Sol. Let $P(n): 1 + 3 + 5 \ldots + (2n - 1) = n^2$

when $n = 1$, L.H.S. $= 1$

R.H.S. $= 1^2 = 1 \Rightarrow$ P(1) is true.

Let $P(k)$ be true

$\therefore \quad 1 + 3 + 5 \ldots + (2k - 1) = k^2$

We have to show that $P(k + 1)$ is true.

$P(k + 1): 1 + 3 + 5 + \ldots + (2k - 1)(2k + 1) = (k + 1)^2$

$$\begin{aligned} \text{L.H.S.} &= 1 + 3 + 5 + \ldots + (2K - 1) + (2k + 1) \\ &= k^2 + 2k + 1 \qquad [\text{Using } P(k)] \\ &= (k + 1)^2 = \text{R.H.S.} \end{aligned}$$

$\Rightarrow \quad P(k + 1)$ is true.

Hence $P(n)$ is true for all natural numbers n.

☞ **Example 2. If we take any three consecutive natural numbers, prove that the sum of their cubes is always divisible by 9.** [*T.B.Q. 2*]

Sol. Let three consecutive natural numbers be $n, (n + 1), (n + 2)$.

Let $P(n): n^3 + (n + 1)^3 + (n + 2)^3$ is always divisible by 9.

For $n = 1$, $\quad$ P(1) is a statement :

$1^3 + (1 + 1)^3 + (1 + 2)^3 = 1 + 8 + 27 = 36$,

which is divisible by 9

$\Rightarrow \quad$ P(1) is true.

i.e., $\quad k^3 + (k + 1)^3 + (k + 2)^3$ is always divisible by 9

We have to show that $P(k + 1)$ is true

i.e., $\quad P(k + 1): (k + 1)^3 + (k + 2)^3 + (k + 3)^3$

is always divisible by 9

Consider $\quad (k + 1)^3 + (k + 2)^3 + (k + 3)^3$

$$\begin{aligned} &= (k + 1)^3 + (k + 2)^3 + k^3 + 27 + 9k^2 + 27k \\ &= k^3 + (k + 1)^3 + (k + 2)^3 + 27 + 9k^2 + 27k \end{aligned}$$

Now $k^3 + (k + 1)^3 + (k + 2)^3$ is divisible by 9 because $P(k)$ is true. Also $27 + 9k^2 + 27k$ is clearly divisible by 9 because every term contains 9.

$\therefore \quad P(k + 1)$ is true.

Hence $P(n)$ is true for all natural numbers.

☞ **Example 3. Prove by induction the inequality $(1 + x)^n \geq 1 + nx$ wherever x is positive and n is a positive integer.**

$x > -1 \ (x \neq 0)$ [*T.B.Q. 3*]

Sol. Let $P(n)$ be the statement

$(1 + x)^n > 1 + nx, \ x > -1, \ x \neq 0$

We have to prove the truth of $P(n)$ for $n \geq 2$, so we start induction from $n = 2$.

P(2) is true if $(1 + x)^2 > 1 + 2x$

If $1 + x + x^2 > (1 + 2x)$

If $x^2 > 0$, which is true because x is a real non-zero number

Let $P(k)$ be true

$$\begin{aligned} \therefore \quad & (1 + x)^k > 1 + kx \qquad \ldots(i) \\ \text{From } (i) \quad & (1 + x)^{k+1} > (1 + kx)(1 + x) \\ &= (1 + x)^{k+1} > 1 + x + kx + kx^2 \\ &= (1 + x)^{k+1} > 1 + (k + 1)x \quad (\because kx^2 > 0) \\ \therefore \quad & (1 + x)^{k+1} > 1 + (k + 1)x \end{aligned}$$

$\therefore \quad P(k + 1)$ is true

$\therefore$ By principle of mathematical induction

$P(n)$ is true for $n \geq 2$

Hence $(1 + x)^n > (1 + nx), \ n \geq 2$

Example 4. If $P(n)$ is the statement $n^2 - n + 41$ is prime, prove that P(1), P(2) and P(3) are true. Prove also that P(41) is not true. How does this not contradict the principle of induction ? [*T.B.Q. 4*]

Sol. Let $P(n): n^2 - n + 41$ is prime number

Then $P(1): 1^2 - 1 + 41 = 41$ is a prime number. It is true.

$\Rightarrow \quad$ P(1) is true.

P(2) : $2^2 - 2 + 41 = 43$, is a prime number, it is true.

P(3) : $3^2 - 3 + 41 = 47$, is a prime number, it is true.

P(41) : $(41)^2 - 41 + 41 = (41)^2$, is a prime number.

But $41 \times 41 = 1681$, which is not true so P(41) is false statement.

This does not contradict the principle of mathematical induction P(41) has not been proved to be true.

Example 5. Prove by induction that $(2n + 7) < (n + 3)^2$ for all natural numbers n. Using this, prove by induction that $(n + 3)^2 < 2^{n+3}$ for all natural numbers n. [T.B.Q. 5]

Sol. (*i*) Let P(n) be the statement "$2n + 7 \le (n + 3)^2$"

Then P(1) is the statement

"$2 \times 1 + 7 \le (1 + 3)^2$ or $9 \le 16$"

which is true. Suppose P(k) is true, then

$2k + 7 \le (k + 3)^2$

P(k + 1) is the statement "$2(k + 1) + 7 \le (k + 3)^2$"

$$\begin{aligned} \text{Now } 2(k + 1) + 7 &= (2k + 7) + 2 \\ &\le (k + 3)^2 + 2 \quad [\because P(k) \text{ is true}] \\ &= k^2 + 6k + 11 \\ &= (k^2 + 8k + 16) - 2k - 5 \\ &= (k + 4)^2 - (2k + 5) \\ &< (k + 4)^2 \end{aligned}$$

since $(2k + 5) > 0$ for all $k \in N$

$\Rightarrow$ P(k + 1) is true.

$\therefore$ By the principle of mathematical induction, P(n) is true for all $n \in N$.

(*ii*) Let P(n) be the statement "$(n + 3)^2 \le 2^{n+3}$".

Then P(1) is the statement "$(1 + 3)^2 \le 2^{1+3}$ or $16 \le 16$ which is true.

Suppose P(r) is true, then $(k + 3)^2 \le 2^{k+3}$

P (k + 1) is the statement "$(k + 4)^2 \le 2^{k+4}$"

$$\begin{aligned} \text{Now } (k + 4)^2 &= [(k + 3) + 1]^2 \\ &= (k + 3)^2 + 2(k + 3) + 1 \\ &\le 2^{k+3} + (2k + 7) \\ &\le 2^{k+3} + (k + 3)^2 \\ &\qquad [\because 2k + 7 \le (k + 3)^2 \ \forall\ n \in N] \\ &\le 2^{k+3} + 2^{k+3} \quad [\because P(k) \text{ is true}] \\ &= 2 \cdot 2^{k+3} = 2^{k+4} \Rightarrow P(k + 1) \text{ is true} \end{aligned}$$

By PMI P(n) is true for all $n \in N$

Example 6. Prove that for $n \in N$

$10^n + 3.4^{n+2} + 5$ is divisible by 9. [T.B.Q. 6]

Sol. We shall prove the result by using principle of mathematical induction. Let P(n) be the statement "$10^n + 3.4^{n+2} + 5$ is divisible by 9"

when $n = 1$, $10^n + 3.4^{n+2} + 5 = 10^1 + 3.4^{1+2} + 5$
$= 207 = 9 \times 23$

$\therefore$ $10^1 + 3.4^{1+2} + 5$ is divisible by 9

$\therefore$ P(1) is true

Let P(k) be true

$\therefore$ $10^k + 3.4^{k+2} + 5$ is divisible by 9

Let $10^k + 3.4^{k+2} + 5 = 9M$ ...(*i*)

$$\begin{aligned} \text{when } n = k + 1, &= 10^n + 3.4^{n+2} + 5 \\ &= 10^{k+1} + 3.4^{k+3} + 5 \\ &= 10(10^k) + 3.4^{k+3} + 5 \\ &= 10(9M - 3.4^{k+2} - 5) + 3.4^{k+3} + 5 \quad [\text{by } (i)] \\ &= 90M - 30.4^k.16 - 50 + 3.4^k.64 + 5 \\ &= 90M = 4^k(480 - 192) - 45 \\ &= 9(10M - 32.4^k - 5) \\ &= a \text{ multiple of } 9 \end{aligned}$$

$\therefore$ $10^{k+1} + 3.4^{k+1+2} + 5$ is divisible by 9

$\therefore$ P(k + 1) is true whenever P(k) is so

$\therefore$ By PMI, P(n) is true for $n \in N$

$\therefore$ $10^n + 3.4^{n+2} + 5$ is divisible by 9 for all natural numbers.

Example 7. Prove that $10^{2n-1} + 1$ is divisible by 11 for all $n \in N$. [N.M.O.C. 1994 (Set B)] [T.B.Q. 7]

Sol. Let $a = 10^{2n-1} + 1$

For $n = 1$, $a = 10^{2-1} + 1 = 10 + 1 = 11$

As $\frac{11}{11} = 1 \Rightarrow$ T(1) is true

Let T(k) be true

i.e., $\frac{11}{10^{2k-1} + 1}$ ...(*i*)

Let us consider a for $n = k + 1$

$$\begin{aligned} \text{Thus } a &= 10^{2(k+1)-1} + 1 \\ &= 10^{2k+2-1} + 1 \\ &= 10^{2k-1}.10^2 + 1 \\ &= 100(10^{2m-1} + 1) - 99 \end{aligned}$$

Now $\frac{11}{10^{2m-1} + 1}$ [by (*i*)]

and therefore $\frac{11}{100(10^{2m-1} + 1)}$

Also $\frac{11}{99}$

$\therefore$ $\frac{11}{100(10^{2m-1} + 1) - 99}$

$\frac{11}{a}$ for $n = m + 1$

$\therefore$ $T(m + 1)$ holds

Hence by principle of mathematical induction,

$$\frac{11}{10^{2n-1}+1} \quad \forall \quad n \in N$$

Example 8. Prove that

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1}, \; n \in N$$

[*T.B.Q. 8*] [*V. Imp.*]
[*A.I.S.S.E. 1983 ; Pb Board, 1987 ; H.P. Board, 1988*]

Sol. We shall prove the result by using P.M.I. Let $P(n)$ be the statement

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1}$$

$P(1)$ is true, if $\frac{1}{1.2}=\frac{1}{1+1}$, which is true

$\therefore$ $P(1)$ is true

Let $P(k)$ be true

$$\therefore \quad \frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots+\frac{1}{k(k+1)}=\frac{k}{k+1} \quad \ldots(i)$$

Now $P(k + 1)$ is true if

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots+\frac{1}{(k+1)(k+1+1)}=\frac{(k+1)}{k+1+1}$$

$$\text{If} \quad \left(\frac{1}{1.2}+\frac{1}{2.3}+\ldots+\frac{1}{k(k+1)}\right)+\frac{1}{(k+1)(k+2)}=\frac{k+1}{k+2}$$

$$\text{If} \quad \frac{k}{k+1}+\frac{1}{(k+1)(k+2)}=\frac{k+1}{k+2} \quad [\text{by } (i)]$$

$$\text{If} \quad \frac{1}{(k+1)}\left\{k+\frac{1}{(k+2)}\right\}=\frac{k+1}{k+2}$$

$$\text{If} \quad \frac{1}{(k+1)}\left\{\frac{k^2+2k+1}{k+2}\right\}=\frac{k+1}{k+2}$$

$$\text{If} \quad \frac{1}{(k+1)}\left\{\frac{(k+1)^2}{(k+2)}\right\}=\frac{k+1}{k+2}, \text{ which is true}$$

$\therefore$ $P(k + 1)$ is true whenever $P(k)$ is so

$\therefore$ By PMI, $P(n)$ is true for $n \in N$

$$\therefore \quad \frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots+\frac{1}{n(n+1)}=\frac{n}{(n+1)}, \; n \in N$$

Example 9. Prove that

$$1^3+2^3+3^3+\ldots+n^3=\frac{n^2(n+1)^2}{4}$$

for every +ve integer *n*. [*M. Imp.*] [*T.B.Q. 9*]

Sol. Let

$$P(n)=1^3+2^3+3^3+\ldots+n^3=\frac{n^2(n+1)^2}{4} \quad \ldots(i)$$

Putting $n = 1$, we have

$$1^3=\frac{1^2(1+1)^2}{4}=1$$

Thus $P(1)$ holds

Let $P(k)$ be true

$$i.e., \quad 1^3+2^3+3^3+\ldots+k^3=\frac{k^2(k+1)^2}{4} \quad \ldots(ii)$$

We shall prove that $P(k + 1)$ is also true

Adding $(k + 1)^3$ to both sides of *(ii)*, we have

$$1^3+2^3+3^3+\ldots+k^3+(k+1)^3=\frac{k^2(k+1)^2}{4}+(k+1)^3$$

$$\text{or} \quad P(k+1)=\frac{(k+1)^2}{4}[k^2+4(k+1)]=\frac{(k+1)^2(k+2)^2}{4}$$

which is the same expression as obtained by putting $n = k + 1$ is *(i)*. Thus $P(k + 1)$ is true. Thus by principle of mathematical induction $P(n)$ is true for every natural number.

MISC. PRACTICE EXERCISE ON CHAPTER 2

1. Use the principle of mathematical induction to prove the following statements for all $n \in N$.

$$x+4x+7x+\ldots+(3n-2)x=\frac{1}{2}n(3n-1)x$$

2. If $P(n)$ is the statement the arithmetic mean of the numbers n and $(n + 2)$ is the same as their geometric mean, prove that $P(1)$ is not true. Prove also that if $P(n)$ is true, then $P(n + 1)$ is also true. How does this not contradict the principle of induction ?
3. Using P.M.I., prove that $2^n > n$, for all $n \in N$.
4. Using P.M.I., prove that "$3n > 2^n$, for all $n \in N$".
5. Use the principle of mathematical induction to prove the following statements for all $n \in N$.

$$1.2+2.3+3.4+\ldots+n(n+1)=\frac{n(n+1)(n+2)}{3}$$

6. Using P.M.I., prove that

$$1^2+3^2+5^2+\ldots+(2n-1)=\frac{n(2n-1)(2n+1)}{3}$$

7. Use the principle of mathematical induction to prove the following statements for all $n \in N$.

ADDITIONAL SOLVED EXAMPLES
SECTION—A
[2 marks questions]

Example 1. If P(n) is the statement "$n^3 + n$ is divisible by 3". Is the statement P(3) true ? Is the statement P(4) true ?

Sol. P(n) : $n^3 + n$ is divisible by 3.

P(3) : $3^3 + 3 = 27 + 3 = 30$, which is divisible by 3.

Hence the given statement is true.

Again P(4) : $4^3 + 4 = 64 + 4 = 68$.

which is not divisible by 3.

Hence the given statement is not true.

Example 2. Let P (n) be the statement C(n, r) ≤ n ! for all ≤ r ≤ n". Is P(3) true ?

Sol. P(n) "C(n, r) ≤ $\underline{|n}$"

∴ P(3) is "$(3, r) \le \underline{|3}$" ∀ "$1 \le r \le 3$"

Now $C(3, 1) = 3 \le \underline{|3}$

$C(3, 2) = 2 \le \underline{|3}$

$C(3, 3) = 1 \le \underline{|3}$

∴ $C(3, r) \le \underline{|3}\ \forall\ 1 \le r \le 3$

Hence P(3) is true.

Example 3. (*a*) If P(n) is the statement "$n(n + 1)$ is even", then what is P(4) ?

Sol. Let P(n) be the statement "$n(n + 1)$ is even".
Then P(4) : $4(4 + 1) = 20$, which is even.

∴ P(4) is even.

(*b*) Let P(n) be the statement "$3^n > n$". What is P($n + 1$) ?

Sol. P(n) : $3^n > n$

P($n + 1$) is the statement "$3^{n+1} > n + 1$".

Example 4. If P(n) is the statement "$9^n - 8^n - 1$ is a multiple of 8", then (*i*) evaluate P(1), P(3) and P(6), (*ii*) Is P(2) true ? (*iii*) Is P(3) false ?

Sol. We have

P(n) : "$9^n - 8^n - 1$ is a multiple of 8".

(*i*) P(1) : "$9^1 - 8^1 - 1 = 0$ is a multiple of 8".

P(3) : "$9^3 - 8^3 - 1 = 216$ is a multiple of 8"

P(6) : "$9^6 - 8^6 - 1 = 269296$ is a multiple of 8".

(*ii*) When $n = 2$, $9^n - 8^n - 1 = 9^2 - 8^2 - 1 = 16 = 8.2$.

∴ P(2) : "$9^2 - 8^2 - 1$ is a multiple of 8" is true.

(*iii*) P(3) : "$9^3 - 8^3 - 1 = 216 = 8.27$ is multiple of 8".

∴ P(3) is not false.

Example 5. If P(n) is the statement "$2^{3n} - 1$ is an integral multiple of 7", then prove that P(5) is true.

Sol. When $n = 5$

$2^{3n} - 1 = 2^{15} - 1 = 32767 = 7.4681$

∴ The statement P(5) : "$2^{3(5)} - 1$ is an integral multiple of 7" is true.

ADDITIONAL PRACTICE EXERCISE 2 (*a*)

1. If P(n) is the statement "$n(n + 1)(2n + 1)$ is an integral multiple of 6". Prove that P(2), P(5) and P(7) are true.
2. If P(n) is the statement "$12n + 3$ is a multiple of 5", then prove that P(3) is false whereas P(6) is true.
3. If P(n) is the statement "$n^2 + 2$ is a multiple of 5", then show that *P(4) is not true*.
4. If P(n) is the statement

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \left(\frac{n(n + 1)}{2}\right)^2,$$

then verify that P(3), P(7) are both true.

5. Let P(n) be the statement given in problem 4 above, what is P($n + 1$) ?

ADDITIONAL SOLVED EXAMPLES
SECTION—B
[4 marks questions]

Example 1. If P(n) is the statement that the sum of first n natural numbers is divisible by $n + 1$, prove that if P(r) is true, then P($r + 2$) is true.

Sol. P(n) : $1 + 2 + 3 + \ldots + n$ is divisible by $n + 1$

$\Rightarrow$ $\dfrac{n(n + 1)}{2}$ is divisible by $n + 1$

∵ P(r) is true

∴ $1 + 2 + 3 + \ldots + r$ is divisible by $r + 1$.

i.e., $\dfrac{r(r + 1)}{2}$ is divisible by $r + 1$.

Now, P($r + 2$) : $1 + 2 + 3 + \ldots + (r + 2)$ is divisible by $r + 3$

Consider $1 + 2 + 3 + \ldots + r + (r + 1) + (r + 2)$

$$= (1 + 2 + 3 + \ldots + r) + (2r + 3)$$

$$= \frac{r(r + 1)}{2} + (2r + 3) \qquad [\text{using P}(r)]$$

$$= \frac{r^2 + r + 4r + 6}{2} = \frac{r^2 + 5r + 6}{2}$$

$$= \frac{(r + 2)(r + 3)}{2}$$

which is divisible by $r + 3$

Hence P($r + 2$) is true

Example 2. Write down the binomial expansion of $(1+x)^{n+1}$ when $x = 8$. Deduce that $9^{n+1} - 8n - 9$ is divisible by 64, whenever n is a positive integer.

Sol. $(1+x)^{n+1} = (1+8)^{n-1}$

$$= 1 + {}^{n+1}C_1 . 8 + {}^{n+1}C_2 . 8^2 + {}^{n+1}C_3 . 8^3 + \ldots + {}^{n+1}C_{n+1} . 8^n$$

or $$9^{n+1} = 1 + 8(n+1) + {}^{n+1}C_2\ 64 + {}^{n+1}C_3 . 8^3 + \ldots + {}^{n+1}C_{n+1} . 8^n$$

$$\Rightarrow 9^{n+1} = 1 + 8n + 8 + {}^{n+1}C_2 . 64 + {}^{n+1}C_3 . 8^3 + \ldots + {}^{n+1}C_{n+1} . 8^n$$

$$\Rightarrow 9^{n+1} - 8n - 9 = {}^{n+1}C_2 . 64 + {}^{n+1}C_3 . 8^3 + \ldots + {}^{n+1}C_{n+1} . 8^3$$

R.H.S. has 64 as a factor of every term, so R.H.S. is divisible by 64.

Hence L.H.S. *i.e.*, $9^{n+1} - 8n - 9$ is also divisible by 64.

Example 3. For every natural numbers n, prove by mathematical induction $4^n + 15n - 1$ is divisible by 9. *[Roorkee Entrance, 1994]*

Sol. Let $P(n) = 4^n + 15n - 1$,

We have $P(1) = 4 + 15 - 1 = 18 = 9.2$

i.e., $P(1)$ is divisible by 9.

Now assume that for some positive integer m, $P(m)$ is divisible by 9.

i.e., $4^m + 15m - 1 = 9k$, where k is some integer ...(*i*)

Then $P(m+1) = 4^{m+1} + 15(m+1) - 1$

$= 4.4^m + 15m + 14$

$= 4.[9k - 15m + 1] + 15m + 14$, by (*i*)

$= 36k - 45m + 18$

$= 9(4k - 5m + 2) = 9$ some integer.

Thus $P(m+1)$ is divisible by 9 if $P(m)$ is divisible by 9. But as already shown, $P(1)$ is divisible by 9.

Hence by principle of mathematical induction $P(n)$ is divisible by 9 for all positive integers n.

Example 4. Prove by the principle of mathematical induction that :

(*a*) $2 + 4 + 6 + 8 + \ldots + 2n = n(n+1), \forall\ n \in N$. *[N.M.O.C. 1996, (Set A)]*

(*b*) $1 + 3 + 5 + 7 + \ldots + (2n-1) = n^2, \forall\ n \in N$. *[N.M.O.C. 1996, (Set B)]*

Sol. (*a*) Let $P(n) : 2 + 4 + 6 + 89 \ldots + 2n = n(n+1)$

Put $n = 1$, $P(1)$:

R.H.S. $= 1 \times (1+1) = 1 \times 2 = 2 =$ R.H.S.

$\therefore$ $P(1)$ is true.

Let us suppose that $P(r)$ is true *i.e.*,

$$2 + 4 + 6 + 8 + \ldots + 2r = r(r+1)$$

We shall prove that $P(r+1)$ is also true. *i.e.*,

$(2 + 4 + 6 + 8 + \ldots + 2r) + (2r+2) = (r+1)(r+2)$

Now, L.H.S. $= (2 + 4 + 6 + 8 + \ldots + 2r) + (2r+2)$

$= r(r+1) + (2r+2)$

$= r(r+1) + 2(r+1)$

$= (r+1)(r+2) =$ R.H.S.

$\therefore$ $P(r+1)$ is also true.

Hence, by the principle of mathematical induction the given statement is true for all natural numbers n.

Proved.

(*b*) Let $P(n) : 1 + 3 + 5 + 7 + \ldots + (2n-1) = n^2$

Put $n = 1$, $P(1)$: R.H.S. $= (1)^2 = 1 =$ L.H.S.

$\therefore$ $P(1)$ is true.

Let us suppose that $P(r)$ is true, *i.e.*,

$$1 + 3 + 5 + 7 + \ldots + (2r-1) = r^2$$

We shall prove that $P(r+1)$ is also true, *i.e.*

$1 + 3 + 5 + 7 + \ldots + (2r-1) + (2r+1) = (r+1)^2$

Now,

L.H.S. $= \{1 + 3 + 5 + 7 + \ldots + (2r-1)\} + (2r+1)$

$= r^2 + 2r + 1 = (r+1)^2 =$ R.H.S.

$\therefore$ $P(r+1)$ in also true.

Hence, by the principle of mathematical induction the given statement is true for all natural numbers n.

Proved.

ADDITIONAL PRACTICE EXERCISE 2 (*b*)

1. Prove $3^{2n} - 1$ is divisible by 8 for all $n \in N$.
2. Prove that $10^{2n-1} + 1$ is divisible by 11 for all $n \in N$.
3. Prove that $8^n - 3^n$ is divisible by 5 for all $n \in N$.
4. Prove that $4^n - 3^n - 1$ is a multiple of 9 for all $n \in N$.
5. Prove that $9^n - 8n - 1$ is a multiple of 64 for all $n \in N$.
6. Prove that $n(n+1)(2n+1)$ is divisible by 6 for all $n \in N$.
7. Prove that $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9 for every natural number.
8. Prove that $n(n+1)(n+5)$ is divisible by 6 for all $n \in N$.
9. Show that if the statement $P(n)$,
$$2 + 4 + 6 + \ldots\ldots + 2n = n(n+1) + 2$$
is true for $n = k$, then it is true for $n = k+1$. Can we conclude that $P(n)$ is true for every natural number.
10. If $P(n)$ be the statement, "A.M. between n and $n+2$ is equal to G.M. between n and $n+2$", prove that $P(n)$ is not true for all natural numbers.
[**Hint :** $P(1)$ is not true.]

11. Using principle of mathematical induction prove the following for all $n \in N$.

(i) $(2.1+1)+(2.2+1)+(2.3+1)+\ldots+(2n+1)=(n+1)^2-1$

(ii) $2+2^2+2^3+\ldots+2^n=2(2^n-1)$

(iii) $3+3^2+3^3+\ldots+3^n=\frac{3}{2}(3^n-1)$

(iv) $5+15+45+\ldots+5.3^{n-1}=\frac{5}{2}(3^n-1)$

(v) $1.\underline{|1}+2.\underline{|2}+3.\underline{|3}+\ldots+n\underline{|n}=\underline{|n+1}-1$

12. Using principle of mathematical induction prove that $n(n^2+20)$ is divisible by 48 for every even natural number n. [*M. Imp.*]

13. Use the P.M.I. to prove each of the following statements. [*V. Imp.*]

(i) $1+2+3+\ldots+n<\frac{1}{8}(2n+1)^2$

(ii) $1+4+12+\ldots+(3n-2)=\frac{n(3n-1)}{2}$

(iii) $4+8+12+\ldots+4n=2n(n+1)$

[**Hint :** $P(k)+4(k+1)=2k(k+1)+4(k+1)=2(k+1)(k+2)=P(k+1)$]

(iv) $2.5+5.8+8.11+\ldots$ to n terms $=n(3n^2+6n+1)$

(v) $1.3+2.4+3.5+\ldots+n(n+2)=\frac{n(n+1)(2n+7)}{6}$, $\forall\ n \in N$

14. Using the principle of mathematical induction prove that following statements : [*Imp.*]

(i) n^2-n-41 is prime

(ii) Any natural number equals it successor *i.e.*, $P(n): n=n+1$.

(iii) $11^{n+2}+12^{2n+1}$ is divisible by 133. [*Roorkee 1982*]

(iv) $5^{n+2}-24n-25$ is divisible by 576.

(v) $1+2.2^2+3.2^3+\ldots+n.2^n=(n-1)2^{n+1}+2$

(vi) $1+2.2+3.2^2+\ldots n.2^{n-1}=1+(n-1)2^n$

(vii) $2^{n+1}>2n>1$.

ADDITIONAL SOLVED EXAMPLES
SECTION—C
[6 marks questions]

Example 1. **Prove by the principle of mathematical induction that**

(a) $7^{2n}+2^{3(n-1)}.3^{n-1}$ **is always divisible by 25, $\forall\ n \in N$.** [*N.M.O.C. 1996 (Set A)*]

(b) 12^n+25^{n-1} **is always divisible by 13, $\forall\ n \in N$.** [*N.M.O.C. 1996, (Set B)*]

Sol. (a) Let $P(n): 7^{2n}+2^{3n-3}.3^{n-1}$

Put $n=1$, $P(1): 7^2+2^{3-3}.3^{1-1}=49+2^0.3^0$
$=49+1=50$ which is divisible by 25.

$\Rightarrow$ P(1) is true.

Let us assume that $P(k)$ is true. *i.e.*,

$7^{2k}+2^{3k-3}.3^{k-1}$ is divisible by 25.

$\therefore\quad 7^{2k}+2^{3k-3}.3^{k-1}=25r$, for some $r \in N$ ...(i)

Now, $P(k+1)=7^{2(k+1)}+2^{3(k+1)-3}.3^{(k+1)-1}$

$=7^{2k}.7^2+2^{3k-3}.2^3k^{-1}.3$

$=49.7^{2k}+24.2^{3k-3}.3^{k-1}$

$=(50-1)7^{2k}+(25-1).2^{3k-3}.3^{k-1}$

$=(50.7^{2k}+25.2^{3k-3}.3^{k-1})-(7^{2k}+2^{3k-3}.3^{k-1})$

$=25(2.7^{2k}+2^{3k-3}.3^{k-1})-(7^{2k}+2^{3k-3}.3^{k-1})$

$=25(2.7^{2k}+2^{3k-3}.3^{k-1})-25r$ [by (i)]

= Divisible by 25 – divisible by 25

= Divisible by 25

$\therefore$ $P(k+1)$ is also true.

Hence, by the principle of mathematical induction the given statement in true for all positive number n. **Proved.**

(b) Let $P(n): 12^n+25^{n-1}$

Put $n=1$, $P(1): 12+25^{1-1}=12+25^0$
$=12+1=13$ which is divisible by 13.

$\Rightarrow$ P(1) is true.

Let us assume that $P(k)$ is true *i.e.*,

12^k+25^{k-1} is divisible by 13.

$\therefore\quad 12+25^{k-1}=13r$, for some $r \in N$ ...(i)

Now, $P(k+1)\ 12^{k+1}+25^{(k+1)-1}$

$=12^k.12+25^{k-1}.25$

$=(13-1).12^k+(26-1).25^{k-1}$

$=(13.12^k+26.25^{k-1})-(12^k+25^{k-1})$

$=13(12^k+2.25^{k-1})-(12^k+25^{k-1})$

$=13(12^k+2.25^{k-1})-13r$ [by (i)]

= Divisible by 13 – divisible by 13

= Divisible by 13

$\Rightarrow$ $P(k+1)$ is also true.

Hence, by the principle of mathematical induction the given statement is true for all $n \in N$. **Proved.**

Example 2. **Prove by the principle of mathematical induction that :**

$$6+66+666+\ldots+\underbrace{666\ldots6}_{n\ \text{digits}}=\frac{2}{27}(10^{n+1}-9n-10)$$

[*N.M.O.C. 1995 (Set B)*]

Sol. Let $P(n) : 6 + 66 + 666 + ... + \underbrace{(6666 ... 6)}_{n \text{ digits}}$

$$= \frac{2}{27}(10^{n+1} - 9x - 10)$$

Basic step :

To prove : P(1) in true

Proof : For $n = 1$,

$$\text{R.H.S.} = \frac{2}{27}(10^2 - 9 \times 1 - 10)$$

$$= \frac{2}{27}(81) = 6 = T_1$$

$\therefore$ P(1) in true.

Induction step :

Given $P(k)$ is true. Or

$$6 + 66 + 666 ... + \underbrace{666 ... 6}_{k-1 \text{ digits}} = \frac{2}{27}[10^{k+1} - 9k - 10]$$

To prove : $P(k + 1)$ is true *i.e.*,

$$6 + 66 + 666 ... + \underbrace{666 ... 6}_{k+1 \text{ digits}}$$

$$= \frac{2}{27}[10^{k+2} - 9(k + 1) - 10]$$

Proof. L.H.S. $= 6 + 66 + 666 ... + \underbrace{666 .. .6}_{k+1 \text{ digits}}$

$$= \frac{2}{27}[10^{k+1} - 9k - 10] + 6 \quad \underbrace{[11111 ... 1]}_{k+1 \text{ digits}}$$

$$= \frac{2}{27}[10^{k+1} - 9k - 10] + \frac{6}{9}[10^{k+1} - 1]$$

$$= \frac{2}{27}[10^{k+1} - 9k - 10] + \frac{2}{27}[9.10^{k+1} - 9]$$

$$= \frac{2}{27}[10^{k+1} + 9.10^{k+1} - 9k - 9 - 10]$$

$$= \frac{2}{27}[10^{k+1} - 9(k + 1) - 10] = \text{R.H.S.}$$

$\therefore$ $P(k + 1)$ is true.

Hence $P(n)$ is true.

Example 3. Prove the following by mathematical induction :

$$\mathbf{1 + 5 + 9 + ... + (4n - 3) = n(2n - 1).}$$

[Annual Exam. 1995]

Sol. Let $P(n)$ be the statement

$P(n) : 1 + 5 + 9 + ... + (4n - 3) = n(2n - 1)$ when $n = 1$,

$P(1) : 1 = 1(2 - 1) = 1$, which is true.

$\therefore$ P(1) is true.

Let us assume that it is true for $n = k$,

i.e. $1 + 5 + 9 + ... + (4k - 3) = k(2k - 1)$.

Now, we shall prove that it is true for $n = k + 1$.

i.e. $1 + 5 + 9 + ... + (4k - 3) + (4k + 1)$

$$= (k + 1)[2(k + 1) - 1]$$

$$\begin{aligned}
\text{L.H.S.} &= [1 + 5 + 9 + ... + (4k - 3)] \\
&\qquad + (4k + 1) \\
&= k(2k - 1) + 4k + 1 \\
&= 2k^2 - k + 4k + 1 \\
&= 2k^2 + 3k + 1 \\
\text{R.H.S.} &= (k + 1)[2(k + 1) - 1] \\
&= (k + 1)(2k + 2 - 1) \\
&= (k + 1)(2k + 1) \\
&= 2k^2 + 3k + 1 \\
\therefore \quad \text{L.H.S.} &= \text{R.H.S.}
\end{aligned}$$

$\therefore$ $P(k + 1)$ is also true.

Hence, by mathematical induction, the given statement is true for all natural numbers. **Proved.**

☞ **Example 4. Prove the following by the principle of mathematical induction : $(3^{2n} - 1)$ is an integral multiple of 8.** *[Annual Exam. 1994]*

Sol. Let $P(n)$ be the statement that $(3^{2n} - 1)$ is an integral multiple of 8.

When $n = 1$, then $3^2 - 1 = 9 - 1 = 8$ is an integral multiple of 8, which is true.

$\therefore$ P(1) is true.

Now, suppose $P(k)$ is true, *i.e.*, $(3^{2k} - 1)$ is an integral multiple of 8.

Then, to prove that $P(k + 1)$ is also true.

$$\begin{aligned}
3^{2(k+1)} - 1 &= 3^{2k}.3^2 - 1 \\
&= 3^{2k}.9 - 1 \\
&= 3^{2k}.9 - 1 - 8 + 8 \\
&= (3^{2k} - 1).9 + 8 \\
&= (\text{an integral multiple of } 8) + 8 \\
&[\because (3^{2k} - 1) \text{ is an integral multiple of } 8] \\
&= \text{an integral multiple of } 8.
\end{aligned}$$

$\therefore$ $P(k + 1)$ is also true. Hence, by the principle of mathematical induction, $P(n)$ is true for all natural numbers n. **Proved.**

Example 5. Prove the following by the principle of mathematical induction :

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + ... + \frac{1}{m(m + 1)} = \frac{m}{m + 1}$$

[Annual Exam. 1993]

Sol. Let $P(m)$ be the statement

$$P(m) : \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + ... + \frac{1}{m(m + 1)} = \frac{m}{m + 1}$$

When $m = 1$,

$$\text{L.H.S.} = \frac{1}{1(1+1)} = \frac{1}{1.2} = \frac{1}{2}$$

$$\text{R.H.S.} = \frac{1}{1+1} = \frac{1}{2}$$

$$\because \quad \text{L.H.S.} = \text{R.H.S.}$$

$\therefore$ P(1) is true.

Let us suppose that the statement is true for $m = k$.

i.e., $$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \dots + \frac{1}{k(k+1)} = \frac{k}{k+1}$$

We shall prove that the statement is true for $m = k + 1$.

i.e., $$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \dots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} = \frac{k+1}{(k+1)+1}$$

$$\text{L.H.S.} = \left[\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \dots + \frac{1}{k(k+1)}\right] + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} = \frac{k(k+2)+1}{(k+1)(k+2)}$$

$$= \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}$$

$$\text{R.H.S.} = \frac{k+1}{(k+1)+1} = \frac{k+1}{k+2}$$

$\therefore$ L.H.S. = R.H.S.

$\therefore$ P(k + 1) is also true.

Hence, by the principle of mathematical induction, the given statement is true for all positive integers m.

Proved.

Example 6. Using P.M.I., prove that

$$\mathbf{3.2^2 + 3^2.2^3 + 3^3.2^4 + \dots + 3n^n.2^{n+1} = \frac{12}{5}(6^n - 1).}$$

Sol. $P(n) : 3.2^2 + 3^2.2^3 + 3^3.2^4 + \dots + 3^n.2^{n+1} = \frac{12}{5}(6^n - 1)$

$$P(1) : 3.2^2 = \frac{12}{5}(6-1)$$

i.e., $$3 \times 4 = \frac{12}{5} \times 5$$

or $12 = 12$, which is true.

Suppose P(r) is true.

$$\therefore \quad 3.2^2 + 3^2.2^3 + \dots + 3^r.2^{r+1} = \frac{12}{5}(6^r - 1)$$

P(r + 1) is the statement

$$3.2^2 + 3^2.2^3 + \dots + 3^r.2^{r+1} + 3^{r+1}.2^{r+2} = \frac{12}{5}(6^{r+1} - 1)$$

$$\text{L.H.S.} = 3.2^2 + 3^2.2^3 + \dots + 3^r.2^{r+1} + 3^{r+1}.2^{r+2}$$

$$= \frac{12}{5}(6^r - 1) + 3^{r+1}.2^{r+2} \quad [\because \text{P}(r) \text{ is true}]$$

$$= \frac{12}{5}\left(6^r - 1 + \frac{5}{12}.3^r.3.2^r.2^2\right)$$

$$= \frac{12}{5}(6^r - 1 + 5.6^r) = \frac{12}{5}(6.6^r - 1)$$

$$= \frac{12}{5}(6^{r+1} - 1).$$

$\therefore$ P(r + 1) is true.

Hence by the principle of mathematical induction, P(n) is true for all natural numbers n.

Example 7. Prove by principle of mathematical induction that

$$\mathbf{1.4.7 + 2.5.8 + 3.6.9 + \dots + n(n+3)(n+6) = \frac{n}{4}(n+1)(n+6)(n+7).}$$

Sol. Let P(n) denote the given statement

$$P(n) : 1.4.7 + 2.5.8 + 3.6.9 + \dots + n(n+3)(n+6) = \frac{n}{4}(n+1)(n+6)(n+7) \quad \dots(i)$$

Step I. For $n = 1$

$$\text{L.H.S.} = 1.4.7 = 28$$

$$\text{R.H.S.} = \frac{1}{4}(1+1)(1+6)(1+7) = \frac{2 \times 7 \times 8}{2} = 28$$

$$\text{L.H.S.} = \text{R.H.S.}$$

i.e., P(1) is true

Step II. Let us suppose that P(k) is true

$$\therefore \text{P}(k) : 1.4.7 + 2.5.8 + 3.6.9 + \dots + k(k+3)(k+6) = \frac{k}{4}(k+1)(k+6)(k+7) \quad \dots(ii)$$

We shall show that P(k + 1) is true

L.H.S. of P(k + 1)

$$= 1.4.7 + 2.5.8 + 3.6.9 + \dots + k(k+3)(k+6) + (k+1)(k+4)(k+7)$$

$$= \frac{k}{4}(k+1)(k+6)(k+7) + (k+1)(k+4)(k+7)$$

[Using P(k)]

$$= (k+1)(k+7)\left[\frac{k}{4}(k+6) + k + 4\right]$$

$$= \frac{(k+1)(k+7)}{4}[k^2 + 6k + 4k + 16]$$

$$= \frac{(k+1)}{4}(k+7)(k^2 + 10k + 16)$$

$$= \frac{1}{4}(k+1)(k+7)(k+2)(k+8)$$

$$= \frac{1}{4}(k+1)(k+2)(k+7)(k+8)$$

= R.H.S. of P(k + 1).

i.e., P(k + 1) is true.

Hence by principle of mathematical induction P(n) is true for all $n \in$ N.

Example 8. Using the principle of induction, prove that

$$\frac{n^5}{5} + \frac{n^3}{3} + \frac{7n}{15} \quad \forall \quad n \in \text{N}$$

is a natural number.

Sol. Let $\text{P}(n) = \frac{n^5}{5} + \frac{n^3}{3} + \frac{7n}{15}$

Putting $n = 1$, we get

$$\text{P}(1) = \frac{(1)^5}{5} + \frac{(1)^3}{3} + \frac{7.1}{15}$$

$$= \frac{1}{5} + \frac{1}{3} + \frac{7}{15}$$

$$= \frac{3+5+7}{15} = \frac{15}{15} = 1$$

which is a natural number.

$\Rightarrow$ P(1) is true.

Putting $n = 2$, we get

$$\text{P}(2) = \frac{(2)^5}{5} + \frac{2^3}{3} + \frac{7.2}{15} = \frac{32}{5} + \frac{8}{3} + \frac{14}{15}$$

$$= \frac{96 + 40 + 14}{15} = \frac{150}{15} = 10$$

which is a natural number $\therefore$ P(2) is true

Let us now assume that P(k) is a natural number.

Then $\text{P}(k+1) = \frac{(k+1)^5}{5} + \frac{(k+1)^3}{3} + \frac{7(k+1)}{15}$

$$= \left[\frac{k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1}{5}\right] + \left[\frac{k^3 + 3k^2 + 3k + 1}{3}\right] + \frac{7(k+1)}{15}$$

$$= \left[\frac{k^5}{5} + k^4 + 2k^3 + 2k^2 + k + \frac{1}{5} + \frac{k^3}{3} + k^2 + k + \frac{1}{3} + \frac{7k}{15} + \frac{7}{15}\right]$$

$$= \left(\frac{k^5}{5} + \frac{k^3}{3} + \frac{7k}{15}\right) + (k^4 + 2k^3 + 3k^2 + 2k) + \left(\frac{1}{5} + \frac{1}{3} + \frac{7}{15}\right)$$

or $\quad \text{P}(k+1) = \text{P}(k) + (k^4 + 2k^3 + 3k^2 + 2k) + \text{P}(1)$

Now P(1) is a natural number, P(k) is a natural number and $k^4 + 2k^3 + k^2 + 2k$ is a natural number.

($\because$ The sum, product of natural numbers is a natural number)

$\Rightarrow \quad \text{P}(k+1) = \text{P}(k) + \text{P}(1) + k^4 + 2k^3 + 3k^2 + 2k$ is a natural number.

$\therefore$ The truth of P(k) $\Rightarrow$ the truth of P(k + 1).

$\Rightarrow \quad \frac{k^5}{5} + \frac{k^3}{3} + \frac{7k}{15}$ is a natural number for all values of $n \in$ N. **Ans.**

Example 9. By the method of mathematical induction, prove that $3^{4n+2} + 5^{2n+1}$ is a multiple of 14, for all positive integral values of n, including zero.

Sol. Let $\text{P}(n) = 3^{4n+2} + 5^{2n+1}$

Let $n = 0$, then $\text{P}(0) = 3^2 + 5^1 = 9 + 5 = 14$, which is multiple of 14.

Thus, two result is true for $n = 0$.

Let $n = 1$, then $\text{P}(1) = 3^6 + 5^3 = 729 + 125 = 854 = 61 \times 14$ which is a multiple of 14. Thus the result is true for $n = 1$.

Let us assume that the result is true for $n = k$, *i.e.*, P(k) is a multiple of 14. Now we can show that P(k + 1) − P(k) is also a multiple of 14, then the principle of induction is applicable and the result is proved.

Now P(k + 1) − P(k)

$$= \{3^{4(k+1)+2} + 5^{2(k+1)+1}\} - \{3^{4k+2} + 5^{2k+1}\}$$

$$= 3^{4k+2} . 3^4 + 5^{2k+1} . 5^2 - 3^{4k+2} - 5^{2k+1}$$

$$= (3^4 - 1)\, 3^{4k+2} + (5^2 - 1)\, 5^{2k+1}$$

$$= (70 + 10)\, 3^{4k+2} + (14 + 10)\, 5^{2k+1}$$

$$= 70 . 3^{4k+1} + 14 . 5^{2k+1} + 10 . 3^{4k+2} + 10 . 5^{2k+1}$$

$$= 14\,(5 . 3^{4k+1} + 5^{2k+1}) + 10\,(3^{4k+2} + 5^{2k+1})$$

which is a multiple of 14 as 14 appears in the first expression and the second expression has been assumed to be a multiple of 14. Hence the result is true for all positive integral values of n.

Example 10. Using P.M.I., prove that "$n(n+1)(2n+1)$ is divisible by 6.".

Sol. Let P(n) : $n(n+1)(2n+1)$ is divisible by 6.

Let $P(1) = 1(1+1)(2.1+1) = 1.2.3. = 6$ which is divisible by 6.

$\therefore$ $P(n)$ is true for $n = 1$.

Let us assume that $P(n)$ is true for $n = k$.

$\therefore$ $P(k) : k(k+1)(2k+1)$ is divisible by 6. ...(*i*)

Now we shall show that $P(k+1)$ is true, *i.e.*, $(k+1)$ $)k+2)(2k+3)$ is divisible by 6.

Now $(k+1)(k+2)(2k+3)$

$$\begin{aligned}
&= (k+1)(k+2)[(2k+1)+2] \\
&= (k+1)(k+2)(2k+1) + 2(k+1)(k+2) \\
&= (k+2)[(k+1)(2k+1)] + 2(k+1)(k+2) \\
&= k(k+1)(2k+1) + 2(k+1)(2k+1) \\
&\qquad + 2(k+1)(k+2) \\
&= P(k) + 2(k+1)(2k+1+k+2) \qquad \text{[Using } P(k)] \\
&= P(k) + 2(k+1)\,3(k+1) \\
&= P(k) + 6(k+1)^2
\end{aligned}$$

$6(k+1)^2$ is divisible by 6.

$\therefore$ $P(k) + 6(k+1)^2$ being the sum of two divisible by 6 is also divisible by 6.

$\therefore$ $P(k+1)$ is true.

$\therefore$ By the principle of mathematical induction, $P(n)$ is true for all positive integral values of n.

Example 11. Use the principle of mathematical induction to prove that $3^{2n+2} - 8n - 9$ is divisible by 64 for every natural number n.

Sol. $P(n)$ be the statement "$3^{2n+2} - 8n - 9$ is divisible by 64".

When $n = 1$, $3^{2n+2} - 8n - 9$

$$\begin{aligned}
&= 3^{2+2} - 8.1 - 9 \\
&= 81 - 8 - 9 = 64 = \text{a multiple of } 64.
\end{aligned}$$

$\therefore$ $P(1)$ is true.

Let $P(k)$ be true.

$\therefore$ $3^{2k+2} - 8k - 9$ is divisible by 64. ...(*i*)

When $n = k+1)$, $3^{2n+2} - 8n - 9$

$$\begin{aligned}
&= 3^{2(k+1)+2} = 8(k+1) - 9 \\
&= 3^{2k+2}.3^2 - 8k - 8 - k \\
&= (64M + 8k + 9).9 - 8k - 17 \quad \text{[by (1)]} \\
&= 576M + 72k + 81 - 8k - 17 \\
&= 64(9M + k + 1) \\
&= \text{a multiple of } 64
\end{aligned}$$

$\therefore$ $P(k+1)$ is true whenever $P(k)$ is so.

$\therefore$ By P.M.I., $3^{2n+2} - 8n - 9$ is divisible by 64 for all $n \in N$.

Example 12. For all positive integers n, prove that

$$\frac{n^7}{7} + \frac{n^5}{5} + \frac{2n^3}{3} - \frac{n}{105} \text{ is an integer.}$$

[*I.I.T. 1990*]

Sol. Let $P(n) = \frac{n^7}{7} + \frac{n^5}{5} + \frac{2n^3}{3} - \frac{n}{105}$

For $n = 1$,

$$P(1) = \frac{1}{7} + \frac{1}{5} + \frac{2}{3} - \frac{1}{105} = \frac{15 + 21 + 70 - 1}{105}$$

$$= \frac{105}{105} = 1, \text{ which is an integer.}$$

$\therefore$ $P(1)$ is true.

Now suppose $P(k)$ is an integer where $k \in N$

i.e., Let $P(k) = m, m \in I$

We have

$$P(k+1) = \frac{(k+1)^7}{7} + \frac{(k+1)^5}{5} + \frac{2(k+1)^3}{3} - \frac{(k+1)}{105}$$

$$= \frac{k^7}{7} + \frac{k^5}{5} + \frac{3k^3}{3} - \frac{k}{105} + \frac{1}{7}$$

$$[{}^7C_1k^6 + {}^7C_2k^5 + {}^7C_3k^4 + {}^7C_4k^3 + {}^7C_5k^2 + {}^7C_6k + {}^7C_7]$$

$$+ \frac{1}{5}[{}^5C_1k^4 + {}^5C_2k^3 + {}^5C_3k^2 + {}^5C_4k + {}^5C_5]$$

$$+ \frac{2}{3}[{}^3C_1k^2 + {}^3C_2k + {}^3C_3] = \frac{1}{105}$$

$$= m + \frac{1}{7}(\text{multiple of } 7) + \frac{1}{7} + \frac{1}{5}(\text{multiple of } 5)$$

$$+ \frac{1}{5} + \frac{2}{3}(\text{multiple of } 3) + \frac{2}{3} - \frac{1}{105}$$

$$= m + (a \text{ +ve integer}) + \frac{1}{7} + (\text{a positive integer})$$

$$+ \frac{1}{5}(a \text{ +ve integer}) + \left(\frac{2}{3}\right) - \left(\frac{1}{105}\right)$$

$$= (m+1) + (a \text{ +ve integer}) \quad \left[\because \frac{1}{7} + \frac{1}{5} + \frac{2}{3} - \frac{1}{105} = 1\right]$$

= An integer.

Hence $P(k)$ an integer $\Rightarrow$ $P(k+1)$ is an integer.

$\therefore$ By mathematical induction $P(n)$ is an integer for all $n \in N$

Example 13. Using mathematical induction, prove that

$$\sum_{k=0}^{n} k^2 \, {}^nC_k = n(n+1).2^{n-2} \quad \text{for} \quad n > 1.$$

Sol. Let $P(n) : S_n = \sum_{k=0}^{n} k^2 \, {}^nC_k = n(n+1).2^{n-2}$

$$P(1) : S_1 = \sum_{k=1}^{n} k^2 \cdot {}^1C_k = 1 \cdot (1+1) \cdot 2^{1-2}$$

or $\quad (0 + 1^2 \cdot {}^1C_1) = 2 \cdot 2^{-1}$ or $1 = 1$

$\therefore$ P(1) is true.

Let the statement be true when $n = m$.

Then $\quad P(m) : S_n = \sum_{k=0}^{m} k^2 \cdot {}^mc_k = m(m+1)\, 2^{m-2}$

Consider $\quad P(m+1) : S_{m+1} = \sum_{k=0}^{m+1} k^2 \cdot {}^{m+1}C_k$

$$= \sum_{k=0}^{m+1} k^2 \cdot ({}^mC_k + {}^mC_{k-1})$$

$$[\because \ {}^nC_r + {}^nc_{r-1} = {}^{n+1}C_r]$$

$$= \sum_{k=0}^{m+1} k^2 \cdot {}^mC_r + \sum_{k=0}^{m+1} k^2 \cdot {}^mC_{k-1}$$

$$\therefore \quad \sum_{k=0}^{m} k^2 \cdot {}^mC_k + \sum_{k=0}^{m+1} k^2 \cdot {}^mC_{k-1}$$

[$\because$ First summation becomes meaningless for $k = m + 1$ and second for $k = 0$]

$$= \sum_{k=0}^{m} k^2 \cdot {}^mC_k + \sum_{k=0}^{m} (k+1)^2\, {}^mC_k$$

[Changing k into $k + 1$]

$$= S_m + \sum_{k=0}^{m} (k^2 + 2k + 1) \cdot {}^mC_k$$

$$= S_m + \sum_{k=0}^{m} k^2 \cdot {}^mC_k + 2\sum_{k=0}^{m} k \cdot {}^mC_k + \sum_{k=0}^{m} {}^mC_k$$

$$\begin{aligned}
&= S_m + S_m + 2 \cdot (m \cdot 2^{m-1}) + 2^m \\
&= 2S_m + 2m \cdot 2^{m-1} + 2^m \\
&= 2m(m+1)\, 2^{m-2} + 2m \cdot 2^{m-1} + 2^m \\
&= 2^{m-1}\, [m(m+1) + 2m + 2] \\
&= 2^{m-1}\, (m^2 + 3m + 2) \\
&= 2^{m-1} \cdot (m+1)(m+2) = S_{m+1}
\end{aligned}$$

$\therefore$ The statement holds for $n = m + 1$.

Hence by the principle of mathematical induction, the result holds for $n \geq 1$.

Example 14. Prove by principle of mathematical induction that

$$1.3 + 3.5 + 5.7 + \ldots + (2n-1)(2n+1) = \frac{n}{3}(4n^2 + 6n - 1)$$

Sol. $P(n) = 1.3 + 3.5 + 5.7 + \ldots + (2n-1)(2n+1)$

$$= \frac{n}{3}(4n^2 + 6n - 1)$$

For $n = 1$,

L.H.S. $= 1.3 = 3$

and $\quad$ R.H.S. $= \frac{1}{3}(4 + 6 - 1) = 3$

Thus $\quad$ L.H.S. $=$ R.H.S. $= 3$

For $n = 2$,

L.H.S. $= 1.3 + 3.5 = 18$

and $\quad$ R.H.S. $= \frac{1}{4}\{4.2^2 + 6(2-1)\} = 18$

$\Rightarrow$ P(2) is true.

$\therefore$ The relation holds for $n = 1, 2$.

Step I. Assume that the relation to be true for some positive integral value of n, say $n = k$, *i.e.*,

$$P(k) = 1.3 + 3.5 + \ldots + (2k-1)(2k+1) = \frac{k}{3}(4k^2 + 6k - 1) \quad \ldots(i)$$

Add to each side the $(k+1)$th term, viz., $(2k+1)(2k+3)$, we have

$$P(k+1) = 1.3 + 3.5 + \ldots + (2k-1)(2k+1) + (2k+1)(2k+3)$$

$$\begin{aligned}
&= \frac{k}{3}(4k^2 + 6k - 1) + (2k+1)(2k+3) \\
&= \frac{1}{3}\{4k^3 + 6k^2 - k + 3(4k^3 + 8k + 3)\} \\
&= \frac{1}{3}\{4k^3 + 18k^2 + 23k + 9\} \\
&= \frac{1}{3}(k+1)(4k^2 + 14k + 9) \\
&= \frac{1}{3}(k+1)\{(4k+1)^2 + 6(k+1) - 1\}
\end{aligned}$$

which is of the same form as (1) with $(k + 1)$ in place of k. Therefore, the relation is true for $n = k + 1$. If it is true for $n = k$.

Thus we see that if the given relation is true for $n = k$ then it is true for $n = k_{+1}$ and therefore, by the principle of induction P(n) is true $\forall \ n \in N$.

Example 15. Prove by the principle of mathematical induction

$$1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \ldots + \frac{1}{1+2+3+\ldots+n} = \frac{2n}{n+1}.$$

[*N.M.O.C. 1994 (Set A)*]

Sol. Let the given statement be denoted by P(n)

Now P(1) is true because when $n = 1$

L.H.S. $= 1$

and $\quad$ R.H.S. $= 2/2 = 1$

$\therefore \quad$ L.H.S. $=$ R.H.S.

Let use assume that the result is true for $n = k$ *i.e.*, $P(k)$ is true

$$\therefore \quad 1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \ldots + \frac{1}{1+2+3+\ldots+k} = \frac{2k}{k+1}$$

Adding $(k + 1)$ the term, *i.e.*, $\frac{1}{1+2+3+\ldots+(k+1)}$ to both sides, we get

$$1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \ldots + \frac{1}{1+2+3+\ldots+k} + \frac{1}{1+2+3+\ldots+(k+1)}$$

$$= \frac{2k}{k+1} + \frac{1}{1+2+3+\ldots+(k+1)}$$

$$= \frac{2k}{k+1} + \frac{2}{(k+1)(k+2)}$$

$$\left[\because \quad 1 + 2 + \ldots + (k+1) = \frac{(k+1)(k+2)}{2}\right]$$

$$= \frac{2}{(k+1)}\left[\frac{k(k+2)+1}{k+2}\right]$$

$$= \frac{2(k+1)^2}{(k+1)(k+2)} = \frac{2(k+1)}{(k+2)} = P(k+1)$$

Thus the given result is true for $n = k + 1$, whenever it is true for $n = k$. Hence by the principle of induction it is true for all $n \in N$.

Example 16. Prove by Mathematical Induction that

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$$

Sol. Let $P(n) = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^n}$

Putting $n = 1$, we get

$$P(1) = \frac{1}{2} = 1 - \frac{1}{2^1} = 1 - \frac{1}{2}$$

$\Rightarrow$ $P(1)$ is true.

Let $P(k)$ be true

i.e., $$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^k} = 1 - \frac{1}{2^k}$$

We shall show that $P(k + 1)$ is also true.

Now $$P(k+1) = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^k} + \frac{1}{2^{k+1}}$$

$$P(k) + \frac{1}{2^{k+1}} = 1 - \frac{1}{2^k} + \frac{1}{2^{k+1}}$$

$$= 1 - \left(\frac{1}{2^k} - \frac{1}{2^{k+1}}\right)$$

$$= 1 - \left(\frac{2-1}{2^{k+1}}\right) = 1 - \frac{1}{2^{k+1}}$$

$\Rightarrow$ $P(k + 1)$ is true.

$\therefore$ Using principle of mathematical induction, we can say $P(n)$ is true for all $n = N$

i.e., $$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$$

Example 17. If x is not an integral multiple of 2π use mathematical induction to prove that

$$\cos x + \cos 2x + \ldots + \cos nx = \cos \frac{n+1}{2} x \sin \frac{nx}{2} \operatorname{cosec} \frac{x}{2}$$

[*I.I.T. 1994*]

Sol. Let $P(n)$ denote the statement

$$\cos x + \cos 2x + \ldots + \cos nx$$

$$= \cos\left(\frac{n+1}{2}x\right)\sin\frac{nx}{2}\operatorname{cosec}\frac{x}{2} \qquad \ldots(i)$$

For $n = 1$,

The L.H.S. of (1)

$$= \cos x = \cos x \sin\left(\frac{x}{2}\right)\operatorname{cosec}\left(\frac{x}{2}\right)$$

provided cosec $(x/2)$ exists *i.e.*, $x/2$ is not an integral multiple of π.

$$= \cos\left(\frac{1+1}{2}x\right)\sin\frac{1 \cdot x}{2}\operatorname{cosec}\frac{x}{2}$$

$$= \cos x \sin\frac{x}{2}\operatorname{cosec}\frac{x}{2}$$

= The R.H.S. of (1) for $n = 1$

Thus $P(n)$ is true for $n = 1$.

Now assume as our induction hypothesis that $P(n)$ is true for some positive integer m *i.e.*, (1) is true for $n = m$.

Then for $n = m + 1$, then L.H.S. of (1)

$$= (\cos x + \cos 2x + \ldots + \cos mx) + \cos(m+1)x \qquad [\because P(m) \text{ is true}]$$

$$= \operatorname{cosec}\frac{x}{2}\left[\cos\frac{m+1}{2}x\sin\frac{mx}{2} + 2\cos(m+1)x\sin\frac{x}{2}\right]$$

$$= \frac{1}{2}\operatorname{cosec}\frac{x}{2}\left[\left\{\sin\left(mx+\frac{x}{2}\right) - \sin\frac{x}{2}\right\} + \left\{\sin\left(mx+\frac{3x}{2}\right) - \sin\left(mx+\frac{x}{2}\right)\right\}\right]$$

$$= \frac{1}{2} \operatorname{cosec} \frac{x}{2} \left[\sin\left(mx + \frac{3x}{2} \right) - \sin \frac{x}{2} \right]$$

$$= \left(\frac{1}{2} \operatorname{cosec} \frac{x}{2} \right) . 2 \cos \frac{(m+2)x}{x} . \frac{\sin (m+1)x}{2}$$

$$= \cos \left\{ \frac{(m+1)+1}{2} x \right\} \sin \frac{(m+1)x}{2} \operatorname{cosec} \frac{x}{2}$$

The R.H.S. of (1) for $n = m + 1$.

Thus $P(m + 1)$ is true if $P(m)$ is true and as already shown $P(1)$ is true. Hence by mathematical induction $P(n)$ is true for all positive integers n.

Example 18. Prove that $n^2 > 2n$, $\forall \; n \geq 3$, by using the principle of mathematical induction.

Sol. Let $P(n) : n^2 > 2n$.

Putting $n = 3$, we have

$$\text{L.H.S.} = 3^2 = 9$$

and $$\text{R.H.S.} = 2.3 = 6.$$

Thus the statement $P(3)$ is true.

Let $n = 4$, then

$$\text{L.H.S.} = 4^2 = 16$$

and $$\text{R.H.S.} = 2.4 = 8$$

Thus the statement $P(4)$ is true as $4^2 > 8$.

Let us assume the statement be true for $n = m$, *i.e.*,

$$m^2 > 2m. \qquad \ldots(i)$$

Now we shall show that $P(m + 1)$ is also true.

Adding $2m + 1$ to both sides of (i), we have

$$m^2 + 2m + 1 > 2m + 2m + 1$$

or $$(m+1)^2 > 2(m+1) + (2m-1).$$

But $2m - 1$ is a positive quantity for $m \geq 3$.

$$\therefore \quad (m+1)^2 > 2(m+1).$$

$\Rightarrow$ The result is true for $m + 1$ when it holds good for $n = m$

$\therefore$ By the principle of mathematical induction, $P(n)$ is true for all positive integral values of n, $n \geq 3$.

Example 19. Prove by the principle of mathematical induction

$$\mathbf{1.2 + 2.2^2 + \ldots + n.2^n = (n-1)\, 2^{n+1} + 2.}$$

Sol. Let $$P(n) = 1.2 + 2.2^2 + \ldots + n.2^n = (n-1)\, 2^{n+1} + 2$$

The result is true for $n = 1$ because

$$\text{L.H.S.} = 1.2 = 2$$

and $$\text{R.H.S.} = (1-1)^{1+1} + 2 = 0 + 2$$

$\therefore$ L.H.S. = R.H.S. $\Rightarrow$ $P(1)$ is true.

Let the result be true for $n = m$

$$\therefore \quad P(k) = 1.2 + 2.2^2 + \ldots + k.2^k = (k-1)\, 2^{k+1} + 2$$

Adding $(k + 1)\, 2^{k+1}$ on both sides, we have

$$1.2 + 2.2^2 + \ldots + k.2^k + (k+1)\, 2^{k+1}$$
$$= (k-1)\, 2^{k+1} + 2(k+1)\, 2^{k+1}$$
$$= 2k.2^{k+1} + 2$$
$$= k.2^{k+2} + 2$$

This shows that the results is true for $n = k + 1$, *i.e.*, $P(k + 1)$ is true if $P(k)$ is true. Hence by the principle of mathematical induction, $P(n)$ is true for all positive integral values of n.

Example 20. Use the principle of mathematical induction to prove that

$$1^2 + 2^2 + 3^2 + \ldots + n^2 > \frac{n^3}{3}, \quad n \in \mathbb{N}.$$

Sol. Let $P(n)$ be the statement

$$1^2 + 2^2 + 3^2 + \ldots + n^2 > \frac{n^3}{3}$$

$$P(1) \text{ is true, if } \quad 1^2 > \frac{1^3}{3}$$

or if $$1 > \frac{1}{3} \text{ which is true.}$$

$\therefore$ $P(1)$ is true.

Let $P(k)$ be true.

$$\therefore \quad 1^2 + 2^2 + 3^2 + \ldots + k^2 > \frac{k^3}{3}$$

Let $$1^2 + 2^2 + 3^2 + \ldots + k^2 = p + \frac{k^3}{3} \; (p > 0) \qquad \ldots(i)$$

Now $P(k + 1)$ is true, if

$$1^2 + 2^2 + 3^2 + \ldots + (k+1)^2 > \frac{(k+1)^3}{3}$$

If $$(1^2 + 2^2 + 3^2 + \ldots + k^2) + (k+1)^2 = \frac{(k+1)^3}{3} > 0$$

If $$P + \frac{k^3}{3} + (k+1)^2 - \frac{(k+1)^3}{3} > 0$$

If $$P + \frac{k^3 + 3k^2 + 3 + 6k - k^3 - 3k^2 - 3k - 1}{3} > 0$$

[Using (i)]

If $P + \dfrac{(3k+2)}{3} > 0$, which is true because P and $\dfrac{3k+2}{3}$ and both positive.

$\therefore$ $P(k + 1)$ is true whenever $P(k)$ is so.

By P.M.I. $P(n)$ is true for all $n \in \mathbb{N}$. **Ans.**

ADDITIONAL PRACTICE EXERCISE 2 (*c*)

1. Prove by using principle of mathematical induction

$$7 + 77 + 777 + \dots + \underbrace{777 \dots 7}_{n \text{ digits}} = \frac{7}{81}(10^{n+1} - 9n - 10)$$

2. Let $u_1 = 1$, $u_2 = 1$, $u_{n+2} = u_{n+1} + u_n$ for $n \le 1$ use mathematical induction to show that

$$u_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

for all $n \ge 1$.

3. Prove that $x(x^{n-1} - na^{n-1}) + a^n(n-1)$ is divisible by $(x-a)^2$ for all positive integers a greater than 1.
4. Using principle of mathematical induction prove that

$$1 + x + x^2 + x^3 + \dots + x^n = \frac{1 - x^{n+1}}{1 - x},\ x \in \mathrm{N}$$

Using the principle of mathematical induction, prove that :

5. $\dfrac{1}{1.4} + \dfrac{1}{4.7} + \dfrac{1}{7.10} + \dots + \dfrac{1}{(3n-2)(3n+1)} = \dfrac{n}{3n+1}$.
6. $2.1 + 3.2 + 4.2^2 + \dots + (n+1)\,2^{n-1} = n.2^n$.
7. $\dfrac{1}{3.7} + \dfrac{1}{7.11} + \dfrac{1}{11.15} + \dots + \dfrac{1}{(4n-1)(4n+3)} = \dfrac{n}{3(4n+3)}$.

[*N.M.O.C. 1994 (Set B)*]

8. $3.2^2 + 3^2.2^3 + 3^3.2^4 + \dots + 3^n.2^{n+1} = \dfrac{12}{5}(6^n - 1)$.
9. $1.2 + 2.3 + 3.4 + \dots + n(n+1) = \dfrac{n(n+1)(n+2)}{3}$.
10. Using mathematical induction, prove that ${}^mC_0\,{}^nC_k + {}^mC_1\,{}^nC_{k-1} + \dots + {}^mC_k\,{}^nC_0 = {}^{m+n}C_k$, where m, n, k are positive integers, and ${}^pC_q = 0$ for $p < q$.
11. Prove by induction that $2n + 7 < (n+3)^2$ for all natural numbers n. Using this, prove by induction that $(n+3)^2 < 2^{n+3}$ for all natural numbers n. Prove each of the following by the principle of mathematical induction.
12. $1 + 4 + 7 + \dots (3n-2) = \dfrac{1}{2}\,n(3n-1)$.
13. $\dfrac{1}{1.3} + \dfrac{1}{3.5} + \dfrac{1}{5.7} + \dots + \dfrac{1}{(2n-1)(2n+1)} = \dfrac{n}{(2n+1)}$.
14. $\dfrac{1}{1.4} + \dfrac{1}{4.7} + \dfrac{1}{7.10} + \dots + \dfrac{1}{(3n+2)(3n+1)} = \dfrac{n}{(3n+1)}$.
15. $1.3 + 3.5 + 5.7 + \dots + (2n-1)(2n+1) = \dfrac{n(4n^2 + 6n - 1)}{2}$
16. $1.4.7 + 2.5.8 + 3.6.9 + \dots + n(n+3)(n+6) = \dfrac{n}{4}(n+1)(n+6)(n+7)$.
17. $1.3.5 + 2.4.6 + 3.5.7 + \dots + n(n+2)(n+4) = \dfrac{n}{4}(n+1)(n+4)(n+5)$.

Prove by using the principle of mathematical induction

18. $1.3 + 2.3^2 + 3.3^3 + \dots + n.3^n = \dfrac{(2n-1)\,3^{n+1} + 3}{4}$.
19. $2 + 2^2 + 2^3 + \dots + 2^n = 2(2^n - 1)$.
20. $1 + 2 + 3 + \dots + n < \dfrac{1}{8}(2n+1)^2$.
21. Prove by the principle of mathematical induction that $5^{2n} - 1\ \forall\ n \in \mathrm{N}$ is divisible by 24.

[*N.M.O.C. 1993 (Set A) ; 1992 (Set A)*]

22. Prove by the principle of mathematical induction that $4^n + 15n - 1$ is divisible by 9 for all $n \in \mathrm{N}$.

[*N.M.O.C. 1992 (Set B)*]

23. If P(n) is the statement : the arithmetic mean of the numbers n and $n + 2$ is the same as their geometric mean, prove that P(1) is not true. Prove also that if P(n) is true, then P($n+1$) is also true.
24. If $n > 1$, prove that

$$n\,! < \left(\frac{n+1}{2}\right)^n.$$

25. By the principle of mathematical induction prove that for each not natural number n,

$$1 + 2 + 3 + + 4 + \dots + n < (2n+1)^2.$$

26. For each natural number n, $6^{n+2} + 7^{2n+1}$ is divisible by 43.
27. Prove by principle of mathematical induction

$$S_n = n^3 + 3n^2 + 5n + 3$$

is divisible by 3 for any positive integer n.

28. Prove by the principle of induction that $x^{2n} - y^{2n}$ is divisible by $x - y$, where n is a positive integer.
29. Show that if the statement

$$P(n) : 2 + 4 + 6 + \dots + 2n = n(n+1) + 2$$

is true for $n = k$, then it is true for $n = k + 1$ can we conclude that P(n) is true for every natural number n ?

30. Prove by mathematical induction that

$$2^{4n} > 3^{3n},\ \text{for all}\ n \in \mathrm{N}.$$

LIMITS CONTINUITY AND DIFFERENTIATION

by

Mr.B.C.BASTI

1. LIMITS

1.1 Introduction :

We live in a world of change - our values, ideals, hopes and institutions are undergoing constant change. It is interesting to note that certain changes are happening too rapidly, while other changes are not occurring fast enough. This illustrates that, although the topic of change is important, often the concept of rate of change is more relevant. For example, in the study of population growth, it is not sufficient to know that the population changed by doubling. We need to know the rate at which this doubling took place. It is significant that at one time the doubling of the world population took a thousand years, but now the doubling takes only few decades time. The mathematical tool for measuring rates of change is the concept of limits. The concept of limit is needed to pass from the average rate of change to the more useful concept of an instantaneous rate of change. Indeed it is this concept of the limit, that resulted in the invention of Calculus. It may be surprising to discover that Newton did not have a complete understanding of the limit. Many years later Cauchy put the concept of limit on a sound mathematical basis. In this section, the approach to the concept of limit is initially intuitive and later the mathematically elegant Cauchy epsilon-delta approach is given.

There are many topics in school mathematics through which limits can be illustrated. For instance consider the problem of finding circumference of a circle. The circumference of a circle can be taken as the limit of perimeter of inscribed regular polygon as the number of sides tend to infinity. Teachers can also use the action of a bouncing ball. If $\{hn\}_n = 1,2,\ldots$ is a sequence of heights of the bouncing ball, then 0 is the limit of such a sequence.

1.2 Limit of a Function :

Consider the function $f(x) = \frac{x^2-4}{x-2}$ for $x \neq 2$.

f(x) is not defined at 2 because the direct substitution 2 for x results in 0/0 which is an indeterminate form. Let us calculate the values of f(x) for some values x that are very close to but unequal to 2.

From the table it appears that if x is very close to 2, then $f(x) = \frac{x^2-4}{x-2}$ is very near 4. We represent this statement in mathematical shorthand as,

limit of $f(x) = \frac{x^2-4}{x-2}$ as x approaches 2 is 4 or

$$\lim_{x \to 2} f(x) = 4$$

x	$f(x) = \frac{x^2-4}{x-2}$
1.98	3.98
1.99	3.99
2.01	4.01
2.02	4.02

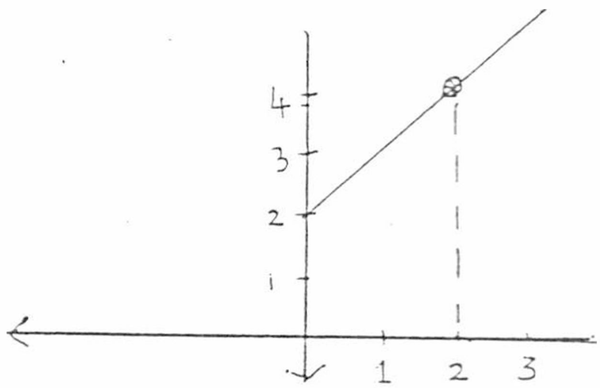

Fig.1

Now we can define f(2) as 4. Here we have used the limit process to define f(2) though originally f(2) was not defined. It is possible to obtain $\lim\limits_{x\to 2} f(x)$ without finding table of values.

Since $f(x) = \frac{x^2-4}{x-2} \quad \frac{(x-2)\,(x-2)}{(x-2)}$ if $x \neq 2$

$= (x+2)$ if $x \neq 2$.

$\lim\limits_{x\to 2} f(x) = \lim\limits_{x\to 2} (x+2) = 2+2 = 4$

Since limit of $(x+2)$ as x tends to 2 can be obtained by substituting x=2 in x+2.

Exercise: Find (i) $\lim\limits_{x\to 3} \frac{x^2-5x+6}{x-3}$

(ii) $\lim\limits_{x\to 3} \frac{x^2-9}{x+3}$

Now we provide intuitive definition of limit of a function.

Definition: If f is a real function defined on a set of real numbers and a in the domain, of f, then we say that limit of f(x) as x a is a real number l if f(x) is very close to l, whenever x is very close to a.

We write this as $\lim\limits_{x\to a} f(x) = l$

If such a l does not exist then we say that $\lim\limits_{x\to a} f(x)$ does not

exi

exist. For instance $\lim_{x \to a} \sqrt{x}$ does not exist.

Next we shall introduce the idea of left hand limit and right hand limit of a function at a point. Let $f(x)$ be a function defined as follows.

$$f(x) = \sqrt{2x+2} \text{ if } x < 2.$$
$$= x+4 \text{ if } x \geq 2$$

We shall examine whether $\lim_{x \to 2} f(x)$ exists.

First suppose $x \to 2$ from the right side of 2 (or $x \to 2$ and $x > 2$) and symbolically it is written as $x \to 2+$.

Then $\lim_{x \to 2+} f(x) = \lim_{x \to 2} x+4 = 2+4 = 6$

This limit is called as right hand limit of $f(x)$ at 2.

Next suppose $x \to 2$ from the left side of 2 (or $x \to 2$ and $x < 2$) and symbolically it is written as $x \to 2-$.

Then $\lim_{x \to 2-} f(x) = \lim_{x \to 2} \sqrt{2x+2} = \sqrt{2 \times 2 + 2} = 3$

$\lim_{x \to 2-} f(x)$ is called as left hand limit of $f(x)$ at 2.

Thus $\lim_{x \to 2+} f(x) \neq \lim_{x \to 2-} f(x)$. In this case we say that $\lim_{x \to a} f(x)$ does not exist. Because $\lim_{x \to a} f(x)$ exists if and only if $\lim_{x \to a+} f(x) = \lim_{x \to a-} f(x)$ when $\lim_{x \to a+} f(x) = \lim_{x \to a-} f(x)$, one of these values is taken as $\lim_{x \to a} f(x)$. Earlier we got $\lim_{x \to 2} \frac{x^2-4}{x-2} = 4$. In this case we notice that $\lim_{x \to 2+} \frac{x^2-4}{x-2} = \lim_{x \to 2-} \frac{x^2-4}{x-2} = 4$

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The definition of limit given earlier is intuitive and suffers from shortcomings. In the first instance, it lacks mathematical rigour and further it is hardly useful in the development of theory of limits. We can examine more closely the idea of limit so as to arrive at Cauchy's mathematical definition.

Let us begin with $\lim_{x \to 3} (2x+1) = 7$. This means that when x is very close to 3, 2x+1 is very close to 7. Since "close to" is not mathematically defined so far, we have trouble in understanding what we mean by these words. Therefore, our first attempt to explain $\lim_{x \to 3} (2x+1) = 7$ is unsatisfactory. In our second attempt to explain $\lim_{x \to 3} (2x+1) = 7$, we mean that the value of 2x+1 can be made as near 7 as we wish to have it by making x near enough to 3. This leads us to the 'Cauchy definition' for limit of a function.

Definition: $\lim_{x \to a} f(x) = L$ iff for every $\varepsilon > 0$ however small there exists $\delta > 0$ such that $|f(x)-L| < \varepsilon$ whenever x is such that $0 < |x-a| < \delta$

Exercise: Use the above Cauchy definition of limit and show that $\lim_{x \to 3} (2x+1) = 7$

Solution: Let $\varepsilon > 0$ be any given number. Then we have to find δ such that $|(2x+1)-7| < \varepsilon$ whenever $0 < |x-3| < \delta$.

Now $|(2x+1)-7| = 2|x-3|$ iff $0 < |x-3| < \varepsilon/2$

Hence choose $\delta = \varepsilon/2$, so that $|(2x+1)-7| < \varepsilon$

for $0 < |x-3| < \delta = \varepsilon/2$.

$\lim_{x \to 3} (2x+1) = 7$

Exercise: Use the Cauchy definition of Limit and show that

$$\lim_{x \to 2} \left[\tfrac{1}{2}x - 4\right] = -3$$

Solution: Let $\varepsilon > 0$ be any given number.

Then $|(\tfrac{1}{2}x-4) - (-3)| < \varepsilon$ iff $|\tfrac{1}{2}x-1| < \delta$

$|(\tfrac{1}{2}x-4) - (-3)| < \varepsilon$ iff $\tfrac{1}{2}\,|x-2| < \delta$

$|(\tfrac{1}{2}x-4) - (-3)| < \varepsilon$ iff $0 < |x-2| < 2\varepsilon$

Choose $\delta = 2\varepsilon$, so that $|(\tfrac{1}{2}x-4) - (-3)| < \varepsilon$

whenever $0 < |x-2| < \delta$

Hence $\lim_{x \to 2} \left[\tfrac{1}{2}x-4\right] = -3$

Now we shall illustrate the use of this definition of limit in proving some of the important properties of limits.

Theorem: $\lim_{x \to a} c = c$ (c is any constant)

(i.e. limit of a constant is constant itself).

Proof: Let $\varepsilon > 0$ be given.

Then $|c-c| = 0 \;\forall\; x$ such that $0 < |x-a| < \delta$ where $\delta > 0$ can be any number. Because $|c-c| = 0$ is always true for any x and so in particular for x such that $0 < |x-a| < \delta$

$$\lim_{x \to a} c = c$$

Theorem: If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$

then $\lim_{x \to a} f(x) + g(x) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = L+M$

(i.e. limit of a sum is sum of limits).

Proof: Let $\varepsilon > 0$ be given. Then $\varepsilon/2 > 0$.

Since $\lim\limits_{x \to a} f(x) = L$ and $\lim\limits_{x \to a} g(x) = M$. By definition of limit there

exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

$|f(x) - L| < \varepsilon/2$ for $0 < |x-a| < \delta_1$ and

$|g(x) - M| < \varepsilon/2$ for $0 < |x-a| < \delta_2$

Let δ be the smaller of δ_1, δ_2 then

$|f(x)-L| < \varepsilon/2$ and $|g(x)-M| < \varepsilon/2$ for $0 < |x-a| < \delta$

$$
\begin{aligned}
\text{Now } |f(x)+g(x) - (L+M)| &= |f(x)-L) + (g(x)-M)| \\
&\quad |f(x)-L| + |g(x)-M| \\
&< \varepsilon/2 + \varepsilon/2 \;\forall\; x \text{ such that } 0<|x-a|<\delta
\end{aligned}
$$

$$\lim_{x \to a} f(x) + g(x) = L+M = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

On the same lines as above some more results on the limits may be proved. These results are given at the end as exercises.

Next we shall explain limits at infinity and infinite limits.

Let $f(x) = 1/x$

Let us examine behaviour of $f(x)$ as x approaches zero from right side. The closer x is to zero, the larger $f(x)$ is. In other words, as $x \to 0+$, $f(x)$ goes on increasing without bound. In this case, we write $\lim\limits_{x \to 0} 1/x = +\infty$ (Read ∞ as "plus infinity").

Similarly as $x \to 0-$, $f(x)$ goes on decreasing without bound and we write $\lim\limits_{x \to 0-} f(x) = \lim\limits_{x \to 0-} 1/x = -\infty$

(Read '$-\infty$' minus infinity).

Here, ∞ is a symbol showing the phenomenon of growing larger and larger without bound. Similarly $-\infty$ is a symbol showing the phenomenon of decreasing without bound. Thus ∞ and $-\infty$ are not numbers.

Next let us consider $\lim\limits_{x \to \infty} 1/x$. As x grows larger and larger the values of $1/x$ are close to zero. Therefore, we write $\lim\limits_{x \to \infty} 1/x = 0$.

Also as $x \to -\infty$, $1/x \to 0$ and so we write $\lim\limits_{x \to \infty} 1/x = 0$

However, we shall not attempt formal definitions of the above type of limits.

<u>Exercises</u> :

Use the Cauchy definition of limit to prove the following results.

1. If $\lim\limits_{x \to a} f(x) = L$ and $\lim\limits_{x \to a} g(x) = M$ then, show that

i) $\lim\limits_{x \to a} f(x) - g(x) = L-M$

ii) $\lim\limits_{x \to a} f(x) \cdot g(x) = L.M$

iii) $\lim\limits_{x \to a} f(x)/g(x) = L/M$ provided $M \neq 0$.

2. If $\lim\limits_{x \to a} f(x) = L$ and K a constant, then show that

$\lim\limits_{x \to a} K f(x) = K.L.$

3. Domination Principle

Let $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} g(x) = L$

Suppose $f(x) \leq h(x) \leq g(x) \quad \forall x$.

Prove that $\lim\limits_{x \to a} h(x) = L$

4. Use $\lim_{n \to \infty} \gamma n = 0$ to prove that i) $\lim_{n \to \infty} \gamma n^2 = 0$

ii) $\lim_{n \to \infty} \gamma n^2+n+1 = 0$

5. Given $h(x) = \dfrac{2x^2-7x+3}{x^2-2x-3}$

Find i) $\lim_{x \to 0} h(x)$ ii) $\lim_{x \to 1} h(x)$ iii) $\lim_{x \to -1} h(x)$

iv) $\lim_{x \to \infty} h(x)$

6. Consider the infinite geometric series $a+ar+ar^2+..+ar^{n-1}+..$

If $S_n = a+ar+...+ar^{n-1}$, define $S = \lim_{n \to \infty} S_n$

If $|r| < 1$, then prove that $S = a/1-r$

7. Consider the circle of radius r. Use the formula for the area $A = \pi r^2$ and show that the circumference C of the circle is given by the formula $C = 2\pi r$.

8. Evaluate the following :

i) $\lim_{x \to 0} \left[\dfrac{(1+x)^3 - (1-x)^3}{x + x^3} \right]$

ii) $\lim_{x \to 0} \left[\dfrac{\sqrt{a+x} - \sqrt{a-x}}{x} \right]$

iii) $\lim_{x \to 3} \left[\dfrac{1}{x^3} - \dfrac{1}{3^3} \Big/ x-3 \right]$

9. Prove that $\lim_{x \to 0} \left(\dfrac{e^x-1}{x} \right) = 1$

10. Show that $\lim_{x \to 0} \dfrac{a^x-1}{x} \neq \log_e a$

2. CONTINUITY AND DISCONTINUITY OF FUNCTIONS

2.1. Closely related to the limit concept is the concept of continuity. We begin with the assumption that you have some idea of continuity. Ourpurpose is to lead you from an intuitively concept to an appropriate mathematical definition through a discussion that primarily follows the historical development of continuity in mathematics.

Consider first the functions $f(x) = x$, and

$g(x) = \frac{|x|}{x}$ for $x \neq 0$. We observe that the graph of $f(x)$ can be drawn with an uninterrupted stroke of the pencil, whereas the graph of $g(x)$ has a gap at 0.

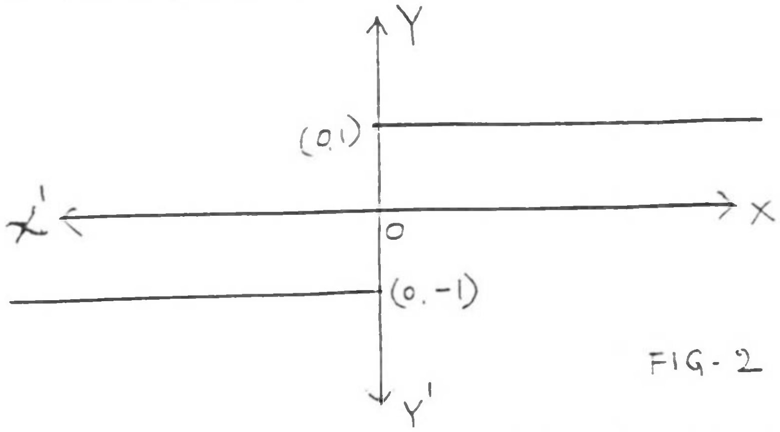


FIG- 2

Intuitively we feel that the graph of $f(x)$ is continuous while the graph of $g(x)$ is not continuous as there is a gap in the graph at 0. In fact $g(0)$ is not defined. Even if we define $g(0) = 0$ still the graph of $g(x)$ is not continuous. The reason is that $\lim_{x \to 0} g(x)$ does not exist. Hence one requirement for continuity of a function say $h(x)$ at a point 'b' is that $\lim_{x \to 6} h(x)$ must exist.

Now consider another function defined as follows :

$f(x) = x$ if $x \neq 0$

$= 2$ if $x = 0$

Here $\lim_{x \to 0} f(x) = 0$. Even though $\lim_{x \to 0} f(x)$ exists the graph of $f(x)$ is not continuous at 0. The reason is that $\lim_{x \to 0} f(x) \neq 2 = f(0)$. If we alter the definition of f at 0 and define $f(0) = 0$, then $f(x)$ becomes continuous at 0. From these illustrations we conclude that a function $f(x)$ is continuous at a point c if

i) $\lim_{x \to c} f(x)$ exists, ii) $f(c)$ is defined and

iii) $\lim_{x \to c} f(x) = f(c)$

Now we are in a position to give the mathematical definition of continuity of function at a point.

<u>Definition</u> : Let $f(x)$ be a function defined in an interval containing the point x_1. Then f is said to be continuous at x_1 iff i) $f(x_1)$ exists, ii) $\lim_{x \to x_1} f(x)$ exists iii) $\lim_{x \to x_1} f(x) = f(x_1)$.

If any one of these three criteria is not met, then f is said to be discontinuous at x_1. Earlier we gave Cauchy definition for limit of a function. Now we shall use this to give another definition of (usually called epsilon delta definition) of continuity.

<u>Definition</u> : Let $f(x)$ be a function defined in an interval containing 'a'. If $f(x)$ exists then f is said to be continuous at a iff given $\varepsilon > 0$ $\exists\ \delta > 0$ such that

$|f(x) - f(a)| < \varepsilon\ \forall\ x$ with $0 < |x-a| < \delta$

2.2 Continuity of a function on an interval

Let $f : I \longrightarrow R$ (R being set of all real numbers) be a function defined on an interval I. Then f is said to be continuous on I iff f is continuous at every point of I. Thus f is not continuous on I iff $\exists\, x \in I$ such that f is not continuous at x.

For instance consider the identity function $f(x) = x$ defined on any interval I, then f is continuous on I. Because if a is any point of I, then $f(a) = a$ and so $f(a)$ exists. Also

$$\lim_{x \to a} f(x) = \lim_{x \to a} x = a.$$

$$\lim_{x \to a} f(x) = a = f(a)$$

f is continuous at a. But a is an arbitrary point of I. Hence f is continuous at every point of I and so f is continuous on I.

Now we shall prove an important result on limits which is quite useful in deciding whether or not a given function is continuous at a point.

Let $f(x)$ be a function defined in an open interval containing a point 'a'. Then when $x \longrightarrow a$, x may approach 'a' through left side of a (or through those values of x for which $x \longrightarrow a$) or x may approach a through right side of a. If x approaches a from left side we write $x \longrightarrow a-$ similarly $x \longrightarrow a+$ means that x approaches a from right side.

Theorem : $\lim_{x\to a} f(x) = L$ (L is a real number)

if and only if $\lim_{x\to a+} f(x) = L = \lim_{x\to a-} f(x)$

Proof: First suppose $\lim_{x\to a} f(x) = L$

Let $\varepsilon > 0$ be given. Then $\exists$ $\delta > 0$ such that

$|f(x) - L| < \varepsilon$ whenever $0 < |x-a| < \delta$

If $a < x < a+\delta$, then $0 < |x-a| < \delta$ and so

$|f(x) - L| < \varepsilon$. Hence $\lim_{x\to a+} f(x) = L$

Similarly, $\lim_{x\to a-} f(x) = L$

Conversely suppose $\lim_{x\to a+} f(x) = \lim_{x\to a-} f(x) = L$

Let $\varepsilon > 0$. There exists $\delta_1 > 0$ such that if $a < x < a+\delta_1$

then $|f(x) - L| < \varepsilon$. Also $\exists$ a δ_2 such that if $a - \delta_2 < x < a$

then $|f(x) - L| < \varepsilon$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then if $|x-a| < \delta$

either $a < x < a+\delta_1$ or $a-\delta_2 < x < a$ so that $|f(x)-L| < \varepsilon$

$\lim_{x\to a} f(x) = L$

2.3 Discontinuous functions

Definition : A function $y = f(x)$ is said to be discontinuous at $x = a$ iff $f(x)$ is not conti nuous at a.

The discontinuity of $f(x)$ at $x = a$ can occur in any one of the following ways.

1. $\lim_{x \to a} f(x)$ does not exist.

2. $\lim_{x \to a} f(x)$ exists but is not equal to $f(a)$.

3. $\lim_{x \to a} f(x)$ is infinite.

Now we shall illustrate these possibilities by means of some examples.

<u>Illustration 1</u> : Let $f(x)$ be a function defined on $[0,2]$ as follows:

$$f(x) = x \quad \forall\, x \in [0,1)$$
$$\quad = x+1 \quad \forall\, x \in (1,2]$$
$$f(1) = 3/2$$

As x approaches 1 from the left side (i.e. $x \to 1-$) we have

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} x = 1$$

As x approaches 1 from right side, we have, $\lim_{x \to 1+} f(x) = \lim_{x \to 1} x+1 = 2$

Thus $\lim_{x \to 1-} f(x) \neq \lim_{x \to 1+} f(x)$

In this case $\lim_{x \to 1} f(x)$ does not exist because if

it exists then $\lim_{x \to 1-} f(x) = \lim_{x \to 1+} f(x) = \lim_{x \to 1} f(x)$

Such a discontinuity is called as ordinary discontinuity or discontinuity of first kind of $f(x)$ at $x = 1$.

Illustration 2

Let $f(x)$ $x = x$ $\forall x \in [0,2]$ and $x \neq 1$

$= 2$ if $x = 1$.

Then $\lim_{x \to 1+} f(x) = \lim_{x \to 1-} f(x) = \lim_{x \to 1} f(x) = 1$

But $f(1) = 2$.

Hence $\lim_{x \to 1} f(x) \neq f(1)$

Hence f is disconti nuous at $x = 1$.

But this discontinuity of f at $x = 1$ can be removed by altering the value of $f(1)$.

Instead of defining $f(1) = 2$ if we define $f(1) = 1$, then f becomes continuous at $x = 1$.

Hence this type of discontinuity of f is called as removable discontinuity.

Illustration 3

If neither $\lim_{x \to a+} f(x)$ nor $\lim_{x \to a-} f(x)$ exist then $f(x)$ is said to have a discontinuity of second kind at $x = a$.

For instance define a function f on $[0,1]$ by,

$f(x) = +1$ if x is rational

$= -1$ if x is irrational.

Then both $\lim_{x \to \sqrt{2}+} f(x)$ and $\lim_{x \to \sqrt{2}-} f(x)$ do not exist.

Hence f has second kind discontinuity at $x = \sqrt{2}$.

Illustration 4

If one of the two limits $\lim_{x \to a+} f(x)$, $\lim_{x \to a-} f(x)$ exists while the other does not exist then the point $x = a$ is called a point of mixed discontinuity for f.

For instance define a function $f(x)$ on $[1,2]$ as follows :

$f(x) = x$ for $0 \leq x < 1$

$$f(x) = \left.\begin{array}{l} 0 \text{ if } x \text{ is rational} \\ 1 \text{ if } x \text{ is irrational} \end{array}\right\} \forall x \in [1,2]$$

Then $\lim_{x \to 1} f(x) = 1$ but $\lim_{x \to 1+} f(x)$ does not exist.

Hence f has mixed discontinuity at $x = 1$.

Illustration 5 If either of the limits $\lim_{x \to a-} f(x)$. $\lim_{x \to a-} f(x)$ is infinite then $f(x)$ is said to have an infinite discontinuity at $x = a$.

Consider $f(x) = 1/x \quad \forall x \in (0,1]$

$= 0$ if $x = 0$

Then $\lim_{x \to 0+} f(x) = \infty$. Therefore, f has an infinite discontinuity at $x = 0$.

EXERCISES :

1. Let $f(x) = \dfrac{2x^4-6x^3+x^2+3}{x-1} = x \neq 1$.

   Is f continuous at $x = 1$?

   Explain the type of discontinuity f has at $x = 1$ if f is discontinuous at $x = 1$.

2. Let $f(x) = \frac{x}{x^2-1}$

Then find out the values of x at which f(x) is continuous.

3. Let $f(x) = \frac{x-|x|}{x}$ for $x \neq 0$, $f(0) = 1$.

Examine the continuity of f(x) at x = 0.

4. Find the points of discontinuity of the function

$$f(x) = \frac{x}{(x-2)(x-4)}$$

5. If f(x) is continuous at 'c', then show that there exists $\delta > 0$, such that f is bounded on $(c-\delta, c+\delta)$.

6. Give an example of a function defined on a closed interval such that the function is discontinuous at every point of that interval.

7. If f(x) is a continuous function on $[a,b]$ then show that f is bounded on $[a,b]$.

8. If f(x) is continuous on $[a,b]$ and $f(a) > 0$, $f(b) < 0$ then show that $f(x) = 0$ for some $x \in (a,b)$.

9. Let $f(x) = 2x+1$ when $x < 1$

$= 3$ when $x = 1$.

$= x+2$ when $x > 1$.

Show that f(x) is continuous at x = 1.

10. Let $f(x) = x$ when $0 \leq x < 1$

$= 3$ when $x = 1$

$= 2x+1$ when $x > 1$

Examine the continuity of f(x) at x = 1.

3. DERIVATIVES

3.1 Introduction :

Newton and Leibnitz had been able to solve independently the two basic problems viz. finding the tangent line to a curve at any given point and finding the area under a curve. The tools that Newton and Leibnitz indepencently invented to solve these two basic problems are now called the 'derivative' and the 'integral'. Moreover, one of the great bonanzas of history is that the derivative and integral which were invented to solve two particular problems, have applications to a great number of different problems in diverse academic fields.

The power of calculus is derived from two sources. First, the derivative and the integral can be used to solve a multitude of problems in many different academic disciplines. The second source of power is found in the relevancy of the calculus to the problems facing mankind. Among the present day, applications of the calculus are the building of abstract models for the study of the ecology of populations, management practices, economics and medicine.

3.2 Gradient of a curve :

The gradient of a curve at any point is defined as the gradient (or slope) of the tangent to the curve at this point. An approximate value for the gradient of a curve at a point can be found by plotting the curve, drawing the tangent by eye and measuring its slope. This method has to be used for a curve when the coordinates of a finite number of points are known, but its equation is not known. When the equation of a curve is known, an accurate method for determining gradients is necessary so that we can further our analysis of curves and functions.

Consider first the problem of finding the gradient of a curve at a given point A. If B is another point on the curve (not too far from A), then the slope of the chord AB gives us an approximate value for the slope of the tangent at A. The closer B is to A, the better is the approximation. In other words, as $B \longrightarrow A$, slope of chord $AB \longrightarrow$ slope of the tangent at A. Let us now consider an example where we can use this definition to find the gradient of a curve at a particular point of the curve.

For this purpose, we introduce the following symbolism. A variable quantity, prefixed by δ, means a small increase in that quantity,

δx is a small increase in x,

δy is a small increase in y.

Here δ is only a prefix and it cannot be treated as a factor.

Now consider the curve $y = x\,(2x-1)$ and the problem of finding gradient at the point on the curve where $x = 1$.

If $x = 1$, $y = 1$, let A be the point (1,1). Let B be a point on the curve very close to A. Then x coordinate of B is $1 + \delta x$ (where δx is very small or close to zero).

$$
\begin{aligned}
y \text{ coordinate of } B &= (1+\delta x)\,\left[2\,(1+\delta x) - 1\right] \\
&= (1+\delta x)\quad(2\delta x + 1)
\end{aligned}
$$

Slope of AB = increase in y/increase in x.

$$= \frac{(1 + \delta x)(2\delta x + 1) - 1}{(1 + \delta x) - 1}$$

$$= \frac{2(\delta x)^2 + 3\delta x}{}$$

$$= 2\delta x + 3$$

As B approaches A, $\delta x \to 0$

Hence gradient of the curve at A = $\lim\limits_{B \to A} \left[\text{slope of AB}\right]$

$$= \lim_{\delta x \to 0} \left[2\delta x + 3\right]$$

$$= 3$$

Now we found that the gradient of the curve $y = x(2x-1)$ is 3 at the point on the curve where $x = 1$. We will now derive a function for the gradient at any point on the curve. Then we can find the gradient at a particular point by substitution into this derived function. Instead of taking a fixed point on the curve, we shall take A as any point (x,y) on the curve. Let B be another point on the curve whose x coordinate is $x + \delta x$.

Then B is the point $(x + \delta x, \left[x + \delta x\right]\left[2x + 2\delta x - 1\right]$

The slope of chord AB $= \frac{(x + \delta x)(2x + 2\delta x - 1) - x(2x-1)}{x}$

$$= \frac{2x^2 + 4x\delta x + 2(\delta x)^2 - dx - x - 2x^2 + x}{\delta x}$$

$$= \frac{4x\delta x - \delta x + 2(-\delta x)^2}{dx}$$

$$= \left[4x - 1 + 2\delta x\right]$$

Then the gradient at any point A on the curve =

$$\lim_{\delta x \to 0} 4x-1 + 2\delta x$$

$$= 4x - 1.$$

So the function $4x-1$ gives the gradient at any point on the curve $y = x(2x-1)$.
We can now find the gradient of the curve at a particular point on $y = x(2x-1)$ by substituting the x coordinate of that point into the function $4x-1$. Thus the gradient of the curve at $x = 1$ is $4.1 - 1 = 3$ which we obtained earlier.

The function $4x-1$ is called the gradient function of $y = x(2x-1)$ and the process of deriving is called differentiation with respect to x. Since $4x-1$ was derived from the function $x(2x-1)$, it is called the derivative or derived function of $x(2x-1)$. Symbolically we write, $d/dx\left[x(2x-1)\right] = 4x-1$ where d/dx stands for "derivative w.r.t. x of". We also write $dy/dx = 4x-1$. Sometimes, we call dy/dx as "differential coefficient of y w.r.t. x". The above method of finding derivatives is called as "finding derivatives from first principles".

3.3 Equations of Tangents and Normals :

Now that we know how to find the gradient of a curve at a given point on the curve, we can find the equation of the tangent or normal to the curve at that point.

Illustration 1

Find the equation of the tangent to the curve $y = x^2-3x+2$ at the point where it cuts the y-axis
$y = x^2-3x+2$ cuts the y-axis where $x = 0$ and $y = 2$.
The slope of the tangent at $(0,2)$ = the value of dy/dx when $x = 0$.

$$= \left[d/dx \left[x^2-3x+2 \right] \right]_{x=0} = \left[2x-3 \right]_{x=0} = -3$$

Thus the tangent is a line with slope -3 and passing through $(0,2)$.

So its equation is $y-2 = -3\ (x-0)$.

Hence the desired equation is $y = -3x+2$.

Illustration 2

Find the equation of the normal to the curve $y = x^2 + 3x-2$ at the point where the curve cuts the y-axis.

As shown in the illustration 1, the slope of the tangent to the curve at $(0,2)$ is -3.

Hence the slope of normal to the curve at $(0,2)$ is $1/3$.

Hence the equation of normal to the curve at $(0,2)$ is given by $y-2 = 1/3\,x$ or $3y = x + 6$.

Exercises :

1. Differentiate the following functions w.r.t. x from first principles.

i) $y = x^2$ ii) $y = 3x^2$, iii) $y = 1/x^2$ iv) $y = x^3+3$

v) $y = x^2 - 2x + 1$

2. Find the equation of the tangent to the curve $y = x^2+5x-2$ at the point where this curve cuts the line $x = 4$.

3. Find the equations of the normals to the curve $y = x^2-5x+6$ at the points where the curve cuts the x-axis.

4. Find the coordinates of the point on $y = x^2$ at which the gradient is 2. Hence find the equation of the tangent to $y = x^2$ whose slope is 2.

5. Find the value of K for which $y = 2x + K$ is a normal to $y = 2x^2-3$.

6. Find the equation of the normal to $y = x^2-3x+2$ whose slope is 2.

7. Find the equation of the tangent to $y = 2x^2-3x$ whose slope is 1.

8. Find the equation of the tangent to $y=(x-5)(2x+1)$ which is parallel to the x-axis.

A P P L I C A T I O N S   O F   D E R I V A T I V E S

1. Mean Value Theorem
2. Derivative as Rate Measurer
3. Differentials and Approximations

APPLICATIONS OF MEAN VALUE THEOREM

The Mean Value Theorem for derivative is of great importance in Calculus because, many useful properties of functions can be deduced from it. A special case of this result known as Rolle's theorem was first proved by Michael Rolle, a French Mathematician in 1691. A formal statement of the Mean Value Theorem is given here for convenience.

(Ref: Th. 4.10 of the Textbook)

Statement : Let f be a real function, continuous on the closed interval $[a,b]$ and differentiable in the open interval (a,b), then, there is a point $C \in (a,b)$ such that

$$\frac{f(b) - f(a)}{b-a} = f^1(c) \qquad (1)$$

C is called a mean value.

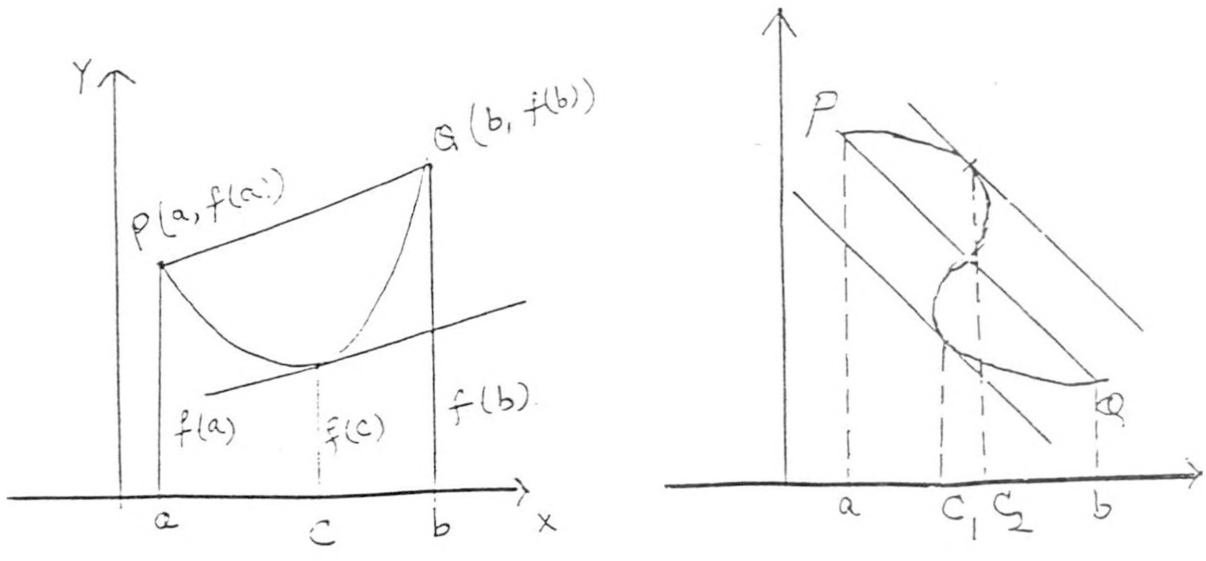


fig. 1

Intuitively (1) can be interpreted thus - If we assume $f(t)$ to be the distance travelled by a movingparticle at time t. Then the lefthand side of (1) represents the mean or average speed in the time interval a,b and the derivative $f^1(t)$ on Rhs represents the instantaneous speed at time t. (1) asserts that at some instant C during the motion of the particle, the average speed is equal to the instantaneous speed.

Geometrically, (1) implies that the slope of the tangent at (C. f(c)), in fig.1. [$(C_1, f(C_1))$ and $(C_2, f(C_2)$ in fig.2], is equal to the slope of the chord PQ.

This is seen in the figure by the fact that the chord PQ is parallel to the tangent line at C (in fig.1) (and at C_1 and C_2 in figure 2).

There may be two or more mean values also on a given interval, depending on the graph of f.

Although the M.V. Theorem guarantees that there will be atleast one mean value for a function whose graph is a smooth curve on a given interval, the theorem gives no information about the exact location of these mean values. We just know that the point C lies somewhere between a and b. Generally, an accurate location of C is difficult. Many useful conclusions can be drawn by simply knowing about the existence of atleast one mean value.

Some Consequences of Mean Value Theorem :

1. A generalization of M.V.Theorem can be obtained by considering the parametric representation of a function whose graph is a smooth curve on $[a,b]$

Let $x = g(t)$, $y = f(t)$; $a \leq t \leq b$ .... (2)

be the parametric form of the given function.

Slope of the chord joining the end points $(g(a).f(a))$ and $(g(b), f(b))$ of the curve $= \dfrac{f(b) - f(a)}{g(b) - g(a)}$ ... (3)

The slope of the tangent to the curve the point C

$$= \frac{f^1(c)}{g^1(c)} \quad \ldots\ldots \quad (4)$$

The Mean Value Theorem asserts that there always exists a mean value C in (a.b) for which

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f^1(c)}{g^1(c)} \quad a < c < b \quad \text{---} (A)$$

$$g^1(c) \neq 0$$

(A) is referred to as Cauchy's M.V. Theorem.

2. <u>Algebraic sign of the first derivative of a function</u> gives useful information about the behaviour of its graph. Using Mean Value Theorem, the algebraic sign of the derivative of a given function can be determined.

Theorem: Let f be continuous on $[a,b]$ and derivable in (a,b), then,

a) If $f^1(x) > 0 \ \forall x \in (a,b)$, then f is strictly increasing on $[a, b]$

b) If $f^1(x) < 0 \ \forall x \in (a,b)$ then f is strictly decreasing on $[a, b]$

c) If $f^1(x) = 0 \ \forall x \in (a,b)$, then f is a constant.

Proof : (a) For any points x_1 and x_2 with $a \leq x_1 < x_2 \leq b$, the Mean Value Theorem applied to $[x_1, x_2]$ gives

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f^1(c); \quad x_1 < c < x_2 \quad --- (5)$$

Since $f^1(c)$ is given to be > 0 and $x_2 - x_1 > 0$, wesee that $f(x_2) - f(x_1) > 0$ implying that $f(x_1) < f(x_2)$ or f is strictly increasing on $[a,b]$.

Proof of (b) is left as an exercise.

Proof of (c) : Put $x_1 = a$ in (5).

We get $$\frac{f(x_2) - f(a)}{x_2 - a} = f^1(c) \quad \ldots \quad (6)$$

Since $f^1(c) = 0$, $(6) \Rightarrow f(x_2) = f(a) \quad \forall x_2 \in [a, b]$

Hence f is a constant on $[a,b]$

Using this result, it is possible to determine the intervals of increase and decrease of functions.

The well-known sufficient condition for the existence of an extrema for a function also follows from the above theorem.

3. The Mean Value Theorem can be used to show that : Any two integrals of the same derived function can differ atmost by a constant.

Proof : Suppose $F(x)$ and $G(x)$ have the same derivative $f(x)$ over some interval $a \leq x \leq b$.

Consider $H(x) = F(x) - G(x)$ ... (1) (1)

Apply Mean Value Theorem to $H(x)$ on $[a,c]$

where C is : $a \leq c \leq b$ to obtain

$H(c) - H(a) = H^1(\xi)(c-a), \quad a \leq \xi \leq c.$

Since $H^1(x) = F^1(x) - G^1(x) = 0$ by hypothesis, $x \in [a, b]$

$H(c) - H(a) = 0$ and so $H(c) = H(a)$

$\Rightarrow F(c) - G(c) = F(a) - G(a)$ where

$F(a) - G(a)$ is a fixed quantity. Let $F(a) - G(a) = C$

Since c is any value of x in $[a,b]$,

we have $F(c) - G(c) = C, \forall c \in [a,b]$

$\therefore F(x)$ and $G(x)$ can differ by a constant C.

Now, $F(x)$ and $G(x)$ which are any two integrals of $f(x)$ can differ only by a constant C.

Differentials and Mean Value Theorem

Recall that the differential dy of a function $y = f(x)$ is defined by the equation

$$dy = f^1(x) \cdot \Delta x = f^1(x)\, dx \text{ for small } \Delta x.$$

Here, dy is an approximate value of Δy, we know that,

$$\Delta y = f(x + \Delta x) - f(x) \quad \ldots \quad (2)$$

Can we improve this approximation ?

Mean Value Theorem helps us to answer this question.

Now, instead of considering x and $x + \Delta x$ let us consider any two values of x say, a and b.

Then we get $\Delta y = f(b) - f(a)$.... (3)

and $dy = f^1(a)\ (b-a)$ .... (4)

Since $dy \approx \Delta y$, $f(b) - f(a) \approx f^1(a)\ (b-a)$ (5)

Can we now find an approximation to $f(b) - f(a)$, which is better than $f^1(a)\ (b-a)$?

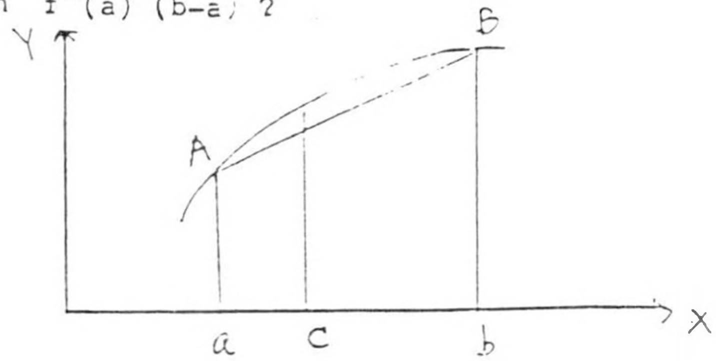


fig. 2.

From the figure, $\dfrac{f(b) - f(a)}{b-a}$, is slope of chord AB. But by Mean Value Theorem there exists a C; $a < c < b$, such that slope of AB = Slope of tangent at $(C, f(c)) = f^1(c)$.

$$\therefore \frac{f(b) - f(a)}{b-a} = f^1(c) \implies f(b) - f(a) = (b-a)\ f^1(c) \quad ...(6)$$
$$a < c < b$$

Comparing (5) and (6) we see that (6) results from (5) when we replace a by c in $f^1(a)$, c being the mean value. Also, (6) is an estimate of $\Delta y = f(b) - f(a)$. In fact (6) gives an exact expression for Δy or $f(b) - f(a)$, whereas (5) gives a mere approximation to

$f(b) - f(a)$ or Δy. Hence we have proved that the approximation of Δy by the differential dy can be bettered by using the Mean Value Theorem. For such an improved approximation of Δy, Δx need not be very small.

(5) If for a given function $y = f(x)$ derivable on (a,b) and continuous on $[a,b]$ we further assume that $f^1(x)$ is continuous on $[a,b]$, then f^1 ought to attain its maximum and minimum values (bounds) atleast once on $[a,b]$. By Mean Value Theorem, we have

$$\frac{f(b) - f(a)}{b - a} = f^1(c), a < c < b \quad .....(*)$$

(*) now implies that $f^1(c)$ cannot exceed max. f^1 nor can it be less than min. f^1 on $[a,b]$. So, we obtain

$$\text{Least value of } f^1 \text{ on } [a,b] \leq \frac{f(b) - f(a)}{b - a} \leq \text{greatest value of } f^1 \text{ on } [a,b]$$

or

$$\underset{x \in [a,b]}{\text{Min}} f^1(x) \leq \frac{f(b) - f(a)}{b - a} \leq \text{Max } f^1(x) \quad x \in [a.b] \quad \ldots \quad (1)$$

(1) can now be used to restate the Mean Value Theorem as follows : The mean value of a continuous function on a closed interval must actually be a value attained by the function.

(1) can also be used to estimate the value of a function at a given point when a and f^1 are known.

Assignment Problems

1. Use Mean Value Theorem to deduce the following inequalities :

a) $|\operatorname{Sin} x - \operatorname{Sin} y| \leq |x-y|$

b) $ny^{n-1}(x-y) \leq x^n - y^n \leq nx^{n-1}(x-y)$

if $0 < y < x$, $n = 1,2,3,\ldots.$

2. The function $y = |4-x^2|$, $-3 \leq x \leq 3$ has a horizontal tangent at $x = 0$ even though the function is not differentiable at $x=-2$ and $x=2$. Does this contradict Mean Value Theorem? Explain.

3. A motorist drove 30 miles during a one hour trip. Show that the Car's speed was equal to 30 miles/hour atleast once during the trip.

4. Show that

$$\frac{d}{dx}\left(\frac{x}{x+1}\right) = \frac{d}{dx}\left(-\frac{1}{x+1}\right)$$

even though $\frac{x}{x+1} \neq \frac{-1}{x+1}$

Explain.

5. Show that the Mean Value Theorem can be given by the equation.

$$\frac{f(x+h) - f(x)}{h} = f^1(x + \theta h), \quad 0 < \theta < 1.$$

Determine θ as a function of x and h when

a) $f(x) = x^2$ (b) $f(x) = e^x$

c) $f(x) = \log x$, $x > 0$

DERIVATIVE AS A RATE MEASURER

Consider a particle P moving in a straight line. Its motion can be described by the function

$S = f(t)$, where S is the position of P at any time instant t. Let V be the Velocity of the moving particle P, at the time instant t. We wish to obtain V as the derivative $f^1(t)$.

Recall that the average velocity of P in a time interval Δt is the difference quotient $\frac{\Delta s}{\Delta t}$ and

$$\frac{\Delta s}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{(t + \Delta t) - t} = \frac{(S + \Delta S) - S}{(t + \Delta t) - t} \quad --- \quad (1)$$

V, the Instantaneous velocity of P at time t. is now computed from the values of $\frac{\Delta s}{\Delta t}$ for progressively smaller values of Δt. This leads to V as $\lim\limits_{\Delta t \to 0} \frac{\Delta s}{\Delta t}$

$$\text{or } V = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{t} = f^1(t) \quad --- \quad (2)$$

(2) Implies that when the position function $S = f(t)$ of a moving particle is known, the rate of motion of the particle w.r.t. time can be given by the derived function $f^1(t)$.

When the motion of P is uniform, the average velocity itself represents the instantaneous velocity, as the velocity of motion remains constant at all instants of time.

If P moves with variable velocity, then average velocity $\frac{\Delta s}{\Delta t}$ differs with differing values of Δt. By taking an instant 't' as a time-interval of length zero, (an instant is at a point of time) $\frac{\Delta s}{\Delta t}$ reduces to $\frac{0}{0}$ for a given instant of time 't', which is meaningless. However, for small values of Δt, $\frac{\Delta s}{\Delta t}$ gives approximate values of instantaneous velocity V. Hence it is reasonable to define V with the aid of the limit concept. Thus,

$$V = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = f^1(t)$$

Note: V is independent of the increment Δt, but depends on the value of t and the type of function f(t).

Variable Physical magnitudes as derivatives: More examples.

1. Acceleration : When the velocity function $v = f(t)$ of a particle performing non-uniform motion is known, the instantaneous rate of change of its velocity (acceleration) is computed by

$$\text{Acceleration} = \frac{dv}{dt} = f^1(t) = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}$$ when the

quotient $\frac{\Delta v}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t}$ is the average acceleration.

2. Heat Capacity: as a derivative

Let $q = H(t)$ give the quantity of heat q, absorbed by a physical body when heated to the temperature t. Heat capacity C is the rate of change of the quantity of heat absorbed w.r.t. temperature. C is expressed as a derivative. If Average Heat Capacity C_{av} is the quotient $\Delta q / \Delta t$,

then, $$C = \lim_{\Delta t \to 0} C_{av} = \lim_{\Delta t \to 0} \frac{\Delta q}{\Delta t} = \lim_{\Delta t \to 0} \frac{H(t + \Delta t) - H(t)}{\Delta t}$$

$$= H^1(t).$$

3. <u>Reaction rate of a chemical reaction</u>

Let the function $m = \phi(t)$ represents the mass of a chemical substance entering into a chemical reaction during time t.

The rate of change of mass of the substance w.r.t. the time t is called the reaction rate. This can be expressed as a derivative.

If Average reaction rate R_{av} for the time interval Δt is given by the quotient

$$\frac{\Delta m}{\Delta t} = \frac{\phi(t + \Delta t) - \phi(t)}{\Delta t}$$

Then the reaction rate R for a given amount of substance at time t is

$$R = \lim_{\Delta t \to 0} \frac{\Delta m}{\Delta t} = \lim_{\Delta t \to 0} R_{av} = \phi^1(t)$$

The above examples show how derivatives are used to express certain variable physical magnitudes as rates of change w.r.t. some other physical magnitudes.

In general, the derivative of a function estimates the rate of change of a given function. Hence

$$f^1(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

gives the measure of the rate at which f(x) changes with respect to x, at a given point x.

Related rates - Problems :

Before attempting to solve some problems, we recall the chain rule, as it is often tailor-made in solving the related rates problems.

If $Z = f(y)$ and $y = g(x)$

then $\dfrac{dz}{dx} = \dfrac{dz}{dy} \cdot \dfrac{dy}{dx}$ ... (1)

where $\dfrac{dz}{dy} = f^1(y)$ and $\dfrac{dy}{dx} = g^1(x)$

(1) Tells us that the rate of change of Z w.r.t. X is the product of the rate of change of Z w.r.t. y and the rate of change of y w.r.t. x.

Problem 1. A variable right triangle ABC in the xy-plane has its right angle at the vertex B, a fixed vertex A at the origin and the third vertex C restricted to lie on the parabola $y = 1 + \frac{7}{36} x^2$. The point B starts at (0,1) at time t = 0 and moves upward along the y axis at a constant velocity of 2 cm/sec. How fast is the area of the triangle increasing when t = 7/2 sec ?

Solution: Clearly the moving vertex C of the expanding triangle has for its coordinates C (x,y) where x is the base and y the height of the triangle, x and y are both variables. C(x,y) satisfies the equation $y = 1 + \frac{7}{36} x^2$. Note that the triangle remains right angled while varying in its size.

The velocity of the moving vertex B along y axis (= dy/dt) is a constant (=2 cm/sec) ∴ The equations relating the variables x, y and t are

(1) ... Area $A = \frac{1}{2} xy$

(2) $y = 1 + \frac{7}{36} x^2$

(3) $y = 1 + 2t$

(4) $7x^2 = 7\ 2t$

We must find $\frac{dA}{dt}$ at $t = 7/2$ sec.

$\frac{dA}{dt} = \frac{1}{2}\ (x.dy/dt + y.dx/dt)$.... (5)

Substituting $x = 6$ and $y = 8$, (found from (3) and (4) for $t = 7/2$)

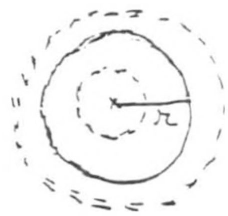
and using the values $\frac{dx}{dt} = \frac{6}{7}$ and $\frac{dy}{dt} = 2$ in the equation (5)

We obtain $\frac{dA}{dt} = \frac{66}{7}$ cm^2/sec at $7/2 = t$

$\therefore$ The triangle is increasing its area at the rate of 66/7 cm^2/sec.

Problem 2. A stone is dropped into a quiet pond and waves move in circles outward from the place where it strikes, at a speed of 3" per second. At the instant when radius of one of the wave rings is three feet, how fast is its enclosed area increasing ?

Solution: Radius r and area A are the variables. The equation relating the variables are



$A = \pi r^2$ so that $\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$

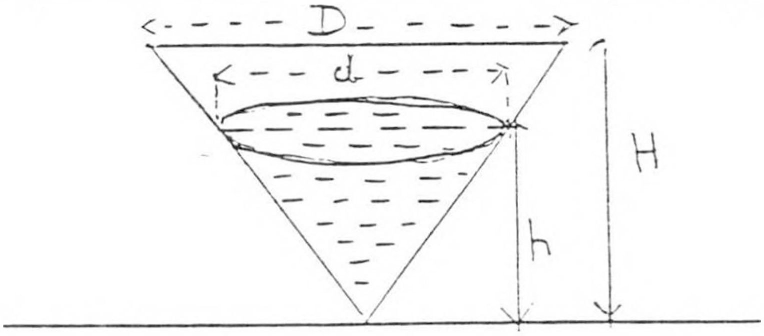
The speed of the wave outward from the center is the rate at which the radius increases $= \frac{dr}{dt}$

$\frac{dr}{dt} = 3''$ /sec. At $r = 3$ the rate of

increase in area $= \frac{dA}{dt} = 2\ \ 3 . \frac{1}{4} = \frac{3}{2} = 4.71$ sq. ft/sec.

Problem 3 : Water runs into conical paraffin paper cup five inches high and 3 inches across the top at the rate of one cubic inch per sec. When it just half filled, how rapidly is the surface of the water rising ?

Solution: The height (H) and the diameter (D) of the conical cup are the given constants. Let h be the height of the surface of water in the conical cup, when the volume of the water already in the cup is V. h and d (the diameter of water in the cup) are both variables.



The rate of increase in the volume of water = rate of inflow of water into the cup = dv/dt = 1 cubic inch per second. The rate of rise in the surface of water in the cone = rate of increase of height h = dh/dt

V = Volume of the conical cup = $\frac{1}{3}\pi \frac{D^2H}{12} = 11.7$

$\frac{1}{2}V = \frac{11.7}{2} = 5.85$ cu. inc. is the volume of water in the cup when first half filled.

We must find dh/dt when v = 5.85 and dv/dt = 1.

V, Volume of water in the cup =

$$V = \frac{\pi d^2 h}{12} \Rightarrow hd^2 = \frac{12}{\pi} V \qquad (1)$$

(1) relates the variables h and V, but also contains 'd'. We must express d in terms of h or V.

We have $\frac{d}{h} = \frac{D}{H} = \frac{3}{5} = .6$

$$d = .6h \qquad (2)$$

Using (2) in (1)

$$h.\ (.36\ h^2) = \frac{12\ V}{\pi} = .36\ h^3$$

$$h = \sqrt[3]{\frac{12}{.36\pi}}\ .\ \sqrt[3]{V}$$

After computing cube roots, we can write $h = 2.2\ V^{1/3}$

$$\frac{dh}{dt} = 2.2\ \frac{1}{3}\ V^{-2/3}\ \frac{dV}{dt}$$

$= \frac{.74}{\sqrt[3]{V^2}} \cdot \frac{dV}{dt}$ when V = 5.85 (half filled) and $\frac{dV}{dt} = 1$

$$\frac{dh}{dt} = \frac{.74}{\sqrt[3]{(5.85)^2}}\ .1 = \frac{.74}{\sqrt[3]{35.2}} = .23\ \text{in/sec.}$$

Problem 4: A balloon is rising vertically from the ground at a constant rate of 15 ft/sec. An observer situated at a point P 160 ft away from the point of lift-off tracks it. Find the rate at which the angle at P and the range r are changing when the balloon is 160 ft. above the ground.

Solution: Variables are angle θ and the range r,

From the figure, $\tan\theta = \frac{h}{160}$ (1)

Differentiating (1) on both sides w.r.t. t

$$\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{1}{160} \cdot \frac{dh}{dt} \quad \ldots \qquad (2)$$

At h = 160 (1) gives $\tan\theta = 1$ $\quad \theta = \pi/4$.

$\sec^2\theta = (\sqrt{2})^2 = 2$; $\frac{dh}{dt} = 15$ ft/sec. (given). $\therefore$ (2) becomes

$2. \frac{d\theta}{dt} = \frac{1}{160} \times 15 \qquad \frac{d\theta}{dt} = \frac{15}{320}$ rad/sec. $= \frac{3}{64}$ radians/sec.

Angle P is increasing at the rate of $\frac{3}{64}$ radians/sec when h = 160 ft.

<u>Now to find the rate of change of the range r</u>

From the figure, $h^2 + 160^2 = r^2$ (3)

(Note h and r are variables)

differentiating (3) w.r.t. t.

$$2h \cdot dh/dt = 2r \cdot \frac{dr}{dt} \qquad (4)$$

when h = 160, $r = \sqrt{160^2+160^2} = 160\sqrt{2}$

$$\frac{dh}{dt} = 15 \text{ ft/sec.}$$

$$\frac{dr}{dt} = \frac{160}{160\sqrt{2}} \cdot 15 = \frac{15}{\sqrt{2}} \quad \frac{15\sqrt{2}}{2} \text{ ft/sec.}$$

Range r is varying at the rate of $\frac{15\sqrt{2}}{2}$ ft/sec.

A step by step guide to solve related rates problems :

1. Draw a figure. Name the variable and constant magnitudes. Label these in the figure.
2. Mark the variable/variables whose rate/rates of change you must find.
3. Form equations relating variable and constants.
4. Substitute known values (if necessary) and differentiate. Obtain a single equation expressing the rate that you want in terms of the rates and quantities already known.

Problems for Assignment :

1. Suppose a rain drop is a perfect sphere. Assume that through condensation, the rain drop accumulates moisture at rate proportional to the surface area. Show that the radius increases at a constant rate.
2. A balloon 200 ft off the ground and rising vertically at the constant rate of 15 ft/s. An automobile passes beneath it travelling along a straight road at the constant rate of 45m/hour. How fast is the distance between them changing one second later ? (Ans. 33.7 ft/sec).
3. A light is at the top of pole 50 ft high. A ball is dropped from the same height from a point 30 ft away from the height. How fast is the shadow of the ball moving along the ground 1/2 second later ? (Ans. 1500 ft/sec.).

4. Two ships A and B are sailing straight away from the point D along routes such that the angle AOB = 120°. How fast is the distance between them changing, if at a certain instant DA=8 miles? Ship A is sailing at the rate of 20 miles/hr and ship B at the rate of 30 miles/hr ? (Hint: Use law of Cosines) 260/37 miles/hr.

5. A particle is moving in the circular orbit $x^2+y^2=25$. As it passes through the point (3,4), its Y-coordinate is decreasing at the rate of 2 units per second. How is the X-coordinate changing ? (Ans: 8/3 units/sec).

Additional Problems for Assignment :

1. Find the height of a right cone with least volume circumscribed about a given sphere of radius R. (Ans.4R)
2. It is required to make a cylinder, open at the top the walls and the bottom of which have a given thickness. What should be the dimensions of the cylinder so thatfor given storage capacity, it will require the least material ? (Ans. R = 3 V/R is the inner radius of the base, V = inner volume).
3. Out of sheet metal having the shape of a circle of radius R, cut a sector such that it may be bent into a funnel of maximum storage capacity. (Ans. The central angle of the sector = $2\pi\sqrt{2/3}$
4. Of all circular cylinders inscribed in a given cube with side a so that their axis coincide with the diagonal of the cube and the circumferences of the base touch its planes. Find the cylinder with maximum volume.

$$h = \frac{a\sqrt{3}}{3}$$

$$r = \frac{a}{\sqrt{6}}$$

5. In a rectangular coordinate system a point (X_0, Y_0) is lying in the first quadrant. Draw a straight line through this point so that it forms a triangle of least area with the positive directions of the axis.
(Ans. $X/2X_0 + Y/2Y_0 = 1$).
6. Given a point in the axis of the parabola $Y^2 = 2px$ at a distance of a from the vertex, find the abscissa of the point of the curve closest to it. (Ans. $X = a-p$).
7. Assuming that the strength of a beam of rectangular cross-section is directly proportional to the width and to the cube of the altitude, find the width of a beam of maximum strength that may be cut out of a log of diameter 16 cms. (Ans. width = 8 cm).
8. A torpedo boat is standing at anchor 9 km from the closest point of the shore. A messenger has to be sent to a camp 15 km (along the shore) from the point of the shore closest to the boat. Where should the messenger land so as to get to the camp in the shortest possible time ? (if he does 5 kms/hr walking and 4 km/hr rowing). (Ans. at a point 5 km from the camp).
9. Show that the volume of the largest right circular cylinder which can be inscribed in a given right circular cone is 4/9 the volume of the cone.
10. If sum of the surface areas of cube and a sphere is constant, what is the ratio of an edge of the cube to the diameter of the sphere when a) the sum of their volumes is a minimum? b) the sum of their volumes is a maximum ?

11. A lamp 50 ft above the horizontal ground and a stone is dropped from the same height from a point 12 ft away from the lamp. Find the speed of the shadow of the stone on the ground when the stone has fallen 10 ft.

12. The volume of a certain mass of a gas under pressure P lbs wt/sq inch is v cu.inches where PV = 1200. If the volume increases at the rate of 40 cubic inches/min. find the rate of change of pressure when vol = 20 c.inches,

(Ans. 120 lbs/min).

13. A circular blot of ink on a blotting paper expands in such a way that the radius r cms at t secs is given by

$$r = t - \frac{1}{8t^2}.$$

Find the rate at which the blot is increasing at the end of 2 seconds. (Ans. $\frac{2079\pi}{512}$).

DIFFERENTIALS AND APPROXIMATIONS.

In this section, we attempt to define derivative as a quotient of two quantities called differentials and see how this definition is useful in carrying out approximate calculations.

Recall that, derivative $f^1(x)$ of a given function $y = f(x)$ is defined as the limit of a quotient,

i.e. $$f^1(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = dy/dx$$

Note that $f^1(x)$ itself is not a quotient.

It is wrong to interpret that dy/dx is obtained by dividing dy by dx, Where $dy = \lim_{\Delta x \to 0} \Delta y$ and $dx = \lim_{\Delta x \to 0} \Delta x$

This interpretation leads to the result 0/0.

However, using the notion of derivative as a limit, it is possible to define a new quantity 'dy' called the differential of y so that the quotient dy/dx will indeed become equal to the derivative $f^1(x)$.

Meaning of differential :

Consider $y = f(x)$, derivable at x.

Then, $$f^1(x) = dy/dx = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \qquad (1)$$

(1) implies that $\frac{\Delta y}{\Delta x}$ differs from $\frac{dy}{dx}$ (or $f^1(x)$) by an infinitesimally small quantity ε.

(Here Δx, ε, are examples of infinitesimals).

$\therefore \frac{\Delta y}{\Delta x} = f^1(x) + \mathcal{E}$ or

$\Delta y = f^1(x).\Delta x + \mathcal{E}.\Delta x$ (1)

The term $\mathcal{E}.\Delta x$ in (1) being the product of the infinitesimals, is much smaller when compared with the term $f^1(x).\Delta x$.
For Δx sufficiently small, we see that $f^1(x).\Delta x$ is a good approximation of Δy if we neglect the term $\mathcal{E}.\Delta x$.

Now let us define the differential dy of y by $dy = f^1(x).\Delta x$
Denoting dx, the differential of x as Δx itself (why ?)
We obtain $dy = f^1(x).dx$ and $dx = \Delta x$ so that,
The derivative $f^1(x)$ = the quotient $\frac{dy}{dx}$

$f^1(x)$ = the quotient of the differentials dy and dx.

<u>Illustration</u>: (1) Consider a square of side x units. An error of .01 has crept into the measurement of its side. Estimate the error in its area.
Let us take x = 12 units.
Error in the measurement of x = .01
$\therefore \Delta x = .01$
If the function in question is $y = x^2$
then, $y = 2x.\Delta x + (\Delta x)^2 = 2 \times 12 \times (.01) + (.01)^2$
whereas $dy = f^1(x).dx = f^1(x).\Delta x = 24x.01$
neglecting $(.01)^2$; $\Delta y \approx dy$.
Error in this estimation is .0001.

(2) To see the advantage gained by approximating

Δy by dy, consider $f(x) = x^4$

$$\Delta y = 4x^3 \cdot \Delta x + 6x^2(\Delta x)^2 + 4x \cdot (\Delta x)^3 + (\Delta x)^4$$

For small Δx, the powers of Δx get progressively smaller.

Replacing Δy by dy,

$dy = f^1(x) . \Delta x = 4x^3 . \Delta x$ is a good approximation to Δy.

It is worth noting here, how much simpler it is to compute dy as compared to Δy.

When the functions under investigation get more complex, the usefulness of approximating Δy by dy becomes even more pronounced.

<u>The geometric meaning of differential</u>.

Refer to the figure 4.22 given in the text book.

A variation of the same figure is supplied here.

Geometrically, the approximation by the differential is the tangent line approximation to the curve $y = f(x)$ at a given point $P(x, y)$. Note that the tangent to a differentiable curve always runs close to the curve near the point of tangency.

From the figure it is clear that Δy and dy are not the same. While Δy gives the actual change in the function $y = f(x)$ as x changes to $x + \Delta x$, dy gives the increment in the function represented by the tangent line to the curve $y = f(x)$ at $P(x,y)$. In other words, if the function $y = f(x)$ were replaced by its tangent line at P, dy would be the increment in the function representing the tangent line corresponding to the increment dx in x. The slope of this tangent line is $f^1(x)$ at $P(x,y)$. The difference in $\triangle y$ and dy is the vertical portion of $\triangle y$ between the tangent line and the graph of $f(x)$. The less the graph curves, nearer is it to the tangent line and better, the approximation is dy to $\triangle y$.

. Errors and approximate calculations :

1. Differentials are used to estimate the square roots, cube roots, fourth roots and so on. (Ref. text).

2. Estimation of small errors: Physical measurments using instruments are subject to small errors. Differentials are used to estimate the accuracy and the error involved in measurements.

For example, when the diameter (d) of a small steel ball is measured by a vernier and if the reading is correct to $\frac{1}{100}$ of an inch. The true measurement differs from the vernier reading by $\frac{1}{100}$ th of an inch.

If Δx is the error in the measurement of a magnitude x, the corresponding error which results in $y = f(x)$ is approximately $\Delta y = f^1(x) . \Delta x = dy$. This error is called the absolute error.

The ratio of this error Δy to the magnitude y is $\frac{\Delta y}{y}$, and is called relative error.

$100 . \frac{\Delta y}{y}$ is called the percentage error in y.

Now, going back to the problem of steel balls, the actual measurement gives the diameter as $d + \Delta x$. The relative error here is $-\frac{\Delta x}{d}$. Now we want to find the corresponding error in the volume of the sphere.

Volume of the sphere $= V(d) = \frac{1}{6} \pi d^3$

$$\Delta V \approx dV = \frac{1}{2} \pi d^2 . \Delta x.$$

Hence the relative error in the volume is

$$\frac{\Delta V}{V(d)} \approx \frac{dV}{V(d)} = \frac{\frac{1}{2}\pi d^2 \cdot \Delta x}{\frac{1}{6}\pi d^3} = \frac{3}{d}\Delta x = 3 \cdot \frac{\Delta x}{d}$$

= 3 times the relative error in the diameter.

Example 1 : If $f(x) = x^4 - 4x^2 + 7x - 5$

find $f(2.99)$.

Here we take $x = 3$, and Δx $-.01$

$f^1(x) = 4x^3 - 8x + 7$

$f^1(3) = 91$, $f^1(x) . \Delta x = f^1(3) . \Delta x = -0.91$

$f(2.99) = f(3) + f^1(3) . \Delta x$

$= 61 + (-0.91) = 60.09 = 60.09$

Example 2 : Find the linear approximation to

$f(x) = \sqrt{1 + 2x}$ near $x = 2$.

We must evaluate $f(2) + f^1(2)\ (x-2)$

taking $\Delta x = (x-2)$

$$f^1(x) = \frac{1}{2}(1+2x)^{-1/2} . 2 = \frac{1}{\sqrt{1+2x}}$$

Its value at $x = 2$ is

$$f^1(2) = \frac{1}{\sqrt{1+2.2}} = \frac{1}{\sqrt{5}}$$

$$f(2) = \sqrt{5}$$

$$\therefore\ f(2) + f^1(2)\ (x-2) = 5 + \frac{1}{\sqrt{5}}(x-2)$$

We have $f(x) \approx \sqrt{5} + \frac{1}{\sqrt{5}}(x-2) = 5 + \frac{x}{\sqrt{5}} - \frac{2}{\sqrt{5}}$

$$= \frac{x}{\sqrt{5}} + \sqrt{5} - \frac{2}{\sqrt{5}} = \frac{x}{\sqrt{5}} + \frac{3}{\sqrt{5}}$$

Linear approximation of $\sqrt{1+2x} = f(x)$ near 2 is

$$f(2) + f^1(2)(x-2) = (x/\sqrt{5}) + \frac{3}{\sqrt{5}}.$$

If $y = f(x)$ is differentiable at x_0 then $f(x) \approx f(x_0) + f^1(x_0)\ (x-x_0)$ for x near x_0.

3) How accurately should we measure the edge x of a cube to compute the volume $v = x^3$ within 1% of its true value.

Solution : We want inaccuracy Δx in our measurement to be small enough to make corresponding increment ΔV in volume to satisfy the inequality

$$|\Delta v| \leq \frac{1}{100} \times V = \frac{1}{100} \cdot x^3$$

Using differentials, $dV = 3x^2. \quad x$

$$V \quad 3x^2 . \quad x$$

$$|3x^2. \quad x| \quad \frac{x^3}{100}$$

$$| \quad x| \quad \frac{x}{3.100} = \frac{x}{3}\ x.\ 01 = \frac{1}{3} \cdot \frac{x}{100}$$

Hence we must measure edge x with an error that is no more than one third of one percent of the true value.

Using differentials $dV = 3x^2 \cdot \Delta x$

$$\Delta V \approx 3x^2 \cdot \Delta x$$

$$\therefore \quad |3x^2 \cdot \Delta x| \leq \frac{x^3}{100} \Rightarrow |\Delta x| \leq \frac{x}{3 \cdot 100} = \frac{1}{3} \cdot \frac{x}{100}$$

$\therefore$ error in the measurement of x (edge) should not exceed a third of one percent of the true value.

Assignments :

1. Estimate $\sqrt[4]{17}$.
2. Calculate Sin 59° approximately, knowing that Sin $60^\circ = \sqrt{3}/2$ Remember that in calculus formulae presuppose radian measure for angles.
3. The width of a river is calculated by measuring the angle of elevation from a point on one bank of the top of a tree 50 feet high and directly across on the opposite bank. The angle is 45° with a possible error of $20'$. Find the possible error in the calculated width of the river.
4. A given quantity of metal is to be cast in the form of a solid right circular cylinder of radius 5" and height 10". If the radius is made 1/20th of an inch too large, what is the error in the height ?
5. The edge of a cube is measured as 10 cm with a possible error of one per cent. The cube's volume is to be calculated from this measurement. About how much error is possible in the volume calculation ?
6. About how accurately must the interior diameter of a 10 meter high storage tank of cylindrical shape be measured to calculate the tank's volume to within an error of one percent of its true value.
7. The radius of a circle is increased from 2.00 to 2.02 meters
   a) estimate the change in area
   b) calculate the error in the estimate in (a) as a percent of the original area.

8. If $f(x) = x^4-2x+3$ and given $f(8) = 4083$ find the value of $f(8.001)$.

9. If $f(x) = x^3+x^2+x-3$, find $f(1.09)$ approximately.

10. Show that the relative error in the volume of a sphere is three times the relative error in the radius.

I N T E G R A T I O N

1. Definite Integral and Properties of Definite Integral
2. Volumes of Solids by Definite Integrals

DEFINITION OF DEFINITE INTEGRAL

Introduction :

Historically, the basic problem of integrals is to find the areas and volumes by certain approximation methods. The first abstract proofs of rules for finding some areas and volumes are said to have been developed by Eudoxus between 400 B.C. and 350 B.C. Later his method of approximation was developed and exploited by Archimedes. This method, called method of exhaustion is at the root of all modern developments in the theory of measure and integral. In the 19th century, this method culminated in the theory of Riemann integration, defined by means of Riemann sums.

In modern times, the method of exhaustion can be stated as follows: Let S be a surface of known area s. Also suppose that S' is a surface of known area s' contained in s and s" is a surface of known area s" containing s. Then s' $\leq$ s $<$ s". The approximating surfaces s' and s" are taken as polygons or sums of slices, mainly trapezoidal or rectangular according to the particular figure s under the method of Eudoxus and Archimedes. In fact, the definition of area as a sum of rectangular areas is in vogue from 16th century A.D.

Calculus (both differential and integral) was invented by both Newton and Leibnitz — independent of each other. Newton, influenced by his teacher Barron used calculus to solve the problems of dynamics. Thus he conceived all functions as functions of a universal independent variable known as time (t). So he had no concept of functions of several variables and partial derivatives.

For Newton, the primary concept was that of fluxion (derivative) and arose from kinematical considerations. Newton did not isolate the concept of integral; nor he introduced one symbol for integration. His first basic problem was to find fluxion (derivatives). Integration was used in a geometric form to find fluents (antiderivatives or indefinite integrals), functions when fluxions (derivatives) are given. Newton based his theory mainly on the fact: The derivative of a variable area $F(x)$ under a curve is the ordinate $f(x)$ of this curve. For Newton, integration was the inverse process of differentiation, as he was mainly interested in the following problem — Given an equality relation containing fluxions, find the relation for fluents, which is the basic problem to solve ordinary differential equations. He solved these by use of series.

On the other hand, Leibnitz thought of the derivative as the slope of a tangent, and the integral as <u>summa omnimum linae</u>. The main purpose of all his work was to devise a universal language, that is, a general formalism for systematisation and organisation of knowledge. To a great extent, he succeeded in creating such a formalism for calculus. In fact, the present formalism in calculus is mainly his including the integral symbol (a stylised form of the letter S standing for <u>summa omnimum</u>). The terms constant, variable, function and integral used in calculus are due to Leibnitz.

G.F.B.Riemann in 1854 (published in 1867) gave necessary and sufficient condition for the existence of integrals called Riemann integral and showed that continuous functions satisfy his condition. The definition of integral as a limit of sum of areas as given in the text books is due to him.

Resume of the key concepts :

Two important ideas underlie the treatment of definite integrals in the text: 1. Definite integral $\int^{b} f(x).\,dx$ as a limit of the sum of areas and 2. Fundamental Theorem of Integral Calculus.

Here, we give an alternative treatment of Fundamental Theorem of Integral Calculus.

Statement Fundamental Theorem of Integral Calculus

If $f(x)$ is integrable in (a,b), $a < b$, and if there exists a function $F(x)$, such that $F'(x) = f(x)$ in (a,b), then

$$\int_a^b f(x)\,.\,dx = F(b) - F(a)$$

Proof : Let $a = x_0 < x_1 < x_2 < \cdots < x_n = b$

Then, by the Mean Value Theorem of Differential Calculus,

$F(x_r) - F(x_{r-1}) = (x_r - x_{r-1})\,F'(\xi_r)$, $x_{r-1} < \xi_r < x_r$

Taking the sum of the respective sides of the above equations, we have

$$\sum_{r=1}^{n} F'(\xi_r)\,\delta_r = \sum_{r=1}^{n} \left[F(x_r) - F(x_{r-1})\right]$$

[where $\delta_r = x_r - x_{r-1}$]

$$= F(b) - F(a) \qquad (1)$$

Suppose that δ is the length of the largest of the sub-intervals (x_{r-1}, x_r). Then as $\delta \to 0$, all the δ_r's will also tend to 0. So we have

$$\underset{\delta \to 0}{Lt} \sum F'(\xi_r)\,\delta_r = F(b) - F(a)$$

Now f(x) and so F'(x) is integrable in (a,b).

Hence

$$\underset{\delta \to 0}{Lt} \sum F'(\xi_r)\,\delta_r = \int_a^b F'(x)\,.\,dx = \int_a^b f(x).dx \qquad (2)$$

From (1) and (2) we have

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

The following points are to be noted regarding the above theorem.

1. This theorem is very useful and important as it gives us an easy method of evaluating the definite integral without calculating the limit of the sum by establishing a connection between the integration as a limit of a sum and the integration as inverse operation of differentiation.

2. $\int_a^b f(x)\,dx$ is a function of lower limit a and upper limit b, and not a function of the variable x.

3. In $\int_a^x f(x).dx$ the upper limit is the variable x. So $\int_a^x f(x).dx$ is not a definite integral, but another form of the indefinite integral. For example,

$$\int f(x).dx = F(x). \text{ Then}$$

$$\int_a^x f(x).dx = F(x) - F(a) = F(x) + \text{a constant} = \int f(x).dx.$$

Extended Definition of $\int_a^b f(x).dx$

The following definition of $\int_a^b f(x).dx$ is an extension of the definition given in the text.

Let f(x) be a bounded function defined in the interval (a,b); and let the interval (a,b) be divided in any manner into n sub-intervals
$(a, x_1), (x_1,x_2),\ldots, (x_{r-1},x_r),\ldots, (x_{n-1},b)$ of lengths $\delta_1, \delta_2, \ldots, \delta_n$ respectively where $a < x_1 < x_2 < \cdots < x_{r-1} < x_r < \cdots < x_{n-1} < b$.
In each of these sub-intervals select an arbitrary point and let these points be such that

$\xi_1 \in (a,x_1)$, $\xi_2 \in (x_1,x_2),\ldots, \xi_r \in (x_{r-1},x_r), \ldots, \xi_n \in (x_{n-1},b)$

Now let $S_n = \sum_{r=1}^{n} \delta_r \cdot f(\xi_r)$.

Now let n increase indefinitely so that the longest of the lengths $\delta_1, \delta_2, \ldots, \delta_n$ tends to 0. In such a case clearly each of $\delta_1, \delta_2, \ldots, \delta_n$ tends to 0. Now, if in such a situation (i.e. max.$(\delta_1, \delta_2, \ldots \delta_n) \to 0$), S_n tends to a finite limit which does not depend on the manner in which (a,b) is divided into sub-intervals and the points $\xi_1, \xi_2, \ldots \xi_n$ are selected; then this limit <u>(if it exists)</u> is defined as the definite integral of f(x) from a to b and symbolically denoted by $\int_a^b f(x).dx$.

In the textbook, for the sake of simplicity, the sub-intervals are supposed to be equal and the points $\xi_1, \xi_2, \ldots, \xi_n$ are taken to be the end-points of the sub-intervals.

Areas of difficulty :

Here are solved some problems the types of which are not discussed in the text.

Problem 1. Evaluate $\int_a^b x^m.dx$ where m is any real number $\neq 1$ and $0 < a < b$.

Solution: Consider the sub-intervals

$(a, ar), (ar, ar^2), (ar^2, ar^3), \ldots, (ar^{n-1}, ar^n)$ of (a,b) where $ar^n = b$ i.e. $r = (b/a)^{1/n}$.

Clearly as $n \to \infty$, $r = (\frac{b}{a})^{1/n} \to 1$ so that each of the lengths of the sub-intervals

$ar - a, ar^2 - ar, \ldots, (ar^n - ar^{n-1})$

i.e. $a(r-1), ar(r-1), \ldots, ar^{n-1}(r-1)$ tends to 0.

Now by the extended definition of $\int_a^b f(x) . dx$.

$$\int_a^b x^m . dx = \lim_{n \to \infty} \left[a^m . a(r-1) + (ar)^m . ar(r-1) + (ar^2)^m (ar^2)(r-1) + \ldots \text{ to } n \text{ terms} \right]$$

$$= \lim_{r \to 1} a^{m+1}(r-1) \{ 1 + r^{m+1} + r^{2(m+1)} + \ldots \text{ to } n \text{ terms}$$

$$= \lim_{r \to 1} \frac{a^{m+1} ((r^{m+1})^n - 1)}{r^{m+1} - 1}$$

as the series in the flower bracket is a G.P. with common ratio r^{m+1} with $m+1 \neq 0$.

Simplifying the last expression we have

$$\int_a^b x^m . dx$$

$$= \underset{r \to 1}{Lt} \; a^{m+1} \left(\frac{r-1}{r^{m+1}-1} \right) \left\{ (r^n)^{m+1} - 1 \right\}$$

$$= \underset{r \to 1}{Lt} \; a^{m+1} . \frac{1}{m+1} \left\{ \left(\frac{b}{a} \right)^{m+1} - 1 \right\}, \quad \text{as } \underset{r \to 1}{Lt} \frac{r-1}{r^{m+1}-1} = \frac{1}{m+1}$$

with $m+1 \neq 0$

$$= \underset{r \to 1}{Lt} \; \frac{b^{m+1} - a^{m+1}}{m+1}$$

$$= \frac{b^{m+1} - a^{m+1}}{m+1},$$ the expression under the limit being independent of r.

<u>Series represented by Definite Integrals</u>

The definition of the definite integral can be used with profit to evaluate easily the limits of the sums of certain series, when the number of terms in the series tends to infinity. The method lies in identifying a definite integral equal to series. In fact,

$$\int_a^b f(x).dx = \underset{h \to 0}{Lim} \; h \sum f\,(a + rh) \text{ where } nh = b-a$$

$$\text{or } \underset{n \to \infty}{lt} \; \frac{b-a}{n} \sum \left(f(a) + r\,\frac{(b-a)}{n} \right) = \int_a^b f(x).dx$$

If $a = 0$, $b = 1$, we have

$$\underset{n \to \infty}{lt} \; 1/n \sum f(r/n) = \int_0^1 f(x) \, . \, dx$$

In the above discussion, r takes the values either $0,1,2,\ldots,n-1$ or $1,2,3,\ldots,n$. These two sets of numbers represent the left and right extremities of the elementary vertical rectangles (columns) in the calculation of area represented by $\int_a^b f(x).dx$. (Refer to the definition of $\int_a^b f(x).dx$ in the text).

The following are illustrative examples.

Problem 2. Evaluate

$$\lim_{n \to \infty} \left[\frac{1}{n+m} + \frac{1}{n+2m} + \ldots\ldots + \frac{1}{n+nm} \right]$$

Solution: The given expression

$$= \lim_{n \to \infty} \left[\frac{1}{n} \left(\frac{1}{1+\frac{m}{n}} + \frac{1}{1+\frac{2m}{n}} + \cdots + \frac{1}{1+\frac{(n-1)m}{n}} + \frac{1}{1+\frac{nm}{n}} \right) \right]$$

$$= \int_0^1 \frac{dx}{1+mx}$$ by definition of the definite integral $\int_a^b f(x).dx$

$= 1/m \log (1 + mx)$

$= 1/m \; \log (1 + m) - \log 1$

$1/m \; \log (1 + m)$

Problem 3. Evaluate

$$\lim_{n \to \infty} \left\{ \left(1 + \frac{1}{n}\right) \left(1 + \frac{2}{n}\right) \ldots\ldots \left(1 + \frac{n}{n}\right) \right\}^{1/n}$$

Solution :

Let $A = \left\{ \left(1 + \frac{1}{n}\right) \left(1 + \frac{2}{n}\right) \ldots\ldots \left(1 + \frac{n}{n}\right) \right\}^{1/n}$

Then $\lim_{n \to \infty} \log A$

$$= \lim_{n \to \infty} 1/n \sum \log \left(1 + \frac{r}{n}\right)$$

$$= \int_0^1 \log (1 + x)$$

Now put $z = 1 + x$

Then $x = 0$ implies $z = 1$ and $x = 1$ implies $z = 2$.

So $\lim\limits_{n \to \infty} \log A$

$= \int_1^2 \log z . dz$

$= \left[z \log z - z \right]_1^2$

$= 2 \log 2 - 2 - 1 \log 1 + 1$

$= 2 \log 2 - 1 = 2 \log 2 - \log e$

$= \log 4/e$

So $\lim\limits_{n} A = \frac{4}{e}$

Assignments :

Using the definition of $\int_a^b f(x).dx$ as a limit of a sum, evaluate the following definite integrals (1 to 10) :

1. $\int_a^b e^{-x}.dx$
2. $\int_0^1 x^2.dx$
3. $\int_a^b (ax+b)\, dx$
4. $\int_0^{\pi/2} \sin x.\, dx$
5. $\int_a^b \cos \theta \, . \, d\theta$
6. $\int_0^1 \sqrt{x} \, . \, dx$
7. $\int_0^1 \frac{1}{\sqrt{x}} \, . \, dx$
8. $\int_2^4 \frac{1}{x^2} dx$
9. $\int_1^3 e^x.dx$
10. $\int_0^{\pi/4} \sec^2 x.dx$

Evaluate the following limits using definite integrals.

11. $\lim\limits_{n \to \infty} \left\{ \frac{1}{n+1} + \frac{1}{n+2} + \dots\dots\dots + \frac{1}{n+n} \right\}$

12. $\lim\limits_{n \to \infty} \left[\frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \dots\dots + \frac{n}{n^2+n^2} \right]$

13. $\lim\limits_{n \to \infty} \left[\dfrac{1^2}{n^3+1^3} + \dfrac{2^2}{n^3+2^3} + \cdots\cdots + \dfrac{n^2}{2n^3} \right]$

14. $\lim\limits_{n \to \infty} \sum\limits_{r=1}^{n} \dfrac{n+r}{n^2+r^2}$

15. $\lim\limits_{n \to \infty} \left[\dfrac{\sqrt{(n+1)} + \sqrt{(n+2)} + \cdots\cdots + \sqrt{(2n)}}{n\sqrt{n}} \right]$

16. $\lim\limits_{n \to \infty} \sum\limits_{r=1}^{n} \dfrac{n}{(n+r)\sqrt{\{r\,(2n+r)\}}}$

17. $\lim\limits_{n \to \infty} \left\{ \left(1 + \dfrac{1^2}{n^2}\right)\left(1 + \dfrac{2^2}{n^2}\right) \cdots\cdots \left(1 + \dfrac{n^2}{2n^2}\right) \right\}^{1/n}$

18. $\mathrm{Lt}_{n \to \infty} \left\{ \left(1 + \dfrac{1}{n^2}\right)^{2/n^2} \left(1 + \dfrac{2^2}{n^2}\right)^{4/n^2} \left(1 + \dfrac{3^2}{n^2}\right)^{6/n^2} \cdots\cdots \left(1 + \dfrac{n^2}{n^2}\right)^{2n/n^2} \right\}$

Answers :

1. $e^{-b} - e^{-a}$
2. $y3$
3. $a/2 + b$
4. 1
5. Sinb - sin a
6. 2/3
7. 2
8. $y4$
9. $e^3 - e$
10. 1
11. log 2
12. $\pi/4$
13. $(y3) \log 2$
14. $\frac{\pi}{4} + (y2) \log 2$
15. $(4/3)\sqrt{2} - 2/3$
16. /3
17. $2e^{(y2)} (\pi - 4)$
18. $4/e$

PROPERTIES OF DEFINITE INTEGRALS

Here we will discuss and clarify certain important properties of definite integrals which have not been discussed in the text.

1. $\int_a^b f(x).dx = \int_a^b f(z).\ dz$

Proof :

Suppose that $\int f(x).dx = \phi\ (x)$

Then, we have by Fundamental Theorem of Integral Calculus

$$\int_a^b f(x).dx = \phi(b) - \phi(a) \qquad (1)$$

Also, $\int f(z).dz = \phi(z)$ and by the Fundamental Theorem of Integral Calculus,

$$\int_a^b f(z).dz = \phi(b) - \phi(a) \qquad (2)$$

From (1) and (2), we have the result.

This property states that a definite integral is independent of the variables with respect to which the integration is performed.

2. $\int_0^{na} f(x).dx = n \int_0^a f(x).dx$ if $f(x) = f(a+x)$

Proof :

$$\int_0^{na} f(x).dx = \int_0^a f(x).dx + \int_a^{2a} f(x).dx + \dots + \int_{(n-1)a}^{na} f(x).dx$$

Set $z + a = x$. Then $dx = dz$

Also, $x = a$ implies $z = 0$ and $x = 2a$ implies $z = a$

So, $\int_a^{2a} f(x).dx = \int_0^a f(z+a).dz = \int_0^a f(a+x)dx$

$= \int_0^a f(x).dx$

Again with the same substitution, $z+a = x$, we can see that

$$\int_{2a}^{3a} f(x).dx = \int_{a}^{2a} f(z+a).dz = \int_{a}^{2a} f(x).dx = \int_{0}^{a} f(x).dx$$

Similarly, we can show that

$$\int_{(n-1)a}^{na} f(x).dx = \int_{(n-2)a}^{(n-1)a} f(x).dx = \ldots = \int_{a}^{2a} f(x).dx = \int_{0}^{a} f(x).dx$$

Hence we get the result.

<u>Illustration</u>: Since $\cos x = \cos (x+\pi)$

we have

$$\int_{0}^{6\pi} \cos x.\, dx = 6 \int_{0}^{\pi} \cos x.\, dx$$

3. $\int_{0}^{2a} f(x).dx = \int_{0}^{a} f(x).dx + \int_{0}^{a} f(2a-x).dx$

Proof :

By formula 7.2 of the textbook

$$\int_{0}^{2a} f(x).dx = \int_{0}^{a} f(x).dx + \int_{a}^{2a} f(x).dx$$

Substitute $2a - z$ for x. Then $dx = -dz$.

Moreover, when $x = a$, $z = a$, and when $x = 2a$, $z = 0$; so

$\int_{a}^{2a} f(x).dx = - \int_{a}^{0} f(2a-z) = \int_{0}^{a} f(2a-z)$ by formula 7.1 of the textbook $= \int_{0}^{a} f(2a-x)$

Hence, $\int_{0}^{2a} f(x).dx = \int_{0}^{a} f(x).dx + \int_{0}^{a} f(2a-x)$

4. i) $\int_{0}^{2a} f(x).dx = 2\int_{0}^{a} f(x).dx$ if $f(2a-x) = f(x)$ and

ii) $\int_{0}^{2a} f(x) = 0$, if $f(2a-x) = -f(x)$

Proof :

i) $\int_0^{2a} f(x).dx = \int_0^a f(x).dx + \int_0^a f(2a-x).dx$ by the previous result.

$= \int_0^a f(x).dx + \int_0^a f(x).dx$

$= 2\int_0^a f(x).dx$

ii) The proof can be written as in 4(i).

5. If f(x) is integrable in the closed interval a,b and if f(x) 0 for all x in [a,b], then $\int_a^b f(x).dx \geqslant 0$ (b > a).

Proof :

Since f(x) is integrable in [a,b], f(x).dx exists. Since $f(x) \geqslant 0$ in [a,b] in the sub-interval (x_{r-1}, x_r) of [a,b] the lower bound $m_r \geqslant 0$, and so the lower sum s for the partition of $[a,b] = \sum m_r \delta_r \geqslant 0$

So l, which is the exact upper bound of the set of numbers s, is $\geqslant 0$.

Now, since $\int_a^b f(x).dx$ exists, $l = \int_a^b f(x).dx$

Hence, $\int_a^b f(x).dx$ exists.

6. If f(x) and g(x) are integrable in a,b) and $f(x) \geqslant g(x)$ for all x in [a,b] then $\int_a^b f(x).dx \geqslant \int_a^b g(x).dx$

Proof:

Let h(x) = f(x) - g(x)

Then as f(x) and g(x) are integrable in [0,1], h(x) is so.

Also, as $f(x) \geqslant g(x)$ in [a,b], $h(x) \geqslant 0$ in [a,b].

Applying the previous result, we find that

$$\int_a^b h(x).\, dx \geqslant 0$$

i.e. $\int_a^b (f(x) - g(x))\ dx \geqslant 0$

i.e. $\int_a^b f(x).dx - \int_a^b g(x).dx \quad 0$

i.e. $\int_a^b f(x).dx \geqslant \int_a^b g(x).dx$

7. If $f(x)$ is integrable in (a,b), then

$$\int_a^b |f(x)|.dx \geqslant \left| \int_a^b f(x).dx \right|$$

Proof:

Let $\{ a = x_0, x_1, x_2, \ldots, x_{n-1}, x_n = b \}$ be a partition of $[a,b]$ and let $\delta_r = x_r - x_{r-1}$.

Then we have

$$|f(\xi_1)\delta_1 + f(\xi_2).\delta_2 + \cdots + f(\xi_n).\delta_n|$$

$$\leqslant |f(\xi_1).\delta_1| + |f(\xi_2).\delta_2| + \cdots + |f(\xi_n).\delta_n|$$

$$= |f(\xi_1)|\,|\delta_1| + |f(\xi_2)|\,|\delta_2| + \cdots + |f(\xi_n)|\,|\delta_n|$$

$$= |f(\xi_1)|.\delta_1 + |f(\xi_2)|.\delta_2 + \cdots + |f(\xi_n)|.\delta_n$$

where $\xi_r \in [x_{r-1}, x_r]$ and each δ_r is clearly positive.
Now, let $n \longrightarrow \infty$ so that max. $(\delta_1, \delta_2, \ldots, \delta_n)$ $\longrightarrow 0$ i.e. each $\delta_r \longrightarrow 0$

Then clearly

$$\operatorname{lt} \left| \sum f(\xi_r) \cdot \delta_r \right| \leq \operatorname{lt} \sum |f(\xi_r)| \delta_r$$

i.e. $$\left| \int_a^b f(x) \, dx \right| \leq \int_a^b |f(x)| \cdot dx$$

Solved Examples :

The following examples will illustrate the use of the properties of the definite integrals in solving problems.

Example 1 :

Show that

$$\int_0^{\pi/2} \log \sin x. \, dx = \int_0^{\pi/2} \log \cos . \, dx = (\pi/2) \log \sqrt{2}$$

$$\int_0^{\pi/2} \log \sin x. \, dx$$

$$= \int_0^{\pi/2} \log \sin (\pi/2 - x) . \, dx$$

$$= \int_0^{\pi/2} \log \cos x. \, dx \text{ by Formula 7.4 of textbook}$$

Now if each of the definite integrals $\int_0^{\pi/2} \log \sin x . \, dx$ and $\int_0^{\pi/2} \log \cos x. \, dx$ is taken to be I, then

$$2I = \int_0^{\pi/2} \log \sin x. \, dx + \int_0^{\pi/2} \log \cos x . \, dx$$

$$= \int_0^{\pi/2} (\log \sin x + \log \cos x) \, dx = \int_0^{\pi/2} \log (\sin x. \cos x) \, dx$$

$$= \int_0^{\pi/2} \log \frac{\sin 2x}{2} . \, dx = \int_0^{\pi/2} (\log \sin 2x - \log 2) . \, dx$$

$$= \int_0^{\pi/2} \log \sin 2x . \, dx - (\pi/2) \log 2$$

Set $2x = u$. Then $dx = du/2$.

We have

$$\int_0^{\pi/2} \log \sin 2x \,.\, dx = \tfrac{1}{2} \int_0^{\pi} \log \sin u \,.\, du$$

$$= \tfrac{1}{2} \int_0^{\pi} \log \sin x \,.\, dx = \int_0^{\pi/2} \log \sin x \,.\, dx \text{ by result 4(i)}$$

$$= I$$

So, $2I = I - \pi/2 \log 2$

i.e., $I = -(\pi/2) \log 2 = (\pi/2) \log (\tfrac{1}{2})$

Example 2 :

Show that $\displaystyle\int_0^1 \frac{\log (1+x)}{1+x^2} dx = (\pi/8) \log 2$

Set $x = \tan u$

Then $dx = \sec^2 u \; du$

Moreover, $x = 0 \Rightarrow u = 0$

and $x = 1 \Rightarrow u = \pi/4$

So $I = \displaystyle\int_0^{\pi/4} \log (1 + \tan u) \, du$

$$= \int_0^{\pi/4} \log (1 + \tan (\pi/4 - u)) \, du = \int_0^{\pi/4} \log \left(1 + \frac{1 - \tan u}{1 + \tan u}\right) du$$

$$= \int_0^{\pi/4} \log \frac{2}{1 + \tan u} \,.\, du$$

$$= \int_0^{\pi/4} (\log 2 - \log (1 + \tan u)) \, du$$

$$= \int_0^{\pi/4} \log 2 \,.\, du - \int_0^{\pi/4} \log (1 + \tan u) \, du$$

$$= (\pi/4) \log 2 - I$$

So, $2I = (\pi/4) \log 2$

i.e. $I = (\pi/8) \log 2$

Example 3 :

Show that $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\, dx = \frac{\pi^2}{4}$

Put $I = \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\, dx$ (1)

Substituting $\pi - x$ for x

$$I = \int_0^{\pi} \frac{(\pi-x) \sin(\pi-x)}{1+\cos^2(\pi-x)}\, dx$$

i.e. $I = \int_0^{\pi} \frac{(\pi-x) \sin x}{1+\cos^2 x}\, dx$ (2)

Adding (1) and (2) we get

$I + I = \int_0^{\pi} \frac{\pi \sin x}{1+\cos^2 x}\, dx$ i.e. $2I = \pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\, dx$

i.e. $I = \frac{\pi}{2} \int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\, dx$

Set $\cos x = z$. Then $dx = \frac{dz}{-\sin x}$

Also $x = 0 \Rightarrow z = 1$ and $x = \pi \Rightarrow z = -1$

So, $I = \frac{\pi}{2} \int_1^{-1} \frac{\sin x}{1+z^2} \cdot \frac{dz}{-\sin x} = -\frac{\pi}{2} \int_1^{-1} \frac{dz}{1+z^2}$

$= \frac{\pi}{2} \int_{-1}^{1} \frac{dz}{1+z^2}$ by property (1) of the textbook

$$= \frac{\pi}{2} \left[\tan^{-1} z \right]_{-1}^{1}$$

$$= \frac{\pi}{2} (\tan^{-1} 1 - \tan^{-1}(-1))$$

$$= \frac{\pi}{2} \left(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right)$$

$$= \frac{\pi}{2}\left(\frac{\pi}{4}+\frac{\pi}{4}\right) = \frac{\pi}{2}\left(\frac{\pi}{2}\right)$$

$$= \frac{\pi}{4}$$

Example 4 :

Show that $\int_0^{\pi} x \sin x \, dx = \pi \int_0^{\pi/2} \cos x \, . \, dx$

Solution:

$I \equiv \int_0^{\pi} x \sin x \, . \, dx = \int_0^{\pi} (\pi - x) \sin(\pi - x) \, . \, dx$ by result No.4 of the text.

$$= \int_0^{\pi} (\pi - x) \sin x \, . \, dx$$

$$= \pi \int_0^{\pi} \sin x . \, dx - \int_0^{\pi} x \sin x \, . \, dx$$

$$= \pi \int_0^{\pi} \sin x . \, dx - I$$

$= 2\pi \int_0^{\pi/2} \sin x \, . \, dx - I$ by result No.4(i) of this booklet.

$$= 2\pi \int_0^{\pi/2} \sin\left(\frac{\pi}{2} - x\right) dx - I$$

$$= 2\pi \int_0^{\pi/2} \cos x \, . \, dx - I$$

i.e. $2I = 2\pi \int_0^{\pi/2} \cos x . \, dx$

i.e. $I = \pi \int_0^{\pi/2} \cos x . \, dx$

Example 5 :

Show that $\displaystyle\int_0^{\pi/2} \frac{x \sin x \,.\, \cos x}{\cos^4 x + \sin^4 x}\, dx = \frac{\pi^2}{16}$

Solution :

$$I = \int_0^{\pi/2} \frac{x \,.\, \sin x .\, \cos x}{\cos^4 x + \sin^4 x}\, dx$$

$$= \int_0^{\pi/2} \frac{(\pi/2 - x) \cos x \,.\, \sin x}{\sin^4 x + \cos^4 x}$$ by result No.4 of the text

$$= \frac{\pi}{2} \int_0^{\pi/2} \frac{\cos x \,.\, \sin x}{\sin^4 x + \cos^4 x}\, dx - I$$

i.e. $\displaystyle 2I = \frac{\pi}{2} \int_0^{\pi/2} \frac{\cos x - \sin x}{\sin^4 x + \cos^4 x}\, dx$

Now, $\text{Sin}^4 x + \cos^4 x = (\sin^2 x + \cos^2 x)^2 - 2 \sin^2 x \,.\, \cos^2 x$

$$= 1 - \frac{\sin^2 2x}{2} = +1 - \frac{(1 - \cos^2 2x)}{2}$$

So, $\displaystyle 2I = \frac{\pi}{4} \int_0^{\pi/2} \frac{\sin 2x}{1 - \left(\frac{1 - \cos^2 2x}{2}\right)} \,.\, dx$

set $\cos 2x = z$. Then $-2 \sin 2x.dx = dz$,

$x = 0 \Rightarrow z = 1$ and $x = \pi/2 \Rightarrow z = -1$

So, $\displaystyle 2I = \frac{\pi}{4} \int_{+1}^{-1} \frac{-dz}{1 - \left(\frac{1 - z^2}{2}\right)} = \frac{\pi}{4} \int_{+1}^{-1} \frac{-dz}{1 + z^2}$

$$= \frac{\pi}{4} \int_{-1}^{+1} \frac{dz}{1 + z^2}$$

by result No.1 of the text.

$$= \frac{\pi}{4}\left[\tan^{-1} z\right]_{-1}^{+1} = \frac{\pi}{4}\left[\tan^{-1} 1 - \tan^{-1}(-1)\right]$$

$$= \frac{\pi}{4}\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right]$$

$$= \frac{\pi}{4}\left[\frac{\pi}{4} + \frac{\pi}{4}\right]$$

$$= \frac{\pi}{4} \times \frac{\pi}{2} = \pi^2/8$$

i.e. $I = \dfrac{\pi^2}{16}$

Example 6 :

Show that $\displaystyle\int_{-a}^{+a} \frac{t.e^{t^2}}{1+t^2} \,.\, dt = 0$

The given integral

$I = I_1 + I_2$, where $I_1 = \displaystyle\int_{-a}^{0} \frac{t.e^{t^2}}{1+t^2} \,.\, dt$ and $I_2 = \displaystyle\int_{0}^{a} \frac{t.e^{t^2}}{1+t^2}\, dt$

Now $I_1 = \displaystyle\int_{-a}^{0} \frac{t.e^{t^2}}{1+t^2} \,.\, dt$

$= -\displaystyle\int_{a}^{0} \frac{z \,.\, e^{z^2} dz}{1+z^2}$ where $z = -t$ ($t = -a \Rightarrow z = a$)

$= -\displaystyle\int_{0}^{a} \frac{z.\ e^{z^2} dz}{1+z^2}$ by result No.1 of the text

$= -\displaystyle\int_{0}^{a} \frac{t.e^{t^2} .\ dt}{1+t^2}$ by result No.1 of the booklet

$= -I_2$ i.e. $I_1 + I_2 = 0$ i.e., $I = 0$.

<u>Assignments</u> :

1. Show that $\int_a^b f(a+b-x) = \int_a^b f(x).dx$

2. Show that $\int_{a-c}^{b-c} f(x+c) = \int_a^b f(x)$

3. Show that $\int_a^b f(nx)\ dx = 1/n \int_{na}^{nb} f(x).dx$

4. Show that

$$\int_0^{\pi/2} (a\cos^2 x + b\sin^2 x)\ dx = \frac{\pi}{4}(a+b)$$

5. Show that $\int_0^{\pi} xf(\sin x)\ dx = \frac{\pi}{2} \int_0^{\pi} f(\sin x)\ dx$

6. $\int_0^{\pi} t\ .\ \sin^2 t\ dt = \frac{\pi^2}{4}$

7. Show that $\int_0^{\pi} \frac{\sin 4\theta}{\sin \theta}\ .\ d\theta = 0$

8. Show that $\int_0^1 \log \sin\left(\frac{\pi\theta}{2}\right)\ .\ d\theta = -\log 2$

9. Show that $\int_{-a}^{a} t\sqrt{a^2-t^2}\ .\ dt = 0$

10. Show that

$$\int_0^{\pi/2} \frac{\sin^{3/2}\theta}{\sin^{3/2}\theta + \cos^{3/2}\theta}\ d\theta = \frac{\pi}{4}$$

11. Show that

$$\int_0^{\pi/2} f(\sin x)\ .\ dx = \int_0^{\pi/2} f(\cos x)\ dx$$

12. Show that $\int_0^a f(x^2)dx = 1/2 \int_{-a}^{a} f(x^2).dx$

13. Show that $\int_0^1 x^m(1-x)^n dx = \int_0^1 x^n(1-x)^m dx, m > 0, n > 0$

14. Show that $\int_{-\pi/2}^{\pi/2} x^3 . \sin^{-2} x . dx = 0$

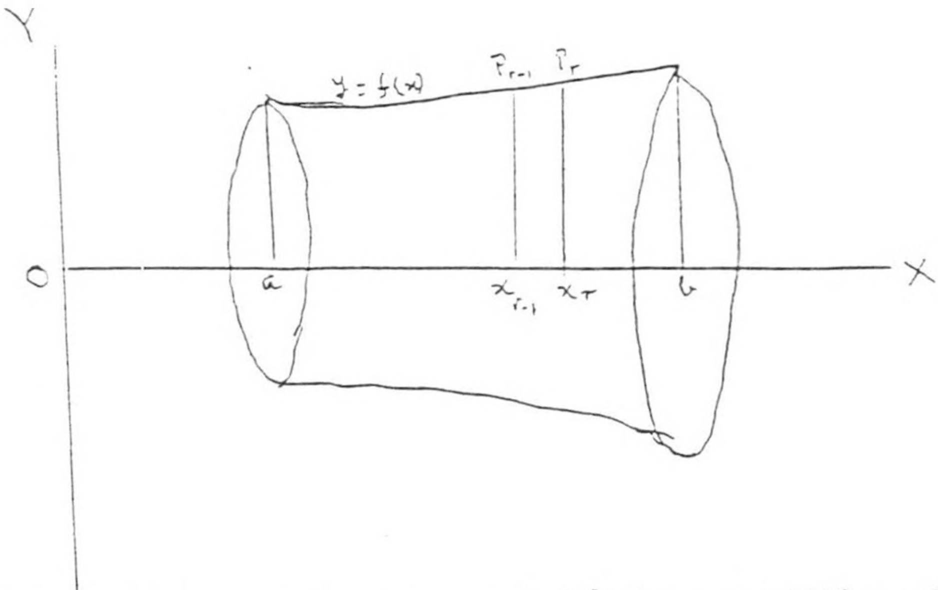
15. Show that $\int_0^{\pi} \frac{x}{a^2\cos^2 x + b^2\sin^2 x} dx = \frac{\pi^2}{2ab}$

EVALUATION OF VOLUMES OF SOLIDS OF REVOLUTION BY DEFINITE INTEGRALS

Key Concepts

1. Volume of a solid by revolution

Let an area be bound by the continuous curve $y = f(x)$, x-axis, the lines $x = a$ and $x = b$. Suppose that this area is revolved about the x-axis. Then a solid of revolution is generated. Here we are to find an expression for the volume of this solid of revolution.



Let $\{a = x_1, x_1, x_2, \ldots . x_{n-1}, x_{n-1}, x_n = b\}$ be a partition of the intervals $[a, b]$ into n sub-intervals. Let $\delta x_r = x_r - x_{r-1}$.
Let P_{r-1}, P_r be the points on the curve $y = f(x)$ corresponding to the points x_{r-1}, x_r respectively on the x-axis. Thus the area under the curve $y = f(x)$ between the points x_{r-1} and x_r generates a disc of thickness δx_r. Clearly, the volume of this disc can be taken as

$\pi [f(x_{r-1})]^2 \delta x_r$ or $\pi [f(x_r)]^2 \delta x_r$

Since δx_r is very small, and f(x) is continuous, the volume of this disc of infinitesimal thickness is given by

$$\delta V = \pi\left[f(t_r)\right]^2 \delta x_r \text{, where } x_{r-1} \leq t_r \leq x_r.$$

Taking the sum of volumes of all such discs, we have

$$V = \sum_1^n \pi\left[f(t_r)\right]^2 \delta x_r \text{, where } x_{r-1} \leq t_r \leq x_r$$

Let $n \longrightarrow \infty$ so that max. $\delta x_r \longrightarrow 0$. Then we have

$$V = \lim_{n\to\infty} \sum_1^n \pi\left[f(t_r)\right]^2 \delta x_r$$

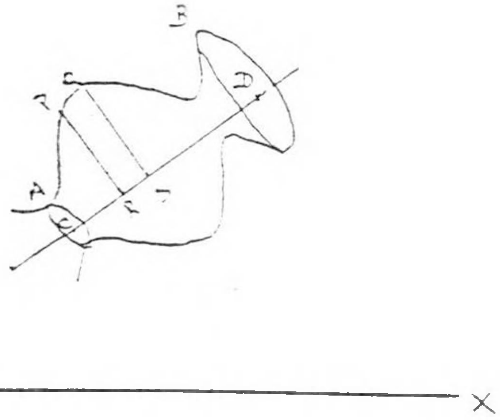
$$= \int_a^b \pi\left(f(x)\right)^2 dx$$

$$= \int_a^b \pi y^2 \, dx$$

2. Suppose that an area is bound by the curve x = g(y), y = c, y = d, and y - axis. Let this area be revolved about y-axis. Then we get a solid of revolution generated by this area. By proceeding as in (1), we can show that the total volume of this solid of revolution is given by

$$V = \int_c^d \pi x^2 . dy$$

3.



Let AB be a curve which is being revolved about a line CD in the plane of the curve. Then a solid of revolution is generated and CD is the axis of this solid of revolution. Now it is required to find an expression for the volume V of this solid of revolution.

Let P and Q be points on the generating curve so that the distance PQ is an infinitesimal. Draw PR and QS perpendiculars on CD such that R and S are feet of the perpendiculars. Then the total volume of the solid of revolution is clearly given by

$$V = \lim \sum \pi \cdot PR^2 \cdot RS = \pi \int_{C}^{CD} PR^2 \cdot d(CR)$$

Solved Examples :

1. Find the volume of the solid of revolution generated by revolving about the x-axis, the area bound by $y = 5x-x^2$ and x-axis.

Solution:

The equation to the curve can be written $y = 5x-x^2$.

i.e., $y = -(x^2 - 5x)$ i.e. $y = -\left[(x - \frac{5}{2})^2 + \frac{25}{4}\right]$

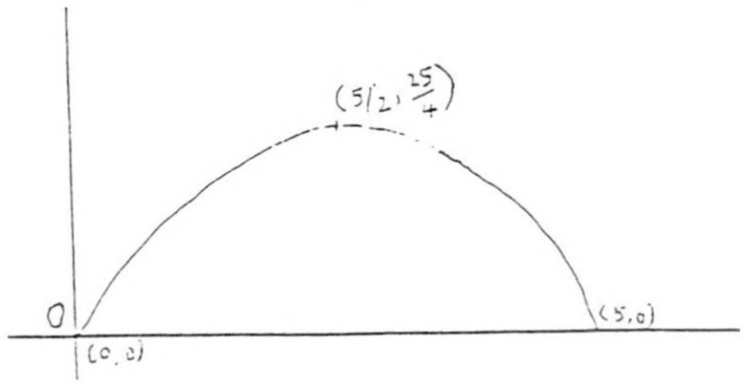
i.e., $y - \frac{25}{4} = -(x - \frac{5}{2})^2$ (1)

The x-coordinates of the points of intersection of this curve with x-axis, i.e. $y = 0$ is given by

$5x - x^2 = 0$ i.e., $x(5-x) = 0$

i.e. $x = 0$ or 5 (2)

Considering the information given by (1) and (2), we can draw the graph of the generating curve as follows :



The generating curve is a parabola with vertex at (5/2, 25/4) and intersecting x-axis at (0,0) and (5,0). So the total volume of the solid of revolution is given by

$$V = \pi \int_0^5 (5x - x^2)^2 \, dx$$

$$= \pi \int_0^5 (25x^2 - 10x^3 + x^4) \, dx$$

$$= \pi \left[25\frac{x^3}{3} - 10\frac{x^4}{4} + \frac{x^5}{5} \right]_0^5$$

$$= \pi = \left[25 \cdot \frac{5^3}{3} - 10 \cdot \frac{5^4}{4} + \frac{5^5}{5} \right] - 0$$

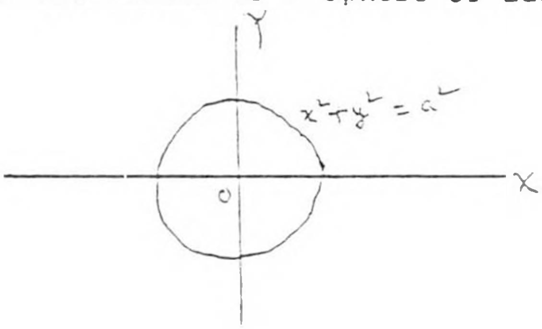
$$= \pi \cdot 5^4 \left[\frac{5}{3} - \frac{5}{2} + 1 \right]$$

$$= 625 \cdot \pi \left[\frac{10 - 15 + 6}{6} \right]$$

$= 625 . \pi . \frac{1}{6}$

$= \frac{625\pi}{6}$

2. Show that the volume of a sphere of radius a is $\frac{4}{3}$ a^3.



A sphere is generated by revolving the region bounded by the circle

$$x^2+y^2 = a^2 \tag{1}$$

about the y-axis.

So, the volume of the sphere

$= \int_0^a x^2 - dy \qquad =$

$= \pi \int_{-a}^{a} (a^2-y^2)dy = \pi \left[a^2y - \frac{y^3}{3}\right]_{-a}^{a}$

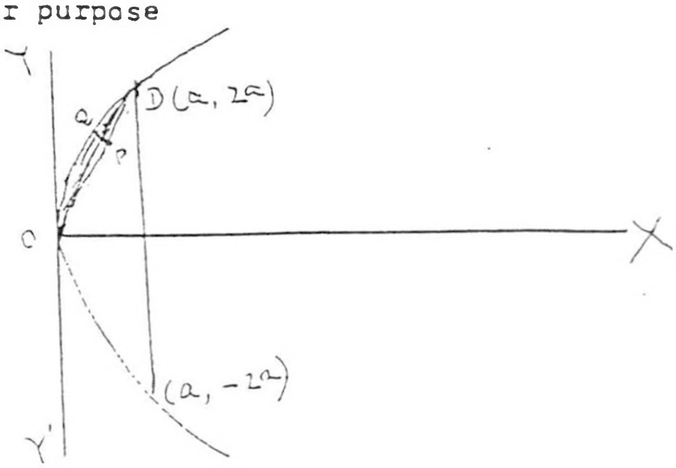
$= \pi \left[a^3 - \frac{a^3}{3} + a^3 - \frac{a^3}{3}\right]$

$= \pi \left(2a^3 - \frac{2a^3}{3}\right)$

$= \frac{4}{3}\pi a^3.$

3. The area cut off from the parabola $y^2 = 4ax$, by the chord joining the vertex to an end of the latus rectum rotates about the chord. Find the volume of the solid so formed.

Solution: The equation to the latus rectum of the parabola $y^2 = 4ax$ is $y = 2a$. So the latus rectum intersects the parabola $y^2 = 4ax$ at points whose x-coordinates are given by $(2a)^2 = 4ax$ i.e. $4a^2 = 4ax$ i.e., $x = a$. Correspondingly, y-coordinates of the points of intersection are given by $y^2 = 4a^2$ i.e., $y = \pm 2a$. So the points of intersection are $(a, 2a)$ and $(a, -2a)$. Let us consider the point $D\ (a, 2a)$ for our purpose



Now, OD is the line joining $O\ (0,0)$ the origin and $D(a, 2a)$. The equation to OD is given by

$$\frac{y}{x} = \frac{2a}{a} \quad \text{i.e. } y = 2x, \text{ i.e. } y - 2x = 0.$$

Let $P\ (x', y')$ be a point on the parabola $y^2 = 4ax$ and PQ be perpendicular to OD with Q on OD. Clearly, the length PQ is given by

$$PQ = \frac{y' - 2x'}{\sqrt{5}}$$

Now the area shaded in the figure is rotated about CD and the volume of the solid so formed is to be evaluated.

The elem-entary length along CD is $\sqrt{5}\,.\,dx$.

So the volume V of the solid of revolution is given by

$$V = \pi \int_0^a PC^2 . \sqrt{5} . dx$$

$$= \pi \int_0^a \left(\frac{y - 2x}{5}\right)^2 \sqrt{5}\ dx \text{ , suppressing the dashes in } x^1, y^1$$

$$= \pi \int_0^a \frac{y^2 - 4xy + 4x^2}{5} \sqrt{5}\ dx$$

$$= \pi \int_0^a \frac{(4ax - 8\sqrt{a}x^{3/2} + 4x^2)}{\sqrt{5}}\ dx$$

$$= \frac{\pi}{\sqrt{5}} \left[\frac{4ax^2}{2} - \frac{2}{5} . 8\sqrt{a}\, x^{5/2} + \frac{4x^3}{3} \right]_0^a$$

$$= \frac{\pi}{\sqrt{5}} \left[\frac{4a^3}{2} - \frac{16}{5}a^3 + \frac{4a^3}{3} \right]$$

$$= \frac{\pi a^3}{\sqrt{5}} \left(2 - \frac{16}{5} + \frac{4}{3}\right)$$

$$= \frac{\pi}{\sqrt{5}} a^3 \left(\frac{30 - 48 + 20}{15}\right) = \frac{2\pi a^3}{15\sqrt{5}}$$

Assignments :

Find the volumes of solids generated by revolving about the x-axis, the areas bounded by the following curves and lines.

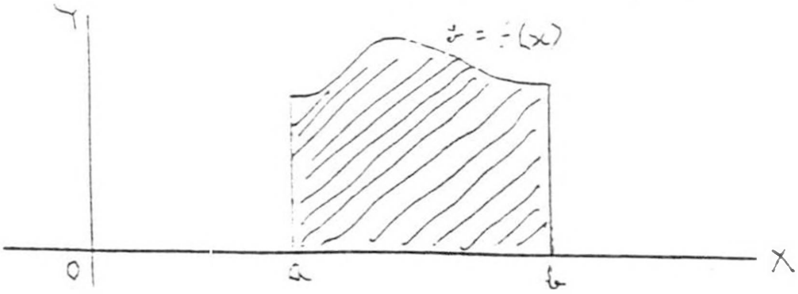
1. $y = \sin x;\ x = 0,\ x = \pi$
2. $y = 5x - x^2,\ \ x = 0,\ x = 4$
3. $y^2 = 9x,\ x = 4$
4. $x^2+y^2 = 4,\ \ x = 1,\ y = 0$
5. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
6. Prove that the volume of a right circular cone of height h and base of radius r is $\frac{1}{3}\pi r^2h$.
7. An arc of a parabola is bounded at both ends by the latus rectum of length 4a. Find the volume of the solid generated by rotating the arc about the latus rectum.
8. The area cut off by the line $x+y = 1$ from the parabola $\sqrt{x} + \sqrt{y} = 1$ is revolved about the same line. Find the volume of the solid so generated.
9. Show that the volume of the solid of revolution generated by revolving the cycloid $x = a\ (\theta + \sin \theta),\ \ y = a(1 + \cos \theta)$ about its base is equal to $5\pi^2a^3$.
10. Show that the volume of the solid generated by revolving the cardioide $r = a(1 - \cos \theta)$ about the initial line is equal to $\frac{8}{3}\pi a^3$.

9. Evaluation of Plane Areas by Definite Integrals

Key Concepts

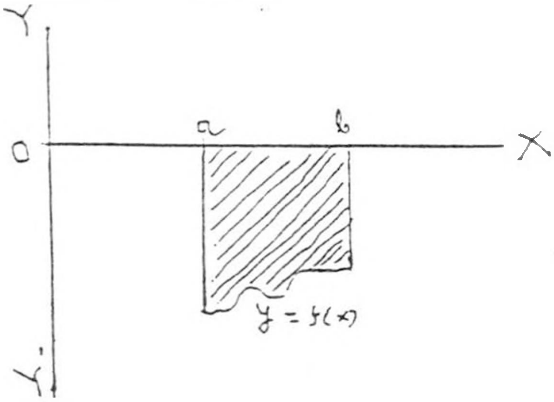
1. Let a region by bounded by the graph of $y = f(x)$, x-axis, the lines $x=a$ and $x=b$, $(a < b)$. Then area A of this region is given by

$$A = \int_a^b f(x).dx \text{ if } f(x) \geq 0 \text{ for } a \leq x \leq b$$



2. If $f(x) \leq 0$ for all $x \in [a,b]$, then $-f(x) \geq 0$ for all x in $[a,b]$ and the area A bounded by the graph of this function, $x = a$, $x = b$ and x - axis $(a < b)$ is given by

$$A = -\int_a^b f(x).\, dx$$

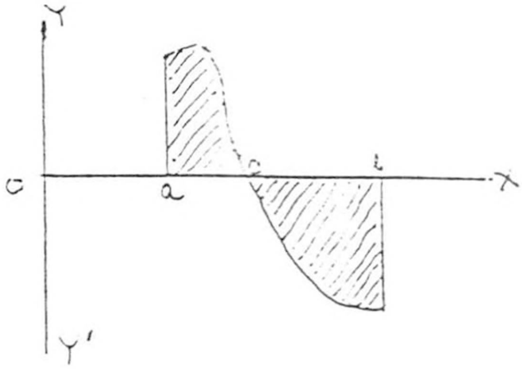


The proofs of the above two assertions are very much similar to the extended definition of $\int_a^b f(x).dx$ given in lesson 1 and the reader can frame the proofs themselves based on the definition of $\int_a^b f(x).\, dx$.

The above two assertions immediately lead to the following :

3. If $f(x) \geqslant 0$ for $x \in [a,c]$ and $f(x) \leqslant 0$ for $x \in [c, b]$ then the total area A bounded by $y = f(x)$, $x = a$, $x = b$ and y-axis is given by

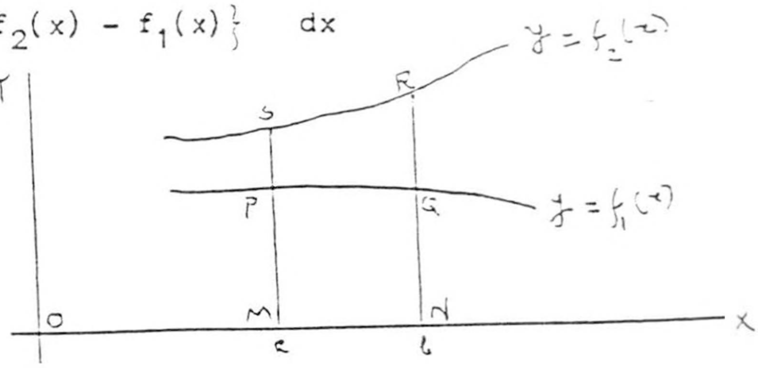
$$A = \int_a^c f(x)\,dx - \int_c^b f(x)\,dx$$



Similar results can be stated for the function $x = g(y)$.

4. The area A bounded by the graphs of the functions $y = f_1(x)$ and $y = f_2(x)$, and the ordinates $x = a$ and $y = b$, $(a < b)$ where $f_1(x) \leq f_2(x)$ for all $x \in [a,b]$ is given by

$$A = \int_a^b \{f_2(x) - f_1(x)\}\,dx$$



The figure is self-explanatory.

Clearly, area PQRSP

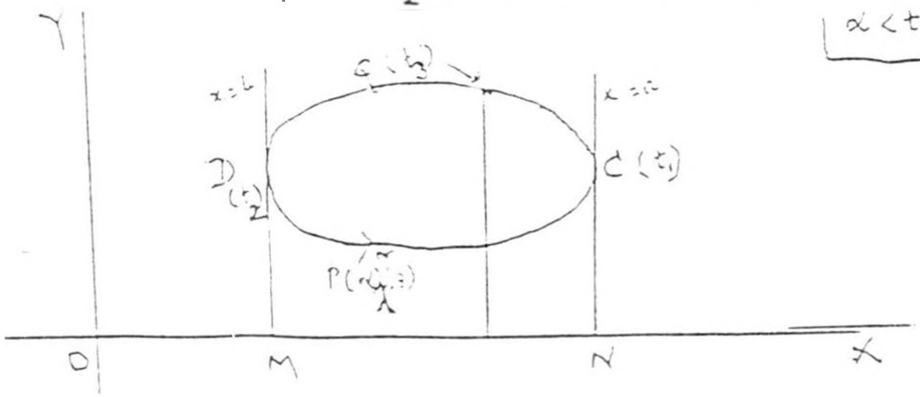
$= $ area MNRSM $-$ area MNQPM

$$= \int_a^b f_2(x)\,dx - \int_a^b f_1(x)\,dx$$

$$= \int_a^b \{f_2(x) - f_1(x)\}\,dx$$

5. <u>Area enclosed by a plane curve (equations given in parametric form):</u>

Let a closed curve be given by $x = f(t)$, $y = g(t)$, $\alpha \le t \le \beta$ so that $f(\alpha) = f(\beta)$ and $g(\alpha) = g(\beta)$. Let us suppose that the closed curve starts (corresponding to α) and ends (corresponding to β) at the point P. Let any line parallel to y-axis (intersecting the curve) intersect the curve in exactly two points. Let the lines $x = a$ and $x = b$ touch the curve in points D and C, where these points correspond to t_1 and t_2 (values of t) respectively so that $\alpha < t_1 < t_2 < \beta$.



Let Q be a point on the curve corresponding to t_3 such that $t_1 < t_3 < t_2$.

Now the area of the region

= area of region MNCQDM - area of region MNCPDM

$= S_2 - S_1$

where S_2 = area of region MNCQDM

where S_1 = area of region MNCPDM

Also $S_2 = \int_{\alpha}^{\alpha} y\, dx$, covering the region MNCQDM

$$= \int_{t_2}^{t_1} y(t) \frac{dx}{dt}\, dt + \int_{t_3}^{t_1} y(t) \cdot \frac{dx}{dt} \cdot dt$$

Similarly,

$$S_1 = \int_{t_2}^{\beta} y(t) \cdot \frac{dx}{dt}\, dt + \int_{t_1}^{\alpha} y(t) \cdot \frac{dx}{dt} \cdot dt ,$$

considering the areas under the arcs DP and PC respectively.

So, $S = S_2 - S_1$

$$= \left(\int_{t_2}^{t_3} y \cdot \frac{dx}{dt}\, dt + \int_{t_3}^{t_1} y \cdot \frac{dx}{dt}\, dt \right) - \left(\int_{t_2}^{\beta} y \cdot \frac{dx}{dt} \cdot dt - \int_{\alpha}^{t_1} y \cdot \frac{dx}{dt} \cdot dt \right)$$

$$= - \int_{\alpha}^{t_1} y \cdot \frac{dx}{dt} \cdot dt - \int_{t_1}^{t_3} y \cdot \frac{dx}{dt} \cdot dt - \int_{t_3}^{t_2} y \cdot \frac{dx}{dt} \cdot dt - \int_{t_2}^{\beta} y \cdot \frac{dx}{dt} \cdot dt$$

$$= - \int_{\alpha}^{\beta} y \cdot \frac{dx}{dt} \cdot dt \quad \text{- - - - - - - - - - - - (1)}$$

Similarly, considering tangents to the closed curve parallel to x-axis, we can show that

$$S = \int_{\alpha}^{\beta} x \cdot \frac{dy}{dt} \cdot dt \qquad (2)$$

Adding (1) and (2), we get

$$2\,S = \int_{\alpha}^{\beta} x \frac{dy}{dt} \cdot dt - \int_{\alpha}^{\beta} y \cdot \frac{dx}{dt} \cdot dt$$

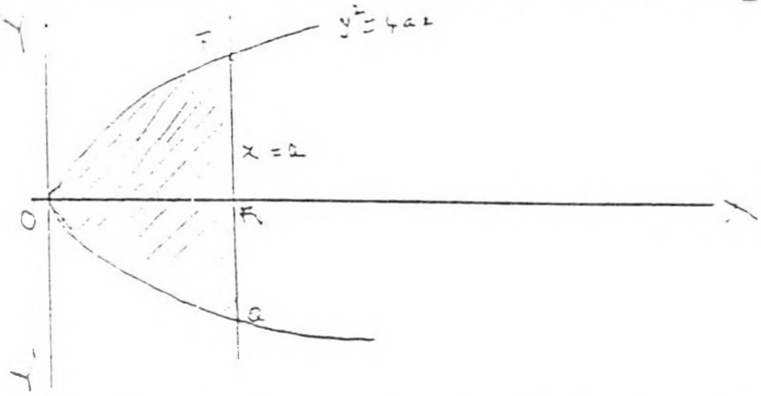
$$= \int_{\alpha}^{\beta} \left(x \frac{dy}{dt} - y \frac{dx}{dt}\right) dt$$

Hence the area enclosed in the closed curve

$$= 1/2 \int_{\alpha}^{\beta} x \frac{dy}{dt} - y \frac{dx}{dt} \quad dt$$

Solved Examples :

1. Determine the area bounded by the parabola $y^2 = 4ax$ and $x = b$.



The required area is the shaded portion in the figure which is self-explanatory. The parabola $y^2=4\,a\,x$ is symmetrical about x-axis. So, the required area

$= 2 \times$ area QPR

$= 2 \int_0^b y \,.\, dx$

$= 2 \int_0^b \sqrt{4ax} \,.\, dx$ (y is taken as the positive side of the area is considered here)

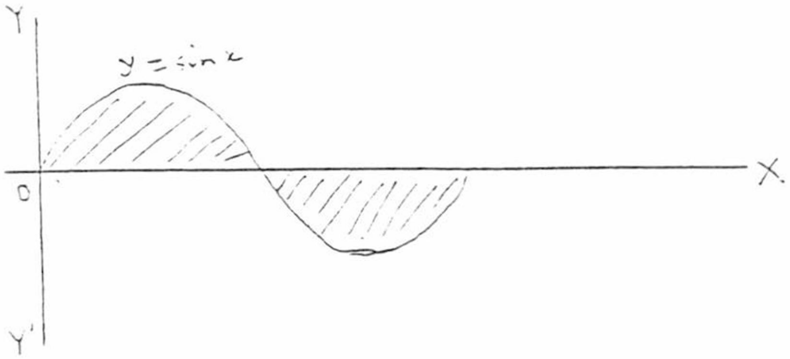
$= 2.2 \sqrt{a} \int_0^b x^{1/2} \,.\, dx$

$= 4\sqrt{a}\ \frac{2}{3} \left[x^{3/2} \right]_0^b \qquad = 4\,\frac{2}{3}\sqrt{a}.\ b^{3/2}$

$= \frac{8}{3}\ \sqrt{a}\,.\ b^{3/2}$

$= \frac{8}{3}\ \sqrt{a\, b^{3/2}}$

2. Find the area under the curve $y = \sin x$ between 0 and 2 .



Here, we note that between 0 and π, $\sin x \geq 0$; and between π and 2π, $\sin x \leq 0$.

So the required area

$$= \int_0^{\pi} \sin x \, . \, dx - \int_{\pi}^{2\pi} \sin x . \, dx$$

$$= \Big[\cos x\Big]_0^{\pi} - \Big[\cos x\Big]_{\pi}^{2\pi}$$

$$= (1-0) - (0-1)$$

$$= 1 + 1 = 2$$

3. Find the area enclosed by a loop of the curve

$$a^2y^2=x^2\ (a^2-x^2)$$

Solution: Here the equation of the curve is

$$a^2y^2 = x^2\ (a^2-x^2) \qquad (1)$$

The curve (1) intersects $y = 0$ in the points given by
$0 = x^2(a^2-x^2)$ i.e. $x = 0$, $x = \pm a$.

The tangents at the origin is given by

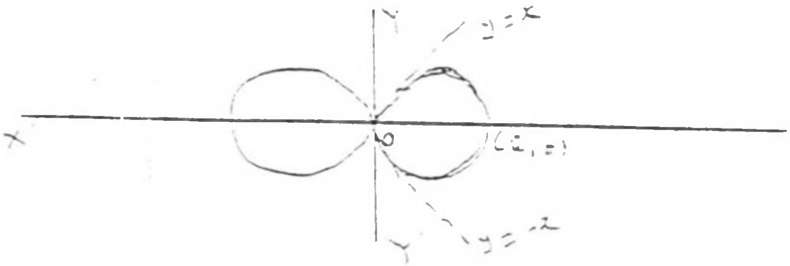
$$x^2-y^2 = 0$$

which shows that the origin is a node.

So, a loop of the curve is

$$a^2y^2 = x^2\ (a^2-x^2), \quad 0 \leq x \leq a$$

Also the loop is symmetric about x- axis



Thus the area of the loop is

$$= 2 \int_0^a y \ dx = \frac{2}{a} \int_0^a x \sqrt{a^2 - x^2} \ . \ dx$$

$$= \ 2/a \int_0^{\pi/2} a \sin \theta, \ \ a \sin \theta, \ \ a \cos \theta \ \ d\,\theta$$ by putting

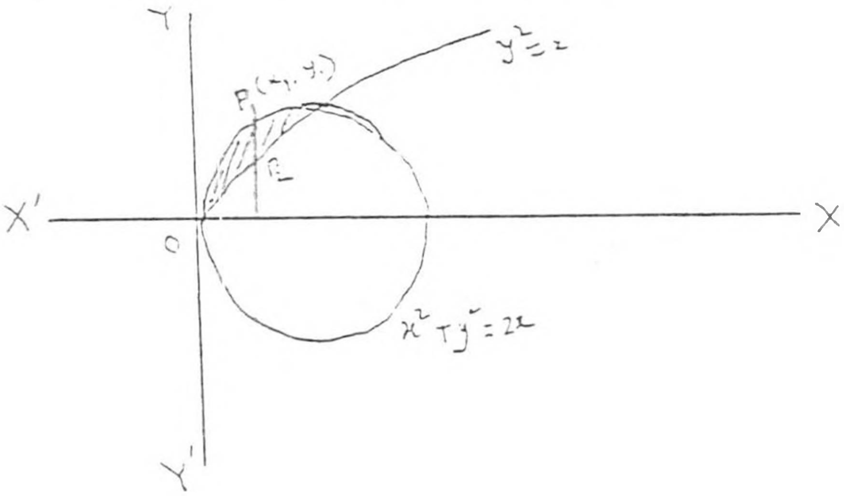
$$x = a \sin \theta$$

$$= \ 2 \ . \ a^3 \int_0^{\pi/2} \cos^2\theta, \ \sin \theta \ . \ d\theta$$

$$= 2a^2 \Big[- \cos^3\theta \Big]_0^{\pi/2}$$

$$= \ \frac{2a^2}{3}$$

4. Find the area above the x-axis, of the region bounded by the parabola y^2=x and the circle x^2+y^2 = 2x.



The x-coordinates of points of intersection of the parabola $y^2 = x$ and the circle $x^2+y^2 = 2x$ are given by

$x^2+x = 2x$ i.e., $x^2-x = 0$ i.e. $x(x-1) = 0$ i.e.
$x = 0$ and $x = 1$.

So we have to find the area bounded by the given curve above the x-axis so that for the points of the region

$0 \leq x \leq 1$

Thus the required area

$$= \int_0^1 (y_1-y_2)\, dx, \text{ where } y_1 = 2x-x^2 \text{ and } y_2^2 = x$$

$$= \int_0^1 \left(\sqrt{2x - x^2} - \sqrt{x}\right) dx$$

$$= \int_0^1 \sqrt{2x - x^2} \,.\, dx - \int_0^1 \sqrt{x} \,.\, dx$$

For integrating $\int_0^1 \sqrt{2x - x^2} \,.\, dx$, set $x = 2 \sin^2 \theta$. Then
$dx = 4 \sin \theta \cos\theta \,.\, d\theta$

and $x = 0 \Longrightarrow \theta = 0$,
$x = 1 \Longrightarrow \theta = \frac{\pi}{4}$

Then $\int_0^1 \sqrt{2x-x^2} \,.\, dx$

$$= \int_0^{\pi/4} \sqrt{(2.2 \sin^2\theta - 4 \sin^4\theta)}\ 4 \sin \theta \,.\, \cos \theta \,.\, d\theta$$

$$= \int_0^{\pi/4} 2 \sqrt{\sin^2\theta\ (1-\sin^2\theta)}\ 4 \sin \theta \cos \theta \,.\, d\theta$$

$$= \int_0^{\pi/4} 3\ \sqrt{\sin^2\theta - \cos^2\theta} \,.\, \sin \theta \,.\, \cos \theta \,.\, d\theta$$

$$= \int_0^{\pi/4} 8 \sin^2\theta \cos^2\theta \,.\, d\theta = \int_0^{\pi/4} 2 \sin^2 2\theta \,.\, d\theta$$

$$= \int_0^{\pi/4} (1-\cos 4\theta)\, d\theta = \left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\pi/4} = \frac{\pi}{4}$$

Also, $\int_0^1 \sqrt{x} \,.\, dx = \left[\frac{2}{3} x^{3/2}\right]_0^1 = \frac{2}{3}$

Therefore, the required area $= \frac{\pi}{4} - \frac{2}{3}$

5. Find the area enclosed by the curve given by
$x\,(1 + t^2) = 1 - t^2$, $y\,(1 + t^2) = 2t$

Solution :

Here it is a variable parameter taking its values from to . So we can set $t = \tan\theta$ where

Then $x = \dfrac{1 - t^2}{1 + t^2} = \dfrac{1 - \tan^2\theta}{1 + \tan^2\theta} = \cos 2\theta$

and $y = \dfrac{2t}{1 + t^2} = \dfrac{2\tan\theta}{1+\tan^2\theta} = \sin 2\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

Note that the parametric equation represents a closed curve.
Hence the required area

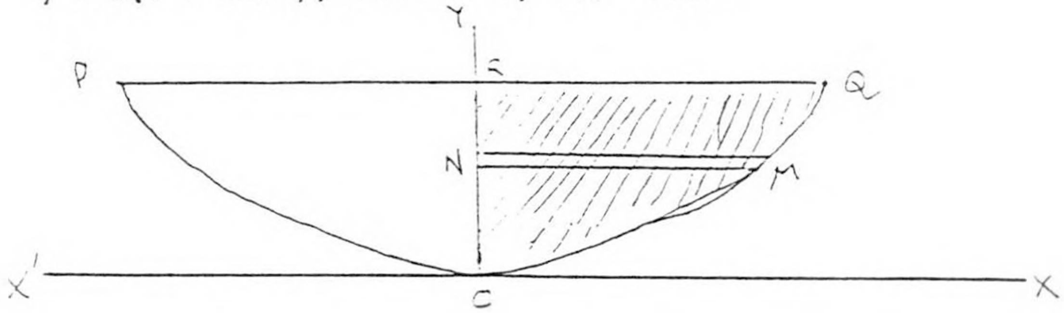
$$= \frac{1}{2} \int_{-\pi/2}^{\pi/2} \left(x \frac{dy}{d\theta} - y \frac{dx}{d\theta}\right) d\theta$$

$$= \frac{1}{2} \int_{-\pi/2}^{\pi/2} (\cos 2\theta \,.\, 2\cos 2\theta - \sin \theta \,.\, (-2\sin\theta))\, d\theta$$

$$= \frac{1}{2} \cdot 2 \int_{-\pi/2}^{\pi/2} (\cos^2 2\theta + \sin^2 2\theta)\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\pi/2} d\theta \quad = \left[\theta\right]_{-\frac{\pi}{2}}^{\pi/2} \quad = \pi/4$$

6. Find the whole area of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$, bounded by its base.



Here the area of half the cycloid i.e., the shaded portion in the figure is the region bounded by the cycloid, $y = 0$ and $y = 2a$. Hence the total area of the cycloid

= 2 (area of the shaded portion in the figure).

$$= \quad 2 \times \int_{0}^{2a} dy$$

$$= 2 \int_{0}^{\pi} a(\theta + \sin\theta),\ a \sin\theta \, . \, d\theta$$

$$\left[\begin{aligned} &\text{for } x = a(\theta + \sin\theta), \\ &\quad dy = a.\cos\theta \, . \, dx, \\ &\quad y = 0 \Rightarrow \theta = 0, \\ &\quad y = 2a \Rightarrow \theta = \pi \end{aligned}\right]$$

$$= 2a^2 \int_{0}^{\pi} (\theta \, . \sin\theta + \sin^2\theta)\ d\theta$$

$$\int_0^\pi \theta \sin\theta \, d\theta = -\theta\cos\theta - \int_0^\pi (-\cos\theta)\, d\theta$$

$$= \left[-\theta\cos\theta + \sin\theta\right]_0^\pi$$

$$\int_0^\pi \theta \cdot \sin^2\theta \, d\theta = \int_0^\pi \frac{(1-\cos 2\theta)}{2}\, d\theta$$

$$= \left[\frac{\theta - \frac{\sin 2\theta}{2}}{2}\right]_0^\pi$$

Hence the required area

$$= 2a^2 \left[-\theta\cos\theta + \sin\theta + \tfrac{1}{2}\left(\theta - \tfrac{\sin 2\theta}{2}\right)\right]_0^\pi$$

$$= 2a^2 \left[-\pi\cos\pi + \sin\pi + \tfrac{1}{2}\left(\pi - \tfrac{\sin 2\pi}{2}\right) + 0\cdot\cos 0 - \sin 0 - \tfrac{1}{2}\left(0 \cdot \tfrac{\sin 0}{2}\right)\right]$$

$$= 2a^2\left[\pi + 0 + \frac{\pi}{2} - 0 + 0 - 0 - 0\right]$$

$$= 2a^2 \cdot \frac{3\pi}{2} = 3a^2\pi$$

Note: Here the parametric equations of the cycloid do not represent a closed curve.

Assignments :

1. Find the area of the segment cut off from $y^2 = 4x$ by the line $y = 2x$.

2. Find the area of the portion of the circle $x^2+y^2 = 1$ which lies inside the parabola $y^2 = 1-x$.

3. Find the area bounded by the curves $y^2 - 4x - 4 = 0$ and $y^2 + 4x - 4 = 0$.

4. Find the area included between the ellipses $x^2+2y^2=1$ and $2x^2+y^2 = 1$.

5. Find the areas enclosed by the following curves :

a) $x = a \cos t + b \sin t, \quad y = a^1 \cos t + b^1 \sin t$

b) $x = a \sin 2t, \; y = a \sin t$

c) $x = a\,(1-t^2), \; y = at\,(1-t^2) \quad (-1 \leq t \leq 1)$

d) $x = \dfrac{1 - t^2}{1 + t^2}, \quad y = t\,\dfrac{(1 - t^2)}{(1 + t^2)}, \quad (-1 \leq t \leq 1)$

6. Find the area bounded by the axis x, part of the curve $y = (1 + \dfrac{8}{x^2})$ and the ordinates at $x = 2$ and $x = 4$. If the ordinates at $x = a$ divides the area into two equal parts, find a.

7. Find the area bounded by the curves $x^2+y^2 = 25$, $4y = |4-x^2|$ and $x=0$, above the x-axis.

Answers :

1. 8/3

2. $\left((1/2)\pi + \frac{4}{3}\right)$

3. 16/3

4. $2\sqrt{2}$

5. a) $\pi(ab^1 - a^1 b)$

   b) $\frac{8}{3}a^2$

   c) $\frac{8a^2}{15}$

   d) $2 - \frac{\pi}{2}$

6. Area A = 4 sq. units, $a = 2\sqrt{2}$

7. $4 + 25\sin^{-1}\frac{4}{5}$ sq. units

D I F F E R E N T I A L   E Q U A T I O N S

1. Differential Equations, their Classification and Terminology

2. Methods of Solving First Order Differential Equations

3. Applications of First Order Differential Equations

by

Dr.N.B.BADRINARAYAN

DIFFERENTIAL EQUATIONS

An Introduction :

1. A body is falling freely under gravity.
2. A body is falling under air resistance.
3. The bob of a simple pendulum is pulled aside and let go.
4. A hot body cools according to certain law.
5. A chain of given length hangs over the smooth edge of a table and begins to slide off the table.

Here are a few situations where we need to discuss the problem. The problem may be the motion of the body or the bob of the simple pendulum or the temperature of the cooling body at a given moment or the motion of the chain sliding off the table on which it is lying. A Differential Equation set up to describe each of these problems is the mathematical formulation of the problem itself. Consequently, solving the differential equation is equivalent to solving the problem itself.

Differential equations occur in the context of numerous problems which one comes across in different branches of science and engineering.

Some of them are the problem of determining

a) the motion of a projectile, rocket, satellite or planet.

b) the current in an electric circuit.

c) the conduction of heat in a rod or a slab.

d) the vibrations of a wire or a membrane

e) the flow of a liquid

f) the rate of decomposition of a radioactive substance or the rate of growth of a population.

g) the reaction of chemicals

h) the curves which have certain geometrical properties.

The mathematical formulation of such problems gives rise to differential equations. In each of the situations cited above, the objects involved obey certain laws of nature or scientific laws. These laws involve various rates of change of one or more quantities with respect to other quantities. Such rates are expressed as various derivatives and the scientific laws themselves become mathematical equations involving the derivatives, that is, differential equations.

"The vital ideas of mathematics.... were created by the solitary labour and individual genius of a few remarkable men.... A few of the greatest mathematicians of the past three centuries are Fermat, Newton, the Bernoullis, Euler, Lagrange, Laplace, Gauss, Abel, Hamilton, Liouville, Chebyshev, Hermite, Riemann and Poincare".

An elementary course on differential equations as this, aims at familiarising to its students, basic terminology and methods and techniques of solving first order equations of the type

$$\frac{dy}{dx} = f(x,y) \text{ in easy cases.}$$

Further, a student at the end of this course should be able to apply the concepts and techniques of solving differential equations of first order to problems arising in real life situations, some of which have been mentioned already.

The prerequisites for the course are

i) working knowledge of differentiation and integration

ii) familiarity with plane curves.

Differential Equations and Their Classification - Terminology

An equation involving an unknown function of one or more (independent) variables and the derivatives of the unknown function w.r.t. the independent variable(s) is called a differential equation.

Some examples :

1. $$\frac{d^2y}{dx^2} + x + \left(\frac{dy}{dx}\right)^2 = 0$$

2. $$\frac{d^2y}{dt^2} + 5\cdot\frac{dx}{dt} + 6x = e^t$$

3. $$\frac{\partial v}{\partial s} + \frac{\partial v}{\partial t} = 0$$

4. $$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

5. $$\frac{dx}{dt} = y; \quad \frac{dy}{dt} = -x.$$

A differential equation involving ordinary derivatives of one independent variable w.r.t. the independent variable is called an ordinary differential equation (or equation).

Examples: In the earlier set of examples, equations (1), (2) and (5) are ordinary equations.

In (1) y is the dependent variable or the unknown function of x while x is the lone independent variable.

In (2) x is the dependent variable and t is the independent variable.

In (5) x and y are both dependent variables and t is the independent variable.

A differential equation involving partial derivatives of one dependent variable w.r.t. more than one independent variables is called a <u>partial differential equation.</u>

Examples: In the set of examples already given, equations (3) and (4) are partial differential equations.
In (3) v is the dependent variable and s and t are independent variables. In (4) z is the dependent variable and x,y are independent variables.

More examples of Differential Equations :

1. $$\frac{dy}{dx} = -ky$$

2. $$m.\frac{d^2x}{dt^2} = mg - k.\frac{dx}{dt}$$

3. $$\frac{dy}{dx} + 2xy = e^{-x^2}$$

4. $$\frac{d^2y}{dx^2} - 5.\frac{dy}{dx} + 6x = 0$$

5. $$(1-x^2).\frac{d^2y}{dx^2} - 2x.\frac{dy}{dx} + p(p+1)y = 0.$$

6. $$x^2.\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2-p^2)y = 0.$$

7. $$a^2\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right) = \frac{\partial \omega}{\partial t}.$$

8. $$a^2\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)=\frac{\partial^2 w}{\partial t^2}$$

9. $$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}=0.$$

10. $$L\cdot\frac{d^2Q}{dt^2}+R\frac{dQ}{dt}+\frac{1}{C}Q=E.$$

Some of these equations are classical. (5) and (6) are called Legendre's equation and Bessel's equation respectively.

The equations (7), (8) and (9) are the classical heat equation, wave equation and Laplace's equation respectively.

Readily it is seen that (1) to (6) and (10) are ordinary equations while (7) to (9) are partial equations.

Order and Degree of a Differential Equation :

The order of the highest ordered derivative found in a differential equation is called the order of the equation.

The degree of the highest order derivative in a differential equation which is free from radicals and fractions in its derivatives is called the degree of the equation.

In the examples (1) to (10) we had earlier easily we can recognise the order and degree of each equation.

The equations (1) and (3) are of order 1 and degree 1. The other equations are of order 2 and degree 1.

More examples :

$$\left(\frac{d^2y}{dx^2}\right)^2 + K\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = 0$$

has its order and degree 2 each.

$\left(\frac{dy}{dx}\right)^3 + y = e^x$ has order 1 and degree 3.

The equation $\dfrac{dy}{dx} + \dfrac{1}{dy/dx} = 2x$

has to be rewritten as $\left(\dfrac{dy}{dx}\right)^2 + 1 = 2x\left(\dfrac{dy}{dx}\right)$

Then the order and degree are respectively 1 and 2.

The equation

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

has to be rewritten as

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = \rho^2\left(\frac{d^2y}{dx^2}\right)^2$$

Then the order and degree of the equation are both 2. A Linear Equation of nth order. An ordinary Linear differential equation of nth order is given by

$$a_0(x)\, y^{(n)} + a_1(x)\, y^{(n-1)} + \dots + a_n(x)y = b(x) .$$

$y^{(k)} = \dfrac{d^ky}{dx^k}$ = the kth derivative of y w.r.t. x.

The equation is (1) said to be homogeneous if $b(x) \equiv 0$.

(2) said to be a linear equation with constant coefficients if all the coefficients $a_0(x), a_1(x), \dots, a_n(x)$ are

constants. An equation which is not homogeneous is called a non-homogeneous or inhomogeneous equation.

Examples :

1. $y''' + 3x^2 y'' + 3xy' + 2y = 0$

   is a linear homogeneous equation where $1 = \frac{d}{dx}$, $11 = \frac{d^2}{dx^2}$, etc.

2. $y'' + y' + xy = 0$

   is a homogeneous linear equation with variable coefficients.

3. $y^{(4)} + y'' + y = e^x$

   is a non homogeneous linear equation with constant coefficients.

4. $x^3 y''' + 2x^2 y'' + 3xy' + 4y = \text{Sin } x$

   is a non homogeneous equation with variable coefficients.

5. $y^{11} + xy^2 = 0$ is not a linear equation.

6. $(y^1)^2 + y = e^x$ is also not linear.

Note:

1. y and its derivatives in the linear equation occur in first degree only.
2. Consequently a linear equation is necessarily of first degree.
3. No products of y and/or any of its derivatives are present.
4. No transcendental functions of y and/or its derivatives occur.

More examples :

1. $$\frac{d^2y}{dx^2} + 5.\frac{dy}{dx} + 6y = 0$$

2. $$\frac{d^2y}{dx^2} + x^2.\frac{d^2y}{dx^2} + x^3.\frac{dy}{dx} = xe^x$$

are ordinary linear equations.

An ordinary differential equation which is not linear is called a non linear ordinary differential equation.

. $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y^2 = 0$ is a non linear ordinary equation.

A general ordinary differential equation of nth order is a relation of the type: $F(x, y, y', y'', \ldots, y^{(n)}) = 0$.

Formation of Differential Equations

Problems

1. Suppose that a body of mass m falls freely under gravity. In this case the only force acting on the body is its weight mg. If x is the distance through which the body falls in time t, then its acceleration is $\frac{d^2x}{dt^2}$.

Then the equation of motion of the falling body is

$$m\frac{d^2x}{dt^2} = mg \text{ or } \frac{d^2x}{dt^2} = g \qquad \ldots \qquad (1)$$

2. If there is a resisting force by air (say) proportional to the velocity, then the total force acting on the body is mg - K $\frac{dx}{dt}$ (- because the air resistance opposes the motion). In this case, the equation of motion becomes,

$$m \cdot \frac{d^2x}{dt^2} = mg - k\frac{dx}{dt}$$

or $$m \cdot \frac{d^2x}{dt^2} + k \cdot \frac{dx}{dt} = mg$$

or $$\frac{d^2x}{dt^2} + \left(\frac{k}{m}\right)\frac{dx}{dt} - g = 0 \quad ---(2)$$

3. Consider a pendulum consisting of a bob of mass m at the end of an inelastoc string or rod of negligible mass and of length a. If the bob is pulled aside through an angle and released, then by the principle of conservation of energy

$$1/2\, mv^2 = mg\,(a\cos\theta - a\cos\alpha)$$

$$s = a\theta, \quad \frac{ds}{dt} = v = a \cdot \frac{d\theta}{dt}$$

The equation of motion becomes

$$\frac{1}{2}a^2\left(\frac{d\theta}{dt}\right)^2 = ag\left(\cos\theta - \cos\alpha\right); \quad \alpha > \theta$$

or $$\left(\frac{d\theta}{dt}\right)^2 = \frac{2g}{a}\left(\cos\theta - \cos\alpha\right)$$

Or $$\frac{d\theta}{dt} = \sqrt{\frac{2g}{a}\left(\cos\theta - \cos\alpha\right)} \quad ---(3)$$

4. Assume that a hot body cools at a rate proportional to the difference between the temperatures of the body and the surroundings. This law is known as Newton's law of cooling.

Let θ denote the temperature of the body at any moment t and θ_0 the temperature of the surroundings of the body. Then the rate of cooling is $\frac{d\theta}{dt}$ and this is proportional to $(\theta - \theta_0)$. Then the cooling of the body is governed by the equation

$$\frac{d\theta}{dt} = -k(\theta - \theta_0), \quad K > 0.$$

$$\text{or} \quad \frac{d\theta}{dt} + k\theta = k\theta_0 \quad --- \quad (4)$$

5. A tank contains 50 gal of pure water initially. A brine containing 2 lb of dissolved salt per gallon flows into the tank at the rate of 3 gals./min. The mixture is kept uniform by constant stirring and the well-stirred mixture simultaneously flows out of the tank at the same rate.

Let x denote the amount of salt in the tank at time t. Then the equation for the rate of change of x is

$$\frac{dx}{dt} = \text{Inflow–outflow} \quad \ldots \quad (i)$$

The brine flows at the rate of 3 gals/min and each gallon contains 2 lbs salt.

Then, Inflow = (2lb/gals) x (3 gal/min) = 6 lb/min... (ii)

Since the rate of outflow = the rate of inflow, the tank contains 50 gal of mixture in time t. This 50 gal. contains x lbs of salt in time t. Therefore, the concentration of salt at time t = 1/50 x lb/gal.

Then, the outflow = (x/50 lb/gal) (3 gal/min) =

$\frac{3x}{50}$ lb/min. ..(iii)

Hence, (i), (ii) and (iii)

$$\Rightarrow \frac{dx}{dt} = 6 - \frac{3x}{50} \quad --- \quad (5)$$

which is the equation governing the rate of change of salt content.

The above discussed problems illustrate how a differential equation describes the problem. In other words, in these illustrations, the mathematical formulation of the problem is the differential equation.

In each problem above, we can recognise the following important steps leading to the mathematical formulation of the problem, that is, the differential equation.

1. Identification of the law/laws, operating in the problem.
2. Analysis of the problem.
3. Representing the attributes by symbols.
4. Formation of the equation using the relationships or laws in the problem.

Differential Equations for Families of Curves :

1. Consider the family of concentric circles with their centre at the origin.

The circles are all given by

$$x^2 + y^2 = a^2 \qquad \ldots \qquad (1)$$

As a takes various values, we get different members of the family of circles. We describe a as the parameter of the family of circles.

Differentiating (1) w.r.t. x, we get

$$2x + 2y\frac{dy}{dx} = 0 \quad or \quad x + y\frac{dy}{dx} = 0 \quad --(2)$$

The differential equation (2) represents the family of circles.

We note: 1. that (2) is free from the parameter. In other words, the parameter a is eliminated in getting the differential equation.

2. The number of parameters in (1) is equal to the order of the differential equation (2), each being one.

2. Consider the family of circles through the origin with their centres on the x-axis.

Each circle of the family is given by $x^2 + y^2 = 2cx$. - - (1)

As c takes different values, we get different circles, c is the parameter of the family of circles.

Differentiating (1) w.r.t. x

$$2x + 2y\cdot\frac{dy}{dx} = 2c \quad or \quad x + y\frac{dy}{dx} = c \quad --(2)$$

Eliminating c between (1) and (2), we get

$$x^2 - y^2 = 2\left(x + y\frac{dy}{dx}\right)x$$

$$= 2x^2 + 2xy\frac{dy}{dx}$$

or $$y^2 - x^2 = 2xy \cdot \frac{dy}{dx}$$

Or $$\frac{dy}{dx} + \frac{1}{2}\left(\frac{x}{y} - \frac{y}{x}\right) = 0 \quad \text{-- (3)}$$

This differential equation represents the family of circles. Again we notice that (3) is a first order equation got by eliminating the single parameter c of the family of circles.

3. Consider the family of parabolas : $y = (x+c)^2$... (1)

c being the parameter of the family.

Differentiating (1), $\frac{dy}{dx} = 2\,(x+c)$

Eliminating c from (1) and (2), we get

$$\frac{dy}{dx} = 4\,(x+c)^2 = 4y$$

or $$\frac{dy}{dx} - 4y = 0 \qquad (3)$$

is the differential equation representing the family of parabolas.

Solution of a differential equation :

An Illustration : Consider the function $y = ae^{2x} + be^{-2x}$ (1)

where a,b are arbitrary constants.

Differentiating w.r.t. x we get $y = 2ae^{2x} - 2\,be^{-2x}$

Differentiating w.r.t. x again, $y = 4ae^{2x} + 4be^{-2x}$

$= 4(ae^{2x} + be^{-2x})$

or $y = 4y$ - - (2)

The function (1) satisfies the differential equation (2) for all constants a and b. (1) is a solution of the differential equation (2) for all values of a and b.

Consider an nth order ordinary differential equation

$$F(x, y, y', y'', ---, y^{(n)}) = 0 \quad -- (1)$$

where F is a real function of x, y, y', y",....$y^{(n)}$

$$y^{(n)} = \text{The } n^{th} \text{ derivative of } y \text{ w.r.t. } x = \frac{d^n y}{dx^n}.$$

A real function $y = f(x)$ (2) is called a solution of the differential equation over some interval I if y is differentiable n times and 'satisfies the differential equation'

i.e. $F(x, f(x), f'(x), \ldots, f^{(n)}(x)) = 0$ for all $x \in I$.

The phrase 'satisfies the differential equation' means that when $y, \frac{dy}{dx}, \ldots,$ are replaced by $f(x), f'(x), \ldots f^{(n)}x$ respectively in (1), the equation (1) becomes an identity.

A differential equation is said to be solved if a solution of the equation is found.

Another Illustration :

The differential equation : $\frac{d^2y}{dx^2} + m^2x = 0$ has its solution $y = a \cos mx + b \sin mx$ where a and b are arbitrary constants.

Verification :

$$y = a\cos mx + b\sin mx$$

$$\frac{dy}{dx} = -ma\sin mx + mb\cos mx$$

and $$\frac{d^2y}{dx^2} = -m^2a\cos mx - m^2b\sin mx$$

$$= -m^2(a\cos mx + b\sin mx)$$

$$\frac{d^2y}{dx^2} = -m^2y \quad \text{or} \quad \frac{d^2y}{dx^2} + m^2y = 0$$

In the illustrations, the constants a and b of the solutions can take any values. Such a solution of a differential equation containing arbitrary constants (as a and b) is called the general solution of the differential equation.

A solution got from the general solution for particular values of the arbitrary constants is called a particular solution of the differential equation.

Initial Value Problem :

$y = x^2 + c$, c being an arbitrary constant, is the general solution of $\frac{dy}{dx} = 2x$. The particular solution satisfying the condition $y = 4$ when $x = 1$ is got from the general solution $y = x^2+c$. Putting $x = 1$, $y = 4$, $4 = 1 + c$ or $c = 3$. Hence the particular solution required is $y = x^2 + 3$.

A given differential equation together with an additional condition as in the above is called an Initial Value Problem (I.V.P.)

Thus, $\frac{dy}{dx} = 2x$

together with $y = 4$ when $x = 1$ is an initial value problem.

The above initial value problem is written as

$\frac{dy}{dx} = 2x$ The differential equation

$y(1) = 4$ The initial condition I.V.P.

The condition in the initial value problem is called an initial condition of the problem. For the initial value problem :

$\frac{dy}{dx} = 2x$, $y(1) = 4$, $y = x^2 + 3$ is the solution.

Thus a solution of an initial value problem is a solution of the differential equation of the problem. In addition to this, the solution must satisfy the initial condition also.

Another Example : $\frac{d^2y}{dx^2} + y = 0$ has the general solution $y = a \cos x + b \sin x$. Suppose $y(0) = 2$, $y'(0) = 3$, then $a=2$, $b=3$. Thus, $y = 2\cos x + 3 \sin x$ is a particular solution of the differential equation. This particular solution satisfies the conditions $y(0) = 2$, and $y'(0) = 3$.

Therefore, $\frac{d^2y}{dx^2} + y = 0$

with $y(0) = 2$ and $y'(0) = 3$.

is an initial value problem having the solution

$y = 2\cos x + 3\sin x$.

A general nth order initial value problem is of the type

$F\,(x, y, y', y'', \ldots, y^{(n)}) = 0$ over I. (1)

$y(x_0) = y_0,\; y'(x_0) = y_0',\; \ldots,\; y^{(n-1)}(x_0) = y_0^{(n-1)}$ --(2)

for some value $x = x_0 \in I$.

The set of conditions in (2) is the set of initial conditions of the initial value problem. Here,

$y_0, y_0', y_0'', \ldots, y_0^{(n-1)}$ are given values.

Geometrical Meaning :

A differential equation represents a family of curves. Given a family of curves

$f(x,y,a,b) = 0$ ... (1)

by eliminating a and b, by differentiating (1), we get the differential equation.

$F(x,y, y', y'') = 0$ --- (2)

(1) is the general solution of (2) and represents the family of curves. Each curve of the family is a particular solution of the differential equation (2).

A solution of an initial value problem is a particular curve of the family of curves given by the differential equation (2).

Points to stress while teaching :

1. The difference between

a) the ordinary and partial equations

b) order and degree equations

c) Linear and non-linear equations.

d) Linear homogeneous and non homogeneous equations

e) General solution and particular solutions

f) Formation of an equation and solving an equation

g) Solving an equation and an Initial Value Problem

2. The geometrical meanings of

a) a differential equation : $\frac{dy}{dx} = f(x,y)$

b) the general solution of an equation

c) a particular solution of an equation

3. Information of a differential equation for a physical problem

a) identification of the law/laws operating

b) analysis of the problem

c) symbols and notations

4. Solution of an equation

a) Verification of a function as a solution of a given equation.

b) Formation of the equation from a given solution .

Assignments and Self Test :

I. 1. Classify the differential equations as ordinary or partial differential equations.

2. State the order and the degree.

3. Determine whether the equation is linear or non linear.

4. If the equation is linear, whether it is homogeneous or non-homogeneous.

i) $y' + x^2y = xe^x$

ii) $y''' + 4y'' + 5y' + 3y = \sin x$

iii) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

iv) $x^2dy + y^2dx = 0$

v) $\frac{\partial^2 u}{\partial x^2} + c\frac{\partial u}{\partial t} = 0$

vi) $y^{(4)} + 3y'' + 5y^2 = 0$

vii) $y'' + y \sin x = 0$

viii) $y'' + x \sin y = 0$

ix) $\left(\frac{dr}{ds}\right)^2 = \sqrt{\frac{d^2r}{ds^2} + 1}$

x) $\frac{dy}{dx} + \frac{dx}{dy} = 1$

xi) $xy' = y'\sqrt{1 - x^2y^2}$

xii) $\frac{dv}{dx} = -\frac{xv}{x^2+y^2}$

xiii) $y' = xe^{x^2}$

xiv) $\frac{dy}{dx} + \sqrt{\frac{1-y^2}{1-x^2}} = 0$

xv) $y''' + 4y'' - 5y' + 3y = \sin x$

II. Form the differential equation for the following problems.

a) The population (P) of a bacteria is increasing at a rate proportional to the population at the moment.

b) A moth ball evaporates at a rate proportional to its surface.

c) The air resistance on a falling body exerts a retardation proportional to the square of the velocity.

d) A chain 4 feet long starts sliding off the smooth table when 1 foot of the chain hangs over the edge which is supposed to be smooth (no friction).

e) A tank has 100 gallons of pure water. Brine containing 1 lb/gal. runs into the tank at the rate of 1 gal/min. The mixture is constantly stirred and flows out at the same rate as inflow.

f) An amount of invested money draws interest compounded continuously (i.e. the amount of money increases at a rate proportional to the amount present).

g) A chemical reaction converts a certain chemical into another chemical at a rate proportional to the amount of the unconverted chemical amount present at any time.

h) The rate at which radioactive nuclei decay is proportional to the number of such nuclei that are present in a given sample.

III. Show that the family of curves given by the first equation is represented by the corresponding differential equation.

1. $y = 2 + ce^{-2x^2}$, $\frac{dy}{dx} + 4xy = 8x$.

2. $y = (c+x^3)\, e^{-3x}$, $\frac{dy}{dx} + 3y = 3x^2 e^{-3x}$

3. $y = a\, e^{4x} + be^{-2x}$, $y'' - 2y' - 8y = 0$.

4. $y^2 = 4ax$, $2xy' = y$

5. $y = c_1 \sin 2x + c_2 \cos 2x$, $y'' + 4y = 0$

6. $xy = c$, $xy' + y = 0$

7. $y^2 = 4c\,(x+c)$, $(2x + yy')\, y' = y$

8. $y = c_1 e^x + c_2 e^{-x}$, $y'' = y$

IV. Verify that each function is a solution of the corresponding differential equation.

1. $y = x \operatorname{Tan} x$, $xy' = x^2 + y^2 + y$

2. $y = \log_e x$, $xy' = 1$

3. $y = 1 + 1/x$, $x^2 y' + 1 = 0$

4. $y = ce^{y/x}$, $x\,(y-n)\, y = y^2$

5. $x + y = \operatorname{Tan}^{-1} y$, $1 + y^2 + y^2 y' = 0$

6. $y = cx^n$, $x \frac{dy}{dx} = ny$

7. $y = cx + a/c$, $y = x \frac{dy}{dx} + a \frac{dx}{dy}$

8. $y = x^3 + ax^2 + bx + c$, $y''' = 6$

9. $y = x^2 - cx$, $2xyy' = x^2 + y^2$

10. $y = x + 3e^{-x}$, $y' + y = x + 1$

11. $y = 2e^{3x} - 5e^{4x}$, $y'' - 7y' + 12\,y = 0$

12. $y = e^x + 2x^2 + 6x + 7$, $y'' - 3y' + 2y = 4x^2$

13. $y = (1+x^2)$, $(1+x^2)y'' + 4xy' + 2y = 0$

V. Verify that the function given is a solution of the corresponding initial value problem.

a) $x^2 + y^2 = 25; \quad \frac{dy}{dx} + \frac{x}{y} = 0, \; y(3) = 4$

b) $y = 1/x, \; xy' + y = 0 \; , \; y(1) = 1$

c) $y = (2+x^2)e^{-x}, \quad \frac{dy}{dx} + y = 2xe^{-x}, \; y(0) = 2$

d) $y^2 = 4 \sec 2x, \; \frac{dy}{dx} = y \operatorname{Tan} 2x, \; y(0) = 2.$

e) $y^2 = 16x^3; \; 2xy' = 3y, \; y(1) = 4.$

f) $\operatorname{Sin} y = x; \; y' = \operatorname{Sec} y, \; y(0) = 0$

g) $y = e^{-x}; \quad y' + y = 0, \; y(0) = 1.$

h) $y = \operatorname{Tan}^{-1} x; \; y' = y \; (1+x^2), \; y(0) = 0$

VI. Assuming the given general solution of the differential equation, find the particular solution satisfying the additional (initial) condition.

a) $y' + y = 2xe^{-x}, \; y \; (c+x^2)e^{-x}, \; y(-1) = 3+e$

b) $xy' = 2y, \; y = cx^2, y(1) = 1$

c) $yy' = e^{2x}, \; y^2 = e^{2x} + c, \; y(0) = 1$

d) $y + xy' = x^4 (y')^2, \quad y = c^2 + c/x, \; y(1) = 0$

Key : Ord - ordinary equation, part - partial equation,

1,1 - 1st order, 1st degree

L - Linear, H = homogeneous, NH = non homogeneous, NL = Non linear

i) Ord, 1,1, L, NH

ii) Ord, 3,1, L, NH

iii) Part, 2,1, L, H

iv) Ord, 1,1, N, L

v) Part, 2,1, L H

vi) Ord, 4,1 L, H

vii) Ord, 2,1, L, H

viii) Ord, 2,1, NL

ix) Ord, 2,1, NL

x) Ord, 2,1,NL

xi) **Ord**, 1,1, NL

xii) Ord, 1,1, NL

xiii) Ord, 1,1, L, N, H

xiv) Ord, 1,1, NL

xv) Ord, 3,1, L, NH

II.

a) $dp/dt = kP, \quad k > 0$

b) $m \cdot dv/dt = mg - kv^2$

c) $\dfrac{dv}{dt} = ks, \quad k < 0$

d) $\dfrac{d^2x}{dt^2} = gx$

x = the length of the hanging chain at any moment t.

e) If x lb is the amount of salt present in the tank at time t,

$$\frac{dx}{dt} = 1 - \frac{x}{100}$$

f) $\dfrac{dA}{dt} = KA, \ K > 0$ A = The amount at any moment t.

g) $\dfrac{d}{dt} = K\,(x_o - x), \ x_o$ = The amount of the chemical present initially.

h) $\dfrac{dx}{dt} = Kx$, x = the no. of radioactive nuclei disintegrating.

METHODS OF SOLVING FIRST ORDER DIFFERENTIAL EQUATIONS.

In this lesson, we discuss some first order differential equations and methods of solving them. A first order equation is of the type

$$\frac{dy}{dx} = f(x,y)$$

or the type $Mdx + Ndy = 0$ (1)

where $M = M(x,y)$, $N = N(x,y)$ (i.e. functions of x,y)

Equations with variables separable are of the form

$$Mdx + Ndy = 0$$

Methods of solution

where $M = M(x)$ = a function of x only

where $N = N(y)$ = a function of y only

The solution of the equation of this type is got by direct integration of the equation

The solution of (1) is $\int Mdx + \int Ndy = C$ - - - (1)

C being an arbitrary constant.

Note: (2) is the general solution of the equation (1). The solutions got from (2) by substituting particular values for C are particular solutions of the equation.

Illustrations: Solve the following problems.

1. $(1 + x^2)\,dx + (1 + y^2)\,dy = 0$

The equation is of the type (1) where $M = 1 + x^2$, $N = 1+y^2$

The solution is $\int (1+x)\,dx + \int (1+y^2)dy = C$

or $x + \frac{1}{3}x^3 + y + \frac{1}{3}y^3 = C$

or $x^3+y^3 + 3(x+y) = 3C = K$ (say)

2. $$\frac{d^2y}{dx^2} + \frac{1+y^2}{\sqrt{1-x^2}} = 0$$

The equation can be reduced to an equation in which the variables are separated, by manipulation.

Accordingly we get, $$\frac{dx}{\sqrt{1-x^2}} + \frac{dy}{1+y^2} = 0$$

Integrating $$\int \frac{dx}{\sqrt{1-x^2}} + \int \frac{dy}{1+y^2} = C$$

or $\sin^{-1}x + \tan^{-1}y = c$ is the solution.

3. $y \log x \, dx + x \log y \, dy = 0$

Rewriting the equation, $\left(\frac{\log x}{x}\right) dx + \left(\frac{\log y}{y}\right) dy = 0$.

Integrating $\int \frac{\log x}{x} dx + \int \log y/y \, dy = c$; put $\log x = t$

Now $\int \frac{\log x}{x} dx = \int t \, dt = \frac{1}{2} t^2 = \frac{1}{2}(\log x)^2$

Similarly we get $\int \frac{\log y}{y} dy = \frac{1}{2} (\log y)^2$

Hence, $\frac{1}{2}(\log x)^2 + \frac{1}{2}(\log y)^2 = c$

or $(\log x)^2 + (\log y)^2 = K$ is the solution

4. $\frac{dy}{dx} + Ky = 0$ or $dy + Ky dx = 0$

$\Longrightarrow \int \frac{dy}{y} + k \int dx = c$

$\Longrightarrow \log_e y + kx = c \Longrightarrow \log y = c - kx$

or $y = e^{c-kx} = e^c . e^{-kx} = ae^{-kx}$.

The solution is $y = ae^{-kx}$

a being the constant of integration.

<u>Homogeneous differential equation of the type</u>

$M(x,y) \, dx + N(x,y) \, dy = 0$ - - - (1B)

<u>Homogeneous expressions/functions: Homogeneous equations</u>

Consider (1) $f(x,y) = x^2 + xy + y^2$

We can write $f(x,y) = x^2(1 + y/x + (y/x)^2)$

or $f(x,y) = x \, f(1, y/x)$

Since $f(1, y/x) = 1 + 1. y/x + (y/x)^2 = 1 + y/x + y^2/x^2$

f is a homogeneous function of degree 2 in x and y.

2. $f(x,y) = x^3 + 3x^2y + y^3$

$= x^3(1 + 3 \, y/x + (y/x)^3) = x^3 \, f(1,y/x)$

and $f(x,y)$ is a homogeneous function of degree 3 in x and y.

3. $f(x,y) = x + \sqrt{xy} + y$

$= x\left[1 + \sqrt{y/x} + y/x\right] = x\, f(1,\ y/x)$

so that $f(x,y)$ is a homogeneous function of degree 1 in x and y.

4. $f(x,y) = x \sin(y/x) + y \cos(y/x)$

$= x\left[\sin(y/x) + (y/x)\cos(y/x)\right]$

$= x\, f(1,\ y/x)$

$f(x,y)$ is a homogeneous function of degree 2 in x and y.

In general, a homogeneous function of degree n in x and y, $f(x,y)$ has the property, $f(x,y) = x\ \ f(1,\ y/x)$

Putting $y = v\,x$ or $y/x = v$

$f(x,y) = x\ \ f(1,v)$

Note: In a homogeneous function, each term is of the same degree.
$f(x,y) = x^2 + x + y + y^2$
is not a homogeneous function.

Since $f(x,y) = x^2 + x + y + y^2$

$= x^2\,(1 + 1/x + y/x^2 + y^2/x^2)$

This part is not a function of (y/x). Thus we cannot write $f(x,y) = x^n\ \ f(1,\ y/x)$ for any x.

Definition : $M(x,y)\,dx + N(x,y)\,dy = 0$
is called a homogeneous equation of 1st order if $M(x,y)$ and $N(x,y)$ are homogeneous functions of same degree.

If the differential equation is a homogeneous equation, then we can write the equation as

$$\frac{dy}{dx} = -\frac{M(x.y)}{N(x.y)} = -\frac{x^n\, M(1, y/x)}{x^n\, N(1, y/x)} = -\frac{M(1, y/x)}{N(1, y/x)}$$

or $$\frac{dy}{dx} = f(y/x).$$

Method of solving a homogeneous differential equation :

Given the homogeneous equation

$M(x,y)\,dx + N(x,y)\,dy = 0$ --- (1)

Put $y = vx$ --- (2)

$\frac{dy}{dx} = v.1 + x\frac{dv}{dx}$ or $dy = vdx + xdv$

This substitution converts the equation (1) into an equation in v and x with separated variables. Then the equation can be solved.

<u>Illustrations:</u> Solve the following equations.

1. $x \dfrac{dy}{dx} = x + y$ - - - $*$

By checking the coefficient function, it is easily seen that the equation is a homogeneous equation.

Put $y = vx \quad \therefore \dfrac{dy}{dx} = v + x \cdot \dfrac{dv}{dx}$ in $*$.

$$x\left(v + x \cdot \frac{dv}{dx}\right) = x + vx$$

$$\Rightarrow v + x \cdot \frac{dv}{dx} = 1 + v \Rightarrow x \cdot \frac{dv}{dx} = 1$$

$$\text{or} \quad dv = \frac{dx}{x}$$

On integration of the equation, we get

$v = \log x + c$

or $y/x = \log_e x + c$.

Hence $y = x (\log_e x + c)$ is the solution of the given differential equation.

2. $\dfrac{dy}{dx} = (x^2 + xy)/(y^2 + xy) \Rightarrow (xy + y^2)\dfrac{dy}{dx} = (xy + x^2)$.

x^2 = a homogeneous function of degree 2 and

x + xy + y = a homogeneous function of degree 2.

Hence the equation is a homogeneous equation.

Put $y = vx \quad \dfrac{dy}{dx} = v + x \cdot \dfrac{dv}{dx}$

$$\therefore (v^2x^2 + v^2x^2)\left(v + x \cdot \frac{dv}{dx}\right) = vx^2 + x^2$$

$$\Rightarrow v(1+v)\left(v + x \cdot \frac{dv}{dx}\right) = (v+1)$$

$$\Rightarrow v^2 + vx \cdot \frac{dv}{dx} = 1 \Rightarrow vx \cdot \frac{dv}{dx} = 1 - v^2$$

Separating the variables, we get $\frac{v}{1-v^2}\,dv = \frac{dx}{x}$

$$\text{or. } \frac{dx}{x} + \frac{v.\,dv}{v^2-1} = 0$$

$$\therefore \int \frac{dx}{x} + \int \frac{v\,dv}{v^2-1} = C$$

$$\text{or } \log x + \frac{1}{2}\log(v^2-1) = C$$

$$\text{or } 2\log x + \log(v^2-1) = 2C \Rightarrow \log_e x^2(v^2-1) = 2C$$

$$\text{or } y^2 - x^2 = k = e^{2c}$$

is the solution of the equation.

3. $x^2 \frac{dv}{dx} = x^2 + xy + y^2$

The equation is obviously a homogeneous equation.

Put $y = vx \;\therefore\; \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$x^2\left(v + x\cdot\frac{dv}{dx}\right) = x^2 + x^2 v + x^2 v^2$$

$$\Rightarrow \cancel{v} + x\frac{dv}{dx} = 1 + v^2 + \cancel{v}$$

$$\therefore\; x\cdot\frac{dv}{dx} = 1 + v^2$$

Separating the vari ables, we get

$$\frac{dx}{x} = \frac{dv}{1+v^2} \quad \therefore \quad \int\frac{dx}{x} = \int\frac{dv}{1+v^2} + C$$

On integration

Hence the solution is $\log_e x = \mathrm{Tan}^{-1}(y/x) + C$

or $y = x \,\mathrm{Tan}\,(k + \log_e x)$, k being the constant of integration.

If the given problem is an initial value problem, then we need to find the particular solution of the differential equation which satisfies the initial condition also.

4. $x(1-y') + y(1+y') = 0$
with $y(1) = 0$

$y' = \frac{x+y}{x-y}$ which is a homogeneous equation.

Putting $y = vx$, $y = v + xv$

$v + xv' = \frac{x(1+v)}{x(1-v)}$

$xv' = \frac{1+v}{1-v} - v = \frac{1+v^2}{1-v}$

$dv = \frac{dx}{x}$

On integrating we get

$\mathrm{Tan}^{-1} v - 1/2 \log(1+v^2) = C + \log x$

or $\mathrm{Tan}^{-1}(y/x) = C + \log(x\sqrt{1+v^2})$

or $\mathrm{Tan}^{-1}(y/x) = C + \log\sqrt{x^2+y^2}$ putting $x=1$, $y=0$ so that $C = 0$.

Hence the solution of the equation is $\mathrm{Tan}^{-1}(y/x) = \log\sqrt{x^2+y^2}$

5. $x \sin(y/x)\frac{dy}{dx} = y \sin(y/x) + x$, $y(1) = \pi/2$

Putting $y = vx$, $\frac{dy}{dx} = v + x \cdot \frac{dv}{dx}$

$\sin v\,(v + x\frac{dv}{dx}) = v \sin v + x$

$\cancel{v \sin v} + x \sin v = \cancel{v \sin v} + 1$

$\sin v\,dv = dx/x$

On integrating we get $\log x + c = -\cos v$

Therefore, the general solution $\cos(y/x) + \log x + c = 0$

Putting $x=1$, $y = \pi/2$, $\cos(\pi/2) + \log + C = 0$ $C = 0$

Hence the solution is $\cos(y/x) + \log_e x = 0$

6. $xy' = y + 2x\,e^{-y/x}$ with $y(1) = 0$, putting $y = vx$, $y' = v + xv'$

$x(v + xv') = vx + 2x\,e^{-v}$

$v + xv' = v + 2e^{-v}$ $dv = 2dx/x$

On integrating we get $e^{v} = 2 \log x + c$

The solution is $e^{y/x} = 2 \log x + c$

Put $x = 1$, $y = 0$, $= 1 = 2 \log + C$ $c = 1$

$\exp(y/x) = 2 \log x$ is the solution.

7. $(y + x^2+y^2) - xy' = 0,\ y(1) = 0$

The equation being homogeneous, put $y = vx$, $y' = v + xv'$

$$vx + \sqrt{x^2 + v\,x^2)} - x\,(v + xv') = 0$$

$$\cancel{v} + \sqrt{1 + v^2} - \cancel{v} - x\,v' = 0$$

$$\Rightarrow \frac{dv}{\sqrt{1+v^2}} = \frac{dx}{x}$$

On integrating, $\sin h^{-1} v = C + \log x$

Or $\sin h\ (y/x) = C + \log x$

Putting $x = 1$, $y = 0$, $\sin h^{-1}(0) = C + \log 1$ $\therefore$ $C = 0$

The solution of the initial value problem is

$\sin h^{-1}(y/x) = \log_e x$

or $y = x \sin h\,(\log_e x)$

8. $(x \tan (y/x) + y) = x\dfrac{dy}{dx}$ $\quad y(1) = \pi/2$

Putting $y = vx$, $y' = v + xv'$

$x \tan v + v x = x\,(v + xv')$

$\tan v + v = v + x\,v'$

Separating the variables,

$dx/x = (\cos v/\sin v)dv$

On integrating $\log x = C + \log \sin v$

or $\log x = C + \log \sin (y/x)$

Putting $x = 1$, $y = \pi/2$, we get $C = 0$

The solution of the initial problem is

$\log x = \log \sin (y/x)$

or $y = x \sin^{-1} x$

Equations reducible to homogeneous equations :

1. $$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}, \qquad a_1b_2 \neq a_2b_1$$

In this case, put $x = X + h$, $h = Y + K$ and choose (h,k) such that $a\,h + b\,k + c = 0$

$a\,h + b\,k + c = 0$

With the substitutions, the equation becomes $\dfrac{dy}{dx} = \dfrac{a_1X + b_1Y}{a_2X + b_2Y}$

This equation is homogeneous and can be solved by putting $Y = vX$.

An illustration :

$$\frac{dy}{dx} = \frac{x + y - 5}{x - y + 1}$$

Put $x = X - h$, $y = Y + k$

such that $h + k - 5 = 0$

$h - k + 1 = 0$

$$\frac{dy}{dx} = \frac{X + Y}{X - Y}$$

Solving $h = 2$, $l = 3$,

Hence, $x = X + 2$

$y = Y + 3$

Put $Y = vx$, $\frac{dy}{dx} = v + x\frac{du}{dx}$

$\frac{dy}{dx} = v + x.\frac{dv}{dx}$ so that $v + x.\frac{dv}{dx} = \frac{X + vX}{X - vX} = \frac{1+v}{1-v}$

$\therefore \quad x.\frac{dv}{dx} = \frac{1+v}{1-v} - v = \frac{1+v^2}{1-v} \Rightarrow \frac{dx}{x} = \left(\frac{1-v}{1+v^2}\right)dv$

On separating the variables, we get

$$\frac{dx}{x} = \left(\frac{1-v}{1+v^2}\right)dv \Rightarrow \int\frac{dx}{x} = \int\frac{1-v}{1+v^2}dv = \int\frac{dv}{1+v^2} - \int\frac{v\,dv}{1+v^2}$$

Integrating $\log X = \text{Tan}^{-1} v - \frac{1}{2}\log(1 + v^2) + C$

$\log X + \log\sqrt{1 + v^2} = \text{Tan}^{-1} v + c$

i.e. $\log\sqrt{X^2 + Y^2} = \text{Tan}^{-1}(y/x) + C$

The solution is $\log\sqrt{(x-2)^2 + (y-3)^2} = \text{Tan}^{-1}\left(\frac{y-3}{x-2}\right) + C$

2. $\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$, $a_1b_2 - a_2b_1 = 0$

$a_1b_2 = a_2b_1 \Rightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{1}{k}$ (say)

$\therefore \quad a_2 = ka_1, \quad b_2 = kb_1$

$$\therefore \quad \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2} = \frac{a_1x + b_1y + c_1}{k(a_1x + b_1y) + c_2}$$

$$= \frac{1}{k}\left(\frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2/k}\right) = \frac{1}{k}\left(\frac{a_1x + b_1y + c_1}{a_1x + b_1y + c_3}\right)$$

where $C_3 = C_2/K$.

Note: $(a_2x + b_2y)$ is $(a_1x + b_1y)$ for some constant. Substituting $Z = a_1x + b_1y$ the equation can be solved.

Illustration : $\frac{dy}{dx} = \frac{x+y+4}{2x+2y-5}$, put $x + y = z$

$\therefore 1 + \frac{dy}{dx} = \frac{dz}{dx}$

$$\frac{dz}{dx} - 1 = \frac{z+4}{2z-5} \Rightarrow dx = \left(\frac{2z-5}{3z-1}\right)dz$$

i.e. $\frac{dz}{dx} = \left(A + \frac{B}{3z-1}\right)dz$

WHERE $\frac{2z-5}{3z-1} = A + \frac{B}{3z-1}$

so that $2z-5 = A(3z-1) + B$

Put $z = 0$, $-A + B = -5$, $A = 2/3$

Put $z = 1$, $2A + B = -3$, $B = -13/3$

$\therefore dx = \int\left(A + \frac{B}{3z-1}\right)dz \Rightarrow x + c = Az + \frac{B}{3}\log(3z-1)$

Integrating, $x + c = A\,z + \frac{B}{3}\log(3\,z - 1)$

$x + C = A(x + y) + \frac{B}{3}\log(3x + 3y - 1)$

$x + C = 2/3\,(x+y) - 13/9\,\log(3x + 3y - 1)$

The solution is $9x + k = 6\,(x+y) - 13\log(3x + 3y - 1)$

or $(3x - 6y) + 13\log(3x + 3y - 1) + K = 0$.

__First order Linear Equations :__

Type : $dy/dx + py = Q$ - - - . - . (1)

where $p = P(x)$, $Q = Q(x)$

(i.e. P, Q are functions of x only).

Let $\mu = \mu(x)$ be a function such that (1) becomes an exact*
differential equation on multiplication by .

Multiplying (1) by μ

(1) becomes $\mu \dfrac{dy}{dx} - \mu Py = \mu Q$

or $\mu\, dy + \mu Py\, dx = \mu Q\, dx$ --(2)

By definition of exact equation,
the L.H.S. of the equation (2) can be written as

$d(\phi) = (\mu\, py)\, dx + \mu\, dy$ - - - (3)

By the chain rule,

Hence, $\dfrac{\partial\phi}{\partial x}dx + \dfrac{\partial\phi}{\partial y}dy = (\mu Py)dx + \mu\, dy$

$\dfrac{\partial\phi}{\partial x} = \mu Py, \dfrac{\partial\phi}{\partial y} = \mu$ - - - (4)

$\therefore \dfrac{\partial^2\phi}{\partial y\partial x} = \dfrac{\partial}{\partial y}(\mu Py), \quad \dfrac{\partial^2\phi}{\partial x\partial y} = \dfrac{\partial\mu}{\partial x}$

Since $\dfrac{\partial^2\phi}{\partial y\partial x} = \dfrac{\partial^2\phi}{\partial x\partial y}, \quad \dfrac{\partial}{\partial y}(\mu . Py) = \dfrac{\partial\mu}{\partial x}$ --(5)

Since $\mu = \mu(x), \quad \dfrac{\partial\mu}{\partial x} = \dfrac{d\mu}{dx}$

(5) becomes $\mu P . 1 + y\dfrac{\partial(\mu P)}{\partial y} = \dfrac{d\mu}{dx}$

Since p is a function of x only $\dfrac{\partial}{\partial y}(\mu P) = 0$.

$\therefore \dfrac{d\mu}{dx} = \mu P$

On integration we get $\mu = e^{\int P dx}$ - - - (6)

From (2) and (3) $d\phi = \mu Q\, dx$

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* By exact equation, we mean that the L.H.S. is the total
derivative of some function of n.

Integrating the second equation : $\frac{\partial\phi}{\partial y} = \mu$ w.r.t y

$\phi = \mu y$

$d\phi = d(\mu y) = \mu Q\,dx$

integrating $\mu y = c + \int \mu Q\,dx$

but $\mu = \exp\left(\int P dx\right)$

Hence, $y = e^{-\int P dx}\left[C + \int Q e^{\int P dx} dx\right]$ - - (7)

in the solution of (1).

Working Rule: Given (1)

i) identify P and Q.

ii) compute $\int Pdx$

iii) compute $\exp\int pdx = e^{\int P dx}$

iv) compute $\int Q e^{\int P dx} dx$

v) Fit in (7) to get the solution

A particular case of (1) is got when P is a constant. The equation (1) is then called a <u>first order linear equation with constant coefficients</u>.

Then, $\int pdx = px$

$e^{\int pdx} = e^{px}$

The formula (7) becomes $y = e^{-Px}\left[C + \int Q e^{Px} dx\right]$ - - - (8)

<u>Illustrations</u>: Solve the following equations ;

1. $dy/dx + 2y = 4x$

Here, $P = 2$, $Q = 4x$

$e^{Px} = e^{2x}$

$\int Q e^{Px} dx = \int 4x e^{2x} dx = 4\left[\frac{1}{2} x e^{2x} - \frac{1}{2}\int e^{2x} dx\right]$

$= 2\left[x e^{2x} - \frac{1}{2} e^{2x}\right] = (2x-1)e^{2x}$

The solution $y e^{2x} = C + (2x-1) e^{2x}$

or $y = ce^{-2x} + (2x - 1)$

2. $y' + y = \cos x$

Here $P = 1$, $Q = \cos x$ $\therefore e^{\int P\,dx} = e^x$

$$\int Q e^{\int P\,dx} dx = \int e^x \cos x\, dx$$

Let $I = \int e^x \cos x\, dx$

Then $I = e^x \sin x - \int e^x \sin x\, dx$ (Integrating by parts).

$= e^x \sin x - [e^x(-\cos x) + \int e^x \cos x\, dx]$

$= e^x \sin x + e^x \cos x - I$

or $I = e^x(\sin x + \cos x) - I$

$2I = e^x(\cos x + \sin x)$

$I = \int e^x \cos x\, dx = \frac{1}{2} e^x(\cos x + \sin x)$

The solution is $y\, e^x = C + \frac{1}{2} e^x (\cos x + \sin x)$ or

$y = C e^{-x} + \frac{1}{2}(\sin x + \cos x)$

3. $y' - y = 1$

$P = -1$, $Q = 1$

$e^{Px} = e^{-x}$, $\int Q e^{\int P\,dx} dx = \int 1 \cdot e^{-x} dx = -e^{-x}$

The solution is $y\, e^{-x} = C - e^{-x}$

or $y + 1 = C e^x$

4. $y' + 2y = 6e^x$, $P = 2$, $Q = 6e^x$ $e^{\int P\,dx} = e^{2x}$

$\therefore \int Q e^{\int P\,dx} dx = \int 6e^x e^{2x} dx = 6\int e^{3x} dx$

$\therefore \int Q e^{\int P\,dx} dx = 2e^{3x}$

The solution is $y\, e^{2x} = C + 2e^{3x}$ or $y = C e^{-2x} + 2e^x$.

5. $(1 + \cos x)\, dy/dx = (1 - \cos x)$

$$dy/dx = \frac{1 - \cos x}{1 + \cos x} = \tan^2(x/2)$$

or $dy = \tan^2(x/2)\, dx = (\sec^2(x/2) - 1)\, dx$

Integrating $y = C + \int (\sec^2(x/2) - 1)\, dx$

$y = C + 2\tan(x/2) - x$

is the solution of the equation.

6. $\left(y - x\cdot\frac{dy}{dx}\right) = a\left(y^2 + \frac{dy}{dx}\right)$

$\Rightarrow y - x\cdot\frac{dy}{dx} = ay^2 + a\cdot\frac{dy}{dx}$

$\Rightarrow (y - ay^2) = (x+a)\frac{dy}{dx}$

$\Rightarrow \frac{dx}{a+x} = \frac{dy}{y - ay^2}$

$\therefore \int \frac{dx}{a+x} + C = \int\left(\frac{1}{y} + \frac{a}{1-ay}\right)dy$

$\frac{1}{y-ay^2} = \frac{1}{y(1-ay)}$

$= \frac{1}{y} + \frac{a}{1-ay}$

or $\log(a+x) + c = \log y - \log(1-ay)$

or $\log\left(\frac{y}{1-ay}\right) = \log(a+x) + c = \log(a+x) + \log k$

or $\log\left(\frac{y}{1-ay}\right) = \log\left[k(a+x)\right]$

or $y = K(a+x)(1-ay)$

is the solution.

7. $2x\,y' = 3y$, $y(1) = 4$.

$2x\frac{dy}{dx} = 3y \Rightarrow 2\frac{dy}{y} = 3\frac{dx}{x}$

$2\log y = 3\log x + \log C$

$y^2 = C\,x^3$ put $x = 1$, $y = 4$, $c = 16$

The particular solution satisfying $y(1) = 4$, is $y^2 = 16x^3$

8. $y' = 2e^x y^3$ $y(0) = 1/2$

$\Rightarrow \frac{dy}{y^3} = 2e^x dx \Rightarrow \int\frac{dy}{y^3} = 2\int e^x dx + C$

$\Rightarrow -\frac{1}{2}y^{-2} = C + 2e^x$

Put $x = 0$ $y = 1/2$, $-1/2$

$-1/2\,(1/2)^{-2} = C + 2e^0 = C + 2$

i.e. $C + 2 = -1/2 \times 2^2 = -2$ or $c = -4$

The particular solution required is $-\frac{1}{2y^2} = 2e^x - 4$

or $\therefore y^2 = \frac{1}{8-4e^x}$

9. $\frac{dy}{dx} = x\, e^{x} \quad y(1) = 3$

$dy = x\, e^{x}\, dx$

$\int dy = C + \int x\, e^{x}\, dx$

$C + (x\, e^{x} - \int e^{x}\, 1.dx = C + (x-1)\, e^{x}$

or $y = C + (x-1)\, e^{x}$

Putting $x = 1$, $y = 3$, $3 = C + 0 \implies C = 3$

$y = 3 + (x-1)\, e^{x}$ is the solution of the initial value problem.

10. $\frac{dy}{dx} + x\, e^{x^2-y} = 0, \quad y(0) = 0$

$e^{y}\, dy + xe^{x^2-y}\, dx = 0$

$\int e^{y}\, dy + \int x\, e^{x^2}\, dx = C$

$e^{y} + \tfrac{1}{2}\, e^{x^2} = C$

Put $x = 0$, $y = 0$, $1 + \tfrac{1}{2}.1 = C$ or $C = 3/2$

The solution of the initial problem is $2e^{y} + e^{x^2} = 3$

Equations reducible to the form : $y' + py = Q$, where $P = P(x)$, $Q = Q(x)$.

Bernoulli's equation: $\frac{dy}{dx} + py = Qy^{n}$ - - - (1)

dividing the equation by y^{n}

$\frac{1}{y^n}\frac{dy}{dx} + P\frac{y}{y^n} = Q$

$\implies \frac{1}{y^n}\frac{dy}{dx} + P\left(\frac{1}{y^{n-1}}\right) = Q$

Put $Y = \frac{1}{y^{n-1}}$ - - - (2) $\therefore\ -(n-1)\frac{dy}{dx} = \frac{dY}{dx}$

Then, on substitution, the given equation becomes

$\frac{dY}{dx} - (n-1)\, Y.P = -Q\, (n-1)$ - - - (3)

Put $P_1 = -(n-1)\, P$, $Q_1 = -(n-1)\, Q$

Clearly, $P_1 = P_1(x)$, $Q_1 = Q_1(x)$ and (3) becomes

$\frac{dY}{dx} + P_1 Y = Q_1$ - - (4)

This equation can be solved since it is an equation of the form

$\frac{dY}{dx} + PY = Q.$

1. Solve : $\frac{dy}{dx} + yx = x/y$

$\Rightarrow y\frac{dy}{dx} + y^2x = x$

Put $y^2 = Y$ $\quad 2y\frac{dy}{dx} = \frac{dY}{dx}$

The equation becomes

$Y2\,dY/dx + xY = x$

or $dY/dx + 2xY = 2x$

Comparing it with $dy/dx + py = Q$,

$P = 2x \int Pdx = x^2,\ Q = 2x$

$\int Qe^{Pdx} = \int e^{x^2}\cdot 2xdx = e^{x^2}$

The solution is $Y e^{x^2} = c + e^{x^2}$

or $y = c\,e^{-x^2} + 1$

Since $y = y^2$, the solution is $y^2 = c\,e^{-x^2} + 1$.

2. Solve : $xy' + y = x^4y^3$

Dividing by xy^3, $\frac{1}{y^3}y' + \frac{1}{xy^2} = x^3$

Put $Y = \frac{1}{y^2}$ $\quad \frac{dY}{dx} = -\frac{2}{y^3}y'$ or $\frac{1}{y^3}y' = -\frac{1}{2}\frac{dY}{dx}$

The equation becomes, $\quad -\frac{1}{2}\frac{dY}{dx} + \frac{Y}{x} = x^3$

or $\quad \frac{dY}{dx} - \left(\frac{2}{x}\right)Y = -2x^3$

Comparing this equation with $\quad \frac{dy}{dx} + PY = Q$

$P = -2/x,\ Q = -2x^3$

$\int Pdx = -2\log_e x = \log_e(\gamma x^2)$ $\therefore$ $e^{\int Pdx} = \gamma x^2$

The solution is $Y e^{\int Pdx} = C + \int Q\ e^{\int pdx}\,dx$

$\int Q\,e^{Pdx}\,dx = \int -2x^3\cdot \gamma x^2\,dx = -x^2$

Therefore, $Y.\ \gamma x^2 = C - x^2$

or $Y = x^2(c - x^2)$

or $\gamma y^2 = x^2(c - x^2)$ is the solution or $\therefore y^2 = \frac{1}{x^2(c-x^2)}$

3. Solve $(e^y - 2xy)\, y^1 = y^2$

Here it is necessary to treat x as the dependent variable and y as t
dependent variable.

Noting $y' = dy/dx = \frac{1}{dx/dy}$

The equation becomes $(e^y - 2xy) \Big/ \left(\frac{dx}{dy}\right) = y^2$

$$e^y - 2xy = y^2 \frac{dx}{dy}$$

Here $P = P(y) = 2/y$, $Q = Q(y) = e^y / y$

$\int P dy = \int 2/y \, dy = 2 \log y$

$e^{\int P dx} = y^2$

$\therefore \int Q\, e^{\int P dy} dy = \int \frac{e^y}{y^2} \cdot y^2 dy = e^y$

The solution is $x\, e^{\int P dy} = C + \int Q\, e^{\int P dy} dy$

The solution is $xy^2 = C + e^y$

4. Solve the initial value problem :

$$\frac{dy}{dx} + y = xy^3, \quad y(0) = \sqrt{2}$$

$$\Rightarrow \frac{1}{y^3}\frac{dy}{dx} + \frac{1}{y^2} = x$$

Put $1/y^2 = Y$

$dY/dx - 2Y = -2x$

$P = -2$, $Q = -2x$

$\int P dx = -2x \qquad e^{\int P dx} = e^{-2x}$

$$\int Q\, e^{\int P dx} dx = \int -2x e^{-2x} dx$$

$$= -2\left[x\left(\frac{e^{-2x}}{-2}\right) - \int \frac{e^{-2x}}{-2} \cdot 1\, dx\right]$$

$$= x e^{-2x} + \frac{1}{2} e^{-2x} = \left(x + \frac{1}{2}\right) e^{-2x}$$

The solution is $Y e^{-2x} = C + (x + 1/2)\, e^{-2x}$

or $1/y^2 = x + 1/2 + ce^{+2x}$.

Using the initial condition

$y(0) = \sqrt{2}$

$\frac{1}{2} = \frac{1}{2} + c \qquad c = 0$

The solution is $\frac{1}{2} y^2 = x + \frac{1}{2}$

5. $\cos y \dfrac{dy}{dx} + \dfrac{\sin y}{x} = 1$

Put $\sin y = Y \qquad \cos y \dfrac{dy}{dx} = \dfrac{dY}{dx}$

$$\frac{dY}{dx} + \frac{Y}{x} = 1$$

$P(x) = 1/x, \; Q(x) = 1$

$\int P(x)\, dx = \log_e x \Rightarrow \int e^{P(x) dx} = x$

$\int Q\, e^{P(x)}\, dx = \int 1.\, x \; dx = \frac{1}{2} x^2$

Hence the solution is $Y = e^{-\int P(x)\, dx} \left[C + \int Q\, e^{P(x)}\, dx \right]$.

or $\sin y = 1/x \,(c + \frac{1}{2} x^2)$

or $\sin y = c/x + x/2$

6. $(y+1)\, dy/dx + (y^2+2y)\, x = x$

Put $y^2 + 2y = Y$

$(2y+2)\, dy/dx = dY/dx$ using these in the given d.e.

$\frac{1}{2}\, dY/dx + Yx = x$

or $\dfrac{dY}{dx} + (2x)\, Y = 2x$

$P = 2x, \; Q = 2x$

$e^{\int P(x)\, dx} = e^{x^2}$

$\int Q\, e^{P(x)}\, dx = \int 2x \; e^{x^2}\, dx = e^{x^2}$

Hence the solution is $Y = e^{-\int P(x)\, dx} \left[C + \int Q\, e^{\int P(x)\, dx} \right]$

i.e. $(y^2 + 2y) = e^{-x^2} \; (c + e^{x^2})$

or $y^2 + 2y = c\, e^{-x^2} + 1$

Theoretical Problems on linear first order equations :

1. If f and g are two solutions of $dy/dx + py = 0$ ---(1) then $c_1 f + c_2 g$ is also a solution of the equation for any arbitrary constants c_1 and c_2.

Proof : f is a solution. $\Rightarrow$ $df/dx + p.f. = 0$ $\times c_1$

g is a solution $\Rightarrow dg/dx + p.g. = 0$ $\times c_2$

$$c_1 \frac{df}{dx} + c_2 \frac{dg}{dx} + p(c_1 f + c_2 g) = 0$$

or $d/dx\ (c_1 f + c_2 g) + P\ (c_1 f + c_2 g) = 0$

Hence $c_1 f + c_2 g$ is also a solution of $dy/dx + py = 0$.

Note : The result can be extended. Accordingly, for any solution, $f, g, h, \ldots$ of the equation, $c_1 f + c_2 g + c_3 h + \ldots$ is also a solution of the equation for any arbitrary constants $c_1, c_2, c_3, \ldots$

2. Consider the differential equation $dy/dx + py = 0$ where $P = P(x)$. Show that

a) $f(x) \equiv 0$ for all x is a solution of the equation.

b) if $f(x)$ is a solution of the equation such that $f(x_0) = 0$ for some value $x = x_0$, then $f(x) \equiv 0$ for all x.

c) if f and g are solutions such that $f(x_0) = g(x_0)$ for some $x = x_0$, then $f(x) \equiv g(x)$ for all x.

Note: The solution $f(x) \equiv 0$ of the equation (1) is called the zero solution or trivial solution. Any other solution than this is called a non zero or non trivial solution of (1).

a) Putting $y = 0$ in the equation, the equation is satisfied. Hence $f(x) \equiv 0$ is a solution of the equation.

b) consider $dy/dx + py = 0$

Separating the variables, $dy/y + P(x)\,dx = 0$

Integrating the equation, we get $y = c\, e^{\int P(x)\,dx}$

is the general solution (i.e. all solutions are of this form).

Let $f(x)$ be a solution. Then for some c, $f(x) = c\, e^{\int P(x)\,dx}$.

Let $f(x_0) = 0$ for some $x = x_0$, then putting $x = x_0$

$f(x_0) = c\, e^{\int P(x)\,dx} = 0 \Rightarrow c = 0$

Then $f(x) \equiv 0$ for all x.

c) Let $f(x)$, $g(x)$ be two solutions such that $f(x_0) = g(x_0)$.

Then $f(x) = c_1 e^{\int P(x)\,dx}$

$g(x) = c_2 e^{\int P(x)dx}$

$f(x_0) = g(x_0)$

$\Rightarrow f(x) - g(x) = 0$. Also $f(x) - g(x)$ is also a solution of the equation.

Hence from (b), $f(x) - g(x) \equiv 0$ for all x

$f(x) \equiv g(x)$ for all x.

3. Let $f_1(x)$ be a solution of $\frac{dy}{dx} + P(x)\, y = Q_1\,(x)$ (1)

and $f_2(x)$ be a solution of $\frac{dy}{dx} + P(x)\, y = Q_2(x)$ (2)

Then prove that $f_1(x) + f_2(x)$ is a solution of

$$\frac{dy}{dx} + P(x)\, y = Q_1\,(-x) + Q_2(x\,) \qquad (3)$$

Since $f_1(x)$ and $f_2(x)$ are solutions of the differential equations (1) and (2) respectively.

$$\frac{df_1}{dx} + P(x)\, f_1 = Q_1(x)$$

$$\frac{df_2}{dx} + P(x)\, f_2 = Q_2(x)$$

Adding: $d/dx\,(f_1 + f_2) + P(x)\,(f_1 + f_2) = Q_1(x) + Q_2(x)$ which shows that $f_1(x) + f_2(x)$ is a solution of the differential equation (3).

<u>A Uniqueness Theorem</u> :

4. If $P(x)$ and $Q(x)$ are continuous functions of x, then show that $dy/dx + P(x)\, y = Q(x)$.
has a unique solution $y(x)$ satisfying the initial condition.
$y\,(x_0\,) = y_0$

Let $y_1(x)$ and $y_2(x)$ be two solutions of the initial value problem

$$dy/dx + P(x)\, y = Q(x)$$

$$y(x_0) = y_0$$

Then $y_1(x) - y_2(x)$ is a solution of $dy/dx + P(x) = 0$

Also, $y_1(x_0) - y_2(x_0) = y_0 - y_0 = 0$

Hence, $y_1(x) - y_2(x)$ is a solution of the homogeneous equation $dy/dx + P(x)\, y = 0$ satisfying the condition $y_1(x_o) - y_2(x_o) = 0$.

Hence, $y_1(x) - y_2(x) \equiv 0$ (i.e. for all x)

$$y_1(x) \equiv y_2(x) \text{ for all } x$$

Hence the solution is unique.

Assignment and Self Test :

1. Solve the differential equations.

a) $(x-4)\, y^4 dx - (y^2-3)\, x^3 dy = 0$

b) $x \sin y\, dx + (x^2+1) \cos y\, dy = 0$

c) $4xy + (x^2+1)\, y^1 = 0$

d) $(e^y + 1) \cos x + e^y (\sin x + 1) \frac{dy}{dx} = 0$

e) $\tan \theta\, dr + 2\, r\, d\theta = 0$

f) $(x+y)\, dx - x dy = 0$

g) $(2xy + 3y^2) - (2xy + x^2 y^1 = 0$

h) $(x^2-2y^2) + xyy^1 = 0$

i) $x^2 \frac{dy}{dx} = 3\,(x^2+y^2) \tan^{-1} (y/x) + xy$

j) $(xy^1 - y) \sin (y/x) = x$

k) $xy^1 = y + 2x\, e^{-y/x}$

l) $(x \tan (y/x) + y)\, dx - x\, y = 0$

2. Solve the Initial Value Problem

a) $(y+2\ )\, dx + y\, (x+u)\, dy = 0, \ y(-3) = -1$

b) $(x^2+3y^2)\, dx - 2xy\, dy = 0, \ y(2) = 6$

c) $(2x-5y)dx + (4x-y)\, dy = 0, \ y(1) = 4$

d) $(3x+8)\, (y^2+4)\, dx - 4y\, (x^2+5x+6)\, dy = 0, \ y(1) = 2$

e) $(3x^2+9xy + 5y^2) - (6x^2 + 4xy) \frac{dy}{dx} = 0, \ y(2) = -6.$

3. Solve :

a) $(x+2y-3) \frac{dy}{dx} + (2x-y-1) = 0$

b) $(x+y-1)\,dx + (2x-y-8)\,dy = 0$

c) $(x+y) \frac{dy}{dx} + (2+y-x) = 0$

d) $(x+y+1)\,dy + (2-x-y)\,dx = 0$

e) $(x+2y+3) \frac{dy}{dx} + (2x+4y+3) = 0$

4. Solve :

a) $\frac{dy}{dx} + \frac{3y}{x} = 6x^2$

b) $x \frac{4dy}{dx} + 2x^3 y = 1$

c) $\frac{dy}{dx} + 3y = 3x^2 e^{-3x}$

d) $\frac{dy}{dx} + 4xy = 8x$

e) $\frac{dx}{dt} \quad \frac{x}{t^2} = \frac{1}{t^2}$

f) $(u^2+1) \frac{du}{du} + 4u\,v = 34$

g) $x \frac{dy}{dx} + \frac{2x+1}{x+1} y = x-1$

h) $(x^2 + x - 2) \frac{dy}{dx} + 3(x+1)y = (x-1)$

i) $\frac{dr}{d\theta} + r\,\mathrm{Tan}\,\theta = \mathrm{Cos}\,\theta$

j) $\frac{dy}{dx} - \frac{y}{x} + \frac{y^2}{x} = 0$

k) $x \frac{dy}{dx} + y + 2x^6 y^4 = 0$

l) $\frac{dy}{dx} + x\left(4y - 8/y^3\right) = 0$

m) $\frac{dx}{dt} + \left(\frac{t+1}{2t}\right) x = \frac{t-1}{xt}$

n) $x \frac{dy}{dx} - 2y = 2x^4, \; y(2) = 8$

o) $\frac{dy}{dx} + 3x^2 y = x^2, \; y(0) = 2$

p) $\frac{dy}{dx} + \frac{y}{2x} = \frac{x}{y^3}, \; y(1) = 2$

q) $x \frac{dy}{dx} + y = (xy)^{3/2}, \; y(1) = 4$

r) $\frac{dy}{dx} + y = f(x)$, when $f(x) = \begin{cases} 2, & 0 \leq x < 1 \\ 0 & x \geq 1 \end{cases}$
$y(0) = 0$

s) $(x+2) \frac{dy}{dx} + y = f(x)$ when $f(x) = \begin{cases} 2x, & 0 \leq x < 2 \\ 4 & x \geq 2 \end{cases}$
$y(0) = $

APPLICATIONS OF FIRST ORDER EQUATIONS

Geometrical Applications - orthogonal Trajectories

Given a first order equation

$$\frac{dy}{dx} = f(x,y) \qquad (1)$$

the general solution of (1) is given by

$$F(x,y,c) = 0 \quad - - - (2)$$

c being an arbitrary constant.

(2) represents a family of curves (a one-Parameter family) in the x y plane.

(1) gives the slope of a curve of the family at (x,y).

Definition : Given a C_1 - family of curves, a C_2 - family of curves is called an orthogonal family of curves to C_2 if each curve of C_2 cuts every curve of C_1 orthogonally (i.e. at right angles).

Note : Since orthogonality (i.e. Perpendicularity) is a symmetric relation, if C_2 - family is orthogonal to C_1 - family, then C_1 - family is orthogonal to C_2 - family.

Given the family of curves C_1 by the differential equation (1), the orthogonal trajectories to C_1 are got by

$$\frac{dy}{dx} = -\frac{1}{f(x,y)} \quad - - (3)$$

(Recall that for two curves to be orthogonal, the Product of slopes= -1).

Procedure for finding the orthogonal trajectories of a given family of curves :

Step 1 : From the equation (given) $F(x,y,c) = 0$ (i) of the given family of curves, find the differential equation of the family : $\frac{dy}{dx} = f(x,y)$ (ii)

Step 2 : Replace in (ii), $f(x,y)$ by $-1/f(x,y)$ to get $\frac{dy}{dx} = -1 \quad f(x,y)$ --- (iii)

This is the differential equation of the orthogonal trajectories of (1).

Step 3 : Solve the equation (iii) to get the equation of the family of orthogonal trajectories - a one - parameter family of curves

$$G(x,y,c) = 0 \qquad (iv)$$

Caution : In step 1, in finding the equation (ii) be sure of eliminating C.

Illustration :

1. Obtain the orthogonal trajectories of the family of circles :

$x^2 + y^2 = C^2$ -- (1)

(1) represents the family of concentric circles centred at the origin.

Differentiating (1) we get

$$2x + 2y \frac{dy}{dx} = 0$$

$$\text{or} \quad \frac{dy}{dx} = -\frac{x}{y} \quad \text{-- (2)}$$

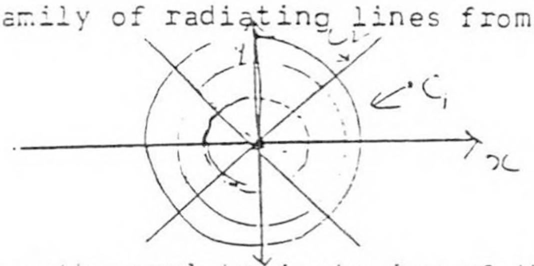
changing $-x/y$ by $-\left(-\frac{1}{x/y}\right) = y/x$

The orthogonal trajectories of (1) are given by

$$\frac{dy}{dx} = \frac{y}{x} \quad \text{--- (3)}$$

Separating the variables in (3) and integrating (3) we get $y = c\,x$. (4)

This is the family of radiating lines from the origin.



(2). Find the orthogonal trajectories of the family of Parabola $Y = Cx^2$ -- (1)

Differentiating $\frac{dy}{dx} = 2\,cx$

Eliminating C, $\frac{dy}{dx} = 2\,y/x$ --- (2)

The orthogonal trajectories are given by

$$dy/dx = -\frac{x}{2y} \quad \text{...(3)} \Longrightarrow x\,dx + 2y\,dy = 0$$

Integrating this equation

$$2y^2 + x^2 = \text{constant or} \quad x^2 + 2y^2 = C^2 \qquad (4)$$

which are ellipses.

(3) Find the orthogonal trajectories of the curves given by $y^2 = 2\,Cx + C^2$

Consider $Y^2 = 2\,Cx + c^2$ -- (1)

Substituting for C in (1)

$$Y^2 = 2(yy^1)x + y^2y^{1^2}$$

$$y^2 = 2xyy^1 + y^2y^{1^2} \quad \text{---} \quad (2)$$

Replacing y^1 by $-\frac{1}{y'}$

We get $y^2 = -\frac{2xy}{y'} + \frac{y^2}{y'^2}$

$y^2 y'^2 + 2xyy^1 = y^2$ (3)

(2) and (3) are identical. Hence the orthogonal trajectories of the given curves are themselves i.e. given by (1) itself.

<u>Definition</u> : A given family of curves is said to be <u>self -orthogonal</u> if its family of orthogonal trajectories is the same as the given family.

In the above example, the given family of Parabolas

$y^2 = 2cx + c^2$ is self-orthogonal.

<u>Miscellaneous Examples</u> :

(4) Find the curves such that the portion of the tangent intercepted by the axes is bisected at the point of contact.

Let $Y = f(x)$ in a curve with the property.

The equation of the tangent at $P(x_1, y_1)$ is

$$(y-y_1) = \left(\frac{dy}{dx}\right)_P \cdot (x-x_1) \qquad (1)$$

Putting $Y = 0$ in (1) $-y_1 = y^1_P \, (x-x_1)$ (1)

or $x = - x_1 - \frac{y_1}{y'_p}$

$A = \left(x_1 - \frac{y_1}{y'_p}, 0\right)$

Putting $x = 0$ in (1), $y-y_1 = y_p^1 \, (-x_1)$

or $y = y_1 - x_1 y_p^{\prime}$

$B = (\ 0,\ y_1, \ -x_1 y'_p \ )$

Since P is the mid point of AB,

$$(x_1, y_1) = \left[\frac{1}{2}\left(x_1 - \frac{y_1}{y'_p} + 0\right), \ \frac{1}{2}\left(y_1 - x_1 y'_p + 0\right)\right]$$

$$\Rightarrow x_1 = \frac{1}{2}\left(x_1 - \frac{y_1}{y'_p}\right), \quad y_1 = \frac{1}{2}\left(y_1 - x_1 y'_p\right)$$

$$\Rightarrow 2x_1 = x_1 - \frac{y_1}{y'_p}, \quad 2y_1 = y_1 - x_1 y'_p$$

$$x_1 y'_p + y_1 = 0$$

Then the curves with the property are given by the differential equation $x y' + y = 0$

Solving it by separating the variables, we get the curves to be $x y = c^2$.

(5) Find the curves for which the subnormal at any point of the curve is of length 1.

The sub normal at any point $P(x,y)$ is given by $y y^1$.

Therefore, the differential equation of the curves whose subnormal is 1

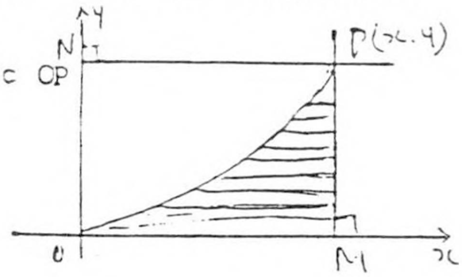
is $y y^1 = 1 \Rightarrow y \frac{dy}{dx} = 1$

Integrating $y^2 = 2x + c$

are the required curves.

6. A curve rises from the origin in the xy plane in the first quadrant. The area under the curve from $(0,0)$ to (x,y) is one-third the area of the rectangle with the points on opposite points. Find the equation of the curve.

Given: $1/3$ (Area of OMPN) = Area under the arc OP



The curve is such that the area under the arc OP, $P(x,y)$

$= 1/3$ (The area of the rectangle OMPN)

$= 1/3\ xy$

The area under the curve $= \int_0^1 y\,dx$

Hence, $\int_0^1 y\,dx = 1/3\ xy$

Differentiating w.r.t. x

$$y = \frac{1}{3} \frac{d}{dx}(x.y)$$

or $3y = x y^1 + y$

or $2y = xy^1 \Rightarrow \frac{2dx}{x} = \frac{dy}{y}$

This is the differential equation of the curve.
Integrating the equation we get its equation to be $y = x^2$ or $2x$.

Falling Body Problems/Pendulum

(a) Free ball: If m is the mass of a falling body and a is the acceleration of the body, then the force acting on the body is given by F = ma by the second law of motion. Accordingly, if v is the velocity of a freely falling body which has fallen through a distance x, then the equation of motion is $m \frac{dv}{dt} = mg$ or $\frac{dv}{dt} = g$ (1)

Integrating the equation $v = v_0 + gt$ (1)
v_0 being the initial velocity (at t=0).

Since $v = \frac{dx}{dt}$ (2) becomes $\frac{dx}{dt} = v_0 + gt$

Integrating again, $x = v_0 t + \frac{1}{2} gt^2$ (3)
Since x = 0, when t = 0
the motion of the freely falling body is governed by the equations (1), (2) and (3).

(b) Retarded fall : If we assume that air exerts a force opposing the motion of the falling body and that this opposing force varies directly as the velocity of the body, then the quation of falling body becomes

$$\frac{dv}{dt} = g - cv \quad (1) \qquad (c > 0)$$

or $$\frac{dv}{g - cv} = dt$$

Integrating (1) $-\frac{1}{c}\log(g - cv) = t + c_1$ or $g - cv = c_2 e^{-ct}$

Taking the initial velocity as zero i.e. V(0) = 0

$$C_2 = g$$

$$V = \frac{g}{c}(1 - e^{-ct}) \qquad (2)$$

C is +ve. Hence $v \to g/c$ on $t \to \infty$

This limiting value of v is called the terminal velocity .

Since $v = \frac{dx}{dt}$

(2) becomes $\frac{dx}{dt} = \frac{g}{c}(1 - e^{-ct})$

Integrating again, $x = c_3 + \frac{g}{c}(t + \frac{1}{c}e^{-ct})$

since x = 0 when t = 0 $C_3 + \frac{g}{c^2}1=0$ or $C_3 = -g/c^2$

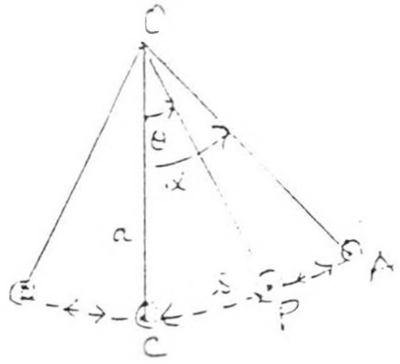
$$x = \frac{g}{c}\left[t + \frac{1}{c}(e^{-ct} - 1)\right] \quad \cdots (3)$$

(c) <u>The motion of a simple pendulum</u> : Let m be the mass of the bob and a the length of the pendulum. The bob is pulled aside through an angle α (measured from the plumb line). If V is the velocity of the bob when the string makes with the plumb line, then by the principle of Conservation of energy

$$\frac{1}{2} mv^2 = mg \quad (a \cos\theta - a \cos\alpha)$$

But $s = a\theta \qquad v = \frac{ds}{dt} = a\frac{d\theta}{dt}$

The equation becomes $\frac{1}{2}a^2\left(\frac{d\theta}{dt}\right)^2 = ag(\cos\theta - \cos\alpha)$ - - - (1)



That is the equation of motion of the pendulum

Differentiating (1) w.r.t. t

$$a\frac{d\theta}{dt}\frac{d^2\theta}{dt^2} = -ag\sin\theta \cdot \frac{d\theta}{dt}$$

or $\dfrac{d^2\theta}{dt^2} = -\dfrac{g}{a}\sin\theta$ - - - (2)

(i) <u>Case of small oscillations</u>

Assuming that the oscillations are small, (i.e. θ is small) We replace : $\sin\theta$ by θ (since θ is small, $\sin\theta$ is almost θ itself)

This equation becomes

$$\frac{d^2\theta}{dt^2} = -\frac{g}{a}\theta \quad \text{- - (3)}$$

This is the equation of motion of a simple pendulum for <u>small oscillations</u>.

Assuming that $\theta = \alpha$ and $\frac{d\theta}{dt} = 0$ when $t = 0$

$$\theta = \alpha\cos\left(\sqrt{g/a}\,t\right) \quad \text{. (4)}$$

<u>Simple electric circuits</u> :

Consider a simple electric circuit consisting of .

(i) a source of electromotive force (emf) E

(ii) a resistor of resistance R which opposes the current producing a drop in emf of magnitude E_R. If I = the current, then $E_R = RI$ (This equation is called Ohm's law).

(iii) An inductor of inductance L, which opposes any change in the current by producing a drop in emf of magnitude

$$E_L = L\frac{dI}{dt}$$

(iv) a capacitor (or condenser) of capacitance C which stores the charge Q. The charge accumulated by the capacitor resists the inflow of additional charge, and the drop in emf arising in this way is

$$E_C = \frac{1}{C} Q.$$

Furthermore, since the current is the rate of flow of charge at which charge builds up on the capacitor, we have $I = \frac{dQ}{dt}$

These elements act in accordance with Kirchoff's Law, which states that the algebraic sum of the emfs around a closed circuit is zero.
This principle yields

$$E - E_L - E_R - E_C = 0$$

or $E - RI - L\frac{dI}{dt} - \frac{1}{C} Q = 0$

or $L\frac{dI}{dt} + RI + \frac{1}{C} Q = E$ (1)

Replacing I by $\frac{dQ}{dt}$

(1) becomes $L\,\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E$ .. (2)

When no capacitor is present the equation (1) becomes the first order differential equation : $L\frac{dI}{dt} + RI = E$ (3)

We solve (3) assuming that initial current I_o and a constant emf E_o is impressed on the circuit at time $t = 0$.
The equation governing the flow of current is

$$L\frac{dI}{dt} + RI = E_o$$

separating the variables

$$\frac{dI}{E_o - RI} = \frac{1}{L}dt$$

on integrating using the initial condition $I = I_o$, when $t = 0$.

We get $\log(E_o - RI) = -\frac{R}{L}t + \log(E_o - RI_o)$

so $I = \frac{E_o}{R} + \left(I_o - \frac{E_o}{R}\right)\exp\left(-\frac{Rt}{L}\right)$

Note that the current I consists of a steady state Part E_o/R and a transient Part $\left(I_o - \frac{E_o}{R}\right)\exp\left(\frac{-Rt}{L}\right)$ that approaches zero as $t \to \infty$. Consequently, Ohm's law $E_o = RI$ is nearly true for large values of t. If $I_o = 0$, then $I = \frac{E_o}{R}(1 - e^{-Rt/L})$

and if $E_0 = 0$, then $I = I_0 e^{-Rt/L}$

OTHER RATE PROBLEMS

a) Mixture Problem: A tank initially contains 50 gallons of pure water. Starting at time $t = 0$ brine containing 2 lb of dissolved salt per gallon flows into the tank at the rate of 3 gallon/minute. The mixture is kept uniform by constant stirring and the mixture simultaneously flows out of the tank at the same rate.

1. How much salt remains in the tank at any time $t > 0$?
2. How much salt remains at the end of 25 minutes ?
3. How much salt remains eventually (after a long time) ?

Let x denote the amount of salt in the tank at five t. The basic equation governing the flow is

$$\frac{dx}{dt} = \text{Inflow} - \text{outflow} \qquad (i)$$

Since the inflow is at the rate 3 gallon/minute and each gallon contains 2 lb of sal t

Thus inflow = 2 lb/gal X 3 gal/min = 6 lb/min (ii)

Since the rate of outflow equals the rate of inflow the tank contains 50 gal. of the mixture at any time.

Then 50 gallons contains x lb of salt.

The concentration of salt at time t = $\frac{x}{50}$ lb/gal

The outflow = $\left(\frac{x}{50} \text{ lb/gal}\right) \times (3 \text{ gal/min}) = \frac{3x}{50}$ lb/min. (iii)

Thus the differential equation governing the flow is

$$\frac{dx}{dt} = 6 - \frac{3x}{50} \qquad (iv)$$

Initially there was no salt in the tank. Hence $x = 0$ when $t = 0$ (or $X(0) = 0$).

To solve (iv) separating the variables, $\frac{dx}{100-x} = \frac{3}{50}\, dt$

Integrating $x = 100 + C\, e^{-3t/50}$

S ince $x(0) = 0$, $C = -100$

$$x = 100\,(1 - e^{-3t/50}) \qquad (v)$$

This answers question (1).

For the question (2), put $t = 25$.

$\therefore x(25) = 100\,(1 - e^{-1.5}) \approx 78$ lb

The question (3) is solved by letting $t \to \infty$

in (v). Then $x = 100$.

b) A certain chemical is converted into another chemical by a chemical reaction. The rate at which the first chemical is converted (into the second) is proportional to the amount of this chemical present at any instant. Ten percept of the original amount of the first chemical has been converted in 5 minutes.

i) What percent of first chemical will have been converted in 20 min?

ii) In how many min. will 60% of the 1st chemical has been converted ?

Let x_o in the amount of first chemical present initially. Let x be the amount of chemical undergoing reaction at the end of time t.
Then $(x_o - x)$ is the amount of the chemical left out at the end of time t.

By the hypothesis, the rate of charge of x is prop. to $(x_o - x)$.

Therefore, the differential equation of the reaction is

$$\frac{dx}{dt} = K\,(x_o - x) \qquad (1)$$

Separating the variables and integrating $x = x_o - C\,e^{-kt}$

Since $x = 0$ when $t = 0$, $c = x_o$

$$\therefore X = x_o\,(1 - e^{-kt}) \qquad (2)$$

Now $x = \frac{x_o}{10}$ when $t = 5$ min.

$$\frac{x_o}{10} = x_o\,(1 - e^{-5k})$$

$$e^{-5k} = 0.9 \qquad e^{k} = (0,9)^{1/5}$$

Hence (2) becomes $x = x_o\,(1 - (0.9)^{t/5})$ . . . (3)

(i) At the end of 20 min. $x = x_o\,(1 - (0.9)^{20/5}) = x_o\,(1, (0,9)^4$

Thus at the end of 20 min. $\frac{100x}{x_o} = 100\,(1 - (0.9)^4)$

Percent of the chemical is converted into the second chemical.

(ii) If $x = \frac{6x_0}{10}$ (60% of the first chemical)

Then from (3) $\frac{6x_0}{10} = x_0\left(1 - (0.9)^{t/5}\right)$

$0.6 = (1 - (0.9)^{t/5}$

$\therefore (0.9)^{t/5} = 0.4$

$\frac{t}{5} = \frac{\log 0.4}{\log 0.9}$

$t = 5\left(\frac{\log 0.4}{\log 0.9}\right)$ min.

c) Assume that the rate at which a hot body cools is proportional to the difference between its temperature and that of the surrounding (this law is called Newton's law of cooling). A body is heated to 110°C and placed in air at 10°C. After one hour its temperature is 60°C. How much additional time is required for it to cool to 30°C ?

Let θ°C be the temperature of the body at time t. from start. Since the temperature of the surrounding air is 10°C, by hypothesis

$$\frac{d\theta}{dt} = k(\theta - 10) \quad (1), \quad k > 0$$

Integrating, $\log_e(\theta - 10) = -kt + \text{constant}$

or $= 10 + c\, e^{-kt}$

when $t = 0$, $\theta = 110$, $110 = 10 + c\, e = 10^7$ C or $C = 100$

$$\theta = 10 + 100\, e^{-kt} \qquad (2)$$

when $t = 1$ hour, $\theta = 60$°C

$60 = 10 + 100\, e^{-k.1}$

$e^{-k} = 0.5$

$\theta = 10 + 100\,(0.5)^t \ldots (3)$

If $\theta = 30°$ C, $30 = 10 + 100\,(0.5)^t$

$$(0.5)^t = \frac{20}{100} = (0.2)$$

$$t = \frac{\log 0.2}{\log 0.5} \text{ hr}$$

The additional time required $\left(\frac{\log 0.2}{\log 0.5} - 1\right)$ hrs $= \left(\frac{\log 5}{\log 2} - 1\right)$ hrs.

d) A sum of money is deposited in a bank that pays interest at an annual rate r compounded continuously.

1. Find the time required for the original sum to double.
2. Find the interest rate that must be paid if the initial amount doubles in 10 years.

If $A = A(t)$, be the amount at any time.

then, $\dfrac{dA}{dt} = \dfrac{r}{100} A$ ... (1)

Integrating $A = A_0 e^{\frac{rt}{100}}$ ... (2)

$A_0 = A(0)$ = the initial deposits.

1. If $t = T$, when $A = 2A_0$ then $2A = A_0 e^{rt/100}$

$rT/100 = \log_e 2$

$T = \dfrac{100}{r} \log_e 2$ ... (3)

2. If $T = 10$, then $10 = \dfrac{100}{r} \log_e 2$

$r = 10 \log_e 2$ (4) is the rate of compound interest.

(e) In a certain chemical reaction a substance A is converted into another substance X. Let a be the initial concentration of A and $x = x(t)$ in the concentration of X at time t. Then $a - x(t)$ in the concentration of A at t. If the reaction is jointly proportional to x and a−x (i.e., the reaction is simulated by the substance being produced, when the reaction is described on auto catalytic) and $x(0) = x_0$, find $x(t)$.

The rate of reaction is governed by

$$\frac{dx}{dt} = Kx(a-x) \quad \text{-- (1)} \qquad K>0$$

$$\frac{dx}{x(a-x)} = K.dt \Longrightarrow \left(\frac{1}{x} + \frac{1}{a-x}\right)dx = akdt$$

$$\log x - \log(a-x) = akt + \text{Const}$$

$$\left(\frac{a-x}{x}\right) = c\exp(-akt)$$

or $\frac{a}{x} = 1 + Ce^{-akt}$

or $x = \frac{a}{[1 + Ce^{-akt}]}$

At $t = 0$, $x = x_0 = \frac{a}{1+c}$

$= C = \frac{a}{x_0} - 1$

$x = \frac{a}{1+(a/x_0 - 1)e^{-akt}} = \frac{ax_0}{x_0 + (a - x_0)e^{-akt}}$

(f) A moth ball whose radius was originally 1/4 inch is found to have a radius 1/8 inch after one month. Assuming that it evaporates at a rate proportional to its surface, find the radius at any time. After how many months will the moth ball disappear altogether.

If x is the radius of the moth ball, V its volume and S its surface a at time t,

$$\frac{dv}{dt} = -kS \qquad k > 0 \quad \cdots (1).$$

$$V = \frac{4}{3}\pi x^3, \quad S = 4\pi x^2$$

$$\frac{dv}{dx} = 4\pi x^2 \frac{dx}{dt} = S\frac{dx}{dt}. \text{ Hence (1) becomes } S \cdot \frac{dx}{dt} = -kS$$

$$\Rightarrow \frac{dx}{dt} = k \quad --(2)$$

Integrating $x = C - Kt$

When $t = 0$, $x = 1/4$, $1/4 = C \qquad x = 1/4 - Kt$

When $t = 1$, $x = 1/8$, $1/8 = 1/4 - K \qquad$ or $K = 1/8$

$$x = \frac{1}{4} - \frac{1}{8}\ t = \frac{2-t}{8}$$

$$x = x(t) = 1/8\ (2-t) \qquad (3)$$

when $t = 2$, $x = 0$ and the moth ball disappears. The moth ball disappears after 2 months.

g) The rate at which radioactive nuclei decay is proportional to the number of such nuclei present in a given sample. Half of the original radio active nuclei have undergone disintegration in 1500 years.

h) What percentage of the original radio active nuclei will remain after 4500 years ?

2. In how many years will only one-tenth of the original nuclei remain ?

If $x = x(t)$ is the amount of radio active nuclei remaining after t years and $x(0) = x_0$, the original amount of the nuclei, then the disintegration of the radioactive nuclei is governed by

$$\frac{dx}{dt} = -Kx \quad (1) \qquad K > 0,$$
$$x(0) = x_0$$

Integrating the equation (1) we get, $x = x(t) = C(\exp(-kt))$.

Putting $t = 0$, $x = x_0$, we get $x = x_0 e^{-kt}$ - - - (2)

It is given that when $t = 1500$, $x = 1/2\, x_0$

$$\frac{1}{2} x_0 = x_0 . e^{-1500\,k}$$

$$\frac{1}{2} = e^{-1500k} \quad \text{or} \quad (e^{-k})^{1500} = \frac{1}{2}$$

$$e^{-k} = -\left(\frac{1}{2}\right)^{\frac{t}{1500}}.$$

(2) becomes $x = x_0 \left(\frac{1}{2}\right)^{\frac{t}{1500}}$ (3)

(1) when $t = 4500$, $x = x_0 \left(\frac{1}{2}\right)^3 = \frac{x_0}{8}$

12.5% of the original amount remains after 4500 years.

(2) when $x = \frac{1}{10} x_0$, $\frac{1}{10} x_0 = x_0 \left(\frac{1}{2}\right)^{\frac{t}{1500}}$

Taking Logarithm

$$\log\left(\frac{1}{10}\right) = \frac{t}{1500}\left(\log\frac{1}{2}\right)$$

$$t = 1500\,\frac{\log\left(\frac{1}{10}\right)}{\log\left(\frac{1}{2}\right)} = 1500\,\frac{\log 10}{\log 2} \approx 4985 \text{ years.}$$

h) The rate at which a certain substance dissolves in water in proportional to the product of the amount undissolved and the difference $C_1 - C_2$ where C_1 is the concentration in the saturated solution and C_2 is the concentration in the actual solution. If saturated, 50 gm of water would dissolve 20 gm of the substance.

If 10 gm of the substance is placed in 50 gm of water and half of the substance is then dissolved in 90 min., how much will be dissolved in 3 hour ?

Since 20 gm of the substance dissolves in 50 gm saturated solution, the concentration in the saturated solution

$$= C_1 = \frac{20}{50} \qquad \text{(i)}$$

Let x gm be the substance dissolved in 50 gm of water at time t. Then (10 - x) gm of the substance is undissolved at time t.

.The concentration of the substance in 50 gm of water at time

$$t = C_2 = \frac{x}{50} \qquad \text{(ii)}$$

The substance dissolves in water according to the law

$$\frac{dx}{dt} = K\,(C_1 - C_2)\,(10-x)$$

$$= K\left(\frac{20}{50} - \frac{x}{50}\right)(10-x)$$

or $\quad \dfrac{dx}{dt} = \dfrac{k}{50}(20-x)\,(10-x)$

Separating the variables

$$\frac{dx}{(10-x)(20-x)} = \frac{k}{50}\,dt \quad \text{or}$$

$$\frac{1}{10}\left(\frac{1}{10-x} - \frac{1}{20-x}\right) = \frac{k}{50}\,dt$$

Integrating $\log_e\left(\dfrac{20-x}{10-x}\right) = \dfrac{k}{5}\,t + \text{const.}$

or $\quad \dfrac{20-x}{10-x} = C\,e^{k/5\,t}$

When t = 0, x = 0, $\quad C = 2$

$$\frac{20-x}{10-x} = 2e^{\frac{k}{5}t} \quad \cdots \quad (1)$$

Since half the substance (i.e. 5 gm) is dissolved in 90 m in

Putting x = 5, $\quad$ t = 90

$$3 = 2e^{18k}, \quad \frac{3}{2} = e^{18k} \quad \text{or} \quad e^{k} = \left(\frac{3}{2}\right)^{\frac{1}{18}}$$

$$\frac{20-x}{10-x} = 2e^{\frac{k}{5}t} = 2(e^{k})^{t/5} = 2\left(\frac{3}{2}\right)^{\frac{t}{90}}$$

Now we express x in terms of t.

$$20-x = 2\left(\frac{3}{2}\right)^{t/90}(10-x)$$

$$20-20\left(\frac{3}{2}\right)^{t/90} = x\left[1-2\left(\frac{3}{2}\right)^{\frac{t}{90}}\right]$$

$$x = 20\left[\frac{\left(\frac{3}{2}\right)^{t/90}-1}{2\left(\frac{3}{2}\right)^{t/90}-1}\right] \quad ---(2)$$

When t = 3 hrs = 180 min.

$$x = 20\left[\frac{\left(3/2\right)^{2}-1}{2\left(3/2\right)^{2}-1}\right] \text{gm} = 20\left[\frac{9-4}{18-4}\right] = \frac{100}{14} \quad \text{or} \quad x = 7.14 \text{ gm}.$$

EXERCISES
ON

Differential Equations

Formation of Differential Equations

I. Form the differential equation by eliminating the arbitrary constants in the following equations.

1. $y = a\cos mx + b\sin mx$ where a and b are arbitrary constants.
2. $y = ae^{mx} + be^{-mx}$, a, b are constants.
3. $Ax^2 + By^2 = 1$
4. $y = ae^{bx}$
5. $y = a\cos(x + b)$
6. $y^2 = m(a^2 - x^2)$, m, a are constants
7. $c(y + c)^2 = x^3$
8. $y = c(x - c)^2$
9. $y = a\sin(bx + c)$ where a and c are constants.
10. $xy = Ae^x + Be^{-x}$
11. $y^2 - 2ay + x^2 = a^2$
12. $(x - a)^2 + (y - b)^2 = r^2$, where a and b are constants.
13. Form the differential equation representing a family of straight lines passing through the origin.
14. Form the differential equation representing a family of concentric circles $x^2 + y^2 = a^2$.
15. Form the differential equation representing the family of parabolas having x-axis as the axis and focus at the origin.
16. Obtain the differential equation representing a family of rectangular hyperbolas which have co-ordinate axes as asymptotoes.
17. Form the differential equations by eliminating a and b.
    - i) $y = e^x(a\cos x + b\sin x)$
    - ii) $y = a\sec x + b\tan x$
    - iii) $y = a\sin x + b\cos x - x\sin x$

Variables separable form :

Solve the following equations:

1. $\sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy = 0$
2. $\sqrt{1 - y^2}\, dx + \sqrt{1 - x^2}\, dy = 0$
3. $(2y - 1)\, dx + (2x + 3)\, dy = 0$
4. $\dfrac{dy}{dx} + xy = xy^2$
5. $(y^2 + y)\, dx + (x^2 + x)dy = 0$
6. $y - x\dfrac{dy}{dx} = a\left(y^2 + \dfrac{dy}{dx}\right)$

7. $x\sqrt{1+y^2}\,dx + y\sqrt{1+x^2}\,dy = 0$

8. $(e^y + 1)\cos x\,dx + e^y \sin x\,dy = 0$

9. $\log \frac{dy}{dx} = ax + by$

10. $\frac{dy}{dx} = (y-a)(y-b)$

11. $\cos y \log(\sec x + \tan x)\,dx = \cos x \log(\sec y + \tan y)\,dy$

12. $\frac{dy}{dx} = \frac{x(2\log x + 1)}{\sin y + y\cos y}$

13. $(y^2 + y + 1)\,dx + (x^2 + x + 1)\,dy = 0$

14. $y - x\frac{dy}{dx} = a\left(x^2\frac{dy}{dx} + 1\right)$

15. $\frac{dy}{dx} = (x+y)^2$

16. $\frac{dy}{dx} = \sin(x+y)$

17. $\frac{dy}{dx} + 1 = e^{x+y}$

18. $(x+y+1)\frac{dy}{dx} = 1$

19. $\frac{dy}{dx} = 1 + e^{x-y}$

20. $\frac{dy}{dx} = \frac{x+y+1}{2x+2y+3}$

21. $\left(\frac{x+y-a}{x+y-b}\right)\frac{dy}{dx} = \frac{x+y+a}{x+y+b}$

22. $\frac{dy}{dx} = y\tan 2x$ with $y(0) = 2$

23. $2x \dfrac{dy}{dx} = 3y$ given $y(1) = 4$

24. $\dfrac{dy}{dx} = \sec y$ given $y(0) = 0$

25. $\dfrac{dy}{dx} = 2e^{x} y^{3}$ given $y(0) = \frac{1}{2}$

26. $\sin\left(\dfrac{dy}{dx}\right) = a$ given $y(1) = 2$

27. $e^{dy/dx} = x + 1$ given $y(0) = 3$

28. $(1 + x^2)\dfrac{dy}{dx} + (1 + y^2) = 0$ given $y(0) = 1$

29. $(1 + y^2)\,dx - xy\,dy = 0$ given $y(0) = 1$

30. $e^{x} \dfrac{dy}{dx} = 3y^{3}$ given $y(0) = \frac{1}{2}$

Homogeneous Equations

Show that the following equations are homogeneous and solve them:

1. $xy' = x - y$
2. $x^2 y' = x^2 + xy + y^2$
3. $(3xy + y^2)\,dx + (x^2 + xy)\,dy = 0$
4. $\dfrac{dy}{dx} = \dfrac{x - 3y}{x - y}$
5. $\dfrac{dy}{dx} = \dfrac{x - y}{x - y}$
6. $2xy \dfrac{dy}{dx} = x^2 + y^2$
7. $(x^2 + xy)\,dy = (x^2 + y^2)\,dx$
8. $(y - x^2)\,dx - 2xy\,dy = 0$
9. $x\,dy - y\,dx = \sqrt{x^2 + y^2}\;dx$
10. $x\dfrac{dy}{dx} = y - x\tan\dfrac{y}{x}$
11. $\left(x\tan\dfrac{y}{x} - y\sec^2\dfrac{y}{x}\right)dx - x\sec^2\dfrac{y}{x}\,dy = 0$

12. $x \cos \frac{y}{x} (y\,dx + x\,dy) = y \sin \frac{y}{x} (y\,dx - x\,dy)$

13. $x \frac{dy}{dx} = y[\log y - \log x + 1]$

14. $(1 + e^{x/y})\,dx + e^{x/y}\left(1 - \frac{x}{y}\right) dy = 0$

15. $x \frac{dy}{dx} = y + x \sec^2 \frac{y}{x}$

16. $x\,dx + \sin^2\left(\frac{y}{x}\right)(y\,dx - x\,dy) = 0$

17. $(2\sqrt{xy} - x)\,dy + y\,dx = 0$

18. $\frac{dy}{dx} = 1 + \frac{y}{x} + \frac{y^2}{x^2}$

Linear Differential Equations

Find the general solution of the following equations.

1. $(x^2 + 1)\frac{dy}{dx} + 2xy = 1$

2. $\frac{dy}{dx} + y \cot x = \cos x$

3. $\cos^2 x \frac{dy}{dx} + y = \tan x$

4. $(x + 2y^3)\frac{dy}{dx} = y$

5. $(1 + y^2)\,dx = (\tan^{-1} y - x)\,dy$

6. $(x - \tan y)\,dy = \sin 2y\,dx$

7. $\frac{dy}{dx} + y \tan x = \sec x$

8. $x(x - 1)\frac{dy}{dx} - y = x^2 (x - 1)^2$

9. $x \frac{dy}{dx} + 2y = \log x$

10. $(x + y + 1)\frac{dy}{dx} = 1$

11. $(2x - 10y^2)\dfrac{dy}{dx} + y = 0$

12. $x\cos x\dfrac{dy}{dx} + y\,(x\sin x + \cos x) = 1$

13. $(x+1)\dfrac{dy}{dx} - y = (x+1)^2\, e^{3x}$

14. $x\left(\dfrac{dy}{dx} + y\right) = 1 - y$

15. $\dfrac{dy}{dx} + \dfrac{y}{(1-x)\sqrt{x}} = 1 - \sqrt{x}$

Reduce the following equations to the linear form and hence solve.

Equations of the form $f'(y)\dfrac{dy}{dx} + f(y)P = Q$ can be reduced to linear form by using the substitution f(y) = u so that $f'(y)\dfrac{dy}{dx} = \dfrac{du}{dx}$

1. $\dfrac{dy}{dx} + x\sin 2y = x^3\cos^2 y$

2. $\dfrac{dy}{dx} - \dfrac{\tan y}{1+x} = (x+1)\,e^x \sec y$

3. $x\dfrac{dy}{dx} + y\log y = xye^x$

4. $\dfrac{dy}{dx} - \dfrac{y\log y}{x} = \dfrac{y(\log y)^2}{x^2}$

5. $\dfrac{dy}{dx} + \dfrac{1}{x} = \dfrac{e^y}{x^2}$

6. $\tan y \dfrac{dy}{dx} + \tan x = \cos y \cos^3 x$

7. $x \dfrac{dy}{dx} = y\,(\log y - \log x + 1)$

8. $\dfrac{dy}{dx} + \dfrac{1}{x} \tan y = \dfrac{1}{x^2} \tan y \sin y$

9. $\dfrac{dy}{dx} = e^{x-y}\,(e^x - e^y)$

10. $\dfrac{dy}{dx} = (\sin x - \sin y)\,\dfrac{\cos x}{\cos y}$

11. $\sin y \dfrac{dy}{dx} = \cos y\,(1 - x \cos y)$

12. $\dfrac{dy}{dx} + (2x \tan^{-1} y - x^3)(1 + y^2) = 0$

Differential Equations of the form $\dfrac{dy}{dx} = \dfrac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$.

Find the general solutions of the following equations.

1. $\dfrac{dy}{dx} = \dfrac{x + 2y - 3}{2x + y - 3}$

2. $\dfrac{dy}{dx} = \dfrac{y - x + 1}{y + x + 5}$

3. $\dfrac{dy}{dx} = \dfrac{2x - 2y - 1}{3x + y - 5}$

4. $\dfrac{dy}{dx} = \dfrac{x + 3y - 4}{3x - y - 2}$

5. $(3y - 7x + 7)\,dx + (7y - 3x + 3)\,dy = 0$

6. $(y + x - 2)\,dx - (y - x - 4)\,dy = 0$

7. $\dfrac{dy}{dx} = \dfrac{6x - 4y + 3}{3x - 2y + 1}$

8. $(5x - y + 2)\,dy + (3x - 7y - 1)\,dx = 0$

9. $(2x - y + 1)\,dy - (x + 2y + 3)\,dx = 0$

10. $(x + y + 1)\,dx - (2x + 2y + 3)\,dy = 0$

11. $(2x - 4y + 3)\,dx + (x - 2y + 1)\,dy = 0$

12. $(x + 2y)(dx - dy) = dx + dy$

Bernoulli's equations

A differential equation of the form $\frac{dy}{dx} + Py = qy^n$ where P and Q are functions of **x** is called a Bernoulli's differential equation.

Find the general solution of the following equations.

1. $x\frac{dy}{dx} + y = x^3 y^6$

2. $\frac{dy}{dx} + \frac{4x}{1+x^2} y = 2x\sqrt{y}$

3. $y(2xy + e^x)\,dx - e^x\,dy = 0$

4. $(x^2 y^3 + xy)\frac{dy}{dx} = 1$

5. $x\frac{dy}{dx} + y = y^2 \log x$

6. $\frac{dy}{dx} - y\tan x = -y^2 \sec x$

7. $x^3 \frac{dy}{dx} - x^2 y + y^4 \cos x = 0$

8. $\frac{dy}{dx} + 2y\tan x = y^2 \tan^2 x$

9. $2\frac{dy}{dx} - y\sec x = y^3 \tan x$

10. $xy - \frac{dy}{dx} = y^3 e^{-x^2}$

11. $(x - y^2)\,dx + 2xy\,dy = 0$

12. $y(2xy + e^x)\,dx - x e^x\,dy = 0$

13. $\frac{dy}{dx} - \frac{xy}{1-x^2} = x\sqrt{y}$

14. $2xy\,dy - (x^2 + y^2 + 1)\,dy = 0$

15. $\frac{dy}{dx} - 2y \tan x = y^2 \tan^2 x$

Differential Equations of the type $\frac{d^2 y}{dx^2} = f(x)$ *and* $\frac{d^2 y}{dx^2} = g(y)$.

Solve :

1. $\frac{d^2 y}{dx^2} = \frac{1}{x}$

2. $\frac{d^2 y}{dx^2} = \cos nx$

3. $\frac{d^2 y}{dx^2} = \sec^2 x$

4. $\frac{d^2 y}{dx^2} = x^2 \sin x$

5. $\frac{d^2 y}{dx^2} = \sin^{-1} x$

6. $\frac{d^2 y}{dx^2} = xe^x$

7. $\frac{d^2 y}{dx^2} = \cos 3x - \sin 3x$

8. $\frac{d^2 y}{dx^2} = \sin^2 x$

9. $\frac{d^2 y}{dx^2} = \log x$

10. $\frac{d^2 y}{dx^2} = x \sin x$

11. $\frac{d^2 y}{dx^2} = 0$

12. $\dfrac{d^2 y}{dx^2} = \left(\sqrt{\dfrac{1-\cos 2x}{1+\cos 2x}}\right)$

13. $\dfrac{d^2 y}{dx^2} = \dfrac{x-1}{(2x-x^2)^{3/2}}$

14. $\dfrac{d^2 y}{dx^2} = \operatorname{cosec}^2 x + \log x$

15. $\sec x\left(\dfrac{d^2 y}{dx^2} + 6e^{2x}\right) = 2 - x$

16. $\dfrac{d^2 y}{dx^2} = x - \sin x$ given $y(0) = 0$ and $y'(0) = -1$

17. $\dfrac{d^2 y}{dx^2} = \log x$ given $y(1) = 1$, $y'(1) = -1$

18. $\dfrac{d^2 y}{dx^2} = x^2 \sin x$ given $y(0) = 0$, $y'(0) = 0$

19. $x\dfrac{d^2 y}{dx^2} = 1$ given $y(1) = 1$, $y'(1) = 0$

20. $\dfrac{d^2 y}{dx^2} = e^x(\sin x - \cos x)$ given $y(0) = 1$, $y'(0) = 0$

Application of Differential Equations

1. A horizontal beam of length $2l$ metres carrying a uniform load of ω kg per metre of length is freely supported at both the ends satisfying the differential equation :

$$EI\frac{d^2 y}{dx^2} = \frac{1}{2}\omega x^2 - \omega l x$$

y being the deflection at a distance x from one end. If $y = 0$ at $x = 0$ and $\dfrac{dy}{dx} = 0$ at $x = l$ find the deflection at any point.

2. A particle starting with velocity 'u' is falling freely under gravity with a constant acceleration. Find the velocity v and the distance s travelled by the particle at time 't'.

3. The velocity v of a particle vertically satisfies the equation :

$$v\frac{dv}{dx} = g\left(1 - \frac{v^2}{k^2}\right)$$

where g and k are constants. If both v and x are zero initially find v in terms of x.

4. The decay rate of radium at any time is proportional to its mass at that time. The mass is m_o at t = 0. Find the time when the mass will be halved.

5. The equation of electromotive forces for an electric circuit containing resistance and self induction is :

$$E = R_i + L\frac{di}{dt}$$

where E is the electromotive force given to the circuit; R, the resistance; L the coefficient of induction. Find the current 'i' at time 't' when (i) $E = 0$ and (ii) $E = a$ non-zero constant.

6. A particle falls towards the earth, starting from rest at a height 'h' above the surface. If the acceleration of the earth varies inversely as the square of the distance from its centre, find the velocity of the particle on reaching the earth's surface, given 'a' the radius of the earth and 'g' the value of acceleration due to gravity at the surface of the earth.

L I N E A R   P R O G R A M M I N G

1. Linear Inequations and Convex Sets

2. Formulation of L.P. Problems

3. Applications of Linear Programming

by

Dr.G.RAVINDRA

Exercises :

1. A random variable X has a binomial distribution with parameters $n = 4$ and $p = 1/3$.

i) Describe the probability mass function and sketch its graph.

ii) Compute the probabilities $P(1 < X \leq 2)$ and $P(1 \leq X \leq 2)$.

2. In a binomial distribution consisting of 5 independent trials, probabilities of 1 and 2 successes are 0.4096 and 0.2048 respectively. Find the parameter p of the distribution.

3. The probability of a man hitting a target is $1/3$.

i) If he fires 5 times what is the probability of hitting the target at least twice ?

ii) How many times must he fire so that the probability of hitting the target at least once is more than 90% ?

4. The random variable X has a binomial distribution with $n = 4$, $p = 0.5$. Find $P\{|X-2| \geq 1\}$.

Answers :

1. ii) 8/27, 56/81.
2. 0.2
3. i) 131/243
   ii) 6
4. 5/16

LINEAR PROGRAMMING

Introduction :

Mathematical Programming constitutes one of the most important problem areas of Operational Research (OR). It encompasses a wide variety of optimization problems. The basic problem of Mathematical Programming is to find the optimum (maximum or minimum) of a non-linear/linear function (called the objective function variously known as cost function, gain, measure of efficiency, return function, performance index, utility measure, etc. depending on the context) in a domain determined by a given system of non-linear and linear inequalities and equalities (called constraints).

Linear Programming (LP) is a Mathematical Programming problem where the objective function and the constraints are all (at least approximated) Linear functions of the unknown variables.

In practical terms, mathematical programming is concerned with the allocation of scarce resources - men, materials, machines and money (commonly known as the 4 M's in OR) - for the manufacture of one or more products so that the products meet certain specifications and some objective function (cost/profit) is minimized or maximized. Whenever the objective function is a linear function of the decision variables and the restrictions on the utilization or availability of resources are expressible as a system of linear equations or inequations, we have a Linear Programming Problem (LPP). For example, in the case of manufacturing a variety of products on a group of machines, the production problem is to determine the most efficient utilization of available machine capacities to meet the required demand. The programming problem is to allocate the available machine resources to the various products so that the total production cost is minimum. To solve this problem, we need to know the unit production cost (cost for producing one item), unit production time, machine capacity and production requirements. This is an LPP (for more clarification see Section 3 on formulation of Linear Programming Problems for a similar example).

The standard technique of solving an LPP is by Simplex Method (due to George, B.Dantzig, 1947) which is quite complicated and is beyond the scope of this unit. However, LPP's involving two variables can be solved graphically. Moreover, there are certain special types of LPP's such as transportation and assignment problems which admit easier methods of solution. Recently, there have been some spectacular developments in the area of LP due to an Indian, Narendra Karmakar of Bell Telephone Labs, U.S.A, where he is able to reach the solution of an LPP considerably faster than simplex method.

In this unit, we confine our attention to formulation of LPP's and their solution by graphical method.

LINEAR INEQUATIONS AND CONVEX SETS :

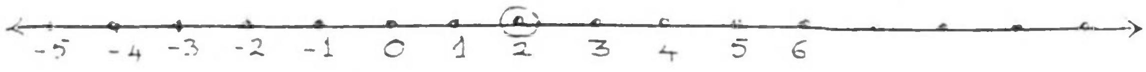
The restrictions on the utilization (demand) or availability of resources in a linear programming problem (LPP) are expressed as a system of linear equations or linear inequations, and the set of feasible solutions of an LPP is convex set. Though any LPP (in any number of variables) could be solved by the famous Simplex Algorithm, the LPP in two variables can be solved in an easier way by graphical method essentially identifying the intersection of graphs of various linear inequations and testing the objective function for maximum or minimum at the vertices of such a graph. The graph of a linear inequation is essentially a convex set. Thus the concept of Linear Inequations (and their graphs) and convex sets play an important role in the study and the solution of Linear Programming Problem (especially in the two variables case).

Linear Inequation :

Consider the relation $2x=4$ in exactly one variable x on real number line. In this equation, the highest power of x is 1 and so it is a linear equation in one variable. The graph of the equation is the set of all those points on x axis (Real line, R) satisfying the condition $2x=4$. Since there is exactly one point satisfying the condition namely $x=2$, the graph of the equation consists of just one point namely $x=2$ and it divides the x-axis into exactly two parts A and B, where A is the set of points on the axis satisfying $2x \leq 4$ and B is the set of points on the axis satisfying $2x \geq 4$, $2x \leq 4$ and $2x \geq 4$ are linear inequations in one variable and their graphs are respectively A and B, which are two opposite rays with end point $x = 2$.

The following illustrates the graphs of equation $2x=4$ and inequations $2x \leq 4$ and $2x \geq 4$.

A : $x \leq 2$ $\qquad$ $x = 2$ $\qquad$ B : $x \geq 2$



In general, $ax= b$, where a and b are real numbers, is a linear equation in one variable and its graph is just the point $x = b/a$ on x-axis (real line). Also, the point $x = b/a$ is common to the rays $ax \leq b$ and $ax \geq b$.

Consider another relation $2x+3y = 6$ in two variables. This is a linear equation in two variables. The graph of the equation is the set of all the points (x,y) in the cartesian plane (i.e. R^2 or xy-plane) which satisfy the equation $2x+3y = 6$. $(3,0)$ and $(0,2)$ are respectively the points of x-axis and y-axis satisfying $2x+3y = 6$. Thus the graph of $2x+3y = 6$ intersects the x-axis and y-axis respectively at $(3,0)$ and $(0,2)$. We know that the equation of the line passing through $(3,0)$ and $(0,2)$ is $2x+3y = 6$. Thus, the graph of $2x+3y = 6$ is essentially the straight line which intersects x-axis and y-axis respectively at $(3,0)$ and $(0,2)$. Further, the graph of $2x+3y = 6$ is the common edge of the two regions C and D where C is the set of points satisfying the inequation $2x+3y \leq 6$ and D is the set of points satisfying the inequation $2x+3y \geq 6$. C and D are called the graphs of $2x+3y \leq 6$ and

$2x + 3y \geq 6$ respectively. More precisely, we observe that the xy-plane has the following partitions.

1. The set of points satisfying $2x+3y < 6$.
2. The set of points satisfying $2x+3y = 6$.
3. The set of points satisfying $2x+3y > 6$.

Thus, if (x,y) is a point in the xy-plane, then it belongs to

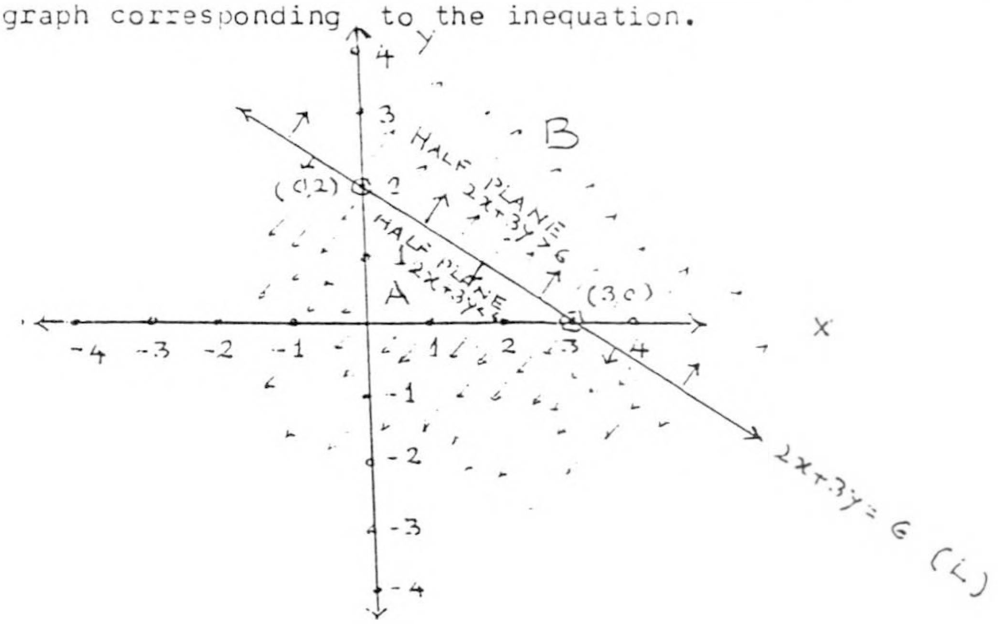
<u>either</u> i) the graph of $2x+3y < 6$

<u>or</u> ii) the graph of $2x+3y = 6$

<u>or</u> iii) the graph of $2x + 3y > 6$

This is the basic philosophy in identifying the graph of an inequation. We illustrate the same as follows :

Suppose we wish to identify the graph of the inequation $2x+3y < 6$. In the following figure, L represents the graph of the line $2x+3y = 6$. The graph of $2x+3y < 6$ could be either A or B (but not a portion of both). We have to mark which one of them is the exact graph corresponding to the inequation.



Here A and B are mutually disjoint. Choose a point which does not belong to L. (0,0) is one such point. The point (0,0) satisfies the inequation $2x+3y < 6$. Hence A is the graph of the inequation.

AUL is the graph of the inequation $2x+3y \leq 6$. Suppose we wish to identify the graph $2x+3y > 6$. Since (0,0) which is in A does not satisfy the inequation, A cannot be the graph of the inequation. Therefore, B is the graph of the inequation. Also B U L is the graph of the inequation $2x+3y \geq 6$.

In general, the graph of the linear equation $ax+by = c$ (in two variables) is the set of points on the line intersecting x-axis at (c/a, 0) and y-axis at (0,c/b). Further, the graph divides the xy-plane into two parts E and F, one of which is the graph of $ax+by \leq c$ and the other is the graph of $ax+by \geq c$. If a point in E (which is not on L) satisfies $ax+by \leq c$, then E is the graph of the inequation $ax+by \leq c$ and F is the graph of the inequation $ax+by \geq c$. Otherwise, E is the graph of the inequation $ax+by \geq c$ and F is the graph of the inequation $ax+by \leq c$.

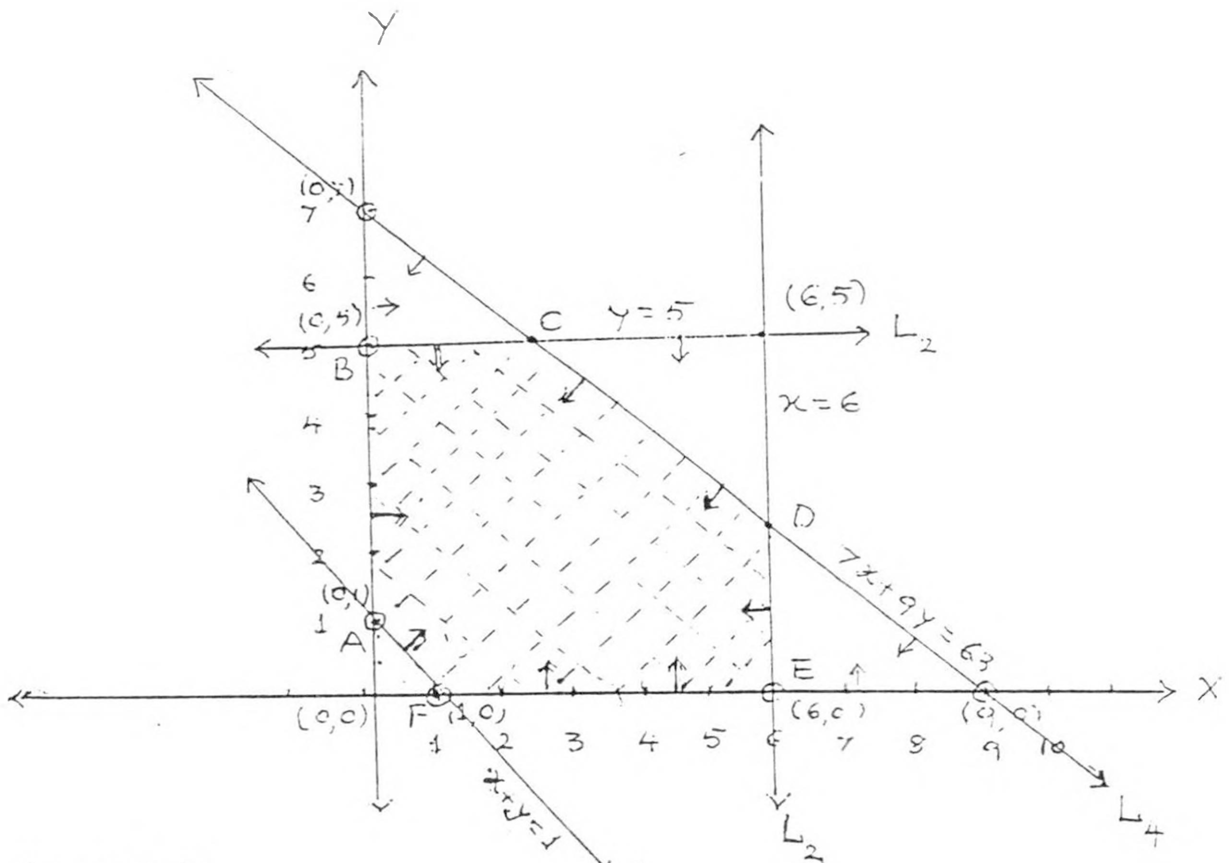
Consider the linear equation $ax+by+cz = d$ in three variables. The graph of this is a plane in the space R^3 and is common to the two parts A and B where A is a set of points (x,y,z) in R^3 satisfying $ax+by+cz \leq d$ and B is the set of the points (x,y,z) satisfying $ax+by+cz \geq d$. A and B are called <u>half planes</u>.

In general, the graph of $a_1x_1 + a_2x_2 + \dots + a_nx_n = b$ is called Hyper plane in the space R^n (i.e. n-dimensional Euclidean space) giving rise to two parts A and B where A is the set of points $(x_1, x_2, \dots, x_n)$ in R^n such that $a_1x_1 + a_2x_2 + \dots + a_nx_n \leq b$ and B is the set of points such that $a_1x_1 + a_2x_2 + \dots + a_nx_n \geq b$. A and B are called Half spaces.

In what follows, we shall mainly confine our discussion to equations and inequations in two variables only.

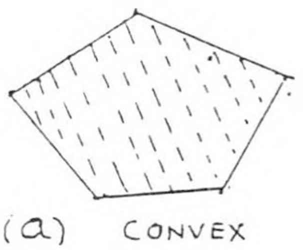
Example: Identify the intersection of graphs of the following linear inequations : $x + y \geq 1$, $y \leq 5$, $x \leq 6$, $7x + 9y \leq 63$, $x, y \geq 0$.

In the following figure, we have drawn arrow marks along the line L_1 representing $x+y = 1$ in such a way that the pointers of the arrows lie in the graph (region) of $x+y \geq 1$. The same is repeated for the rest of the inequations. The intersection of the graphs of these inequations is identified as that region which includes pointers corresponding to all the lines L_1, L_2, L_3, L_4, X and Y. The region enclosed by the polygon ABCDEF is such a region and hence it is the required graph satisfying all the six inequations simultaneously. Note that the region S enclosed by CDF is not the required region as no pointer corresponding to L_4 lies in it. Note that the arrows corresponding to all the lines L_1, L_2, L_3, L_4, X and Y converge in the graph satisfying all the six inequations.

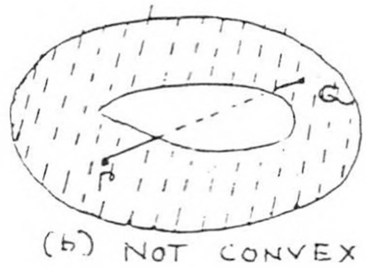


Convex Sets

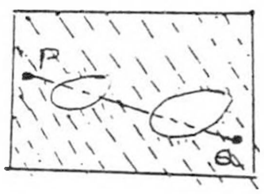
Examine the following figures.



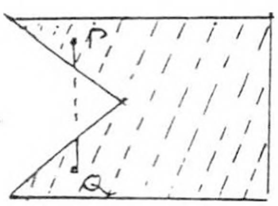
(a) CONVEX



(b) NOT CONVEX



(C) NOT CONVEX



(d) NOT CONVEX

The figure (a) is distinctly different from the other three. In the figure, the linear segment joining any two points is entirely within it, while the regions (b), (c) and (d) do not have the same property. For example, in (b) the line segment joining X and Y is not entirely in it, in (c) the line segment PQ is not in it and in (d) the line segment joining R and S is not entirely in it. Note that the dotted portion of the lines in (b), (c), (d) are not inside the regions. The figures like that of (a) are of special significance in the solution of LPP's and they are said to be <u>convex</u>. Speaking more precisely, a set of points C in the xy-plane (or R^n in general) is called a convex set if the line segment joining any two of its points is entirely contained in C.

<u>Examples of Convex Sets</u> :

i) xy-plane is a convex set.

ii) Circular region in xy-plane is convex but a circle is not convex. (by a circle, here we mean the set of points in xy-plane each of which is equidistant from a given point in the plane).

iii) Sphere, cube, cone, ellipsoid, paraboloid, etc. are convex sets in R^3.

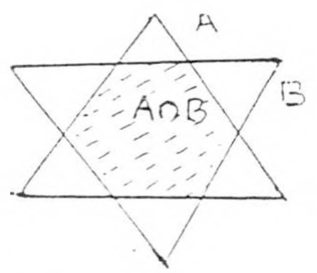
iv) Torus is not a convex set in R^3.

v) Hyperboloid is not a convex set in R^3.

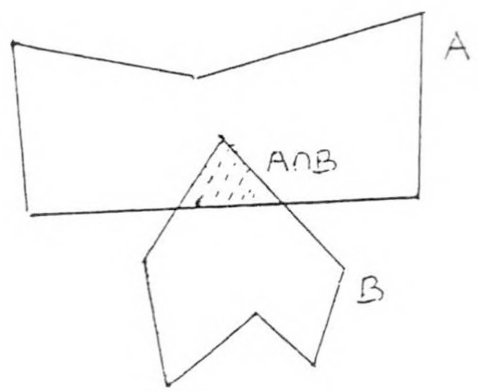
vi) The graphs of the inequations $ax+by \leq c$ and $ax+by \geq c$ are convex, i.e. half planes are convex.

vii) Half spaces in R^n are convex.

Now suppose A and B are any two sets with a given property P. The intersection of A and B may or may not have the property P, though it is part of the both. For example, if A and B are triangular regions in xy-plane their intersection is not necessarily a triangular region in the xy-plane. Similarly, if A and B are two sets in xy plane which are 'not convex', their intersection need not have the same property, that is, it could be convex. The following figures illustrate this.



The Intersection A B is not a triangular region.



The Intersection A B is a convex set.

If A and B are convex, will the intersection of A and B also be convex ? We will verify whether this is true or false.

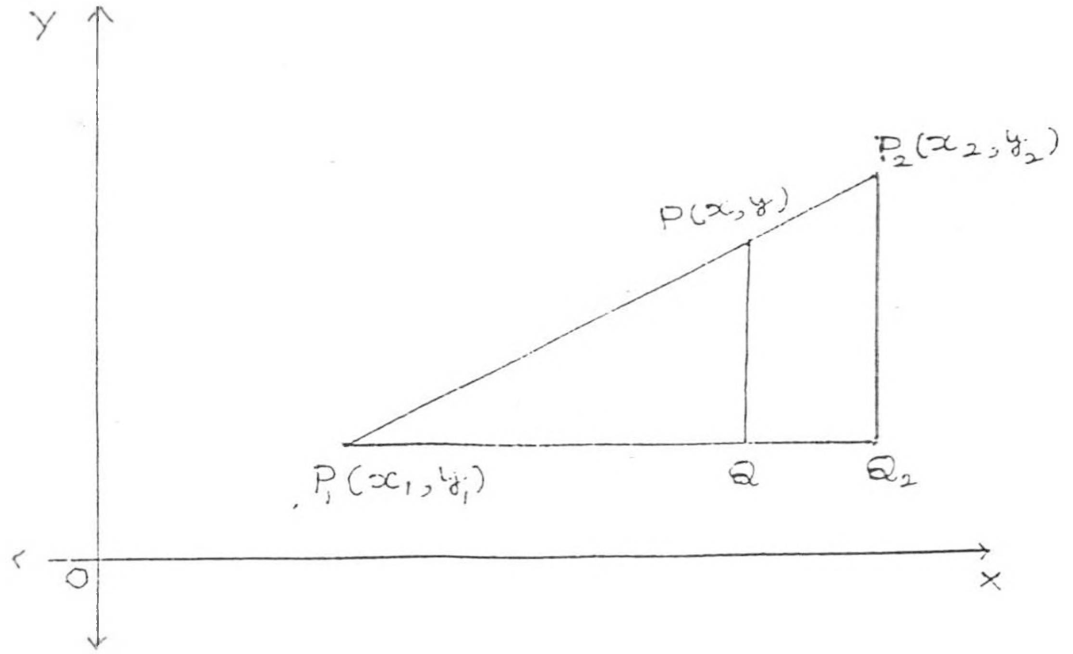
Let C be the intersection of A and B. Let P and Q be any two points in C. Let L be the line segment joining P and Q. Since A is convex, the points of L are contained in A. Since B is convex, the points of L are also contained in it. Thus the points of L are in both A and B. That is, the line segment joining any two points P and Q is entirely in C. This implies that C is a convex set. That is, the intersection of A and B is a convex set. We list the interesting result as

FACT : The intersection of any number of convex sets is also convex. Justification for this essentially follows from the above arguments, replacing sets A and B by any number of sets.

We now look for another way of defining convex sets which often helps in proving results concerning convex sets.

We know from coordinate geometry that (x,y) is a point on a line segment joining the points (x_1,y_1) and (x_2,y_2) if and only if $x = (1-t)\, x_1 + tx_2$ and $y = (1-t)\, y_1 + ty_2$, where $0 \leq t \leq 1$. Justification for the statement follows by considering the similar triangles P_1QP and $P_1P_2Q_2$ and their implication viz.

$$\frac{P_1Q}{P_1Q_2} = \frac{QP}{Q_2P_2}$$



Let $X = (x,y)$, X_1 a (x_1, y_1), $X_2 = (x_2, y_2)$, $t_1 = 1-t$, $t_2 = t$.
Using these symbols, the above statement can be restated as follows:
X is a point on the line segment joining X_1 and X_2 if and only if $X = t_1X_1 + t_2X_2$ such that $t_1 + t_2 = 1$, $t_1, t_2 \geq 0$.
(Since $X = (x,y) = ((1-t)\ x_1 + tx_2,\ (1-t)\ y_1 + ty_2)$ $= ((1-t)\ x_1,\ (1-t)y_1) + (tx_2,\ ty_2) = (1-t)\ (x_1, y_1) + t\ (x_2, y_2) =$ $(1-t)\ X_1 + tX_2$). The point X so expressed is said to be a <u>convex combination</u> of the points X_1 and X_2 in xy-plane.

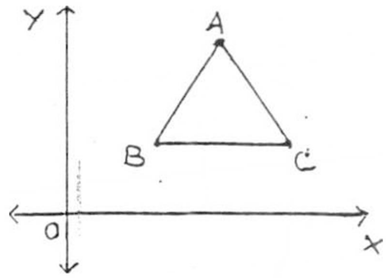
A <u>convex combination</u> of points $X_1, X_2, \ldots, X_n$ in xy-plane (or R^n in general) is a point $X = t_1X_1 + t_2X_2 + \ldots\ldots + t_nX_n$ where t_i's are non-negative real numbers and, $t_1 + t_2 + \ldots + t_n = 1$. As seen already, a point $X = (x_1, y_1)$ belongs to the line segment joining $X_1 = (x_1, y_1)$ and $X_2 = (x_2, y_2)$ if and only if X is a convex combination of X_1 and X_2. Thus a convex set can also be defined as follows:

A set C in xy-plane (or R^n) is a convex set if convex combination of any two points in C is also in it.

In fact, for a given convex set C any convex combination of any number of points in C is also in C.

Not every point in C is a convex combination of some points in C. For example, consider the triangle ABC in xy-plane (The following figure). There are no two distinct points in the triangle such that the line segment joining them contains A. That is, A is not an 'intermediate' point of any line segment in the triangle. Though A is a point on the line segment AB, it is not an intermediate point but one of the extreme points. Thus, A is not a convex combination of any other two distinct points in the triangle. Similarly, the points B and C have the same property. But any other point in the triangle is an intermediate point of same line segment in C. That is any point in the triangle other than A, B and C is a convex combination of some other two distinct points. The points A,B,C are <u>extreme points</u> in comparison with other points in the triangle.

A point X in a convex set is called an <u>extreme point</u> if X cannot be expressed as a convex combination of any other two distinct points in C.



Note that in the above example, the vertices A, B and C are the only extreme points of the triangle.

Examples :

1. The end points of a line segment are extreme points.
2. Vertices or corners of a cube in R^3 are extreme points.
3. Every point of the boundary of a circular region is an extreme point.
4. All the interior points of a circular region are not extreme points.
5. No point of a xy-plane is extreme in the plane.
6. The extreme points of a polygonal region are its vertices.
7. Any point in xy-plane is an extreme point of the singleton set containing the point.
8. The point of intersection of two line segments is not an extreme point of the line segments.

The extreme points play a very significant role in the solution of a LPP. In fact, the objective function of a LPP attains its optimum at at least one of the extreme points of its feasible region which is always convex.

Exercises :

1. Which of the given points belong to the graph of the given inequations ?

i) $x + y < 5$ $(0,0)$; $(3,2)$

ii) $x - y > 6$ $(4,3)$; $(11,4)$

iii) $3x+y \leq 2$ $(0,0)$; $(0,4)$

2. State whether the solution set of the following system of linear inequations is a null set or not.

i) $x \leq 0$ and $x \leq 2$

ii) $x < 2$ and $x > 2$

iii) $y > 1$ and $y > -1$

3. State true or false.

i) The line $y = 10x + 50$ separates the xy-plane in two half planes.

ii) A half plane is the graph of the inequation.

iii) The graph of a linear inequation is a convex set.

iv) The union of two convex sets in xy-plane is also a convex set in xy-plane.

v) The intersection of two convex sets in xy-plane is a convex set in xy-plane.

vi) If A and B are two sets in R^2 which are not convex, their intersection is also not convex in R^2.

vii) Vertices of a cube are extreme points.

viii) If m is the number of linear inequations in two variables and if the intersection of their graphs is a polygonal region with n sides then m = n.

ix) If a point (x,y) in xy-plane is a convex combination of two points (r,s) and (p,q) in the plane, then it lies on the line joining the two points (p,q) and (r,s).

x) The converse of the above statement is generally not valid.

xi) The intersection of two convex sets could possible be disjoint union of two convex sets.

xii) Union of two convex sets is convex.

xiii) Every point in a convex set is a convex combination of two other points in it.

4. Find two points in xy-plane that satisfy each of the following.

i) $y = 5x$, ii) $y < 5x$ iii) $y > 5x$

5. Mark the region which represents the graph of following inequations.

a) $x < 3$ b) $y > 3$ c) $2x + 4y \leq 8$

d) $x + y \leq 4$

6. State whether the region representing the following is bounded or unbounded.

$x \geq 0$, $y \geq 0$ and $x+y \leq 8$.

7. Let ABCD is a square in the first quadrant of xy-plane.

i) If $x + y = 1$ is the equation of the side AB, find the equations of the sides, BC, CD and DA.

ii) Write the inequations whose intersection is the <u>interior</u> of the square.

8. Let ABCDEF be a regular hexagon with length of each of its sides equal to 1 unit. Write the inequations whose intersection is the given hexagon.

9. Prove or disprove :

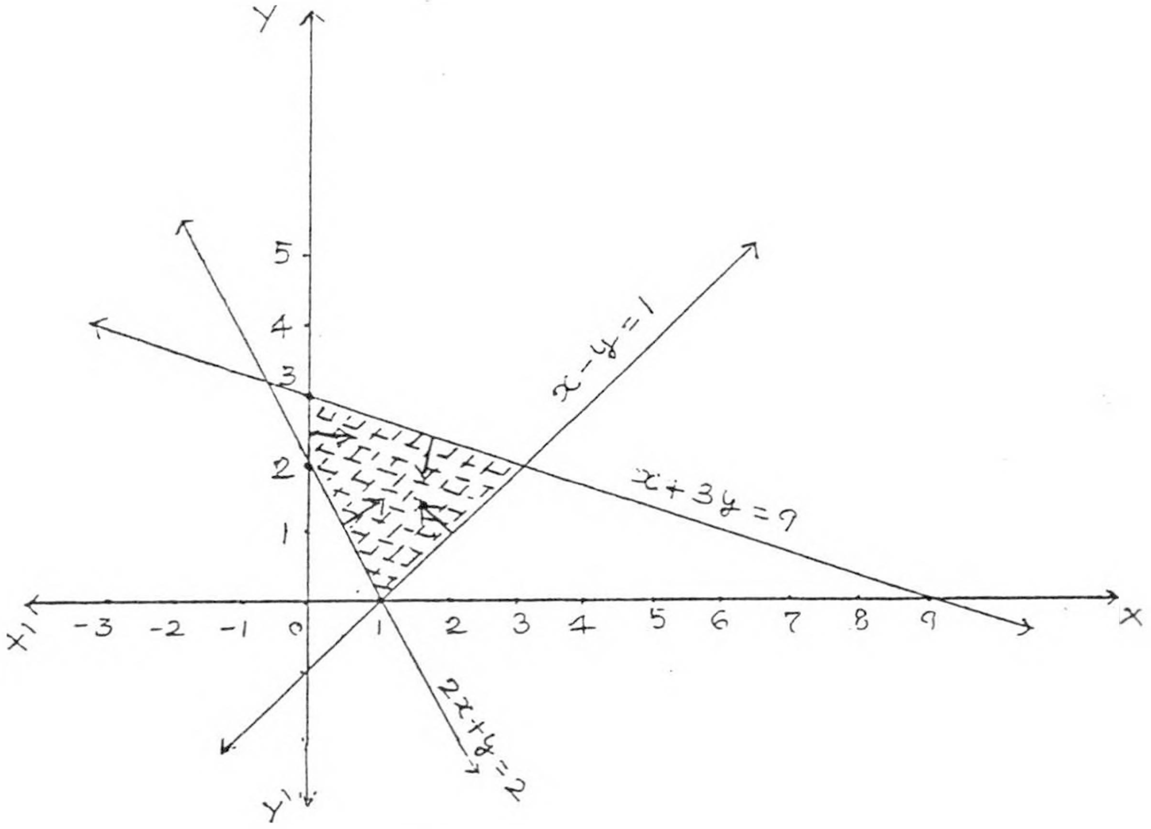
i) The circle $x^2+y^2 = a^2$ (a is a given real number) is a convex set.

ii) Every point on the boundary of a circular region is an extreme point.

iii) If G is the graph satisfying m linear inequations simultaneously, then G is a polygonal region having m sides.

iv) A set consisting of single element of R^2 is a convex set in R^2.

10. Find the linear constraints for which the shaded region in the following figure is the solution set.



FORMULATION OF LINEAR PROGRAMMING PROBLEMS

A large class of problems can be formulated as LP models. While formulating an LP model it is worth-while to remember the following 3-way rule suggested by Dantzig..

i) Identify the unknown activities to be determined and represent them by suitable algebraic symbols. Identify the inputs and outputs associated with each activity.

ii) Identify the restrictions (constraints) in the problem and express (at least approximate) them as linear algebraic equations/inequations.

iii) Identify the objective function and express it as a linear function of the unknown variables.

Proper definitions of the variables (step (i)) is a key step and will largely facilitate the rest of the work.

Let us illustrate the formulation by a few examples.

Example : Suppose we are concerned with a problem encountered by a man who sells oranges and apples in a running train. He has only Rs.120 with him and he decided to buy atleast 5 kgs of each item. One kg of apple costs Rs.10 and 1 kg of orange costs Rs.5. He can carry to the train only a maximum load of 15 kgs which his bag would hold. He expects a profit of Rs.2 per kg from applies and Rs.1 per kg from oranges. How much each of these two items should he buy (if he is wise enough) so as to get a maximum profit ?

Here, the ultimate goal or objective of the fruit seller is to get the maximum profit in his business, i.e. he wants to maximise his profit. To achieve this, he cannot purchase the items at random. The problem is to find out in what combinations should he buy apples and oranges so that the profit is maximum. Let us try to find out the possible combinations. The man can buy a total of 15 kgs of apples and oranges. Can he buy 15 kgs of oranges? Of course, not, because he has to buy at least 5 kgs of apples, i.e., he can buy a maximum of 10 kgs of oranges. Can he buy 15 kgs of apples ? He cannot because he should buy at least 5 kgs of oranges i.e. he can buy a maximum of 10 kgs of apples. He can purchase oranges from 5 kgs to 10 kgs and so also apples. We can list all the possible combinations of his purchase of apples and oranges and calculate the profit in each case. See the table below.

PURCHASE (in kgs)		COST			PROFIT		
Orange	Apple	Orange Rs.5	Apple Rs.10	Total	Orange Re.1	Apple Rs.2	Total
5	10	25.00	100.00	125.00	Not possible		
6	9	30.00	90.00	120.00	6.00	18.00	24.00
7	8	35.00	80.00	115.00	7.00	16.00	23.00
8	7	40.00	70.00	110.00	8.00	14.00	22.00
9	6	45.00	60.00	105.00	9.00	12.00	21.00
10	5	50.00	50.00	100.00	10.00	10.00	20.00

Look at the last column. The maximum profit is Rs.24. He gets this profit when he purchases 6 kgs of oranges and 9 kgs of apples.

This is the solution of the problem which maximises or optimises the profit. So we call it an optimal solution of the problem.

Optimal solution = 9 kgs of apples and 6 kgs of oranges.

Optimum profit = Rs.24.

After investigating the next example, where we maximise the profit as in this example, we will be able to see if we can arrive at the optimal solution by trial and error method. Before that let us formulate the above example in Mathematical terms (see Dantzig's 3-way rule).

i) Definition of variables

Let x be the number of kgs of oranges and y be the number of kgs of apples bought.

ii) Constraints : Since one cannot buy negative number of oranges or apples it is clear that $x \geq 0$ and $y \geq 0$.

Since one kg of orange costs Rs.5, x kgs of orange will cost Rs.5x. Similarly, y kgs of apple costs Rs.10y. Therefore, the total cost will be $5x + 10y$. Since he has only Rs.120 with him we have,

$$5x + 10y \leq 120.$$

Since he has decided to buy atleast 5 kgs of each item,

$$x \geq 5, \quad y \geq 5.$$

As he cannot carry more than 15 kgs

$$x + y \leq 15.$$

iii) The objective function :

Since he expects a profit of Rs.2 per kg from apples and Re.1 per kg from oranges, his total profit would be x+2y which has to be maximised. The L.P. model is : Maximise $Z = x+2y$ subject to $x \geq 5$, $y \geq 5$, $5x+10y \leq 120$, $x+y \leq 15$; and $x,y \geq 0$. In this problem, the non-negativity restrictions are not necessary in view of the constraints $x,y \geq 5$.

Example : A company sells two different types of radios - 3 band types and 2 band types. Company has a profit of Rs.50 for each of the former type and Rs.30 for each of the second type. The production process has a capacity of 80,000 man hours in total. It takes 10 man hours labour to assemble 3-band type and 8 man hours for 2-band type. It is expected that a maximum of 6000 numbers of the former type and a maximum of 8000 of latter type can be sold out. How many of each type should be produced so as to maximise the profit ?

In this problem, the company aims at getting the maximum profit. i.e. profit is to be maximised. The problem is to find out in what combination should he produce 2-band radios and 3-band radios in order to achieve this objective. We know that the company gets more profit from the 3-band radios. Naturally, we can think of a possibility where all the radios produced are 3-band type. This could not be done since the maximum number of 3-band type radios should be six thousand. The other possibility is to think of another way. The man hours needed to produce a 2- band radio is smaller compared to 3-band radios. In that case, he should increase the number of 2-band radios, which should not exceed 8000. Naturally, a third question arises - can the company produce 6000, 3-band radios and 8000, 2-band radios. In that case, we have to take into consideration the man hours available. The man hours required for producing 8000, 2-band radios is 8x8000 = 64000. The total man hours required to produce 6000 3-band type and 8000 2-band type is 124000 which is greater than the man hours available. From the above discussion, we found that the number of 3-band radios can extend from 0 to 6000 and that of 2-band radios from 0 to 8000. To get a solution for this problem, we have to enumerate all the cases from 0 to 6000 and 0 to 8000, which evidently is laborious. Therefore, we have to find out an easier method to solve such problems.

We will now think of evolving an easy method to solve such problems. Before entering into the details of this method, let us explain the problem mathematically. In other words, let us try to write the LP formulation of the problem.

In the above problem, what we are expected to find is the number of 3-band radios and 2-band radios to be produced so as to get the maximum profit. Let us assume that the number of 3-band radios produced is 'x' and the number of 2-band radios produced is 'y'.

Number of 3-band radios = x
Number of 2-band radios = y

Once we know the number of each type of radios, we can calculate the total profit of the company. Profit from a 3-band radio is Rs.50 and the profit from a 2-band radio is Rs.30.

Total profit = $50x + 30y$.

The objective of the company is to get the maximum profit i.e. $50x + 30y$ should be maximum. We call this the objective function of the problem. Now the problem reduces to finding the maximum values of $50x + 30y$. In other words, we have to maximise $50x + 30y$.

What are the conditions to be satisfied ?
We know that 'x' and 'y' are the numbers of radios produced. So we can say that x and y cannot be negative. Mathematically, we put it as

$x \geq 0$ and $y \geq 0$

x is the number of 3-band radios. The maximum number of 3-band radios produced is 6000.

i.e. $x \leq 6000$

Similarly $y \leq 8000$

The total man hours available is only 80000. Man hours required to produce one 3-band radio is 10.

Man hours required for x radios = $10 \times X$
$= 10X$

In a similar way, man hours needed for y, 2-band radios = 8y.
The total man hours should not exceed 80000

i.e. $10x + 8y \leq 80000$

Thus the restrictions or conditions to be satisfied are

1. $x \geq 0$
2. $y \geq 0$
3. $x \leq 6000$
4. $y \leq 8000$
5. $10x + 8y \leq 80000$

These conditions are generally called <u>constraints</u> of the problem. The first two viz. $x \geq 0$ and $y \geq 0$ are called <u>non-negativity restrictions</u>. Each of these constraints is an inequation of degree 1. Hence, they are called <u>linear constraints</u>.

The mathematical formulation of the problem is as follows :

Maximise $50x + 30y$

subject to $x \geq 0$

$y \geq 0$

$x \leq 6000$

$y \leq 8000$

and $10x + 8y \leq 80000$

Here the objective function as well as the constraints are all linear (first degree).

<u>A typical LP Model</u> :

Suppose a company with two resources (labour and material) wishes to produce two kinds of items A and B.

Let t_1, t_2 units of time (hours or minutes) be respectively time required to produce one unit of A and B, m_1 and m_2 be the amount of unit material (in Kg or pounds or any unit of weight) respectively required for one unit of A and B, and Rs.p_1 and Rs.p_2 profit per unit of A and B. Suppose the daily availability of manpower (labour) is T hours and the supply of raw material is restricted to M Kgs per day. The problem of the company is :

How many items of kind A and how many items of kind B be produced everyday, so that the total profit is maximum ?

This kind of problem is generally known as <u>Product-Mix Problem</u>.

The entire information of the problem can be stored in matrix (tabular) form as follows :

Resources	Kinds of Items A	B	Supply/availability
Labour (hours/unit)	t_1	t_2	T
Material (Kgs/unit)	m_1	m_2	M
Profit (Rs./unit)	p_1	p_2	

In view of the 3-way rule suggested earlier we have

Step 1 : Let x = Daily production of kind A

y = Daily production of kind B

Step 2 : Constraint corresponding to the first row :

$$t_1x + t_2y \leq T$$

Constraint corresponding to second row :

$$m_1x + m_2y \leq M$$

Non negativity conditions :

$$x, y \geq 0$$

Step 3 :

The third row corresponds to the objective function and is given by

$$Z = p_1x + p_2y$$

Thus the mathematical formulation of the problem is :

(I) - Find numbers x, y which will <u>maximize</u>

$$Z = p_1x + p_2y$$

subject to the constraints

$$t_1x + t_2y \leq T$$

$$m_1x + m_2y \leq M$$

and $x, y \geq 0$

Note that in the mathematical formulation (I) above we deal only with numbers, equations, inequations and the given situation (that is company's problem) is no langer under consideration.

The above typical problem can be adopted in many real life situations and thus a teacher can find a problem of linear programming according to the nature of the students (urban, rural, etc). For example, if the 'company' is an industry like "ORKAY" A could be taken as Idly mix and B could be taken as Dosa mix. The relevant information concerning resources and profit (possibly in terms of cost price and selling price) can be obtained in the form of a matrix. Such matrix will help in identification of the problem as well as in its mathematical formulation. If we consider a comfy in kitchen appliance,

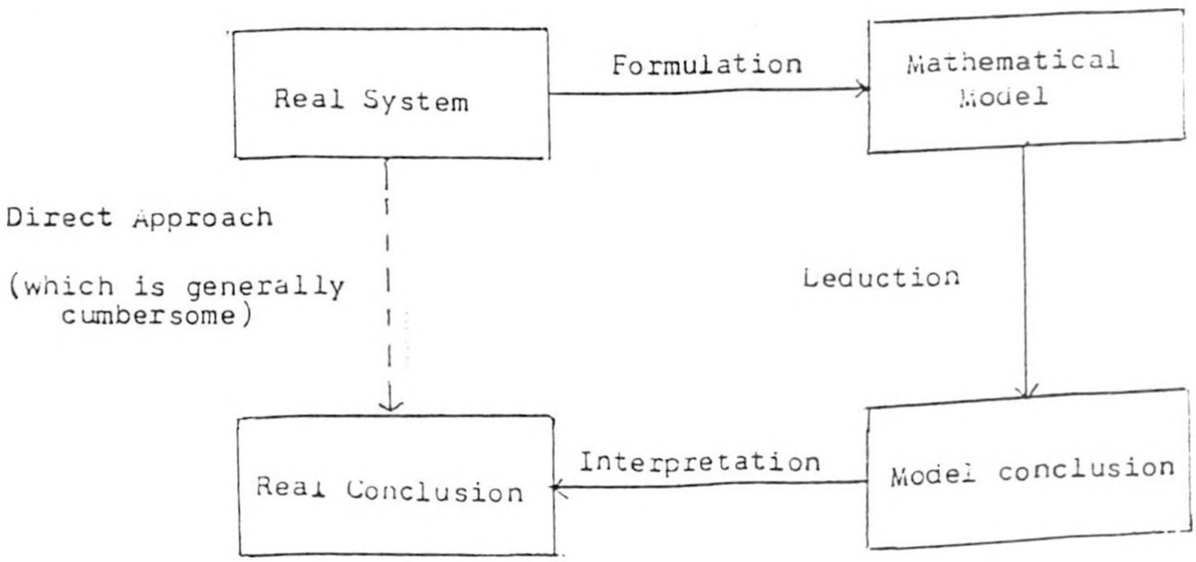
A could be considered as a pressure cooker

B could be considered as pressure pan.

If we want to have a farmer's problem, we can take A and B respectively to be areas of a given field for production of wheat and gram. The resource corresponding to material could be fertilizer. Here we will have an extra constraint viz. $x+y \leq a$ where 'a' is the area of the given field. Note that there could be any number of resources (and hence constraints) depending upon the situations.

Linear Programme, a Mathematical Model :

A mathematical model is a symbolic representation of a real situation. The process of mathematical modelling is depicted in the following figure.



In example 1, the real situation is 'selling of oranges and apples'. In example 2, the real problem (situation) is 'to evolve a selling policy of two kinds of radios' and in the product mix problem the real situation is 'productive scheduling'. In all these problems, mathematical formulation is mathematical model. The mathematical models in the above examples consist of objective function and constraints which are expressed quantitatively or mathematically as functions of decision variables 'Mathematical conclusion' and 'Real conclusion' constitute the solution of a linear programming problem, which we would be dealing within the next section.

EXERCISES :

1. A company makes two kinds of leather belts A, B. Belt A is of higher quality and belt B is of lower quality. The respective profits are Rs.4 and Rs.3 per belt. Each belt of type A requires twice as much time as a belt of type B, and, if all belts were of type B, the company, could make 1000 per day. The supply of leather is sufficient for only 800 belts per day (both A and B combined). Belt A requires a fancy buckle and only 400 per day are available. There are only 700 buckles a day are available for belt B. Formulate this as a linear programming model.

2. Give an example of a real situation (other than those mentioned in this lesson) whose mathematical model is a linear programming model.

3. Give an example of a mathematical model which is not a linear programming model.

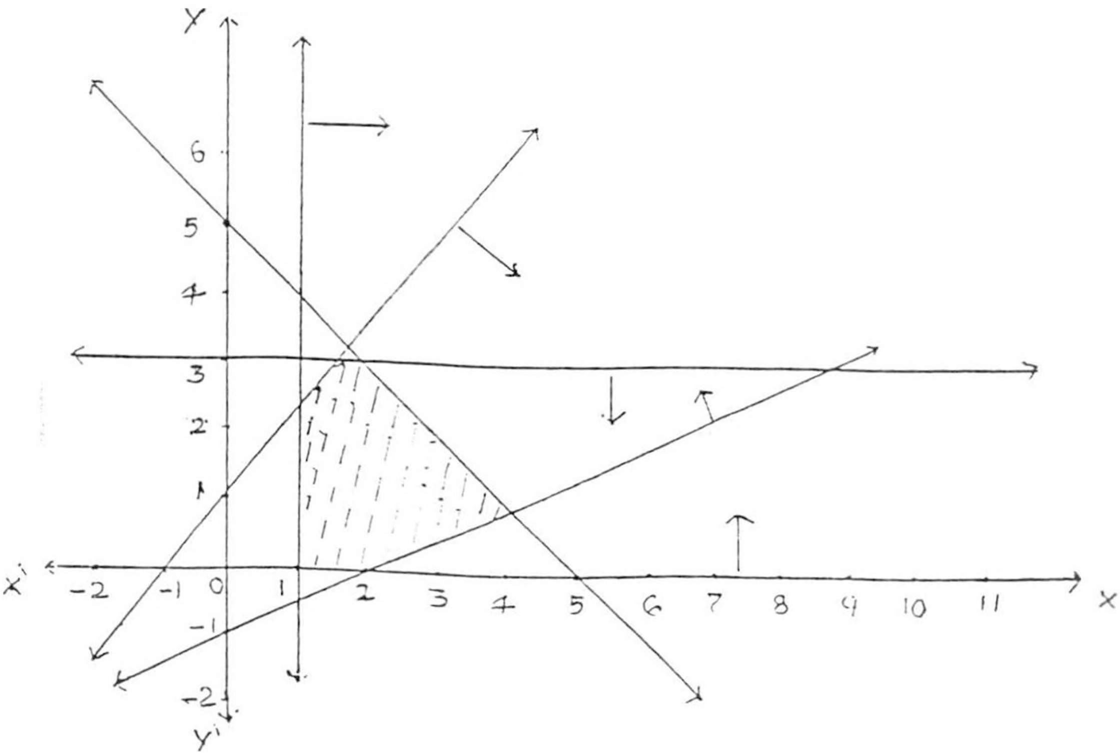
4. An Advertising company wishes to plan an advertising campaign in three different media - television, radio and magazines. The purpose of the advertising company is to reach as many potential customers as possible. Results of the market study are given below :

	Television Day Time	Television Prime Time	Radio	Magazine
Cost of an advertising unit	Rs. 40,000	Rs. 75,000	Rs. 30,000	Rs. 15,000
Number of potential customers reached per unit.	400,000	900,000	500,000	200,000
Number of women customers reached per unit	300,000	400,000	200,000	100,000

The company does not want to spend more than Rs.800,000 on advertising. It further requires that (i) atleast 2 million exposures take place among women, (ii) advertising on television be limited to Rs.500,000, (iii) atleast 3 advertising units be bought on day time television and two units during prime time; and (iv) the number of advertising units on radio and magazine should each be between 5 and 10.

Find different types of advertising units which minimize the total number of potential customers reached is maximum.
(Note: The problem involves four decision variables).

5. Write the constraints associated with the solution space shown in the following figure and identify all redundant constraints.



SOLUTION OF LINEAR PROGRAMMING PROBLEM BY GRAPHICAL METHOD :

Let us consider another example of an optimisation problem. We can examine whether this is a linear programming problem by formulating a mathematical model of the problem. We can also try to find the solution of the problem by graphical method.

Example 1 : A contractor has 30 men and 40 women working under him. He has contracted to move at least 700 bags of cement to a work site. Due to the peculiar nature of the work site he could employ at the maximum of 50 workers at a time. A man will carry 25 bags in a day and a woman will carry 20 bags in a day. A man demands Rs.45 a day and a woman demands Rs.35 a day as their wages. In what ratio should the contractor employ men and women so that the cost of moving the cement to the work site is minimum ?

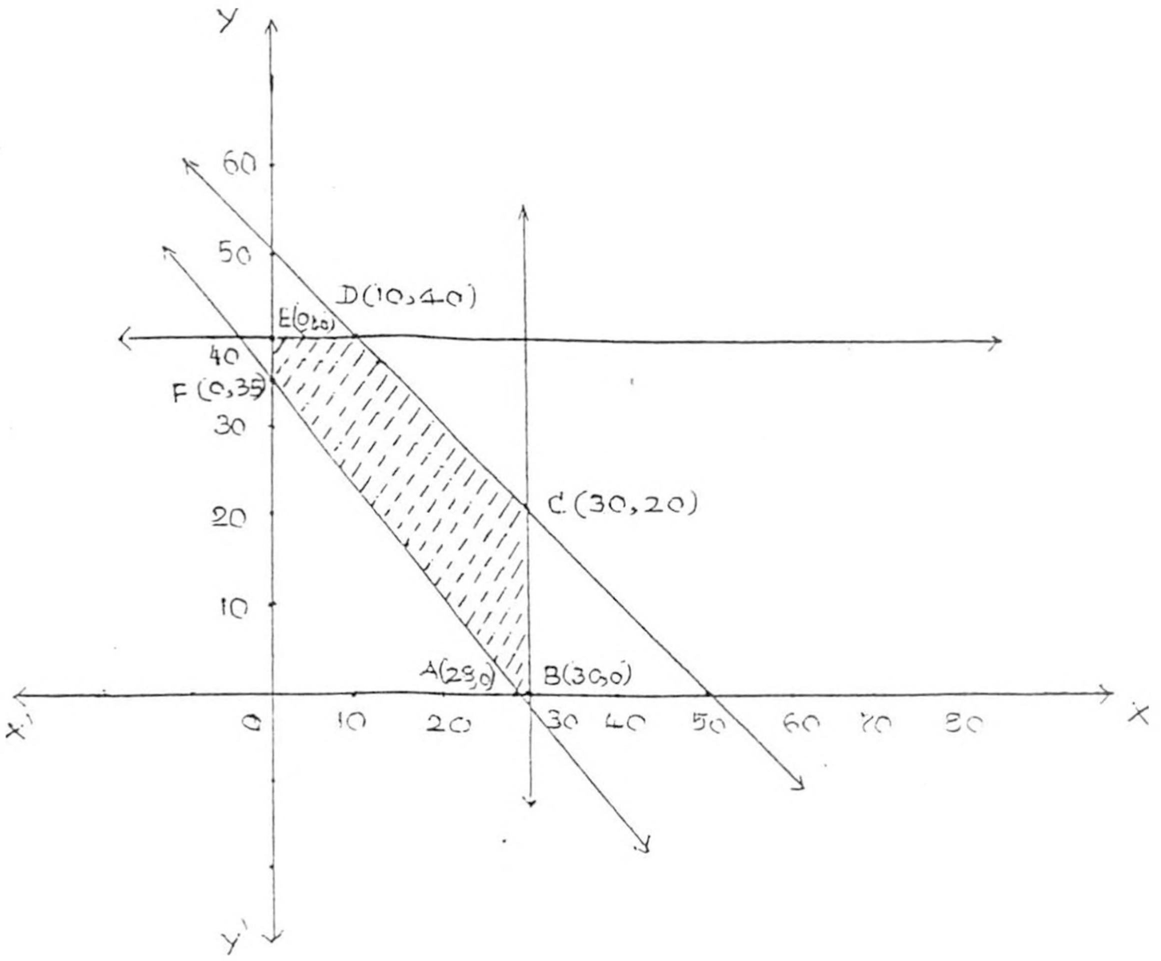
Now the mathematical model of the problem is :
Minimize $Z = 45x + 35y$

subject to the conditions
$x \geq 0$
$y \geq 0$
$x \leq 30$
$y \leq 40$
and $5x + 4y \geq 140$
(x and y are respectively the number of men and women employed and z is the total wage for them).

The above problem is an optimisation problem. The objective function as well as the constraints are linear. Hence, it is a LPP.

The next step is to find a value for x and a value for y such that $45x + 35y$ is minimum subject to the conditions laid down in the problem.

We first draw the graph of the inequations and see how the graph will give the solution of the problem.



The intersection of the graphs of the inequations is the region of the polygon ABCDEF, called the <u>feasible region</u>. Any point p (x,y) in the feasible region is a <u>feasible solution</u> of the LPP. The coordinates of such a point will satisfy all the inequations. Let us consider a point p(20, 20) in this region. We can easily verify that it satisfies all inequations. So we can consider the x-coordinate of P as a value of x and y - coordinate of P as a value of y. i.e. x = 20 (x- coordinate of P) and y = 20 (y - coordinate of P) is a feasible solution of the LPP. If we select another point say D (10,40) in the region, x = 10 and y = 40 is another feasible solution of the problem. We know that there are infinite number of points in the region ABCDEF. The coordinates of each point will give a feasible solution of the problem i.e.

the number of feasible solutions are infinite. The problem is to decide which one of these is optimal. For this, we make use of the following key result.

THEOREM . If there exists an optimal solution to an LPP, the objective function of the LPP always attains its optimum (minimum or maximum) at at least one of the corners (extreme points) of the feasible region.

Proof: We prove the validity of the theorem for two variables (coordinates) and in fact the same arguments can be extended to prove the theorem for any number of variables.

Let K be the set of feasible solutions of a linear programming problem. Let $(x_1, y_1), (x_2,y_2),\ldots, (x_n,y_n)$ be the extreme points (corners), of the feasible region corresponding to K. Let $Z(x,y) = C_1x + C_2y$ be the objective function of the linear programming problem.

Suppose for $x = x_0$ and $y = y_0$ the objective function attains its minimum.

That is, $Z(x_0, y_0) = C x_0 + C y_0$ is the minimum value of the objective function. Let $m = Z(x_0, y_0)$.

If (x_0, y_0) is one of the extreme points (corners) of the region representing K, the theorem is true. Therefore, we assume that (x_0, y_0) is not an extreme point. Hence, by the definition of extreme point, (x_0, y_0) can be expressed as a convex combination of extreme points of K.

That is,

$$(x_0, y_0) = t_1(x_1, y_1) + t_2(x_2, y_2) + \ldots + t_n(x_n, y_n) \qquad (1)$$

where $t_1 + t_2 + \ldots t_n = 1$ and $t_i \geqslant 0$.

This implies that

$$m = Z(x_0, y_0) = t_1\, z(x_1, y_1) + t_2\, z(x_2, y_2) + \ldots + t_n\, z(x_n, y_n)$$

Suppose $Z(x_r, y_r)$ be minimum along $Z(x_1, y_1), \ldots, z(x_n, y_n)$ so that

$$Z(x_i, y_i) \geq Z(x_r, y_r), \quad 1 \leq i \leq n \qquad (2)$$

Now (1) and (2) together imply that

$m \geq t_1 z(x_r, y_r) + t_2 z(x_r, y_r) + \dots + t_n z(x_r, y_r)$

(Since t_i' s are non negative).

That is,

$m \geq (t_1 + t_2 + \dots + t_n) \; Z(x_r, y_r)$

or $m \geq Z(x_r, y_r)$ (Since $t_1 + t_2 + \dots + t_n = 1$) (3)

By definition of minimum

$m \leq Z(x,y)$ for every (x,y) in K and in particular

$m \leq Z(x_r, y_r)$. (4)

(3) and (4) together imply that

$m = Z(x_r, y_r)$ where (x_r, y_r) is an extreme point. Thus Z (the objective function) attains its minimum at an extreme point of the feasibility region.

<u>Remark</u> :

Let for $x = x^1$ and $y = y^1$, $z(x,y)$ (the objective function) attain its maximum. Then by definition of maximum

$z^1 \geq z(x,y)$ for every x,y in K (where $z^1 = z(x^1, y^1)$)

$\Rightarrow -z^1 \leq -z(x,y)$

$\Rightarrow -z^1$ is the minimum value of $-z(x,y)$

That is, $-z^1 = \min(-z(x,y))$

or $-(\max z(x,y)) = \min(-z(x,y))$

or $\max z(x,y) = -\min(-z(x,y))$

Thus minimisation problem can be converted to maximization problems by considering negative of the objective function $z(x,y)$. And accordingly, the above theorem is true in the case of maximisation problems also.

In view of the above theorem, it is sufficient to concentrate our attention only on the corner points of the polygon ABCDEF. Evaluating the objective function at each of the vertices of ABCDEF and selecting the minimum of these values, we get the minimum value of the objective function. The coordinates of the corresponding vertices will constitute an optimal solution. The details are shown in the table given below :

Corner Point	Value of the objective function $Z = 45x + 35y$	
A (28, 0)	$45 \times 28 + 35 \times 0$	$= 1260$
B (30,0)	$45 \times 30 + 35 \times 0$	$= 1350$
C (30,20)	$45 \times 30 + 35 \times 20$	$= 2050$
D (10,40)	$45 \times 10 + 35 \times 40$	$= 1850$
E (0,40)	$45 \times 0 + 35 \times 40$	$= 1400$
F (0,35)	$45 \times 0 + 35 \times 35$	$= 1225$

Thus, it is clear that when the contractor employs 35 women and no men the cost of moving cement to work-spot is minimum and the minimum cost is Rs.1225. Now let us solve a maximisation problem by graphical method.

Example2: If a young man rides his motor cycle at 25 km per hour, he has to spend Rs.2 per km on petrol; if he rides at faster speed of 40 km per hour, the cost increases to Rs.5 per km. He has Rs.100 to spend on petrol. What is the maximum distance he can travel within one hour ?

Let x = distance travelled by the young man in one day at the speed of 2 km/hour,

and y = distance travelled by the young man in one day at the speed of 40 km/hour.

Let $Z = X+Y$

Objective Function : $Z = x+y$ (with the objective to maximize Z)

Constraints:

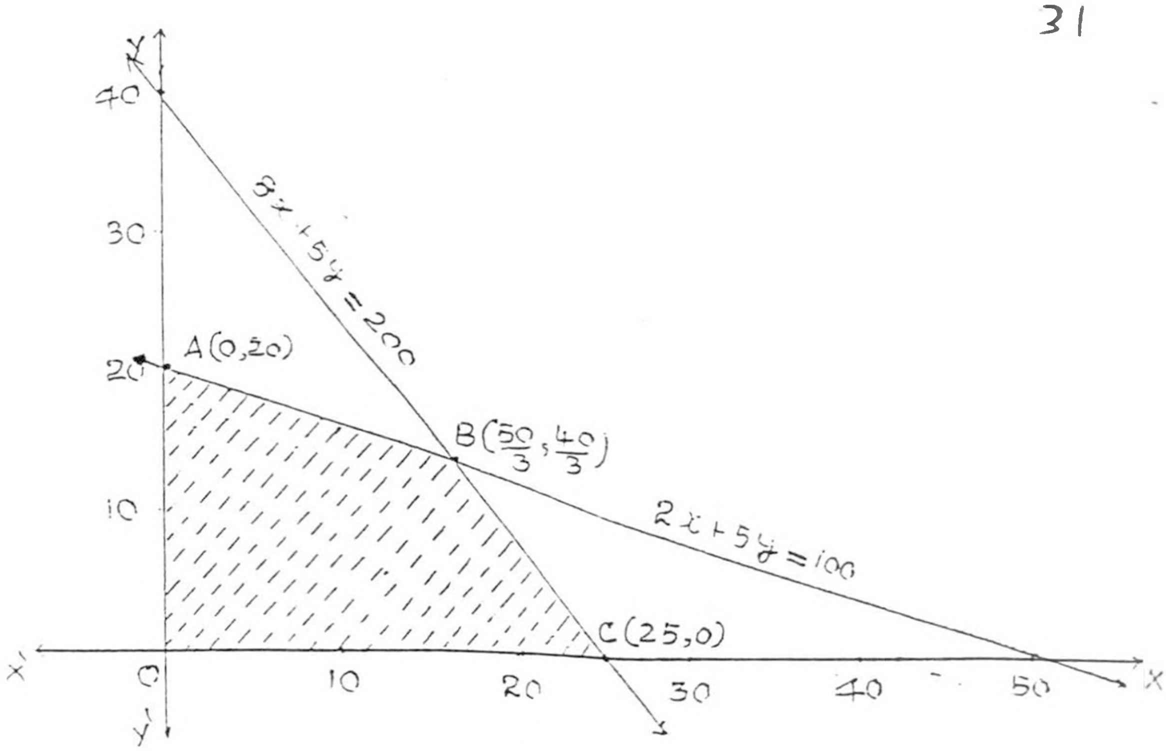
i) money spent on petrol $= 2x+5y \leq 100$ (constraint due to money)

ii) total time of travel $= \frac{x}{25} + \frac{y}{40} \leq 1$ (constraint due to time)

or $8x + 5y \leq 200$

iii) non negativity conditions : $x \geq 0$, $y \geq 0$

We now draw the graph corresponding to the constraints.



The feasible region is the shaded region of the polygon OABC.

Corner point	Value of $z = x+y$
O (0,0)	0
A(0,20)	20
B $\left(\frac{50}{3}, \frac{40}{3}\right)$	30
C (25,0)	25

Therefore, 30 = Max z = the maximum distance the young man can travel in one day.

The procedures that we follow in solving a LPP (in two variables) by graphical method is summarised below :

1. Mark the feasible region. (This is the intersection of the graphs of constraints).

2. Evaluate the objective function at each of the corner points of the feasible region and pick out the point which gives the minimum (maximum) value for the objective function as the case may be.

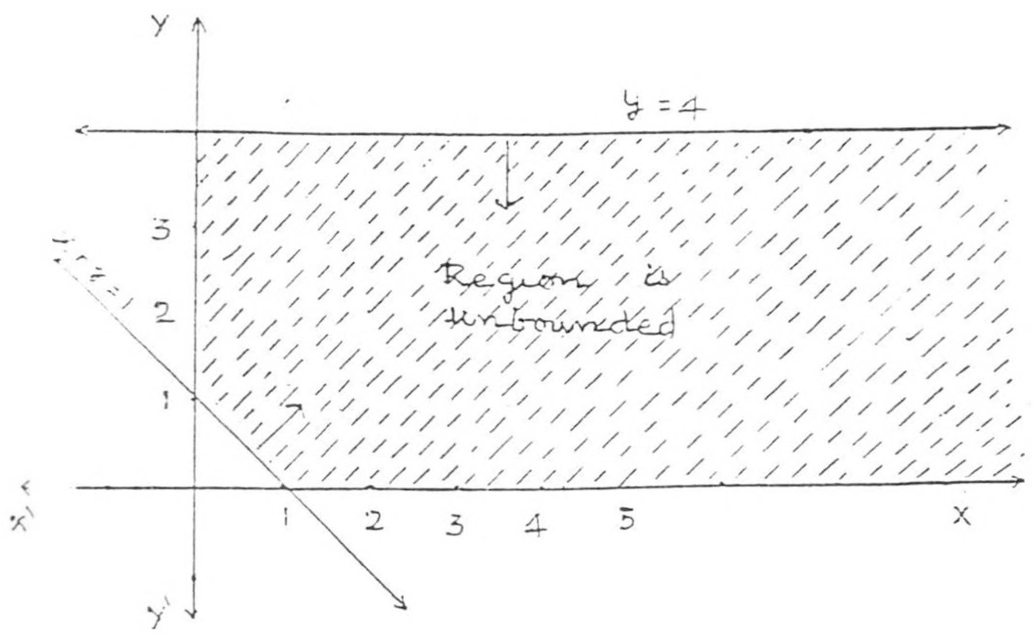
Theorem holds true if there exists an optimal solution to a LPP. There may be cases where the objective function has no finite optimal value. For example, Maximise $Z = x + 2y$

subject to $x + y \geq 1$

$x \geq 0$, $y \geq 0$

$y \leq 4$

The shaded region in the following figure is the feasible region of the problem. Note that the feasible region is not a polygonal region, but is unbounded.



In this case, moving farther away from the origin increases the value of the objective function $Z = x + 2y$ and the maximum value of Z would tend to $+\infty$ i.e., Z has no finite maximum. Whenever a LPP has no finite optimal value (maximum or minimum), we say that it has an <u>unbounded solution</u>. Further, there could be a <u>linear programming</u> problem such that it has no feasible solution.

For example,

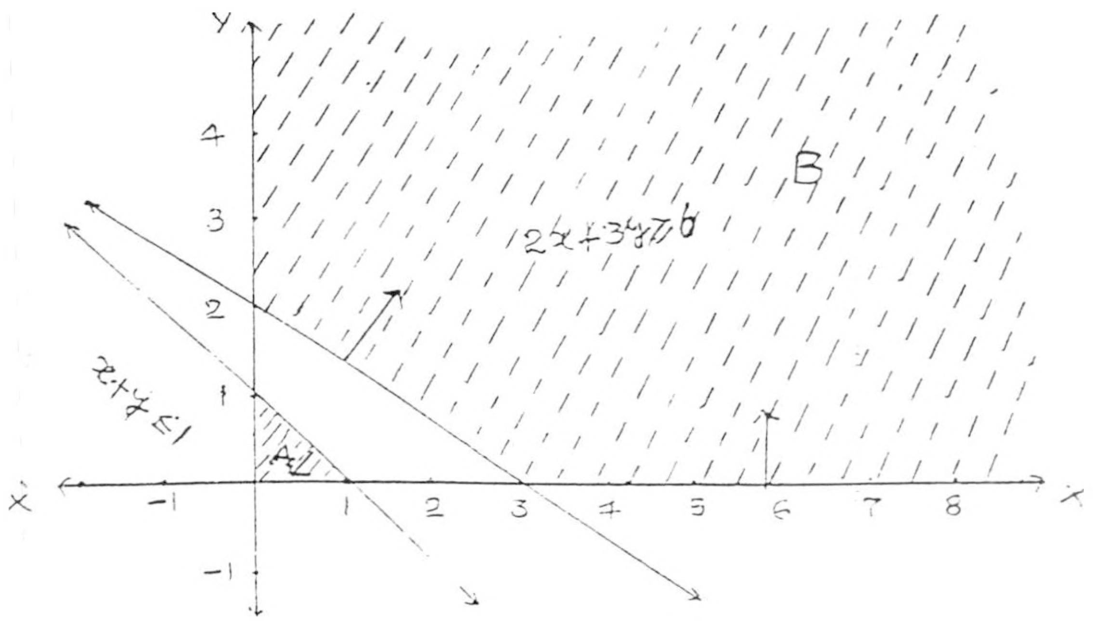
Maximise $Z = 4x + 3y$

subject to $x + y \leq 1$

$2x + 3y \geq 6$

$x \geq 0, y \geq 0$

The shaded regions A and B in the following figure indicate the graphs of the inequation $x+y \leq 1$ and the graph of the inequation $2x + 3y \geq 6$ respectively.



Obviously, the intersection of A and B is empty. Hence the LPP has no feasible solution.

The following LPP has or does not have a feasible solution depending upon the value of L.

Maximise $Z = x$

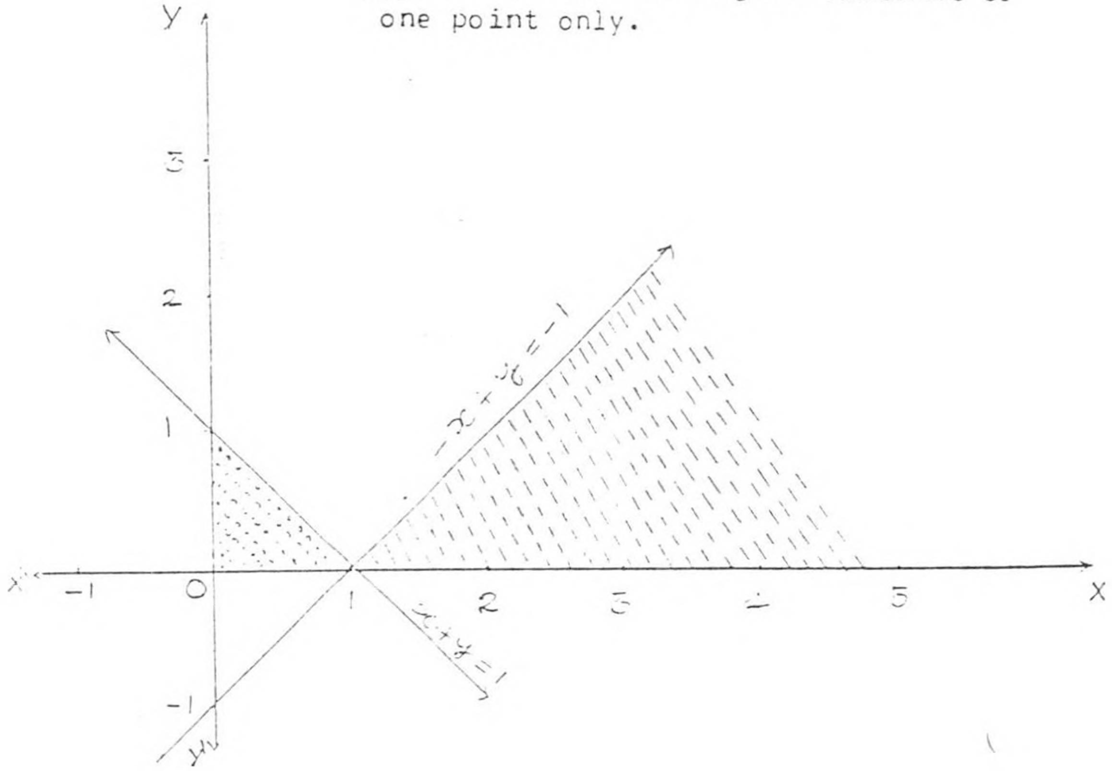
subject to $x+y \leq L$

$-x + y \leq -1$

$x \geq 0 , \quad y \geq 0$

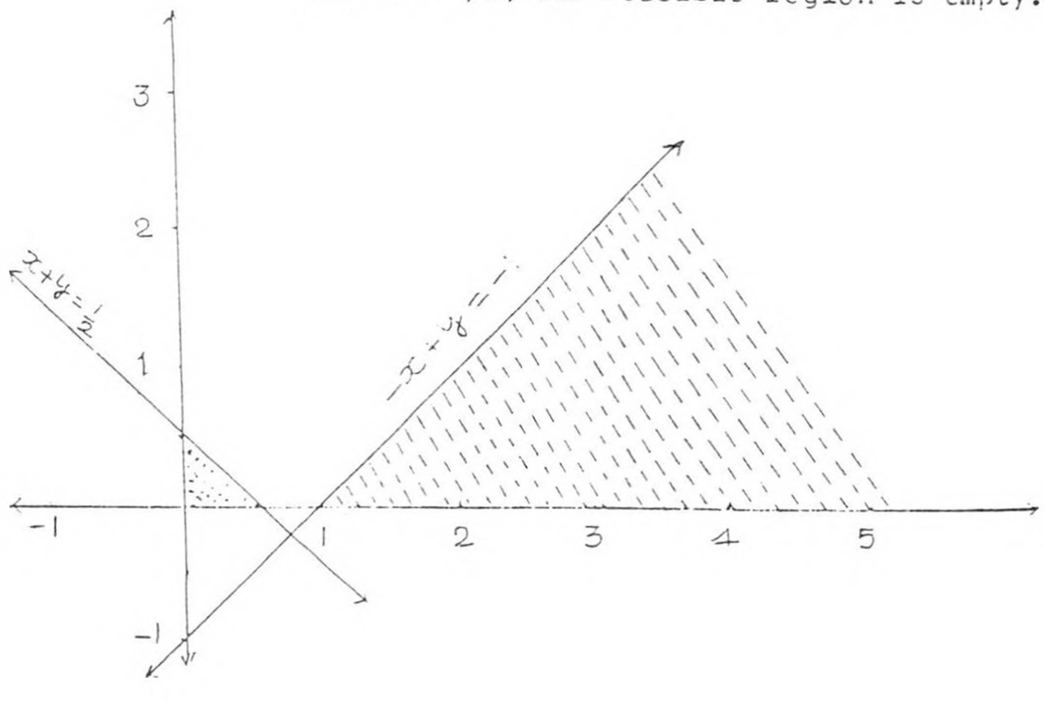
If $L = 1$, the feasible region of the problem consists of just one point $(1,0)$ (See figure shown below).

when L=1, feasible region consists of one point only.



If L = 1/2, the feasible region is empty since there are no points satisfying the non-negativity restrictions.

when L = 1/2, the feasible region is empty.



In fact, for all values of $L < 1$ the feasible region corresponding to the given constraints is empty.

The above fact can also be verified analytically. For $L < 1$, suppose there exists a point (x_1, y_1) satisfying the constraints of the problem.

That is $x_1 + y_1 < 1$, since L is strictly less than 1

$-x_1 + y_1 \leq -1$

and $x_1 \geq 0$, $y_1 \geq 0$

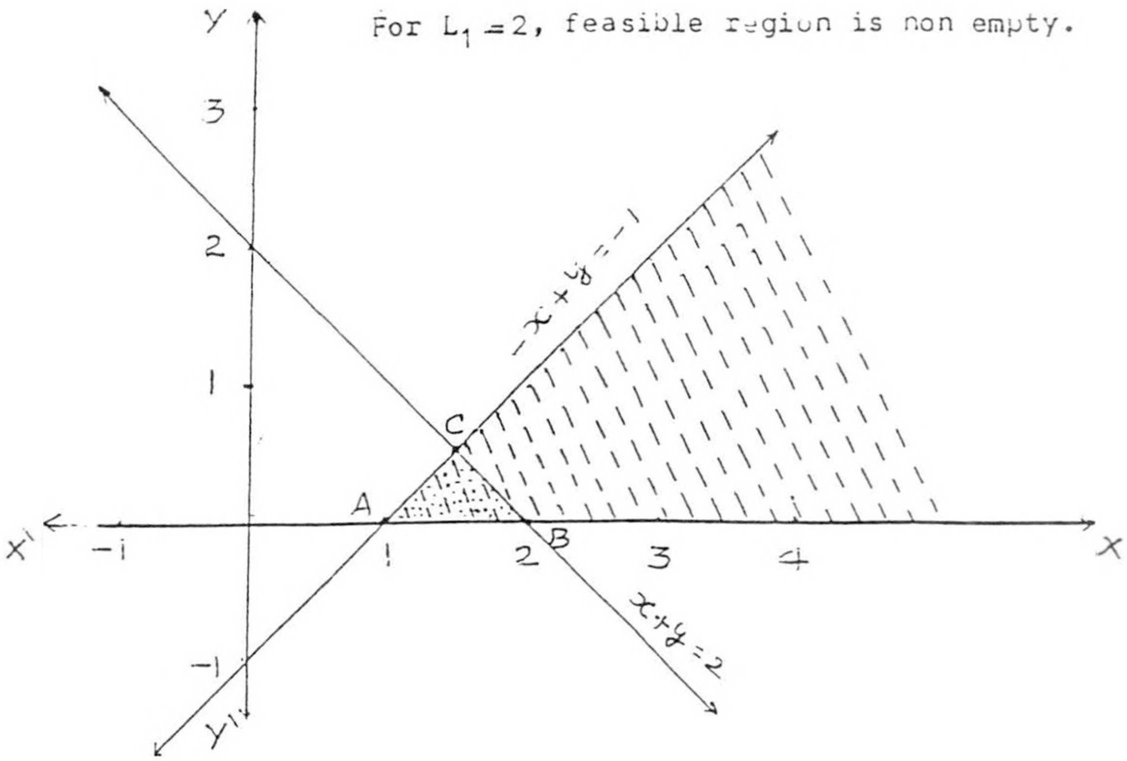
The first two inequalities imply (by adding them), that $2y_1 < 0$.

In other words, $y_1 < 0$ which contradicts the fact that $y_1 \geq 0$.

Thus we conclude that there is no point (x,y) which satisfies the given constraints whenever $L < 1$.

If $L = 2$, the feasible region is the shaded region ABC of the figure which is non empty.

For $L_1 = 2$, feasible region is non empty.



From the foregoing discussion, it is clear that the feasible region is non-empty for all values of $L \geq 1$.

If $L = 1$, it consists of just one point. If $L > 1$ it consists of infinitely many points.

We can verify this analytically also. Given constraints are

$$x + y \leq L$$
$$-x + y \leq -1$$
$$x \geq 0, \ y \geq 0$$

First two inequalities (by adding them) imply that

$2y \leq L-1$ or $L-1 \geq 2y$

This implies that

$L-1 \geq 0$, (since $y \geq 0$)

In other words, $L \geq 1$

If $L \geq 1$, choose non negative numbers x_1 and y_1 such that

$$2x_1 = L + 1$$

and $2y_1 = L - 1$

(This is possible since $L-1 \geq 0$)

These equations imply that

$$2x_1 + 2y_1 = 2L \text{ and } -2x_1 + 2y_1 = -2$$

That is, $x_1 + y_1 = L$ and $-x_1 + y_1 = -1$ obviously, such x_1 and y_1 satisfy the given constraints.

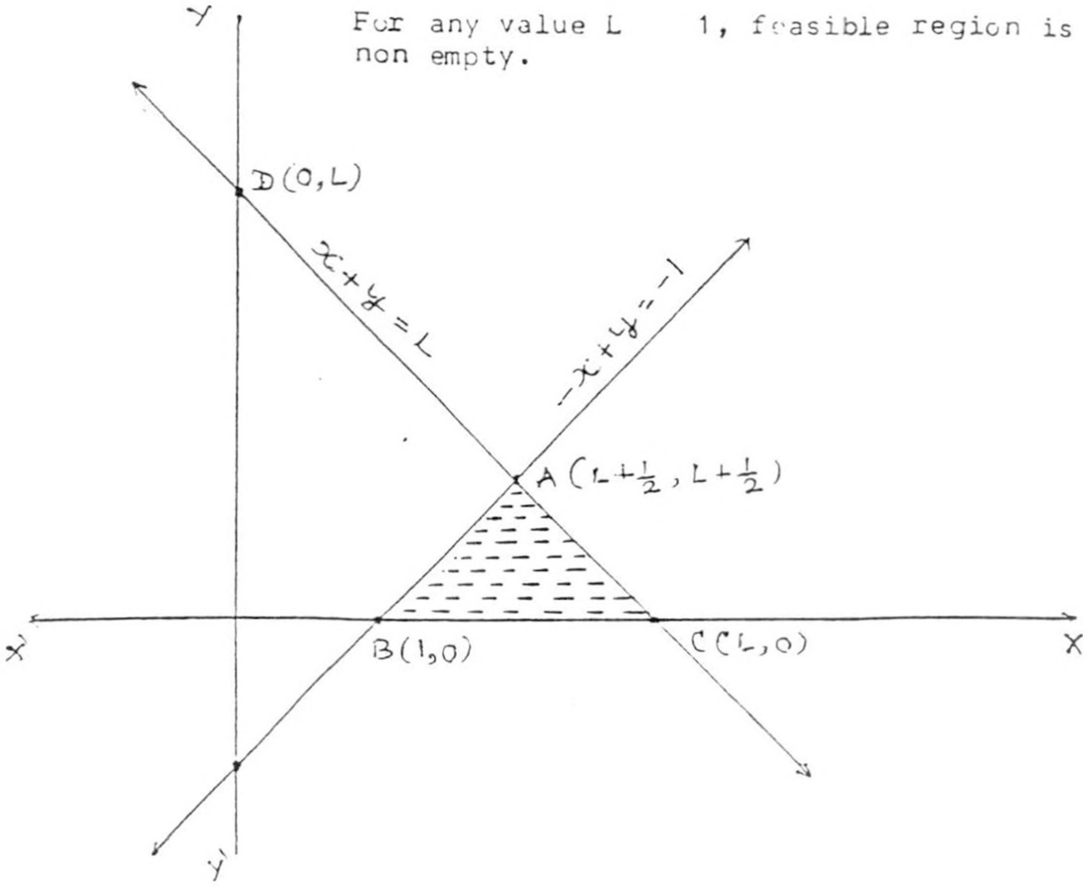
Thus we conclude that there exist numbers $x=x_1$ and $y=y_1$ satisfying the given constraints <u>if and only if</u> $L \geq 1$. That is the given L.P.P. has a feasible solution if and only if $L \geq 1$.

We now solve the given L.P.P.

If $L < 1$, then the given problem has no feasible solution. Therefore, let $L \geq 1$.

If $L = 1$, the feasible solution has just one point $(1,0)$ and so the maximum value of Z is 1. The feasible region for any value of $L > 1$ will look like the shaded region ABC of the following figure.

For any value L   1, feasible region is non empty.



The coordinates of A are obtained by solving $x+y = L$ and $-x+y = -1$.

i.e., $x = \frac{L+1}{2}$, $y = \frac{L-1}{2}$

Now,

the value of Z at A = $\frac{L+1}{2}$

The value of Z at B = 1

The value of Z at C = L

Since $L \geq 1$, $L+1 \geq 2$ and so $\frac{L+1}{2} \geq 1$

Also, since $L \geqslant 1$, $2L \geqslant L+1$ and so $L \geqslant \frac{L+1}{2}$

Thus, we have

$$1 \leq \frac{L+1}{2} \leqslant L$$

Therefore, max $\left\{ 1, \frac{L+1}{2}, L \right\} = L$

That is maximum value of Z is L, and Z attains the maximum at C.

If the problem is to minimize $Z=x$ with the same constraints, minimum value of Z is 1 and it is attained at B.

Exercises :

1. Choose the most appropriate answer.

i) The set of feasible solutions of a linear programming problem is

a) convex b) not a convex set c) convex or concave
b) bounded and convex

ii) The minimum number of inequations needed to find a feasible region in a linear programming problem is

a) 1, b) 2, c) 3, d) 4

iii) The maximum value of the objective function of a linear programming problem always occurs

a) exactly at one vertex of the feasibility region.
b) everywhere in the feasibility region.
c) at all the vertices of the feasibility region.
d) at some vertices of the feasibility region.

iv) The feasible region of a linear programming problem intersects

a) first quadrant b) second quadrant
c) third quadrant d) fourth quadrant

v) A factory has an auto lathe which when used to produce screws of larger size produces 400 items per week and when used to produce screws of smaller size produces 300 items per week. Supply of rods used in making these screws limits the total production of both types/week to 380 items in all. The factory makes a profit of 25 paise per large screw and 10 paise per small screw. How much of each type should be produced to get a maximum profit ? (Ans. 80,300)

vi) Using graphical method
maximise $Z = 3x + 4y$

subject to $4x + 2y \leqslant 80$

$2x + 5y \leqslant 180$

$x \geqslant 0, \quad y \geqslant 0,$

(Ans: $x = 2.5$; $y = 35$; maximum value $= 147.5$)

vii) Using graphical method

minimise $Z = 4x + 2y$

subject to $x + 2y \geq 2$

$3x + y \geq 3$

$4x + 3y \geq 6$

$x \geq 0, \quad y \geq 0$

(Ans: $x = .6$, $y = 1.2$, minimum value = 4.8)

viii) Consider the following problem :

Maximize $Z = 6x_1 - 2x_2$

subject to $x_1 - x_2 \leq 1$; $3x_1 - x_2 \leq 6$; $x_1, x_2 \geq 0$.

Show graphically that at the optimal solution the variables x_1, x_2 can be increased indefinitely, while the value of the objective function remains constant.

ix) Consider the following LPP :

Maximize $Z = 4x + 4y$

subject to $2x + 7y \leq 21$; $7x + 2y \leq 49$; $x, y \geq 0$.

Find the optimal solution (x, y) graphically. What are the ranges of variation of the coefficients of the objective function that will keep (x, y) optimal ?

x) Consider the following problem.

Maximize $Z = 3x + 2y$ subject to $2x + y \leq 2$, $3x + 4y \geq 12$, $x, y \geq 0$.

Show graphically that the problem has no feasible extreme points. What can one conclude concerning the solution of the problem ?

xi) Prove or disprove :

a) For some LPP, the set of feasible solutions is a disjoint union of convex sets.

b) The set of feasible solutions of every LPP is non empty.

c) Every L.P.P. is a mathematical model.

Applications of L.P.

LP is a powerful and widely applied technique to solve problems related to decision making. It was employed formally in three major categories - military applications, inter industry economics and zero sum two-person games. But, now the emphasis has been shifted to the industrial area. The following are a few of the applications of L.P.

1. Agricultural applications :
   Farm economics and Farm management - the first is related to the economy of a region whereas the second is related to individual farm.
2. Industrial applications :
   a) Chemical Industry - Production and inventory control - chemical equilibrium problem.
   b) Coal industry
   c) Airline operations
   d) Communication industry - optical design and utilisation of communication network
   e) Iron and steel industry
   f) Paper industry - for optimum newsprint production
   g) Petroleum industry
   h) Rail road industry
3. Economic analysis - Capital budgeting
4. Military - Weapon Selection and Target analysis
5. Personal assignment
6. Production scheduling - inventory control and planning cost controlled production
7. Structural designs
8. Traffic analysis
9. Transportation problem and network theory
10. Travelling salesman problem
11. Logical design of electrical network
12. Efficiency in the operations of a system of Dams

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P R O B A B I L I T Y

1. Basic Terminology
2. Some theorems on Probability
3. Random Variables and Probability Distribution

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PROBABILITY

In our day-to-day life we perform certain activities to verify certain known facts or to observe certain phenomena. Such activities usually we call as experiments. In certain experiments, we can predict results exactly before conducting the experiment and in other it will not be possible. The experiments where the results can be predicted exactly are known as deterministic experiments and the experiments where the prediction is not exact are known as non-deterministic or random or probabilistic experiments. For example, a train is running at a uniform speed of sixty k.m. per hour, then we can predict with hundred percent surety that it will cover one hundred twenty kilometers after two hours, assuming that it never stopped during these hours. Similarly, for a perfect gas, PV = constant (P is pressure, V is volume).

In case of non-deterministic experiments, we cannot make predictions with complete reliability. The results are based on some 'chance element'. For example, if we toss a coin, will it show 'head up' or 'tail up'? Although we cannot predict anything with complete surety, yet if we throw the coin a large number of times, it is very likely that the head will turn up fifty percent of the times and also it is very unlikely that the head turns up in every case.

Consider another example of a trained parachuter who is ready to jump. When he jumps then either his parachute will open or it will not. But experience says that most of the time it opens, though there are occasions on which it does not i.e. the uncertainty associated with the head or tail coming up when we toss a coin.

How will you proceed in answering the following questions ?

1. How should a businessman order for replenishment (filling once again) of his stocks (inventory) so that he has not carried very large stocks, yet the risk of refusing customers is minimized ? (Inventory problem).
2. At what intervals should a car owner replace the car so that the total maintenance expenses are minimized ? (Replacement problem).
3. How many trainees should a large business organisation recruit and train them in certain intervals so that at any time it does not have a large number of trained persons whom it cannot employ and yet the risk of its being without sufficient persons when needed is minimized ?
4. How should the bus service in a city be scheduled so that the queues do not become too long and yet the gains by the bus company are maximized ? (Queing problem).
5. How many booking counters should be provided at a station to serve in the best way the interests of both the railways and the travelling public ? (Queing problem).

6. What should be the strength of a dam (or a bridge) so that its cost is reasonable and yet the risk of its being swept away by the floods is minimized ?
7. How many telephone exchanges should be established in a given city so as to give the best service at a given cost ?
8. Which variety is the best out of given varieties of wheat, on the basis of yields from experimental fields ?
9. What should be the minimum premia charged by an insurance company so that the chance of its running into loss is minimized ?
10. How to decide whether a given batch of items is defective when only a sample of the batch can be examined ?

Answers for all such questions are based upon certain facts and then try to measure the uncertainty associated with some events which may or may not materialise. The theory of probability deals with the problem of measuring the uncertainty associated with various events rather precisely, making it these by possible today, to a certain extent of course, to control phenomena depending upon chance.

The 'measure of uncertainty' is known as probability.

History of Probability Theory

Probability had its birth in the seventeenth century and over the last three hundred years, it has progressed rapidly from its classical heritage of simple mathematical and combinatorial methods to its present rigorous development based on modern functional analysis. The probability had its origin in the usual interest in gambling that pervaded France in the seventeenth century. Eminent mathematicians were led to the quantitative study of games of chance. The Chevalier de Mere, a French nobleman and a notorious gambler, posed a series of problems to B Pascal (1623-1662) like the following :

> Two persons play a game of chance. The person who first gains a certain number of points wins the stake. They stop playing before the game is completed. How is the stake to be divided on the basis of the number of points each has got ?

Though Galileo (1564-1642) had earlier solved a similar problems, this was the beginning of a systematic study of chance and regularity in nature. Pascal's interest was shared by Fermat (1601-1665), and in their correspondence the two mathematicians laid the foundation of the theory of probability. Their results aroused the interest on the Dutch physicist Huyghens (1629-1695) who started working on some difficult problems in games of chance, and published in 1654 the first book on the theory of probability. In this book, he introduced the concept of mathematical expectation which is basic to the modern theory of probability. Following this, Jacob Bernoulli (1654-1705) wrote his famous 'Art Conjectandi' the result of his work of over twenty years. Bernoulli approached this subject from a very general point of view and clearly foresaw the wide applications of the theory. Important contributions were made by Abraham de Moivre (1667-1754) whose book 'The Doctrine of Chance' was published in 1718. Other main contributors were T.Bayse (Inverse Probability), P.S. Laplace (1749-1827) who after extensive research published 'Theoric Analytique des probabilities' in 1812. In addition to these Levy, Mises and R.A.Fisher were the main contributors. It was, however, in the work of Russian mathematicians Tschebyshev (1821-1874), A Markov (1856-1922), Liapounov (Central Limit theorem), A Kintchine (Law of Large Numbers) and A. Kolmogorov that the theory made great strides. Kolmogoroff was the person who axiomised the calculus of probability.

The probability theory itself has developed in many directions, but at present the dominant area is the stochastic processes, which has wide applications in physics, chemistry, biology, engineering, management and the social sciences.

Calculus of Probability

In our day-to-day vocabulary we use words such as 'probably', 'likely', 'fairly good chances', etc. to express the uncertainty as indicated in the following example. Suppose a father of a XII class student wants to know his son's progress in the studies and asks the concerned teacher about his son. Teacher may express to the father about the student's progress in any one of the following sentences.

It is certain that he will get a first class.
He is sure to get a first class.
I believe he will get a first class.
It is quite likely that he will get a first class.
Perhaps he may get a first class.
He may or he may not get a first class.
I believe he will not get a first class.
I am sure he will not get a first class.
I am certain he will not get a first class.

Instead of expressing uncertainty associated with any event with such phrases, it is better and exact if we express uncertainty mathematically. The measure of uncertainty or probability can be measured in three ways and these are known as the three definitions of probability. These methods are

Mathematical or Classical or Priori Probability
Statistical or Empirical Probability and
Axiomatic Probability

Before discussing those methods, we define some of the terms which are useful in the definition of probability.

Experiment : An act of doing something to verify some fact or to obtain some result. (Ex Throwing a die to observe which number will come up (Die is a six-faced cube).

Trial : Conducting experiment once is known as the trial of that experiment. Ex. Throwing a die once.

Outcomes : The results of an experiment are known as outcomes. Ex. In throwing a die, getting '1' or '2' or '6' are the outcomes.

Events: Any single outcome or set of outcomes in an experiment is known as an event.

Ex: 1. Getting '1' in throwing of a die is an event.

Also getting an even number in throwing a die is also an event.

Ex: 2. Drawing two cards from a well shuffled pack of cards is a trial and getting of a king and a queen is an event.

Exhaustive Events : The total number of possible outcomes in any trial are known as exhaustive events.

Ex: 1. In tossing a coin there are two exhaustive events.

2. In throwing a die, there are six exhaustive cases viz (1,2,3,4,5,6).

Favourable Events (Cases): The number of outcomes which entail the happening of an event are known as the favourable cases (events) of that event.

Ex: In throwing two dice, the number of cases favourable for getting a sum of 5 are (1,4), (2,3), (3,2) and (4,1).

Mutually Exclusive Events : Events are said to be mutually exclusive or incompatible if the happening of any one of them precludes or excludes the happening of all others.

Ex: In tossing a coin, the events head and tail are mutually exclusive (because both cannot occur simultaneously).

Mathematical or Classical or 'a priori' probability

If a trial results in 'n' exhaustive, mutually exclusive and equally likely cases and 'm' of them are favourable to the happening of an event E, then the probability 'p' of happening of E is given by

p = Favourable number of outcomes / Total No of outcomes = m/n

We write p = P(E).

Ex:1. Probability of getting head in tossing of a coin once is ½ because the number of exhaustive cases are 2 and these are mutually exclusive and equally likely (assuming the coin is made evenly) and of these only 1 case is favourable to our event of getting head.

Ex: 2. The probability of getting a number divisible by 3 in throwing of a fair (evenly made) die is 2/6 because the favourable cases are 3 (viz. 3 and 6) and exhaustive cases are 6.

The probability 'q' that E will not happen is given by

$$q = \frac{n-m}{n} = 1 - \frac{m}{n} = 1 - p$$

Always $0 \leq p \leq 1$.

If $p = P(E) = 1$, E is called a certain event and if $P(E) = 0$, E is called an impossible event.

In this method, the mathematical ratio of two integers is giving the probability and therefore, this definition is known as mathematical definition. Here we are using the concept of probability in the form of 'equality likely cases' and therefore, this definition is a classical definition. Before using this definition, we should know about the nature of outcomes (viz. Mutually exclusive, exhaustive and equally likely) and therefore, it is also known as 'a priori' probability definition.

The definition of mathematical or classical probability definition breaks down in the following cases: 1. If the various outcomes of the trial are not equally likely. 2. If the exhaustive number of cases in a trial is infinite.

Ex:1. When we talk about the probability of a pass of a candidate, it is not ½ as the two customers 'pass' and 'fail' are not equally likely.
Ex: 2. When we talk about the probability of a selected real number is to be divided by 10, the number of exhaustive cases are infinite.

In such above mentioned circumstances it is not possible to use mathematical probability definition. Therefore, probability is defined in the other way as below :

Statistical or Empirical Probability :

If a trial is repeated a number of times under essentially homogeneous and identical conditions, then the limiting value of the ratio of the number of times an event happens to the number of trials, as the number of trials becomes indefinitely large, is called the probability of happening that event.

Mathematically, we write

$$p = P(E) = \lim_{n \to \infty} \left(\frac{m}{n}\right)$$

Here n is the number of trials and m is the number of times of the occurrence of event E. The above limit should be finite.

Ex: When you throw a die 10000 times and if you get 1600 times the number '1', then the probability of getting '1' is 1600/10000. This ratio is nothing but the relative frequency of '1'.

But this definition is also not applicable always because it is very difficult to maintain the identical conditions throughout the experiment. Therefore, the probability is defined in another way by using certain axioms. This definition is known as 'Axiomatic Probability' definition.

Here we define some of the terms which are useful in the 'Axiomatic Probability' definition.

Sample Space: The set of all possible outcomes of an experiment is known as the sample space of that experiment. Usually we denote it by S. Ex: In tossing a coin, S = { H, T }.

Sample Point : Any element of a sample space is known as a sample point.

Ex: In tossing a coin experiment, H or T is a sample point.

Event: Any subset of a sample space is an event.
Ex: In throwing a die, (1,3,5), (2,4,6) or (5,6) are the events where S = {1,2,3,4,5,6}.

If A and B are any two events then $\overline{A}$, $\overline{B}$, AUB, $A \cap B$ are also events because they are also subsets of S.

The event S (entire sample space) is known as certain event and the event Φ (empty set) is known as impossible event.

Mutually Exclusive Events : Events are said to be mutually exclusive if the corresponding sets are disjoint.
Ex: In throwing of a die experiment, if A = (1,3,5) and B = (2,4,6) then A and B are mutually exclusive because we cannot get both odd number and even number simultaneously. That is, if $A \cap B = \Phi$, then A and B are mutually exclusive events.

Axiomatic Probability :

Let S be a sample space and ξ be the class of events. Also let P be a real valued function defined on ξ. Then P is called a probability function and P(A) is called the probability of the event A if the following axioms hold :

i) For every event A, $0 \leq P(A) \leq 1$.
ii) P(S) = 1.
iii) If A and B are mutually exclusive events, then P(AUB) = P(A) + P(B).
iv) If $A_1, A_2, \ldots$ is a sequence of mutually exclusive events, then
$P(A_1 U A_2 \ldots) = P(A_1) + P(A_2) + \ldots\ldots$

In the above definition axiom (iv) may seem to be not necessary. But it is necessary to stress that axiom (iii) should be extended to more than two events.

Theorem 1 : If Φ is the empty set, then $P(\Phi) = 0$.

Proof : We know that $S = S \cup \Phi$ and $P(S) = P(S \cup \Phi) = P(S) + P(\Phi)$.
(because S and Φ are disjoint and according to axiom (iii)). But $P(S) = 1$ and therefore,
$1 = 1 + P(\Phi)$.

$\therefore P(\Phi) = 0$.

Theorem 2 : If $\bar{A}$ is the complement of an event A, then

$P(\bar{A}) = 1 - P(A)$.

Proof : $A \cup \bar{A} = S$.

$P(A \cup \bar{A}) = P(A) + P(\bar{A}) = P(S)$ (A and $\bar{A}$ are disjoint).

But $P(S) = 1$, therefore,
$P(A) + P(\bar{A}) = 1$
Or $P(\bar{A}) = 1 - p(A)$.

Theorem 3 : If $A \subseteq B$, then $P(A) \leq P(B)$.
Proof : We know that if $A \subseteq B$, then
$B = A \cup (B - A)$ (here we may use the notation B/A)
So, $P(B) = P(A) + P(B-A)$
But from axiom i, $P(B-A) \geq 0$
$\therefore P(B) \geq P(A)$.

Theorem 4 : If A and B are any two events, then
$P(A-B) = P(A) - P(A \cap B)$

Proof: We can write, $A = (A \cap B) \cup (A - B)$

But $(A \cap B)$ and $(A - B)$ are disjoint and according to axiom (iii).
$P(A) = P(A \cap B) + P(A - B)$.
Or $P(A - B) = P(A) - P(A \cap B)$.

Theorem 5 : (Addition Theorem)

If A and B are any two events, then
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Proof: We can write, AUB = BU(A-B). But B and (A-B) are disjoint and therefore, by axiom (iii),
P(AUB) =P(B) + P(A-B).

Also, from theorem 4, P(A-B) = P(A) – P(A $\cap$ B)

Hence, P(AUB) = P(B) + P(A-B)
= P(B) + P(A) – P(A $\cap$ B)

This theorem is known as addition theorem and it can be extended to any number of events as follows :

Theorem 6 : (Addition Theorem in case of n events)

If $A_1, A_2, \ldots, A_n$ are any n events, then

$$P(A_1 \cup A_2 \cup \ldots A_n) = \sum_{i=1}^{n} P(A_i \cap A_j) - \sum_{i,j,k=1;\ i<j<k}^{n} P(A_i \cap A_j \cap A_k) + \ldots + (-1)^{n-1} P(A_1 \cap A_2 \cap \ldots \cap A_n)$$

Proof: This theorem can be proved by the method of induction. For the events A_1 and A_2 we have from theorem 5,

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$$

$$= \sum_{i=1}^{2} P(A_i) + (-1)^{2-1} P(A_1 \cap A_2)$$

Hence the theorem is true for n = 2.
Now, suppose the theorem is true for n = r, say

Then,

$$P(A_1 \cup A_2 \cup \ldots \cup A_r) = \sum_{i=1}^{r} P(A_i) - \sum_{i,j=1,\ i<j}^{r} P(A_i \cap A_j) + \ldots + (-1)^{r-1} P(A_1 \cap A_2 \cap \ldots \cap A_r)$$

Now,

$$P(A_1 \cup A_2, \ldots, \cup A_r \cup A_{r+1}) = P(A_1 \cup A_2 \cup \ldots \cup A_r) \cup A_{r+1})$$

$$= P(A_1 \cup A_2 \cup \ldots \cup A_r) + P(A_{r+1}) - P((A_1 \cap A_{r+1}) \cup (A_2 \cap A_{r+1}) \ldots \cup (A_r \cap A_{r+1})$$

$$= \sum_{i=1}^{r} P(A_i) - \sum_{i,j=1,\ i<j}^{r} P(A_i \cap A_j) + \ldots (-1)^r P(A_1 \cap A_2 \cap \ldots A_{r+1})$$

$$+ P(A_{r+1}) - \sum_{i=1}^{r+1} P(A_i \cap A_{r+1}) + \sum_{i,j=1,\ i<j}^{r+1} P(A_i \cap A_j \cap A_{r+1}) + \ldots + (-1)^r P(A_1 \cap A_2 \cap \ldots \cap A_{r+1})$$

$$= \sum_{i=1}^{r+1} P(A_i) - \sum_{i,j=1,\ i<j}^{r+1} A(A_i \cap A_j) + \ldots + (-1)^r P(A_1 \cap A_2 \cap \ldots A_{r+1})$$

Hence, if the theorem is true for n=r, it is also true for n=r+1. But we have proved that the theorem is true for n = 2. Hence by the method of induction, the theorem is true for all positive integer values of n.

Corollary 1 : If A and B are two mutually exclusive events, then,
$P(A \cup B) = P(A) + P(B)$.

Corollary 2 : If $A_1, A_2, \ldots A_n$ are n mutually exclusive events,

Then $P(A_1 \cup A_2 \ldots \cup A_n) = P(A_1) + P(A_2) + \ldots + P(A_n)$

Conditional Probability :

So far, we have assumed that no information was available about the experiment other than the sample space while calculating the probabilities of events. Sometimes, however, it is known that an event A has happened. How do we use this information in making a statement concerning the outcome of another event B ?

Consider the following examples.

Ex.1: Draw a card from a well-shuffled pack of cards. Define the event A as getting a black card and the event B as getting a spade card. Here $P(A) = ½$ and $P(B) = ¼$. Suppose the drawn card is a black card then what is the probability that card is a spade card? That is, if the event A has happened then what is the probability of B given that A has already happened? This probability symbolically we write as P(B/A). In the given example,

$$P(B/A) = \frac{1}{2} = \frac{P(A \cap B)}{P(A)} = \frac{(1/4)}{(1/2)}$$

Because probability of simultaneous occurrence of A and B is ¼ and probability of A is ½.

Ex.2: Let us toss two fair coins. Then the sample space of the experiment is S = { HH, HT, TH, TT}. Let event A = { both coins show same face } and B = { at least one coin shows H }. Then P(A) = 2/4. If B is known to have happened, this information assures that TT cannot happen, and P {A, conditional on the information that B has happened } =

$$P(A/B) - 1/3 - \frac{1/4}{3/4}$$

$$= \frac{P(A \cap B)}{P(B)}$$

In the above two examples, we were interested to find the probability of one event given the condition that the other event has already happened. Such events based on some conditions are known as conditional events. In the above examples B/A and A/B are the conditional events. The probability of a conditional event is known as conditional probability of that event. We write the conditional probabilities as P(A/B), P(E/F), etc.

Definition of conditional probability : The conditional probability of an event A, given B, is denoted by P(A/B) and is defined by

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

Where A, B and $A \cap B$ are events in a sample space S, and $P(B) \neq 0$.

From the definition of conditional probability we know that

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

Therefore, we can write from the above

$$P(A \cap B) = P(B)\, P(A/B)$$

Also, we know that $P(A \cap B) = P(B \cap A)$ and

$$P(B \cap A) = P(A)\, P(B/A)$$

Hence we can write

$$P(A \cap B) = P(A)\, P(B/A) \text{ or } P(B)\, P(A/B).$$

The above result is known as multiplication law of probabilities in case of two events.

Multiplication Theorem of Probabilities : If A and B are any two events of a sample space S, then

$$P(A \cap B) = P(A)\, P(B/A) \text{ or } P(B)\, P(A/B).$$

The above theorem can be extended to any n events as follows :

If $A_1, A_2, \ldots, A_n$ are any n events, then

$$P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_1)\, P(A_2/A_1)\, P(A_3/A_1 \cap A_2) \ldots P(A_n/A_1 \cap A_2 \cap \ldots \cap A_{n-1}).$$

This theorem can be proved by method of induction or generalization.

Baye's Theorem : If $E_1, E_2, \ldots, E_n$ are mutually exclusive events with $P(E_i) \neq 0$, $(i = 1,2,\ldots n)$ then for any arbitrary event A which is a subset of $\bigcup_{i=1}^{n} E_i$ such that $P(A) > 0$, we have

$$P(E_i / A) = \frac{P(E_i) P(A/E_i)}{\sum_{i=1}^{n} P(E_i) P(A/E_i)} \quad \text{for all i.}$$

Proof: Since $A \subset \bigcup_{i=1}^{n} E_i$ we have

$$A = A \cap (\bigcup_{i=1}^{n} E_i) = \bigcup_{i=1}^{n} (A \cap E_i)$$ (by distributive law).

Since $(A \cap E_i) \subset E_i$ (for $i = 1,2,\ldots, n$) are mutually exclusive events, we have by additional theorem of probability

$$P(A) = P[\bigcup_{i=1}^{n} (A \cap E_i)] = \sum_{i=1}^{n} P(A \cap E_i) = \sum_{i=1}^{n} P(E_i) P(A/E_i)$$

(By multiplication theorem in case of two events.)

Also, we have

$P(A \cap E_i) = P(A) \quad P(E_i/A)$ and

$$P(E_i / A) = \frac{P(A \cap E_i)}{P(A)} = \frac{P(E_i) P(A/E_i)}{P(A)}$$

Hence, $$P(E_i/A) = \frac{P(E_i) P(A/E_i)}{\sum_{i=1}^{n} P(E_i) P(A/E_i)}$$

This theorem is very useful in calculating the conditional probabilities in certain situations.

If $P(A \cap B) = P(A) P(B)$, then we see that $P(B/A) = P(B)$ and hence we say that the probability of B is not depending upon the happening of A. That is the conditional probability of B is same as the unconditional probability of B. Such events are called independent events.

Two events A and B are independent if and only if

$P(A \cap B) = P(A) \; P(B)$

Ex: Let two fair coins be tossed and let

$A = \{$ head on first coin $\}$, $B = \{$ head on the second coin $\}$.

Then $P(A) = P\{HH, HT\} = ½$

$P(B) = P\{HH, TH\} = ½$ and

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{1/4}{1/2} = 1/2 = P(A)$$

Thus,

$$P(A \cap B) = P(A)\,P(B).$$

and we know that the probability of getting head on the first coin does not depend upon the probability of getting head on the second coin. Hence A and B are independent. Also we see that the condition $P(A \cap B) = P(A)\ P(B)$ is both necessary and sufficient for those events A and B to be independent.

If there are three or more than three events, we will have the situation where every pair of these events are independent or the situation where the events in every set of events are independent. In the first case, we call the events as pairwise independent and in the second case we call as complete or mutual independent events.

Geometric Probability :

Sample space can be countably finite or countably infinite or uncountably finite or uncountably infinite depending upon the situation. If the sample space is countably finite, then it is easy to calculate the probability of any event by using either mathematical probability or axiomatic probability definition. Even if the sample space is countably infinite say $S = (e_1, e_2, \ldots)$ we obtain a probability space assigning to each $e_i \in S$ is a real number p_i called its probability, such that

$$p_i \geq 0 \text{ and } p_1 + p_2 + \ldots\ldots = \sum_{i=1}^{\infty} p_i = 1$$

The probability $P(A)$ of any event A is then the sum of the probabilities of its points.

Consider the sample space $S = \{1, 2, \ldots\}$ of the experiment of tossing a coin till a head appears; here n denotes the number of times the coin is tossed. A probability space is obtained by

$$P(1) = 1/2,\ \ P(2) = 1/4\ ,\ldots,\ P(n) = \frac{1}{2^n}\ ,\ldots$$

But the calculation of probability of events regarding an uncountably finite or infinite sample space is not so easy.

Consider a situation of selecting a point at random on a line segment of length 'l'. Here the sample space is uncountably finite and the procedure to find the probability of any event in case of countable sample space is not applicable.

Consider another example. Suppose that two friends have agreed to meet at a certain place between 9 a.m. to 10 a.m. They also agreed that each would wait for a quarter of an hour and, if the other did not arrive, would leave. What is the probability that they meet ?

In the above example also both the sample space and the given event are uncountable and the ordinary procedures of calculation of probability are not applicable. So we need different procedure in such cases.

If the sample space is uncountably finite, we present that sample space by some geometrical measurement, m(S) such as length, area of volume, and in which a point is selected at random. The probability of an event A, i.e. the selected point belongs to A, is then the ratio of m(A) to m(S) is

$$P(A) = \frac{\text{length of } A}{\text{length of } S} \quad \text{or} \quad P(A) = \frac{\text{area of } A}{\text{area of } S} \quad \text{Or} \quad P(A) = \frac{\text{volume of } A}{\text{volume of } S}$$

Such probability is known as 'geometrical probability'.

Solved Problems :

1. A bag contains 5 red, 4 white and 3 blue balls. What is the probability that two balls drawn are red and blue ?

Sol : Total number of balls = 5 + 4 + 3 = 12

The number of ways of drawing two balls out of 12 balls = $12^{c}2 = \frac{12\times 11}{2} = 66$ ways

The number of ways of drawing 1 red ball out of 5 red balls = 5 ways.

The number of ways of drawing 1 blue ball out of 3 blue balls = 3 ways.

The number of ways of drawing 1 red ball out of 5 red balls and 1 blue ball out of 3 blue balls = 5 x 3 = 15 ways.

The required probability = 15/66 = 5/22, by using Mathematical probability definition.

2. If the letters of the word 'STATISTICS' are arranged at random to form words, what is the probability that three S's come consecutively ?

Sol: Total no. of letters in the word 'STATISTICS' = 10. Total no. of arrangements of these 10 letters in which 3 are of one kind (viz. S) , 3 are of second kind (viz. T), 2 are of third kind (viz. I), 1 of fourth kind (viz. A) and 1 of fifth kind (viz. C).

$$= \frac{10!}{3!\ 3!\ 2!\ 1!\ 1!}$$

Following are the 8 possible combinations of 3 S's coming consecutively.

i) in the first three places

ii) in the second, third and fourth places

iii) in the eighth, ninth and tenth places

Since in each of the above cases, the total number of arrangements of the remaining 7 letters, viz. TTTIIAC of which 3 are of one kind, 2 of second kind, 1 of third kind and 1 of fourth kind

$$= \frac{7!}{3!\ 2!\ 1!\ 1!}$$

and the required number of favourable cases = $\frac{8 \times 7!}{3!\ 2!\ 1!\ 1!}$

Hence the required probability

$$= \frac{\text{Favourable Cases}}{\text{Total No of cases}} = \frac{8 \times 7!}{3!\ 2!\ 1!\ 1!} \Big/ \frac{10!}{3!\ 3!\ 2!\ 1!\ 1!}$$

$$= \frac{8 \times 7! \times 3!}{10!} = \frac{1}{15}$$

3. What is the probability that a leap year selected at random will contain 53 Sundays ?

Sol : In a leap year, there are 366 days of 52 complete weeks and 2 days more. In order that a leap year selected at random should contain 53 Sundays, one of these extra 2 days must be Sunday. But there are 7 different combinations with these two extra 2 days viz. Sunday and Monday, Monday and Tuesday, etc. Out of these 7 possible ways, only in 2 ways we are having an extra Sunday.

∴ Required probability =2/7.

4. Two dice are thrown simultaneously. What is the probability of obtaining a total score of seven?

Sol: Six numbers (1,2,3,4,5,6) are on the six faces of each die. Therefore, there are six possible ways of outcomes on the first die and to each of these ways, there corresponds 6 possible number of outcomes on the second die.

Hence the total number of ways, n = 6 x 6 = 36. Now we will find out of these, how many are favourable to the total score of 7. This may happen only in the following ways (1,6), (6,1), (2,5), (5,2), (3,4) and (4,3) that is, in six ways where first number of each ordered pair denotes the number on the first die and second number denotes the number on the second die.

m = 6.

Hence required probability = $\dfrac{Favourable\ No\ of\ Cases}{Total\ No\ of\ cases}$

$$= \frac{m}{n} = \frac{6}{36} = \frac{1}{6}$$

5. Two digits are selected at random from the digits 1 through 9. If the sum is even find the probability, p that both numbers are odd.

Sol: If both numbers are even or if both numbers are odd, then the sum is even. In this problem, there are 4 even numbers (2,4,6,8) and hence there are $4^{C}2$ ways to choose two even numbers. There are 5 odd numbers (1,3,5,7,9) and hence there are $5^{C}2$ ways to choose two odd numbers. Thus there are $4^{C}2 + 5^{C}2 = 16$ ways to choose two numbers such that their sum is even. Since 10 of these ways occur when both numbers are odd, the required probability,

$$p = \frac{10}{16} = \frac{5}{8}$$

6. Six boys and six girls sit in a row randomly. Find the probability that a) the six girls sit together, b) the boys and girls sit alternately.

Sol: a) Six girls and six boys can sit at random in a row in 12 ways. Consider six girls as one object and the six boys as six different objects. Now these seven objects can be arranged in 7! different ways. But the six girls in the first object can be arranged in 6! ways. Thus the favourable number of cases to the event of sitting all girls together is 7! 6! ways.

Therefore, the required probability = $\dfrac{Favourable\ No\ of\ Cases}{Total\ No\ of\ Cases} = \dfrac{7!\quad 6!}{12\,!} = \dfrac{1}{132}$

b) Since the boys and girls can sit alternately in 6! 6! ways if we begin with a boy and similarly they can sit alternately in 6! 6! ways if we begin with a girl. Thus the total number of ways sitting the boys and girls alternately = 2 6! 6!.

$\therefore$ The required probability = $\frac{\text{Favourable No of Cases}}{\text{Total No of Cases}} = \frac{2\ 6!\ 6!}{12!} = \frac{1}{462}$

7. Out of (2n+1) tickets consecutively numbered, three are drawn at random. Find the chance that the numbers on them are in A.P.

Sol: Suppose that the smallest number among the three drawn is 1. Then the groups of three numbers in A.P. are (1,2,3), (1,3,5), (1,4,7)....., (1, n+1, 2n+1) and they are n in number.

Similarly, if the smallest number is 2, then the possible groups are (2,3,4), (2,4,6),...(2, n+1, 2n) and their number is n-1. If the lowest number is 3, the groups are (3,4,5), (3,5,7),....(3, n+2, 2n+1) and their number is n-1.

Similarly, it can be seen that if the lowest numbers selected are 4,5,6,..., 2n-2, 2n-1, the number of selections respectively are (n-2), (n-2), (n-3), (n-3),..., 2,2,1,1. Thus the favourable ways for the selected three numbers are in A.P.

$= 2\,(1 + 2 + 3 + \ldots + n-1) + n$

$= \frac{2\,(n-1)\,n}{2} + n = n^2$

Also the total number of ways of selecting three numbers out of (2n+1) numbers

$= \binom{2n+1}{3} = \frac{(2n+1)\,(2n)\,(2n-1)}{1\,.\,2\,.\,3} = \frac{n\,(4n^2-1)}{3}$

Hence the required probability = $\frac{\text{Favourable No of cases}}{\text{Total No of cases}} = \frac{n^2}{n\,(4n^2-1)/3} = \frac{3n}{4n^2-1}$

8. If a coin is tossed (m+n) times (m >n), then show that the probability of at least m consecutive heads is $\frac{n+2}{2^{m+1}}$.

Sol: Let us denote by H the appearance of head and by T the appearance of tail and let X denote the appearance of head or tail. Now P(H) = P(T) = 1/2 and P(X) = 1.

Suppose the appearance of m consecutive heads starts from the first throw, we have

(H H H....m times) (X X ...... n times)

The chance of this event = (½ . ½ .... m times) = $\frac{1}{2^m}$

If the sequence of m consecutive heads starts from the second throw, the first must be a tail and we have

T (H H .... m times) (X X ....(n-1) times)

The chance of this event = ½ (½ . ½ .. m times) = $\frac{1}{2^{m+1}}$

If the sequence of m consecutive heads starts from the (r+1)th throw, the first (r-1) throws may be head or tail but rth throw must be tail and we have

(X X,,,, r-1 times) T (H H .... m times) (X X... (m+n-r) times)

The probability of this event = $\frac{1}{2}\ \frac{1}{2^m} = \frac{1}{2^{m+1}}$

In the above case, r can take any value from 1,2,... n. Since all the above cases are mutually exclusive, the required probability when r takes 0,1,2,... n

$$= \frac{1}{2^m} + \left(\frac{1}{2^{m+1}} + \frac{1}{2^{m+1}} + \ldots n \text{ times} \right)$$

$$= \frac{n+2}{2^{m+1}}$$

Hence the result.

9. What is the probability that in a group of N people, at least two of them will have the same birthday ?

Sol: We first find the probability that no two persons have the same birthday and then subtract from 1 to get the required probability. Suppose there are 365 different birthdays possible in a year (excluding leap year).

Any person might have any of these 365 days of the year as birthday. A second person may likewise have any of these 365 birthdays and so on. Hence the total number of ways of N people to have their birthdays = $(365)^N$.

But the number of possible ways for none of these N birthdays to coincide is =

$365 . 364 \ldots (365 - \overline{N-1})$

$$= \frac{(365)!}{(365-N)!}$$

The probability that no two birthdays coincide is

$$\frac{(365)!}{(365-N)!} \Big/ (365)^N$$

Hence the required probability (for at least two people to have the same birthday)

$$= 1 - \frac{(365)!}{(365-N)!\ (365)^N}$$

10. A and B are two independent witnesses (i.e. there is no collusion between them) in a case. The probability that A will speak the truth is x and the probability that B will speak the truth is y. A and B agree in a certain statement. Show that the probability that the statement is true is $xy / (1 - x - y + 2xy)$.

Sol: A and B agree in a certain statement means either both of them speak truth or make false statement. But the probability that they both speak truth is xy and both of them make false statement is $(1 - x)(1 - y)$.

Thus the probability of their agreement in a statement

$$= xy + (1 - x)(1 - y) = 1 - x - y - 2xy$$

Therefore, the conditional probability of their statement is true = $\dfrac{xy}{1 - x - y + 2xy}$

(by using the definition $P(A/B) = \dfrac{P(A \cap B)}{P(B)}$, where A is the event of correct statement and B is the event of common statement).

11. Two friends have agreed to meet at a certain place between nine and ten O' clock. They also agreed that each would wait for a quarter of an hour and, if the other did not arrive, would leave. What is the probability that they meet ?

Sol: Suppose x is the moment one person arrives at the appointed place, and y is the moment the other arrives.

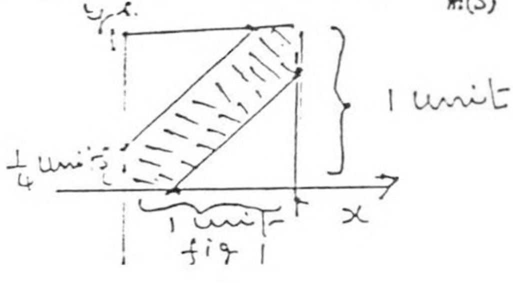
Let us consider a point with coordinates (x,y) on a place as an outcome of the rendezvous.

Every possible outcome is within the area of square having side corresponds to an hour as shown in the figure.

The outcome is favourable (the two meet) for all points (x,y) such that $|x - y| \leq 1/4$. These points are within the shaded part of the square in the above figure 1.

All the outcomes are exclusive and equally possible, and therefore, the probability of the rendezvous equals the ratio of the shaded area to the area of the square. That is, $m(A) = 7/16$ and $m(S) = 1$.

Hence by geometric probability, the required probability = $\dfrac{m(A)}{m(S)} = \dfrac{7/16}{1} = \dfrac{7}{16}$



Exercises :

1. A factor of 60 is chosen at random. What is the probability that it has factors of both 2 and 5 ?

2. The numbers 3,4 and 5 are placed on three cards and then two cards are chosen at random.

   a) The two cards are placed side-by-side with a decimal point in front. What is the probability that the decimal is more than 3/8 ?

   b) One card is placed over the other to form a fraction. What is the probability that the fraction is less than 1.5 ?

   c) If there are 4 cards with numbers 3,4,5 and 6, then what are the probabilities of the above two cases ?

3. A vertex of a paper isosceles triangle is chosen at random and folded to the midpoint of the opposite side. What is the probability that a trapezoid is formed ?

4. A vertex of a paper square is folded onto another vertex chosen at random. What is the probability that a triangle is formed ?

5. Three randomly chosen vertices of a regular hexagon cut from paper are folded to the centre of the hexagon. What is the probability that an equilateral triangle is formed?

6. A piece of string is cut at random into two pieces. What is the probability that the short piece is less than half the length of the long piece ?

7. A paper square is cut at random into rectangles. What is the probability that larger perimeter is more than 1 ½ times the smaller ?

8. The numbers 2, 3 and 4 are substituted at random for a,b,c in the equation $ax + b = c$.

9. Each coefficient in the equation $ax^2 + bx + c = 0$ is determined by throwing an ordinary die. Find the probability that the equation will have real roots.

10. The numbers 1, 2 and 3 are subsituted at random for a,b and c in the quadratic equation $ax^2 + bx + c = 0$.

    a) What is the probability that $ax^2 + bx + c = 0$ can be factored?

    b) What is the probability that $ax^2 + bx + c = 0$ has real roots ?

11. Two faces of a cube are chosen at random. What is the probability that they are in parallel planes ?

12. Three edges of a cube are chosen at random. What is the probability that each edge is perpendicular to the other two ?

13. A point P is chosen at random in the interior of square ABCD. What is the probability that triangle ABP is acute ?

14. Find the probability of the event of the sine of a randomly chosen angle is greater than 0.5.

15. Suppose you ask individuals for their random choices of letters of the alphabet. How many people would you need to ask so that the probability of at least one duplication becomes better than 1 in 2 ?

16. Six boys and six girls sit in a row randomly. Find the probability that i) the six girls sit together, ii) the boys and girls sit alternately ?

17. If the letters of the word 'MATHEMATICS' are arranged at random, what is the probability that there will be exactly 3 letters between H and C ?

18. The sum of two non-negative quantities is equal to 2n. Find the probability that their product is not less than ¾ times their greatest product.

a) What is the probability that the solution is negative ?

b) If c is not 4, what is the probability that the solution is negative ?

19. If A and B are independent events then show that $\bar{A}$ and $\bar{B}$ are also independent events.

20. Cards are dealt one by one from well-shuffled pack of cards until an ace appears,. Find the probability of the event that exactly n cards are dealt before the first ace appears.

21. If four squares are chosen at random on a chess-board, find the chance that they should be in a diagonal line.

22. Prove that if $P(A/B) < P(A)$ then $P(B/A) < P(B)$?

23. If n people are seated at a round table, what is the chance that the two named individuals will be next to each other ?

24. A and B are two very weak students of Mathematics and their chances of solving a problem correctly are 1/8 and 1/12 respectively. If the probability of their making common mistake is 1/1001 and they obtain the same answer, find the chance that their answer is correct.

25. A bag contains an unknown number of blue and red balls. If two balls are drawn at random, the probability of drawing two red balls is five times the probability of drawing two blue balls. Furthermore, the probability of drawing one ball of each colour is six times the probability of drawing two blue balls. How many red and blue balls are there in the bag ?

26. A thief has a bunch of n keys, exactly one can open a lock. If the thief tries to open the lock by trying the keys at random, what is the probability that he requires exactly k attempts, if he rejects the keys already tried ? Find the probability of the same event when he does not reject the keys already tried.

27. A problem in Mathematics is given to three students and their chances of solving it are ½, 1/3 and ¼. What is the probability that the problem will be solved ?

28. A bag A contains 3 white balls and 2 black balls and other bag B contains 2 white and 4 black balls. A bag and a ball out of it are picked at random. What is the probability that the ball is white ?

29. Cards are drawn one-by-one at random from a well-shuffled pack of 52 cards until 2 aces are obtained for the first time. If N is the number of cards required to be drawn, then show that

$$p(N = n) = \frac{(n-1)\ (52-n)\ (51-n)}{50.59.\ \ 17.\ \ 13}$$

Where $2 \leq n \leq 50$.

30. A,B, C are events such that

P (A) = 0.3, P(B) = 0.4, P(C) = 0.8, P(A ∩ B) = 0.08, P (A ∩ C) = 0.28,

P(A ∩ B ∩ C) = 0.09

If P(AUBUC) ≥ 0.75, then show that P(B ∩ C) lies in the interval (0.23, 0.48).

31. A man takes a step forward with probability 0.4 and backwards with probability 0.6. Find the probability that at the end of eleven steps, he is one step away from the starting point.

32. Huyghens Problem. A and B throw alternately a pair of dice in that order. A wins if he scores 6 points before B gets 7 points, in which case B wins. If A starts the game, what is his probability of winning ?

33. A Doctor goes to work following one of three routes A, B, C. His choice of route is independent of the weather. If it rains, the probabilities of arriving late, following A, B, C are 0.06, 0.15, 0.12 respectively. The corresponding probabilities, if it does not rain, are 0.05, 0.10, 0.15.

a) Given that on a sunny day he arrives late, what is the probability that he took route C ? Assume that, on average, one in every four days is rainy.

b) Given that on a day he arrives late, what is the probability that it is a rainy day.

34. Bonferroni's Inequality. Given $n(>1)$ events $A_1, A_2, \ldots A_n$ show that

$$\sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j) \le P\left(\bigcup_{i=1}^{n} A_i\right) \le \sum_{i=1}^{n} P(A_i)$$

35. Show that for any n events $A_1, A_2, \ldots, A_n$

i) $P\left(\bigcap_{i=1}^{n} A_i\right) \ge 1 - \sum_{i=1}^{n} P(\overline{A}_i)$

ii) $P\left(\bigcap_{i=1}^{n} A_i\right) \ge \sum_{i=1}^{n} P(A_i) - (n-1)$

36. If A and B are mutually exclusive and $P(A \cup B) \ne 0$, then prove that

$$P(A/A \cup B) = \frac{P(A)}{P(A) + P(B)}$$

37. If 2n boys are divided into two equal groups, find the probability that the two tallest boys will be a) in different subgroups, and b) in the same subgroup.

38. A small boy is playing with a set of 10 coloured cubes and 3 empty boxes. If he puts the 10 cubes into the 3 boxes at random, what is the probability that he puts 3 cubes in one box, 3 in another box, and 4 in the third box ?

39. The sample space consists of the integers from 1 to 2n, which are assigned probabilities to their logarithms. A) Find the probabilities, b) Show that the conditional probability of the integer 2, given that an even integer occurs is

$$\frac{\log 2}{n\log 2 + \log n!}.$$

40.a) Each of n boxes contains four white and six black balls, while another box contains five white and five black balls. A box is chosen at random from the (n+1) boxes, and two balls are drawn from it, both being black. The probability that five white and three block balls remain in the chosen box is 1/7. Find n.

40b). A point is selected at random inside a circle. Find the probability p that the point is closer to the centre of the circle than to its circumference.

41. What is the probability that two numbers chosen at random will be prime to each other?

42. In throwing n dice at a time, what is the probability of having each of the points 1,2,3,4,5,6 appears at least once ?

43. A bag contains 50 tickets numbered 1,2,3,..., 50 of which five are drawn at random and arranged in ascending order of magnitude ($x_1 < x_2 < x_3 < x_4 < x_5$), what is the probability that $x_3 = 30$?

44. Of the three independent events, the probability that the first only to happen is ¼, the probability that the second only to happen is 1/8 and the third only to happen is 1/12. Obtain the unconditional probabilities of the three events.

45. What is the least number of persons required if the probability exceeds ½ that two or more of them have the same birthday (year of birth need not match) ?

46. If m things are distributed among 'a' men and 'b' women, then show that the chance that the number of things received by men is

$$\frac{1}{2}\ \frac{(b+a)^m - (b-a)^m}{(b+a)^m}$$

47. A pair of dice is rolled until either 5 or a 7 appears. Find the probability that a 5 occurs first.

48. In a certain standard tests I and II, it has been found that 5% and 10% respectively of 10^{th} grade students earn grade A. Comment on the statement that the probability is $\frac{5}{100}\ \frac{10}{100} = \frac{1}{200}$ that a 10^{th} grade student chosen at random will earn grade A on both tests.

49. A bag contains three coins, one of which is coined with two heads while the other two coins are fair. A coin is chosen at random from the bag and tossed four times in succession. If head turns up each time, what is the probability that this is the two headed coin ?

50. A man stands in a certain position (which we may call the origin) and tosses a fa coin. If a head appears he moves one unit of length to the left. If a tail appears he moves one unit to the right. After 10 tosses of the coin, what are his possib. positions and what are the probabilities ?

51. There are 12 compartments in a train going from Madras to Bangalore. Five friend travel by the train for some reasons could not meet each other at Madras statio. before getting aboard. What is the probability that the five friends will be in different compartments ?

52. The numbers 1,2,3,4,5 are written on five cards. Three cards are drawn in successior and at random from the deck, the resulting digits are written from left to righ. What is the probability that the resulting three digits number will be even ?

53. Suppose n dice are thrown at a time. What is the probability of getting a sum 'S' of points on the dice ?

54. A certain mathematician always carries two match boxes, each time he wants a match stick he selects a box at random. Inevitably, a moment comes when he finds a box empty. Find the probability that the movement the first box is empty, the second contains exactly r match sticks (assume that each box contain N match-stick. initially).

55. There are 3 cards identical in size. The first card is red both sides, the second one is black both sides and the third one red one side and black other side. The cards are mixed up and placed flat on a table. One is picked at random and its uppe. (visible) side was red. What is the probability that the other side is black ?

56. N different objects 1,2,...., n are distributed at random in n places marked 1,2,... n. Find the probability that none of the objects occupies the place corresponding to its number.

Answers :

1. ½
2. A) 2/3 b) 5/6 c) 3/4, 3/4
3. 1/3
4. 1/3
5. 1/10
6. 2/3
7. 2/5
8. a) ½ b) 3/4
9. 43/216
10. a) 1/3 b) 1/3
11. 1/5
12. 2/55
13. $1 - \pi/8 = 0.6073$
14. 2/3
15. 7

16. i) $\frac{7! \quad 6!}{12\,!}$ ii) $\frac{2 \quad (6!)^2}{12!}$

17. 7/55

18. ½

20. $\frac{4\ (51-n)\ (50-n)\ (49-n)}{52.51.50.49}$

21. $\frac{91}{158844}$

23. $\frac{2}{n-1}$

24. 13/24

25. Red = 6, Blue = 3

26. 1/n, $1/n \left(1 - \frac{1}{n}\right)^{k-1}$

27. 3/4

28. 7/15

31. $(0.4)^5\ (0.6)^5$

32. 30/61

33. a) 0.5 b) 41/131

37. a) $\frac{n}{2n-1}$ b) $\frac{n-1}{4n-2}$

38. $\frac{3 \quad 10!}{3! \quad 3! \quad 4! \quad 3^{10}}$

39. a) K log 2i b) (log 2i) (n log 2 + log n!)

40. a) 4 b) 1/4

41. $\pi \left(1 - \frac{1}{r^2}\right) = \frac{6}{\pi^2}$

42. $1 - n\left(\frac{5}{6}\right)^n + \binom{n}{2}\left(\frac{4}{6}\right)^n - \binom{n}{3}\left(\frac{3}{6}\right)^n + \binom{n}{4}\left(n\frac{2}{6}\right)^n - \binom{n}{5}\left(\frac{1}{6}\right)^n$

43. $$\frac{\binom{29}{2}\binom{20}{2}}{\binom{50}{2}}$$

44. ½, 1/3, 1/4

45. 23

47. 2/5

49. 8/9

50.

Distance from origin	-10	-8	-6	-4	-2	0	2	4	6	8	10
Prob	$\left(\frac{1}{2}\right)^{10}$	$\binom{10}{1}\left(\frac{1}{2}\right)^{10}$	$\binom{10}{2}\left(\frac{1}{2}\right)^{10}$	$\binom{10}{3}\left(\frac{1}{2}\right)^{10}$	$\binom{10}{4}\left(\frac{1}{2}\right)^{10}$	$\binom{10}{5}\left(\frac{1}{2}\right)^{10}$	$\binom{10}{6}\left(\frac{1}{2}\right)^{10}$	$\binom{10}{7}\left(\frac{1}{2}\right)^{10}$	$\binom{10}{8}\left(\frac{1}{2}\right)^{10}$	$\binom{10}{9}\left(\frac{1}{2}\right)^{10}$	$\left(\frac{1}{2}\right)^{10}$

51. 55/144

52. 1/5

53. $(-1)^k \binom{n}{k} \binom{s-6k-1}{n-1} / 6^n$

54. $\dfrac{\binom{2n-1}{n}}{2^{2n-1}}$

55. ½

56. $\dfrac{1}{2!} - \dfrac{1}{3!} + \dfrac{1}{4!} = \ldots + (-1)^n \dfrac{1}{n!}$

RANDOM VARIABLES AND PROBABILITY DISTRIBUTIONS

In the earlier pages, the idea of a function, subject to certain postulates, which assigned weights called probabilities, to the points of the sample space, was introduced. We then had a probability function which allowed us to compute probabilities for events. Now we deal with the concept of Random Variable.

Random Variable :

Scientific theories on models are our way of depicting and explaining how observations come about. Such theories are simplified statements containing essential features and make for easier comprehension and communication. In statistics, we use a mathematical approach since we quantify our observations. Random variable is the result of such mathematical approach dealing with the probabilities assigning to different events of a random experiment. The set of possible outcomes for a random experiment can be described with the help of a real-valued variable by assigning a single value of this variable to each outcome. For a two coin tossing experiment, the outcomes are two tails, a tail and a head , a head and a tail, or two heads. The sample space can be represented as (TT, TH, HT, HH). Here we express the outcomes by using the number of heads and so assigning the values (0,1,1,2) respectively to those outcomes. Therefore, the outcomes of this experiment can be denoted by the different values of the real-valued variable viz. 0,1,2.

Any function or association that assigns a unique, real value to each sample point is called a chance or random variable. The assigned values are the values of the random variable.

Random variables are symbolised by capital letters, most often X, and their values by lower case letters. The outcome of a random experiment determines a point i.e., the sample space, called the domain of the random variable, and the function transform each sample point to one of a set of real numbers. This set of real numbers is called the range of the random variable. If the sample space is discrete, then the outcomes will be denoted by certain discrete values. The random variable associated with a discrete sample space is known as discrete random variable. Similarly, the random variable associated with continous sample space is known as continuous random variable.

Probability Function :

The association of probabilities with the various values of a discrete random variable is done by reference to the probabilities in the sample space and through a system of relationships or a function is called a probability set function or, simply, a probability function.

Let the discrete random variable X assume the values $x_1, x_2 \ldots x_n$ Then the system of relations can be written as

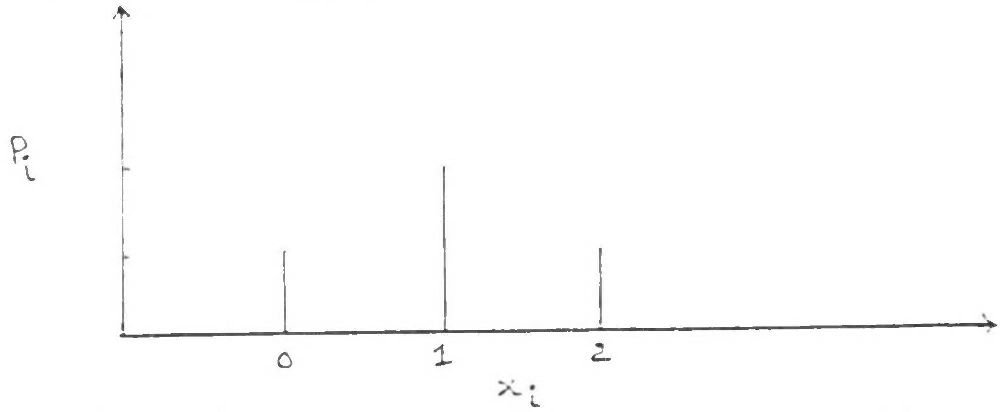
$P(X = x_i) = p_i$

This is read as 'the probability that the random variable X takes the value of x_i is p_i'. The set of ordered pairs (x_i, p_i) constitutes a probability function with numerical values to be provided for the x_i and p_i's such that $p_i \geq 0$ for all i and $\sum_i p_i = 1$.

A discrete probability function is a set of ordered pairs of values of a random variable and the corresponding probabilities.

For a two coin experiment, X takes the values 0,1,2 with the probabilities ¼, ½, ¼ respectively.

Sometimes probability function can be represented by a graph or a mathematical function. In case of above example, the X values and the corresponding probabilities can be represented with the help of the following graph.



Suppose X assume the values 1 and 0 with the probabilities p and 1-p respectively. This information can be given with the help of the following function p(x) defined by

$P(x) = p^x(1-p)^{1-x}, x = 0, 1.$

This type of function which gives the probabilities of the different values assumed by a random variable is known as probability mass function or simply probability function. Therefore, a function p(x) is said to be a probability function of random variable or a distribution if

i) $p(x) \geq 0$ for all x.

$$\sum_x p(x) = 1$$

where p(x) denotes the probability of the events that the random variable X assumes the value x.

Distribution Function :

The law of probability distribution of a random variable is the rule used to find the probability of the event related to a random variables. For instance, the probability that the variable assumes a certain value or falls in a certain interval. The general form of the distribution law is distribution function, which is the probability that a random variable X assumes a value smaller than a given x i.e. $F(x) = P(X \leq x)$.

The distribution function F(x) for any random variable possesses the following properties :

i) $F(-\infty) = 0$

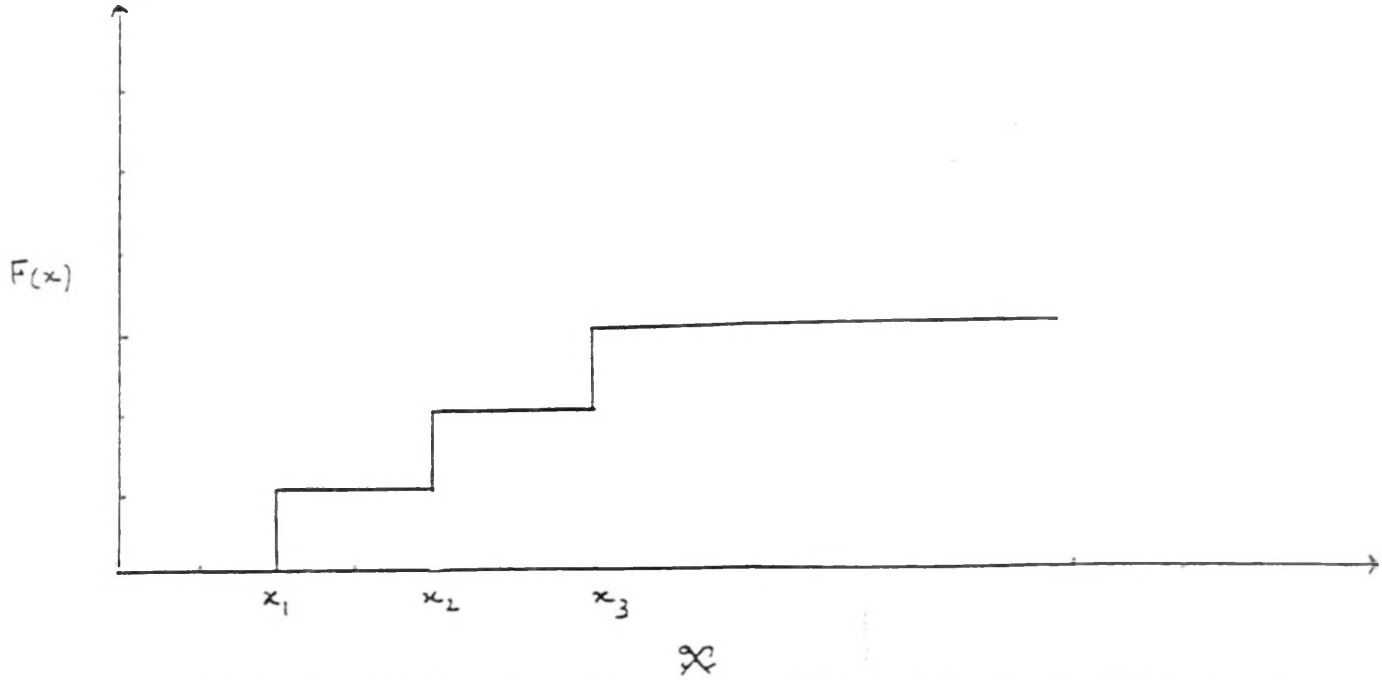
ii) $F(+\infty) = 1$

iii) F(x) does not decrease with an increase in x.

In the case of discrete random variable

$$F(x_k) = \sum_{i=1}^{k} p(x_i)$$

Where $x_1, x_2, \ldots, x_k \ldots$ are the values of the random variable. The graph of F(x) in discrete random variable case is generally as shown below :



It is seen from the above figure that the graph of F(x) is a 'step function' having jump $p(x_i)$ at $x = x_i$ and is constant between each pair of values of x. It can also be proved that

$$F(x_i) - F(x_{i-1}) = p(x_i)$$

Therefore, distribution function can also be used to indicate the distribution of the random variable instead of probability function.

Example :

A student is to match three historical events (Mahatma Gandhi's birth year, India's freedom, and first Word War) with three years (1947, 1914, 1869). If he guesses, with no knowledge of the correct answers, what is the probability distribution of the number of answers he gets correctly ?

Solution : Here the number of correct answers is the random variable, say X. Therefore, X assumes the values 0,1,2,3 because there are three events to match with only three years. Suppose the events are E_1, E_2, E_3 and the corresponding correct years are Y_1, Y_2, Y_3. Student gets the correct answers when he/she matches E_1 to Y_1, E_2 to Y_2 and E_3 to Y_3.

All matchings are wrong only when he/she matches E_1 to Y_2, E_2 to Y_3, E_3 to Y_1 or E_1 to Y_3, E_2 to Y_1, E_3 to Y_2. But the total possible matchings are 6. Therefore, the probability of all matchings to go wrong is 2/6 =1/3. That is, the probability that X to take the value '0' is 1/3.

Similarly X assumes the value '1' with probability 3/6 (= ½) the value '2' with 0 probability and the value '3' with 1/6 probability.

So the probability distribution of the correct answers in the given matching is

No of correct answers (x)	0	1	2	3
Probability	1/3	½	0	1/6

Example : Suppose a number is selected at random from the integers 10 through 30. Let X be the number of its divisors. Construct the probability function of X. What is the probability that there will be 4 or more divisors ?

Solution : X is the number of divisors of randomly selected number from the integers 10 through 30. Therefore, X is a random variable. The possible values that X assumes are :

2, 3, 4, 5,6 depending upon the selected number. For example, if the selected number is either 1,2,3,5,7,11,13,17,19 then X takes the value 2. Similarly when the selected number is 4,6,8,10,14,15 X takes 4. Therefore, the different values of X and the number of their appearances we get the following :

X values	1	2	3	4	5	6
No of appearances out of 20	1	g	3	4	1	3

Now the required probability distribution is

x	1	2	3	4	5	6
p(x)	1/20	g/20	3/20	4/20	1/20	3/20

The probability of X to take 4 or more

$= P\ (x = 4 \text{ or } 5 \text{ or } 6) = P\ (x = 4) + P\ (x = 5) + P\ (x = 6)$

$$= \frac{4}{20} + \frac{1}{20} + \frac{3}{20} = \frac{8}{20} = \frac{2}{5}$$

Mean, Variance, Standard Deviation of the Random Variable.

Let X be a random variable with probability function as follows :

x	x_1	x_2	..	..	..	x_n
p(x)	$p(x_1)$	$p(x_2)$	..	..	..	$p(x_n)$

The mean of X is defined as

$$x_1\ p(x_1) + x_2\ p(x_2) + \dots + x_n\ (n_n)$$

or

$$\sum_{i=1}^{n} x_i\ p(x_i)$$

This is also known as mean of the distribution and generally denoted by μ.

The variance of X is defined as

$$\sum_{i=1}^{n} x_i^2\ p(x_i) - \left[\sum_{i=1}^{n} x_i\ p(x_i) \right]^2$$

$$\sum_{i=1}^{n} x_i^2 \; p(x_i) - \mu^2$$

where μ is the mean of X.

The variance is generally denoted by σ^2.

The standard deviation is the positive square root of variance and is denoted by σ.

Example : A single 6-sided die is tossed. Find the mean and variance of the number of points on the top face.

Solution : Let X represent the number of points on the top face. The probability function of X is

x	1	2	3	4	5	6
p(x)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

The mean, μ is given by

$$\sum_{i=1}^{n} x_i \; p(x_i) = x_1 \, p(x_1) + x_2 \, p(x_2) \ldots + x_n \; p(x_n)$$

$$\text{Here } \mu = 1 \,.\, \frac{1}{6} + 2 \,.\, \frac{1}{6} + 3 \,.\, \frac{1}{6} \,.\, 4 \,.\, \frac{1}{6} + 5 \,.\, \frac{1}{6} + 6 \,.\, \frac{1}{6}$$

$$= \quad \frac{1}{6} \; (1 + 2 + 3 + 4 + 5 + 6)$$

$$= \quad \frac{1}{6} \; \frac{6 \times 7}{2} = \frac{7}{2}$$

Variance, σ^2 is given by

$\sum_{i=1}^{n} x_i^2 \; p(x_i) - \mu^2$ where μ is mean.

Here

$$\sum_{i=1}^{n} x_i^2 \; p(x_i) = 1^2 \,.\, \frac{1}{6} + 2^2 \,.\, \frac{1}{6} + 3^2 \,.\, \frac{1}{6} + 4^2 \,.\, \frac{1}{6} + 5^2 \,.\, \frac{1}{6} + 6^2 \,.\, \frac{1}{6}$$

$$= \frac{1}{6}\left[1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2\right]$$

$$= \frac{1}{6}\ \frac{6\times 7\ \ (2\times 6\times 1)}{6} = \frac{91}{6}$$

Variance = $\sigma^2 = \frac{91}{6} - \left(\frac{7}{2}\right)^2 \qquad (\because \mu = \frac{7}{2})$

$$= \frac{91}{6} - \frac{49}{4} = \frac{35}{12}$$

Exercises :

1. One cube with faces numbers 1,2,3,4,5 and 6 is tossed twice, and the recorded outcome consists of the ordered pair of numbers on the hidden faces at the first and second tosses.

   a) Let the random variable X takes on the value 0 if the sum of the numbers in the ordered pair is even and 1 if odd. What is the probability function for this random variable ?

   b) Let the random variable X takes on the value 2 if both numbers in the ordered pair are even, 1 if exactly one is even, and 0 if neither is even. What is the probability distribution of this random variable ?

   c) Let the random variable X be the number of divisors in the sum of the two faces. What is the probability function of X ?

2. Of six balls in a bag, two are known to be black. The balls are drawn one at a time from the bag and observed until both black balls are drawn. If X is the number of trials (draws) required to get the two black balls. Obtain the probability distribution of X.

3. Suppose that the random variable X has possible values 1,2,3,... and $P(x = j) = \frac{1}{2^j}$, $j = 1,2,...$

   i) compute P(x is even), ii) compute P(x is divisible by 3).

4. The probability mass function of a random variable X is zero except at the points x= 0,1,2,. At these points has the values $p(0) = 3c^3$, $p(1) = 4c - 10c^2$ and $p(2) = 5c - 1$ for some $c > 0$.

i) Determine the value of c.

ii) Compute $P(1 < X \le 2)$.

iii) Describe the distribution function and draw its graph.

iv) Find the largest x such that $F(x) < \frac{1}{2}$

5. Let X denote the profits that a man makes in business. He may earn Rs.3000 with probability 0.5, he may lose Rs.5000 with probability 0.3 and he may neither earn nor lose with probability 0.2. Calculate his average profits.

6. A man wins a rupee for head and loses a rupee for tail when a coin is tossed. Suppose that he tosses once and quits if he wins but tries once more if he loses on the first toss. What are his expected winnings ?

7. Three boxes contain respectively 3 red and 2 black balls, 5 red and 6 black balls and 2 red and 4 black balls. One ball is drawn from each box. Find the average number of black balls drawn.

8. If the random variable, X takes the values 1,2,....n respectively with probabilities $\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}$ find the mean and variance of X.

Answers :

1. a)

X	Prob
0	½
1	½

b)

X	Prob
0	¼
1	½
2	¼

c)

X	Prob
2	15/36
3	12/36
4	8/36
6	1/36

2.

X	Prob
2	1/15
3	2/15
4	3/15
5	4/15
6	5/15

3. i) 1/3 ii) 1/7

4. i) 1/3 ii) 2/3 iii) 1

5. 0

6. 0

7. $\frac{266}{165}$

8. Mean = $\frac{(n+1)}{2}$, Variance = $\frac{n^2-1}{12}$

DISCRETE DISTRIBUTIONS

In the previous pages, we discussed about 'random variable', 'probability function', etc. Here we discuss some theoretical discrete distributions in which variables are distributed according to some definite law which can be expressed mathematically.

Bernoulli Distribution : Suppose you want to study the probability of different events corresponding to tossing of a single coin experiment. The two possible events are getting a head or getting a tail. Define a random variable x assuming the values 1 and 0 corresponding to these two events viz. Head and tail respectively. If the probability of getting a head in tossing that coin is 'p' then the probability that the random variable to take '1' is p and the probability that the random variable to take '0' is 1-p. Therefore, the distribution of the random variable X becomes

X	Prob
1	p
0	1-p

Any experiment where there are only two possible outcomes viz. Success and failure is called as Bernoulli experiment. A single trial of a Bernoulli experiment is known as Bernoulli trial.

Corresponding to any Bernoulli experiment, it is possible to define a random variable X as given above.

A random variable X which takes two values 0 and 1, with probability $q(=1-p)$ and p respectively is called Bernoulli variate and is said to have a Bernoulli distribution.

Binomial Distribution :

Let a Bernoulli experiment be performed repeatedly and let the occurrence of an event in any trial be called a success and its non-occurrence a failure. Consider a series of n independent Bernoulli trials (n being finite), in which the probability 'p' of success in any trial is constant for each trial. Then $q = 1-p$ is the probability of failure inany trial. Let the random variable X be the number of successes in these trials.

The probability of x successes and consequently (n-x) failures in n independent trials, in a specified order (say) SS FF SSS .... FSFF (where S represents success and F failure) is given by compound probability as given below :

$$P(SSFF, \ldots FSFF) = P(S)\ P(S)\ P(F)\ P(F) \ldots P(F)\ P(S)\ P(F)\ P(F)$$
$$= p.p.qq \ldots q\ p\ q\ q$$
$$= pp \ldots p\ qq \ldots q \quad (x \quad p\text{'s and } (n\text{-}x)\ q\text{'s})$$
$$= p^x q^{n-x}$$

But x successes in n trials can occur in $\binom{n}{x}$ ways and the probability for each of these ways is $p^x q^{n-x}$. Hence the probability of x successes in n trials in any order whatsoever is given by the addition of individual probabilities and is given by $\binom{n}{x} p^x q^{n-x}$. The number of successes in n trials will be either 0 or 1 or 2 ... or n in any experiment.

$$p(x) = P(X = x) = \binom{n}{x} p^x q^{n-x}$$

Is true for all x = 0,1,2,...n.

This function $p(x) = \binom{n}{x} p^x q^{n-x}$, x = 0, 1,...., n is called the probability mass function of the Binomial distribution, for the obvious reason that the probabilities of 0,1,2,...n successes, viz. q^n, $\binom{n}{1} q^{n-1} p$, $\binom{n}{2} q^{n-2} p^2$,.... , p^n Are the successive terms of the binomial expansion $(q + p)^n$.

A random variable X is said to follow binomial distribution if its probability mass function is given by

$$P(X = x) = p(x) = \binom{n}{x} p^x q^{n-x}, \; x = 0,1,2,.....n; \; q = 1 - p.$$

The values n and p of this distribution are known as the parameters of the distribution.

Mean and Variance of Binomial Distribution

We know, mean of any discrete distribution

$$= \sum_r r\, p(r)$$

where p(r) is the probability that the random variable X to take the value r. In case of binomial distribution x takes the values r = 0,1,2,...., n and $pr = \binom{n}{r} p^r q^{n-r}$ where n and p are the parameters of the binomial distribution.

$$\therefore \text{Mean} = \sum_{r=0}^{n} r \binom{n}{r} p^r q^{n-r}$$

$$= \sum_{r=0}^{n} r \frac{n!}{r!\,(n-r)!} p^r q^{n-r}$$

$$= np \sum_{r=1}^{n} \frac{(n-1)!}{(r-1)!\,(n-r)!} p^{r-1} q^{n-r}$$

$$= n\,p\left[\,q^{n-1} + (n-1)\,C_1\, q^{n-2} + \ldots + p^{n-1}\right]$$

$$= n\,p\,(q+p)^{n-1}$$

$$= np \quad (\because p+q=1)$$

Also we know Variance $= \sum_r r^2\, p(r) - \left[\sum_r r\, p(r)\right]^2$

$$= \sum_r r^2\, p(r) - (Mean)^2$$

In case of binomial distribution

Variance $= \sum_{r=0}^{n} r^2 \binom{n}{r} p^r q^{n-r} - (n\,p)^2$

$$= \sum_{r=0}^{n} r^2 \frac{n!}{r!\,(n-r)!} p^r q^{n-r} - (np)^2 \qquad (\because Mean = np)$$

$$= \sum_{r=0}^{n} \left[\,r\,(r-1) + r\right] \frac{n!}{r!\,(n-r)!} p^r q^{n-r} - (n\,p)^2$$

$$= \sum_{r=0}^{n} r(r-1) \frac{n!}{r!\,(n-r)!} p^r q^{n-r} + \sum_{r=0}^{n} r \frac{n!}{r!\,(n-r)!} p^r q^{n-r} - (n\,p)^2$$

$$= \sum_{r=2}^{n} \frac{n!}{(r-2)!\,(n-r)!} p^r q^{n-r} + np - (n\,p)^2$$

$$(\because \sum r \frac{n!}{r!\,(n-r)!} p^r q^{n-r} = np \quad provedabove)$$

$$= n(n-1)\,p^2 \left[\sum_{r=2}^{n} \frac{(n-2)!}{(r-2)!\,(n-r)!} p^{r-2} q^{n-r}\right] + np - (n\,p)^2$$

$$= n(n-1)\,p^2 \left[\,q^{n-2} + (n-2)C_1 + (n-2)\,C_2 + \ldots p^{n-2}\right] + np - (np)^2$$

$$= n(n-1)\,p^2\,(q+p)^{n-2} + np - (np)^2$$

$$= n(n-1)\,p^2 + np - (np)^2$$

$$= np\,\left[(n-1)\,p + 1 - np\right]$$

$$= n\,p\,\left[\,np - p + 1 - np\right]$$

$$= np\,[\,1 - p\,] = n\,p\,q$$

So, Mean = np

Variance = npq

Standard Deviation = $\sqrt{Variance} = \sqrt{npq}$

Example : The mean and variance of binomial distribution with parameters n and p are 16 and 8. Find i) P (x = 0) ,   ii) P (x ≥ 2).

Solution : We know mean = np and variance = npq.

$\therefore$ np = 16   and npq = 8

Solving for n and p we get n = 32 and p = ½

Now $P(X=0) = \binom{n}{0} p^0 q^{n-0} = q^n$

(Because $p(x=r) = \binom{n}{r} p^r q^{n-r}$

$\therefore P(x=0) = (1-p)^n = \left(1-\frac{1}{2}\right)^{32} = \left(\frac{1}{2}\right)^{32}$

$(\because n=32,\ q=1-p = 1-\frac{1}{2})$

ii) $P(x \geq 2) = 1-P(x<2) = 1-[P(x=0)+P(x=1)] = 1-P(x=0)-P(x=1)$

But P (x = 0) = $\left(\frac{1}{2}\right)^{32}$ (As obtained above)

and $P(x=1) = \binom{n}{1} p^1 q^{n-1} = 32\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{32-1}$

$\therefore P(x \geq 2) = 1-\left(\frac{1}{2}\right)^{32} - 32\left(\frac{1}{2}\right)^{32} = 1-33\left(\frac{1}{2}\right)^{32}$

Example : A perfect cube is thrown a large number of items in sets of 8. The occurrence of a 2 or 4 is called a success. In what proportion of the sets would you expect 3 successes.

Solution : In this problem we have to find the probability of getting 3 successes out of 8 trials. Tossing of a single cube is our trial. The probability of success, p is getting either 2 or 4. The number of cubes in the set is the number of trials. If we define x as the number of successes in 8 trials, then x is distributed as a binomial variate with parameters 8 and p where p is the probability of success.

The probability of getting either 2 or 4 in tossing of a perfect cube = 2/6 = 1/3.

$\therefore$ p = 1/3

Hence $P(x=r) = \binom{n}{r} p^r q^{n-r}$

and $P(x=3) = \binom{n}{3} p^3 q^{n-3}$ ($\because$ x is a binomial variate)

$$= \binom{8}{3}\left(\frac{1}{3}\right)^{3}\left(\frac{2}{3}\right)^{8-3} \qquad (\because n = 8,\ p = 1/3,\ q = 1-p)$$

$$= 8\times 7\left(\frac{1}{3}\right)^{8} 2^{5}$$

$$= 56\times 32\left(\frac{1}{3}\right)^{8}$$

$= 0.2731$

$\therefore$ The proportion of sets in which we expect 3 successes = 27.31 %.

Example :The probability of a man hitting a target is ¼.

i) If he fires 7 times, what is the probability of his hitting the target at least twice ?

ii) How many times must he fire so that the probability of his hitting the target at least once is greater than 2/3 ?

Solutions :

i) Consider 'firing once' as a Bernoulli trial. Firing 7 times is the Binomial experiment with 7 independent Bernoulli trials. If X is the number of hits in 7 trials, then the required probability of hitting the target at least twice = $P(X \geq 2)$.

We know,

$P(X \geq 2) = 1 - p(X < 2)$

$= 1 - P(X = 0) - P(X = 1)$

and $P(X = x) = \binom{n}{x} p^{x} q^{n-x}$ where $n = 7$, $p = 1/4$, and $q = 1 - p = 3/4$.

$P(X = 0) = (3/4)^7$

$$P(X = 1) = \binom{7}{1}\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^{6} = 7\,\frac{3^{6}}{4^{7}}.$$

The required probability

$$= 1 - \left(\frac{3}{4}\right)^{7} - 7\,\frac{3^{6}}{4^{7}} = \frac{4547}{8192}.$$

ii) $p = 1/4$, $q = 3/4$

We want to find n such that $P(X \geq 1) > 2/3$

Or $1 - P(X < 1) > 2/3$

Or $1 - P(X = 0) > 2/3$

Or $1 - q^n > 2/3$ when $q = 3/4$

$\Rightarrow (3/4)^n < 1/3$

$\Rightarrow n = 4.$

POISSON DISTRIBUTION

There are many situations where we must count the number of individuals possessing a certain characteristic yet have difficulty in defining the basic experiment. In turn, it becomes difficult to say what is the probability of the occurrence of a single event. For example i) number of telephone calls received at a particular telephone exchange, ii) emission of radioactive particles, iii) number of printing mistakes in a book. In all these situations, it is easy to count the events, but what are the non events.

In situations like those mentioned above, we customarily resort to specifying a unit size or a time interval in which to observe the events etc. We find then that we are observing events that fluctuate around some mean value that might be defined in terms of some sort of underlying binomial parameters p and n as np, a product never separable into its component parts and simply give the mean value. Therefore, in such situations, we assume that for a short enough unit of time or space, the probability of an event occurring is proportional to the length of time or size of the space. We also assume that for non overlapping units, the results in one unit are of no value in predicting when or where another event will occur (independently). The above assumptions underlie the probability function given by

$$P(X = x) = \frac{e^{-\lambda} \lambda^x}{x!}, \quad x = 0, 1, 2, 3, \ldots$$

where λ is the average number of times an event occurs in a unit interval and is called the parameter of a Poisson distribution. Poisson Distribution as a limiting case of Binomial Distribution.

The above mentioned Poisson distribution can be viewed as a limiting case of the binomial distribution under the following conditions.

i) n, the number of trials in the binomial experiment is infinitely large i.e. $n \to \infty$.
ii) p, the probability of success in each trial is indefinitely small, i.e. $p \to 0$.
iii) $np = \lambda$ is finite so that $p = \frac{\lambda}{n}$, $q = 1 - \frac{\lambda}{n}$.

We know, if X is a binomial variate with parameters n and p then

$$P(X=x) = p(x) = \binom{n}{x} p^x q^{n-x}, \; x = 0, 1, 2, \ldots n$$

where $n \to \infty$ and $p \to 0$.

Therefore, this probability

$$= \lim_{\substack{n\to\infty \\ p\to 0}} \binom{n}{x} p^x \, q^{n-x}$$

$$= \lim_{n\to\infty} \binom{n}{x} \left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x} \qquad \left(\because p = \frac{\lambda}{n}, \; q = 1-\frac{\lambda}{n}\right)$$

$$= \lim_{n\to\infty} \frac{n!}{x!\,(n-x)!} \left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{\lambda^x}{x!} \lim_{n\to\infty} \frac{n\,.\,(n-1)\; \ldots\; (n-x-1)}{n^x} \; \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^x}$$

$$= \frac{\lambda^x}{x!} \lim_{n\to\infty} \left[1\,.\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) \ldots \left(1-\frac{x-1}{n}\right)\left(1-\frac{\lambda}{n}\right)^n\right]$$

$$\lim_{n\to\infty} \frac{1}{\left(1-\frac{\lambda}{n}\right)^x}$$

$$= \frac{\lambda^x}{x!} \lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^n \qquad \left(\because \lim_{n\to\infty} \frac{1}{\left(1-\frac{\lambda}{n}\right)^x} = 1\right)$$

$$= \frac{\lambda^x}{x!} \lim_{m\to\infty} \left(1+\frac{1}{m}\right)^{-m\lambda}$$

$$= \frac{\lambda^x}{x!} \lim_{m\to\infty} \left[\left(1+\frac{1}{m}\right)^m\right]^{-\lambda}$$

$$= \frac{\lambda^x}{x!} \, e^{-\lambda}$$

$$\therefore P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, 2 \ldots$$

This function is known as the Probability function of the Poisson distribution and λ is the parameter of the distribution.

Mean and Variance of the Poisson Distribution :

$$\text{Mean} = \sum_{x=0}^{\infty} x \; P(x_i)$$

$$= \sum_{x=1}^{\infty} x \; \frac{e^{-\lambda}\lambda^x}{x_i!} \qquad \left(\because P(x) = \frac{e^{-\lambda}\lambda^x}{x!} \text{ In case of Poisson distribution}\right)$$

$$= \sum_{x=0}^{\infty} x \frac{e^{-\lambda} \lambda^x}{x!}$$ ($\because$ the values of Poisson variate are 0,1,2,...)

$$= \lambda e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^{x-1}}{x!}$$

$$= \lambda e^{-\lambda} \left(1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \text{—}\right)$$

$$= \lambda e^{-\lambda} e^{\lambda}$$

$$= \lambda$$

$$\text{Variance} = \sum_{i=1}^{\infty} x_i^2 \, p(x_i) - \left[\sum_{i=1}^{\infty} x_i \, p(x_i)\right]^2$$

Continued in last page.

Exercises :

1. A random variable X has a binomial distribution with parameters n = 4 and p = 1/3

i) Describe the probability mass function and sketch its graph.

ii) Compute the probabilities P(1 < X ≤ 2) and P (1 ≤ X ≤ 2).

2. In a binomial distribution consisting of 5 independent trials, probabilities of 1 and 2 successes are 0.4096 and 0.2048 respectively. Find the parameter p of the distribution.

3. The probability of a man hitting a target is 1/3.

i) If he fires 5 times what is the probability of hitting the target at least twice ?

ii) How many times must he fire so that the probability of hitting the target at least once is more than 90% ?

4. The random variable X has a binomial distribution with n = 4, p = 0.5.
Find { | X – 2 | ≥ 1 }

Answers :

1. 8/27, 56/81
2. 0.2
3. I) 131/243
   ii) 6
4. 5/16

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Problems in Probability

EXERCISE – 1

1. Three points are taken at random on a circle. What is the chance that they determine an acute angled triangle?

2. Two coins C_1 and C_2 have a probability of falling heads p_1 and p_2, respectively. You win a bet if in three tosses you get at least two heads in succession. You toss the coins alternately starting with either coin. If $p_1 > p_2$, what coin would you select to start the game? Give reasons for your answer.

3. A box contains p white balls and q black balls, and beside the box lies a large pile of black balls. Two balls chosen at random are taken out the box. If they are of the same colour, a black ball from the pile is put into the box, otherwise, the white ball is put back into the box. The procedure is repeated until the last two balls are removed from the box and one last ball is put in. What is the probability that this ball is white? (Ans. 1 when p is odd and 0 when p is even)

4. If the probability of success is 0.01, how many trials are necessary in order that probability of at least one success is $> \frac{1}{2}$?

5. Can the following sets serve as sample spaces of some experiments. If yes, give one experiment in each case.

   i) $S = \{(x,y)/\ x, y$ are natural numbers, $1 \le x \le 6,\ 2 \le y \le 6\}$

   ii) $S = \{x\ /\ x = \frac{p}{q}$ where p and q are natural numbers such that $1 \le p \le 6$, $1 \le q \le 6\}$.

6. State and prove multiplication theorem of probability.

7. A sportsman's chance of shooting an animal at a distance r (> a) is $\frac{a^2}{r^2}$. He fires when $r = 2a$ and if he misses he reloads and fires when $r = 3a, 4a, \ldots\ldots$ If he misses at distance na, the animal escapes. What the odds against the sportsman?

8. A local post office is to send M telegrams and to distribute them at random over N communication channels. The channels are enumerated. Find the probability that exactly k_1 telegrams will be sent over the first channel, k_2 telegrams will be sent

over the second channel and so on, k_N telegrams over the N^{th} channel, with $\sum_{1}^{N} k_i = M$.

9. Let A, B and C be the three events with P(B) and P(C) > 0 . If B and C are independent, show that $P(A / B) = P(A / (B \cap C))\, P(C) + P(A/B \cap \bar{C}) P(\bar{C})$. Conversely, if this relation holds, $P(A / (B \cap C)) \neq P(A / B)$ and $P(A) > 0$, then B and C are independent.

10. In the game of tossing a fair coin, the first one to obtain n successes (heads or tails) wins. Show that the game is fair i.e,, each gambler has a probability of winning equal to $\frac{1}{2}$.

11. If the coins are unbiased, the probability of getting exactly 50 heads in tossing of 100 coins is $\frac{1}{2}$. Comment.

12. What is the least number of persons required if the probability exceeds $\frac{1}{2}$ that two or more of them have the same birth day? (Year of birth need not match).

13. A company manufacturing cornflakes puts a card numbered 1 or 2 or 3 ..... or r at random in each package, all numbers being equally likely to be drawn. If n (> r) boxes of cornflakes are purchased, show that the probability of being able to assemble at least one complete set of cards from the packages is

$$1-\binom{r}{1}(1-\frac{1}{r})^n+\binom{r}{2}(1-\frac{2}{r})^n+\dots\dots+(-1)^{r-1}\binom{r}{r-1}(1-\frac{r-1}{r})^n$$

14. The following data was given in a study of 1000 subscribers to a certain magazine. In reference to sex, marital status and education, there were 312 males, 470 married persons, 525 college graduates, 42 male college graduates, 147 married college graduates, 86 married males and 25 married male college graduates. Show that the numbers reported in the study must be incorrect.

15. If 4 married couples are arranged in a row find the probability that no husband sits next to his wife.

16. A man forgets the last digit of a telephone number, and dials the last digit at random. What is the probability of calling no more than three wrong numbers?

17. What is more probable: to get one six with four dice, or to get two sixes in 24 throws of two dice?

18. If $P(E) = 0.9$ and $P(F) = 0.8$, show that $P(E \cap F) \geq 0.7$.

19. Consider an example whose sample space consists of a countable infinite number of points. Also show that not all points can be equally likely.

20. A game is played as follows. The gambler throws two dice. If the first throw he gets 7 or 11 he wins, and if he gets 2, 3 or 12 he loses. For each of the other sums the game is continued in two ways. a) the gambler continues throwing the two dice until he wins with a 7 or he loses with the result of the outcomes of the first throw. b) The gambler continue until he loses with 7 or wins with the result of the first throw. What is the probability of the gambler winning in case (a) and (b) ?

21. A store opens at 9 A.M and closes at 5 P.M . A shopper taken at random walks into this store at time x and out at time y (both x and y being measured in hours on the time axis with 9 A.M as origin) Describe the sample space of (x , y). Also describe, in terms of x and y the following event. The shopper is in the stores less than one hour.

22. A small boy is playing with a set of 10 coloured cubes and 3 empty boxes. If he puts the 10 cubes into the boxes at random, what is the probability that he puts 3 cubes on one box, 3 in another box , and 4 in the third box?

23. Describe how you explain to a layman the meaning of the following statement : ' An insurance company is not gambling with its clients because it knows with sufficient accuracy that will happen to every thousand or ten thousand or a million people even when the company cannot tell that will happen to any individual among them.'

24. Comment on the following statement:

    i) Mutually exclusive events are independent

    ii) Independent events need not be mutually exclusive.

25. Events E_1, E_2, ..., E_n are such that the probability of the occurrence of any specified r of them is p_r , r = 1, 2, ...n. Show that the probability of the occurrence of exactly m of the events E_1, E_2, ...E_n is

$$\binom{m}{n}\binom{n}{m}p_m - \binom{m+1}{m}\binom{n}{m+1}p_{m+1} + \ldots\ldots + (-1)^{n-m}\binom{n}{m}\binom{n}{n}p_n$$

26. When is $P(A / B) + P(A + \bar{B}) = 1$?

27. A box contains n balls numbered 1, 2, ...n. We select at random r balls, a) with replacement b) without replacement. What is the probability that the largest selected number is m?

28. If A and B are two events and the probability, $P(B) \neq 0$, prove that $P(A) >$ or $<$ $P(A / B)$ according as $P(A / \bar{B}) >$ or $< P(A)$.

29. State and prove addition theorem of probability.

30. N players A_1, A_2, ...,A_N throw a biased coin whose probability of heads equals p. A_1 starts (the game), A_2 second etc. The first one to throw heads wins. Find the probability that A_k (k 1, 2, ..., A_N) will be the winner.

31. A and B alternately cut a pack of cards and the pack shuffled after each cut. If A starts and the game is continued until one cuts a diamond, what are the respective chances of A and B first cutting a diamond?

32. If n letters are placed in the corresponding n envelops at random, what is the probability that no letter is placed in the right envelop?

33. A bag contains three coins, one of which is coined with two heads while the other two coins are fair. A coin is chosen at random from the bag and tossed four times in succession. If heads turn up each time, what is the probability that this is the two headed coin?

34. Bertrand's Paradox: A chord AB is chosen at random in a circle of radius r. What is the probability that the length of AB is less than r?

35. Two points are selected at random on a line of length 'a'. What is the probability that none of these three sections in which the line thus divided is less than $\frac{a}{4}$?

36. State and prove Baye's theorem of probability.

37. A group of 2N boys and 2N girls is randomly divided into two equal groups. What is the probability that each group has the same number of boys and girls?

38. A man is equally likely to choose one of three routes A, B, C from his house to the railway station. and his choice of route is not influenced by the weather. If the weather is dry; the probabilities of missing the train by routes A, B, C are respectively $\frac{1}{20}, \frac{1}{10}, \frac{1}{5}$. If he comes out on a dry day and misses the train then what is the probability that the route chosen was C?

On a wet day the respective probabilities of missing the train by routes A, B, C are $\frac{1}{12}, \frac{1}{5}, \frac{1}{2}$. On the average one day in four is wet. If he misses the train, what is the probability that the day was wet?

39. What is wrong with the following procedure ?

To find the probability that an Indian chosen at random was born in a given state, divide the number of favourable cases (1) by the total number of states (say 30), and obtain the answer $\frac{1}{30}$.

40. The letters of the word PEPPER are written on cards. After shuffling thoroughly, four cards are drawn randomly one after the other. What is the probability that the result is PEEP ?

41. A thief has a bunch of n keys, exactly one of which fits a lock. If the thief tries to open the lock by trying the keys at random, what is the probability that he required exactly k attempts if he rejects the keys already tried?

42. A point is chosen at random on a line of length 'l'. What is the probability that the ratio of the shorter to the longer segment is less than $\frac{1}{4}$?

43. Prove that $P[(E_1 \cup E_2) / F)] = P(E_1 / F) + P(E_2 / F) - P[(E_1 \cap E_2) / F)]$

44. Mrs Revathi types 15 letters per day and Mrs Gayathri types 5 letters per day for the department of Science of R.I.E. Mysore. Experience has shown that Mrs Revathi has a probability 0.99 of producing an error free letter and Mrs Gayathri has a probability 0.70 of doing the same. A letter without identification of the typist is placed on the Professor's table for signature. The letter has no error. What is the probability that the letter was typed by Mrs Gayathri?

45. A thief has a bunch of n keys, exactly one of which fits the lock. If the thief tries to open the lock by trying the keys at random, what is the probability that he required exactly k attempts if he rejects the keys already tried?

46. N identical balls are distributed among n boxes. What is the probability that a specified box will contain k balls.

47. Suppose that for the independent events A, B and C we have $P(A) = a$, $P(A \cup B \cup C) = 1 - b$, $P(A \cap B \cap C) = 1 - c$ and $P(\bar{A} \cap \bar{B} \cap \bar{C}) = x$.

Prove that the probability x satisfies the equation

$$Ax^2 + [ab - (1-a)(a-c-1)]x + b(1-a)(1-c) = 0$$

Hence conclude that $c > \dfrac{(1-a)^2 + ab}{1-a}$

Moreover, show that $P(B) = \dfrac{(1-c)(x+b)}{ax}$, $P(C) = \dfrac{x}{x+b}$

48. If m things are distributed among 'a' men and 'b' women, show that the chance that the number of things received by men is odd is

$$\frac{1}{2} \frac{(b+a)^m - (b-a)^m}{(b+a)^m}$$

EXERCISES : 2

1. A factor of 60 is chosen at random. What is the probability that it has factors of both 2 and 5 ?

2. The numbers 3,4 and 5 are placed on three cards and then two cards are chosen at random.

   a) The two cards are placed side-by-side with a decimal point in front. What is the probability that the decimal is more than 3/8 ?

   b) One card is placed over the other to form a fraction. What is the probability that the fraction is less than 1.5 ?

   c) If there are 4 cards with numbers 3,4,5 and 6, then what are the probabilities of the above two cases ?

3. A vertex of a paper isosceles triangle is chosen at random and folded to the midpoint of the opposite side. What is the probability that a trapezoid is formed ?

4. A vertex of a paper square is folded onto another vertex chosen at random. What is the probability that a triangle is formed ?

5. Three randomly chosen vertices of a regular hexagon cut from paper are folded to the centre of the hexagon. What is the probability that an equilateral triangle is formed?

6. A piece of string is cut at random into two pieces. What is the probability that the short piece is less than half the length of the long piece ?

7. A paper square is cut at random into rectangles. What is the probability that larger perimeter is more than 1 ½ times the smaller ?

8. The numbers 2, 3 and 4 are substituted at random for a,b,c in the equation $ax + b = c$.

   a) What is the probability that the solution is negative ?

   b) If c is not 4, what is the probability that the solution is negative ?

9. Each coefficient in the equation $ax^2 + bx + c = 0$ is determined by throwing an ordinary die. Find the probability that the equation will have real roots.

10. The numbers 1, 2 and 3 are substituted at random for a,b and c in the quadratic equation $ax^2 + bx + c = 0$.

    a) What is the probability that $ax^2 + bx + c = 0$ can be factored?

    b) What is the probability that $ax^2 + bx + c = 0$ has real roots ?

11. Two faces of a cube are chosen at random. What is the probability that they are in parallel planes ?

12. Three edges of a cube are chosen at random. What is the probability that each edge is perpendicular to the other two ?

13. A point P is chosen at random in the interior of square ABCD. What is the probability that triangle ABP is acute ?

14. Find the probability of the event of the sine of a randomly chosen angle is greater than 0.5.

15. Suppose you ask individuals for their random choices of letters of the alphabet. How many people would you need to ask so that the probability of at least one duplication becomes better than 1 in 2 ?

16. Six boys and six girls sit in a row randomly. Find the probability that i) the six girls sit together, ii) the boys and girls sit alternately ?

17. If the letters of the word 'MATHEMATICS' are arranged at random, what is the probability that there will be exactly 3 letters between H and C ?

18. The sum of two non-negative quantities is equal to 2n. Find the probability that their product is not less than ¾ times their greatest product.

19. If A and B are independent events then show that $\overline{A}$ and $\overline{B}$ are also independent events.

20. Cards are dealt one by one from well-shuffled pack of cards until an ace appears.. Find the probability of the event that exactly n cards are dealt before the first ace appears.

21. If four squares are chosen at random on a chess-board, find the chance that they should be in a diagonal line.

22. Prove that if P(A/B) < P(A) then P(B/A) < P(B) ?

23. If n people are seated at a round table, what is the chance that the two named individuals will be next to each other ?

24. A and B are two very weak students of Mathematics and their chances of solving a problem correctly are 1/8 and 1/12 respectively. If the probability of their making common mistake is 1/1001 and they obtain the same answer, find the chance that their answer is correct.

25. A bag contains an unknown number of blue and red balls. If two balls are drawn at random, the probability of drawing two red balls is five times the probability of drawing two blue balls. Furthermore, the probability of drawing one ball of each

colour is six times the probability of drawing two blue balls. How many red and blue balls are there in the bag ?

26. A thief has a bunch of n keys, exactly one can open a lock. If the thief tries to open the lock by trying the keys at random, what is the probability that he requires exactly k attempts, if he rejects the keys already tried ? Find the probability of the same event when he does not reject the keys already tried.

27. A problem in Mathematics is given to three students and their chances of solving it are ½, 1/3 and ¼. What is the probability that the problem will be solved ?

28. A bag A contains 3 white balls and 2 black balls and other bag B contains 2 white and 4 black balls. A bag and a ball out of it are picked at random. What is the probability that the ball is white ?

29. Cards are drawn one-by-one at random from a well-shuffled pack of 52 cards until 2 aces are obtained for the first time. If N is the number of cards required to be drawn, then show that

$$p(N = n) = \frac{(n-1)(52-n)(51-n)}{50.59.\ 17.\ 13}$$

Where $2 \leq n \leq 50$.

30. A.B. C are events such that

$P(A) = 0.3$, $P(B) = 0.4$, $P(C) = 0.8$, $P(A \cap B) = 0.08$, $P(A \cap C) = 0.28$,

$P(A \cap B \cap C) = 0.09$

If $P(AUBUC) \geq 0.75$, then show that $P(B \cap C)$ lies in the interval (0.23, 0.48).

31. A man takes a step forward with probability 0.4 and backwards with probability 0.6. Find the probability that at the end of eleven steps, he is one step away from the starting point.

32. .Huyghens Problem. A and B throw alternately a pair of dice in that order. A wins if he scores 6 points before B gets 7 points, in which case B wins. If A starts the game, what is his probability of winning ?

33. A Doctor goes to work following one of three routes A, B,C. His choice of route is independent of the weather. If it rains, the probabilities of arriving late, following A, B, C are 0.06, 0.15, 0.12 respectively. The corresponding probabilities, if it does not rain, are 0.05, 0.10, 0.15.

a) Given that on a sunny day he arrives late, what is the probability that he took route C ? Assume that, on average, one in every four days is rainy.

b) Given that on a day he arrives late, what is the probability that it is a rainy day.

34. Bonferroni's Inequality. Given n(>1) events $A_1, A_2, \ldots A_n$ show that

$$\sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j) \leq P \bigcup_{i=1}^{n} P(A_i) \leq \sum_{i=1}^{n} P(A_i)$$

35. .Show that for any n events $A_1, A_2, \ldots, A_n$

i) $P(\bigcap_{i=1}^{n} A_i) \geq 1 - \sum_{i=1}^{n} P(\overline{A_i})$

ii) $P(\bigcap_{i=1}^{n} A_i) \geq \sum_{i=1}^{n} P(A_i) - (n-1)$

36. If A and B are mutually exclusive and $P(AUB) \neq 0$, then prove that

$$P(A/AUB) = \frac{P(A)}{P(A) + P(B)}$$

37. If 2n boys are divided into two equal groups, find the probability that the two tallest boys will be a) in different subgroups, and b) in the same subgroup.

38. A small boy is playing with a set of 10 coloured cubes and 3 empty boxes. If he puts the 10 cubes into the 3 boxes at random, what is the probability that he puts 3 cubes in one box, 3 in another box, and 4 in the third box ?

39. The sample space consists of the integers from 1 to 2n, which are assigned probabilities to their logarithms. A) Find the probabilities, b) Show that the conditional probability of the integer 2, given that an even integer occurs is

$$\frac{\log 2}{n\log 2 + \log n!}.$$

40.a) Each of n boxes contains four white and six black balls, while another box contains five white and five black balls. A box is chosen at random from the (n+1) boxes, and two balls are drawn from it, both being black. The probability that five white and three block balls remain in the chosen box is 1/7. Find n.

40b). A point is selected at random inside a circle. Find the probability p that the point is closer to the centre of the circle than to its circumference.

41. What is the probability that two numbers chosen at random will be prime to each other?

42. In throwing n dice at a time, what is the probability of having each of the points 1,2,3,4,5,6 appears at least once ?

43. A bag contains 50 tickets numbered 1,2,3,..., 50 of which five are drawn at random and arranged in ascending order of magnitude ($x_1 < x_2 < x_3 < x_4 < x_5$), what is the probability that $x_3 = 30$?

44. Of the three independent events, the probability that the first only to happen is ¼, the probability that the second only to happen is 1/8 and the third only to happen is 1/12. Obtain the unconditional probabilities of the three events.

45. What is the least number of persons required if the probability exceeds ½ that two or more of them have the same birthday (year of birth need not match) ?

46. If m things are distributed among 'a' men and 'b' women, then show that the chance that the number of things received by men is

$$\frac{1}{2}\,\frac{(b+a)^m - (b-a)^m}{(b+a)^m}$$

47. A pair of dice is rolled until either 5 or a 7 appears. Find the probability that a 5 occurs first.

48. In a certain standard tests I and II, it has been found that 5% and 10% respectively of 10^{th} grade students earn grade A. Comment on the statement that the probability is $\frac{5}{100}\,\frac{10}{100} = \frac{1}{200}$ that a 10^{th} grade student chosen at random will earn grade A on both tests.

49. A bag contains three coins, one of which is coined with two heads while the other two coins are fair. A coin is chosen at random from the bag and tossed four times in succession. If head turns up each time, what is the probability that this is the two headed coin ?

50. A man stands in a certain position (which we may call the origin) and tosses a fair coin. If a head appears he moves one unit of length to the left. If a tail appears, he moves one unit to the right. After 10 tosses of the coin, what are his possible positions and what are the probabilities ?

51. There are 12 compartments in a train going from Madras to Bangalore. Five friends travel by the train for some reasons could not meet each other at Madras station before getting aboard. What is the probability that the five friends will be in different compartments ?

52. The numbers 1,2,3,4,5 are written on five cards. Three cards are drawn in succession and at random from the deck, the resulting digits are written from left

to right. What is the probability that the resulting three digits number will be even ?

53. Suppose n dice are thrown at a time. What is the probability of getting a sum 'S' of points on the dice ?

54. A certain mathematician always carries two match boxes, each time he wants a match-stick he selects a box at random. Inevitably, a moment comes when he finds a box empty . Find the probability that the movement the first box is empty, the second contains exactly r match sticks (assume that each box contain N match-sticks initially).

55. There are 3 cards identical in size. The first card is red both sides, the second one is black both sides and the third one red one side and black other side. The cards are mixed up and placed flat on a table. One is picked at random and its upper (visible) side was red. What is the probability that the other side is black ?

56. N different objects 1,2,...., n are distributed at random in n places marked 1,2,...n. Find the probability that none of the objects occupies the place corresponding to its number.

Answers :

1. ½
2. A) 2/3 b) 5/6 c) 3/4, 3/4
3. 1/3
4. 1/3
5. 1/10
6. 2/3
7. 2/5
8. a) ½ b) 3/4
9. 43/216
10. a) 1/3 b) 1/3
11. 1/5
12. 2/55
13. $1 - \pi/8 = 0.6073$
14. 2/3
15. 7
16. i) $\dfrac{7!\ 6!}{12}$ ii) $\dfrac{2\ (6!)^2}{12!}$
17. 7/55
18. ½
20. $\dfrac{4\,(51-n)\,(50-n)\,(49-n)}{52.51.50.49}$
21. $\dfrac{91}{158844}$

23. $\dfrac{2}{n-1}$

24. 13/24

25. Red = 6, Blue = 3

26. 1/n, 1/n $\left(1 - \dfrac{1}{n}\right)^{k-1}$

27. 3/4

28. 7/15

31. $(0.4)^5\,(0.6)^5$

32. 30/61

33. a) 0.5 b) 41/131

37. a) $\dfrac{n}{2n-1}$ b) $\dfrac{n-1}{4n-2}$

38. $\dfrac{3 \quad 10!}{3!\;\; 3!\;\; 4!\;\; 3^{10}}$

39. a) K log 2i b) (log 2i) (n log 2 + log n!)

40. a) 4 b) 1/4

41. $\pi\left(1 - \dfrac{1}{r^2}\right) = \dfrac{6}{\pi^2}$

42. $1 - n\left(\dfrac{5}{6}\right)^n + \binom{n}{2}\left(\dfrac{4}{6}\right)^n - \binom{n}{3}\left(\dfrac{3}{6}\right)^n + \binom{n}{4}\left(n\,\dfrac{2}{6}\right)^n - \binom{n}{5}\left(\dfrac{1}{6}\right)^n$

43. $\dfrac{\binom{29}{2}\binom{20}{2}}{\binom{50}{2}}$

44. ½, 1/3, 1/4

45. 23

47. 2/5

49. 8/9

50.

Distance from origin	-10	-8	-6	-4	-2	0	2
Prob	$\left(\frac{1}{2}\right)^{10}$	$\binom{10}{1}\left(\frac{1}{2}\right)^{10}$	$\binom{10}{2}\left(\frac{1}{2}\right)^{10}$	$\binom{10}{3}\left(\frac{1}{2}\right)^{10}$	$\binom{10}{4}\left(\frac{1}{2}\right)^{10}$	$\binom{10}{5}\left(\frac{1}{2}\right)^{10}$	$\binom{10}{6}\left(\frac{1}{2}\right)^{10}$

4	6	8	10
$\binom{10}{7}\left(\frac{1}{2}\right)^{10}$	$\binom{10}{8}\left(\frac{1}{2}\right)^{10}$	$\binom{10}{9}\left(\frac{1}{2}\right)^{10}$	$\left(\frac{1}{2}\right)^{10}$

51. 55/144

52. 1/5

53. $(-1)^k \binom{n}{k} \binom{s-6k-1}{n-1} | 6^n$

54. $\dfrac{\binom{2n-r}{n}}{2^{2n-r}}$

55. ½

56. $\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} = \ldots + (-1)^n \frac{1}{n!}$.

Compiled by B.C. Basti

Probability

Objective Type Questions

1. The probability that at least one of A and B occurs is 0.6. If A and B occur simultaneously with probability 0.3, then $P(\bar{A}) + P(\bar{B})$ is

a) 0.9 b) 1.15 c) 1.1 d) 1.2

2. For three events A, B and C, P (exactly one of the events A or B occurs) = P(exactly one of the events B or C occurs) = P(exactly one of the events C or A occurs) = p and p(all the three events occur simultaneously) = p^2, where $0 < p < ½$. Then the probability of at least one of the events A, B and C occurring is

a) $\frac{3p + 2p^2}{2}$ b) $\frac{p + 3p^2}{4}$

c) $\frac{p + 3p^2}{2}$ d) $\frac{3p + 2p^2}{4}$

3. Let E and F be two independent events. The probability that both E and F happen is 1/12 and the probability that neither E nor F happens is ½. Then

a) $P(E) = \frac{1}{3}, P(F) = \frac{1}{4}$ b) $P(E) = \frac{1}{2}, P(F) = \frac{1}{6}$

c) $P(E) = \frac{1}{6}, P(F) = \frac{1}{2}$ d) $P(\bar{E}) = \frac{1}{2}, P(\bar{F}) = 1$

4. There are two balls in an urn whose colours are not known (each ball can be either white or black). A white ball is put into the urn. A ball is drawn from the urn. The probability that it is white is

a) $\frac{1}{4}$ b) $\frac{1}{3}$ c) $\frac{2}{3}$ d) $\frac{1}{6}$

5. Let A, B, C be three mutually independent events. Consider the statements S_1 and S_2.
S_1 : A and $B \cup C$ are independent.
S_2 : A and $B \cap C$ are independent.
Then,

a) Both S_1 and S_2 are true. b) Only S_1 is true.
c) Only S_2 is true d) Neither S_1 nor S_2 is true

6. Given that A, B, C are events such that $P(A) = P(B) = P(C) = \frac{1}{5}$, $P(A \cap B) = P(B \cap C) = 0$ and $P(A \cap C) = \frac{1}{10}$. The probability that at least one of the events A, B or C occurs is

a) $\frac{3}{5}$ b) $\frac{1}{2}$ c) 1 d) $\frac{7}{10}$

7. Let A and B be two events such that $P(A \cap \overline{B}) = 0.20$, $P(\overline{A} \cap B) = 0.15$ and $P(A \cap B) = 0.10$, then $P(A \mid B)$ is

a) $\frac{2}{7}$ b) $\frac{5}{7}$ c) $\frac{1}{7}$ d) $\frac{3}{7}$

8. Let A and B be two events such that $P(A) = 0.3$ and $P(A \cup B) = 0.8$. If A and B are independent events than P(B) is

a) $\frac{2}{7}$ b) $\frac{5}{7}$ c) $\frac{1}{7}$ d) $\frac{6}{7}$

9. A speaks the truth in 70 percent cases and B in 80 percent cases. The probability they will contradict each other in describing a single event is

a) 0.36 b) 0.38 c) 0.4 d) 0.42

10. If $\frac{1+4p}{4}$, $\frac{1-p}{3}$ and $\frac{(1-2p)}{2}$ are the probabilities of three mutually exclusive events then

a) $\frac{1}{4} \le p \le \frac{1}{2}$ b) $\frac{1}{3} \le p \le \frac{1}{2}$

c) $\frac{1}{6} \le p \le \frac{1}{2}$ d) None of these

11. Suppose that $P(A) = \frac{3}{5}$ and $P(B) = \frac{2}{3}$. Then

a) $P(A \cup B) \ge \frac{2}{3}$ b) $P(A \cap \overline{B}) = \frac{2}{3}$

c) $P(A \cap B) > \frac{3}{5}$ d) $P(A \mid B) > \frac{9}{10}$

12. A fair die is thrown until a score of less than 5 is obtained. The probability of obtaining not less than 2 on the last throw is

a) $\frac{3}{4}$ b) $\frac{4}{5}$ c) $\frac{5}{6}$ d) $\frac{1}{3}$

13. An urn contains 6 white and 4 black balls. A fair die is rolled and a number of balls equal to that appearing on the die is chosen from the urn at random. The probability that all the balls selected are white is

a) $\frac{1}{6}$ b) $\frac{1}{7}$ c) $\frac{1}{5}$ d) $\frac{1}{8}$

14. Seven digits from the numbers 1,2,3,4,5,6,7,8 and 9 are written in the random order. The probability that this seven digit number is divisible by 9 is

a) $\frac{1}{7}$ b) $\frac{2}{7}$ c) $\frac{1}{9}$ d) $\frac{1}{3}$

15. Ten students are seated at random in a row. The probability that two particular students are not seated together is

a) $\frac{2}{3}$ b) $\frac{3}{4}$ c) $\frac{4}{5}$ d) $\frac{5}{6}$

16. Six boys and six girls sit in a row randomly. The probability that the boys and girls sit alternatively is

a) $\frac{1}{462}$ b) $\frac{1}{132}$ c) $\frac{1}{66}$ d) $\frac{4}{462}$

17. A three digit number is formed using the digits 1,2,3,4,5,6 repetitions being allowed. The probability that the number is divisible by 4 is

a) $\frac{2}{9}$ b) $\frac{1}{4}$ c) $\frac{7}{36}$ d) None of these

18. A three digit number is formed using the digits 1,2,3,4,5,6 without repetition of digits. The probability that the number is divisible by 4 is

a) $\frac{4}{15}$ b) $\frac{7}{30}$ c) $\frac{1}{5}$ d) None of these

19. Two cards are drawn at random from a pack of 52 cards. The probability that both are aces is

a) $\frac{2}{221}$ b) $\frac{1}{221}$ c) $\frac{1}{1326}$ d) $\frac{3}{221}$

20. Two cards are drawn successively with replacement from a pack of 52 cards. The probability that both are aces is

a) $\frac{1}{169}$ b) $\frac{1}{196}$ c) $\frac{1}{221}$ d) None of these

21. A person draws a card from a pack of playing cards, puts it back, shuffles the pack and again draws a card. He continues doing this until a spade card is seen. The chance that he will fail the first two times is

a) $\frac{9}{64}$ b) $\frac{1}{64}$ c) $\frac{1}{16}$ d) $\frac{9}{16}$

22. A card is drawn from an ordinary pack and a gambler bets that it is a spade or an ace. The odds against his winning the bet are

a) 4 to 13 b) 13 to 4 c) 9 to 4 d) 4 to 9

23. Two fair dice are tossed. Let x be the event that the first die shows an even number and y be the event that the second die shows an odd number. The events x and y are

a) mutually exclusive b) independent and mutually exclusive
c) dependent d) none of these

24. The probability in the toss of two dice we obtain the sum 7 or 11 is

a) $\frac{1}{6}$ b) $\frac{1}{18}$ c) $\frac{2}{9}$ d) $\frac{23}{108}$

25. The probability that in the toss of two dice an even sum or sum less than 5 is obtained is

a) $\frac{1}{2}$ b) $\frac{1}{6}$ c) $\frac{2}{3}$ d) $\frac{5}{9}$

26. Two events A and B have probabilities 0.25 and 0.50 respectively. The probability that both A and B occur is 0.14. Then, the probability that neither A nor B occurs is

a) 0.39 b) 0.25 c) 0.11 d) None of these

27. A bag contains 3 red, 6 white and 7 blue balls. Two balls are drawn. The probability that one is white and one is blue is

a) $\frac{13}{20}$ b) $\frac{7}{20}$ c) $\frac{6}{20}$ d) None of these

28. A bag contains 2 red, 5 white, 6 black balls. Three balls are drawn. The probability that all the coloured balls are drawn is

a) $\frac{4}{11}$ b) $\frac{2}{11}$ c) $\frac{1}{11}$ d) None of these

29. The probability that an event A happens in one trial of an experiment is 0.4. Three independent trials of the experiment are performed. The probability that the event happens, at least once is

a) 0.936 b) 0.784 c) 0.904 d) None of these

30. Suppose the probability for the birth of a male child is 0.55 and that two successive births are independent. A woman has 5 children. The probability that she will have children of both sexes is

a) $(0.55)^3$ b) $(0.45)^3$ c) $(0.55)^3\,(0.45)^3$
d) $1 - \{\,(0.55)^5 + (0.45)^5\,\}$

31. A student takes a TRUE or FALSE examination. He is completely unprepared and makes a random guess of the answer. Then the probability that he guesses correctly at least nine times out of 10 times is

a) $\frac{11}{1024}$ b) $\frac{1013}{1024}$ c) ${}^{10}C_9 \left(\frac{1}{2}\right)^{10}$ d) None of these

32. The probability that a marksman will hit a target is 0.25. Then, the probability that he has, at the most, 9 hits out of 10 shots is

a) ${}^{10}C_1\,(0.25)^9\,(0.75)$ b) $(0.75)^{10}$
c) $1-(0.25)^{10}$ d) $1-(0.75)^{10}$

33. India plays two matches each with West Indies and Australia. In any match, the probabilities of India getting 0, 1, 2 points are 0.45, 0.05, 0.50 respectively. Assuming that the outcomes are independent, the probability of India getting at least 7 points is

a) 0.8750 b) 0.0875 c) 0.0625 d) 0.0250

34. A coin is tossed 4 times. The probability of getting 3 heads is

a) $\frac{1}{8}$ b) $\frac{3}{8}$ c) $\frac{1}{2}$ d) None of these

35. 8 coins are tossed simultaneously. The probability of getting at least 6 heads is

a) $\frac{57}{64}$ b) $\frac{229}{256}$ c) $\frac{7}{64}$ d) $\frac{37}{256}$

36. The probability of an event A happening is 0.5 and of B happening is 0.3. If A and B are mutually exclusive events, then the probability of neither A nor B occurring is

a) 0.6   b) 0.5   c) 0.7   d) None of these

37. It is known that at noon at a certain place, the sun is hidden by clouds on an average two days out of three. The probability that at noon on at least four out of five specified future days the sun will be shining is

a) $\frac{11}{243}$ b) $\frac{10}{243}$ c) $\frac{1}{243}$ d) None of these

38. If A and B are two events such that P(A) > 0 and P(B) ≠ 1, then $P(\bar{A} \mid \bar{B})$ is equal to

a) $1 - P(A \mid B)$ b) $1 - P(\bar{A} \mid \bar{B})$
c) $\{1 - P(A \mid B)\} / P(\bar{B})$ d) $P(\bar{A}) \mid P(\bar{B})$

39. If A and B are any two arbitrary events then,

a) $P(A \cap B) \geq P(A) + P(B) - 1$ b) $P(A \cap B) \leq P(A) + P(B) - 1$
c) $P(A \cap B) = P(A) + P(B) - 1$ d) None of these

40. If A and B are independent events, then

a) $P(A \cup B) > 1 - P(\bar{A}) . P(\bar{B})$ b) $P(A \cup B) < 1 - P(\bar{A}) . P(\bar{B})$
c) $P(A \cup B) = 1 - P(\bar{A}) . P(\bar{B})$ d) None of these

41. If A and B are mutually exclusive events, then $P(A \mid A \cup B)$ is

a) $P(A)$ b) $P(A) / \{P(A) + P(B)\}$
c) $P(B) / \{P(A) + P(B)\}$ d) None of these

42. If $A_1, A_2, \ldots, A_n$ are n independent events and $P(A_i) = \dfrac{1}{i+1}$ $(1 \leq i \leq n)$ then the probability that none of the events occurs is

a) $\dfrac{1}{n+1}$ b) $\dfrac{n}{n+1}$

c) $\dfrac{n-1}{n+1}$ d) None of these

43. Three identical dice are rolled. The probability that the same number will appear on each of them is

a) $\frac{1}{6}$ b) $\frac{1}{36}$ c) $\frac{1}{18}$ d) $\frac{3}{28}$

44. The probability that at least, one of the events A and B occurs is 0.6. If A and B occur simultaneously with probability 0.2 then $P(\overline{A}) + P(\overline{B})$ is

a) 0.4 b) 0.8 c) 1.2 d) None of these

45. A purse contains 4 copper coins and 3 silver coins. A second purse contains 6 copper coins and 2 silver coins. A coin is taken out of any purse. The probability that it is a copper coin is

a) $\frac{4}{7}$ b) $\frac{3}{4}$ c) $\frac{3}{7}$ d0 $\frac{37}{56}$

46. If A and B are any two independent events such that $P(\overline{A}) = 0.7$, $P(\overline{B}) = p$ and $P(A \cup B) = 0.8$ then p is

a) $\frac{5}{7}$ b) 1 c) $\frac{2}{7}$ d) $\frac{3}{7}$

47. There are four machines and it is known that exactly two of them are faulty. They are tested one by one in a random order till both the faulty machines are identified. The probability that only two tests are needed is

a) $\frac{1}{3}$ b) $\frac{1}{6}$ c) $\frac{1}{2}$ d) $\frac{1}{4}$

48. A fair coin is tossed repeatedly. If tail appears on first four tosses, then the probability of head appearing on fifth toss is

a) $\frac{1}{2}$ b) $\frac{1}{32}$ c) $\frac{31}{32}$ d) $\frac{1}{5}$

49. One bag contains 3 white and 2 black balls. A second bag contains 5 white and 3 black balls. A ball is drawn out of any bag. The probability that it is white is

a) $\frac{49}{40}$ b) $\frac{37}{40}$ c) $\frac{49}{80}$ d) None of these

50. The probability that out of 2×2 determinants by using 0 and 1 only, the value of determinant chosen is positive is

a) $\frac{1}{8}$ b) $\frac{3}{16}$ c) $\frac{1}{16}$ d) None of these

Key to Objective Questions on Probability

1.	c	2.	a
3.	a	4.	c
5.	a	6.	b
7.	a	8.	b
9.	b	10.	a
11.	a	12.	a
13.	c	14.	c
15.	c	16.	a
17.	b	18.	a
19.	b	20.	a
21.	a	22.	c
23.	d	24.	c
25.	d	26.	a
27.	b	28.	d
29.	b	30.	d
31.	a	32.	c
33.	b	34.	d
35.	d	36.	d
37.	a	38.	c
39.	a	40.	c
41.	b	42.	a
43.	b	44.	d
45.	d	46.	c
47.	b	48.	a
49.	c	50.	b

By N. M. Rao

SYLLABUS : *Vectors as directed line segment, Magnitude and direction of a vector, Equal Vectors, Unit vector, Zero vector, Position vector of a point, localized and free vectors, parallel and collinear vectors, Components of a Vector, Vectors in two and three dimensions, Addition of vectors, Multiplication of a vector by a scalar, position vector of the point dividing a given straight line in a given ratio, Application of vectors in problems of plane geometry.*

POINTS TO REMEMBER

1. Definitions :

(*i*) **Scalars.** A scalar is a physical quantity that is specified by magnitude only. It is represented by a real number along with suitable unit. Thus length, mass, volume, temperature, density, speed are scalars.

(*ii*) **Vectors.** A vector is a physical quantity that is specified by both magnitude and direction. It is represented by a directed line segment. Thus displacement, velocity, acceleration, force are vectors.

2. (*i*) Vector as Directed Line Segment : A directed line segment is a line segment with an arrowhead showing direction. Its two end-points are distinguished as *Initial* and *Terminal*. The directed line segment whose initial point is A and terminal point B is denoted by the symbol $\overrightarrow{AB}$. Its direction is from A to B *i.e.*, from the initial point to the terminal point.

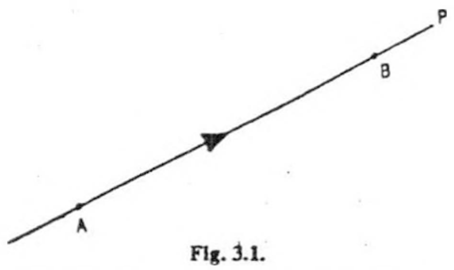

Fig. 3.1.

(*ii*) **Magnitude and Direction of a Vector :** In case we represent the vector $\vec{a}$ by the line segment $\overrightarrow{AB}$, then length or magnitude of $\overrightarrow{AB}$ is given by

$$a = |\vec{a}| = |\overrightarrow{AB}|$$

where A is called the initial point and B is called the terminal point.

The direction of vector $\overrightarrow{AB}$ is defined from A to B.

3. Types of Vectors :

(*i*) **Null Vector :** When A and B coincide, we get a null or zero vector. Thus a vector whose length or magnitude is zero is called a **null** or **zero vector**, denoted by $\vec{0}$. Any non-zero vector is called a **proper vector.**

(*ii*) **Unit Vector :** A vector whose magnitude is unity is called a unit vector. If we divide a vector by its magnitude, we get a unit vector in the same direction.

Thus $\frac{\vec{a}}{a}$ is a unit vector in the direction of $\vec{a}$.

(*iii*) **Equal Vectors :** Two vectors $\vec{a}$ and $\vec{b}$ are said to be equal if they have the same magnitude, same or parallel direction in the same sense. It is written as $\vec{a} = \vec{b}$.

(*iv*) **Collinear Vectors :** Two or more than two vectors are called collinear vectors when they are parallel to the same line.

(*v*) **Coplanar Vectors :** Vectors are said to be coplanar when either they lie in the same plane or are parallel to the same plane.

(*vi*) **Negative (or oppsite) Vectors :** If the line vector $\overline{OA}$ which has the same magnitude but in opposite direction to that the vector of a, then it is called the negative or opposite of a and is denoted by $-\vec{a}$ or $-\overrightarrow{OA}$ (see Fig. 3.2).

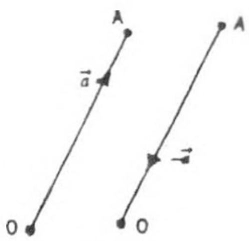

Fig. 3.2.

(*vii*) **Position Vectors :** If the vector $\overline{OA}$ represents the position of a point A relative to a fixed point O, then $\overline{OA}$ is called the **position vector** of the point A with reference to the point O as origin (or origin of reference).

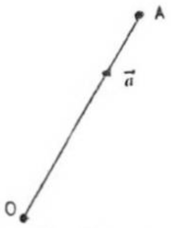

Fig. 3.3.

(*viii*) **Localised Vectors :** A vector drawn parallel to a given vector through a specified point as the initial point, is known as a localised vector.

(*ix*) **Free Vectors :** If the initial point of a vector is not specified, it is said to be a free vector.

(*x*) **Like Vectors :** Two vectors of any magnitude (or modulus) are said to be **like vectors** if their direction is the same.

Thus all vectors drawn in the same direction, whatever their magnitudes may be, are called **like vectors** (see Fig. 3.4).

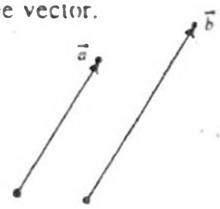

Fig. 3.4.

(*xi*) **Unlike Vectors :** Two vectors of any magnitude are said to be **unlike** vectors if their directions be **opposite** as shown in the adjoining Fig. 3.5.

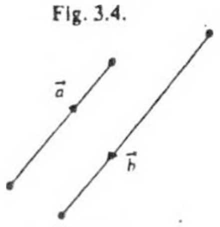

Fig. 3.5.

4. (*i*) **Components of a Vector :** Let i, j, k be the unit vectors along the axes of x, y, z respectively. If $P(x, y, z)$ be any point in space, then

$$\overrightarrow{OP} = xi + yj + zk$$

$$|\overrightarrow{OP}| = \sqrt{(x^2 + y^2 + z^2)}.$$

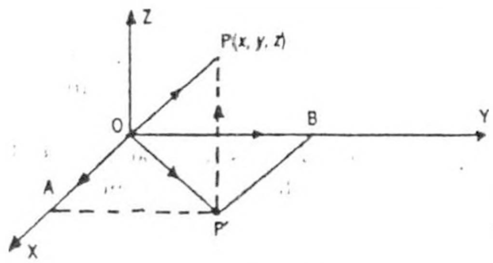

Fig. 3.6.

If $\vec{F} = F_1 i + F_2 j + F_3 k$, then $F_1 i$, $F_2 j$, $F_3 k$ are called components of the vector $\vec{F}$ along OX, OY, OZ respectively. $\vec{F}$ is called resultant of $F_1 i$, $F_2 j$, $F_3 k$.

(*ii*) **Any vector in space :** Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be any two points in space, then

$$\overrightarrow{OP} + \overrightarrow{PQ} = \overrightarrow{OQ} \text{ or } \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$

or $$\overrightarrow{PQ} = (x_2 - x_1)\, i + (y_2 - y_1)\, j + (z_2 - z_1)\, k$$

and $$|\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 + y_1)^2 + (z_2 - z_1)^2}$$

(*iii*) **Magnitude of a vector :** Let PQ be a vector in the plane XOY whose initial point is $P(x_1, y_1)$ and terminal point $Q(x_2, y_2)$. We know that $(x_2 - x_1)$ and $(y_2 - y_1)$ are called the components of vector PQ along the x and y-axis, respectively. $(x_2 - x_1)\, i$ and $(y_2 - y_1)\, j$ are called the component vectors of the vector PQ.

The magnitude of PQ can be determined by applying the Pythagorean theorem. We have

$$|PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Thus, if $a = a_1 i + a_2 j$, then

$$|a| = \sqrt{a_1^2 + a_2^2}$$

Definition : The magnitude or **modulus** of a vector PQ denoted by $|PQ|$, or simply PQ, is the length of the line segment PQ.

Note that **the magnitude of a vector is never negative.** In particular, $|PQ| = |-PQ|$. Modulus of a zero vector is zero.

5. **Operations on Vectors :**

Addition of Vectors :

(*i*) **Triangle Law of Addition of Two Vectors :** The law states that if two vectors are represented by the two sides of a triangle, taken in order, then their sum (or resultant) is represented by the third side of the triangle but in the reverse order.

Let $\vec{a}, \vec{b}$ be the given vectors. Let the vector $\vec{a}$ be represented by the directed segment $\overline{OA}$ and the vector

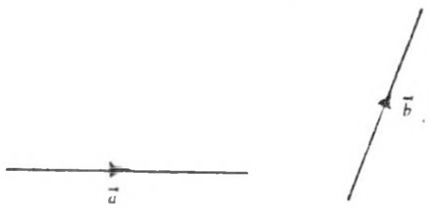

Fig. 3.7.

$\vec{b}$ be the directed segment $\overline{AB}$ so that the terminal point A of $\vec{a}$ is the initial point of $\vec{b}$. Then the directed segment OB (*i.e.*, $\overline{OB}$) represents the sum (or resultant) of $\vec{a}$ and $\vec{b}$ and is written as $\vec{a} + \vec{b}$.

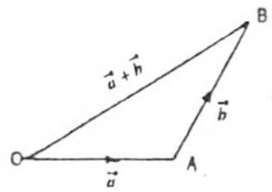

Fig. 3.8.

Thus $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \vec{a} + \vec{b}$.

> **Notes :** 1. The method of drawing a triangle in order to define the vector sum $(\vec{a}+\vec{b})$ is called *triangle law of addition of two vectors.*
>
> 2. Since any side of a triangle is less than the sum of the other two sides.
>
> $\therefore$ Modulus of $\overline{OB}$ is not equal to the sum of the moduli of $\overline{OA}$ and $\overline{AB}$.

(*ii*) **Parallelogram Law of Vectors :** In a parallelogram OABC, if $\overrightarrow{OA}$ and $\overrightarrow{AB}$ represent $\vec{a}$ and $\vec{b}$ respectively, then the diagonal $\overrightarrow{OB}$ represents in magnitude and direction the sum $\vec{a}+\vec{b}$. This is known as parallelogram law of addition of vectors.

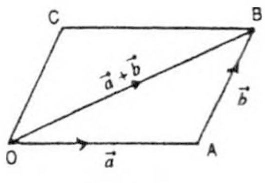

Fig. 3.9.

(*iii*) **Properties of Vector Addition :**

1. **Vector addition is Commutative :** If $\vec{a}$ and $\vec{b}$ be any two vectors, then

$$\vec{a}+\vec{b}=\vec{b}+\vec{a}$$

Proof. Let the vectors $\vec{a}$ and $\vec{b}$ be represented by the directed segments $\overline{OA}$ and $\overline{AB}$ respectively so that $\vec{a}=\overline{OA}, \vec{b}=\overline{AB}$.

Now $\qquad \overrightarrow{OB}=\overrightarrow{OA}+\overrightarrow{AB}$

or $\qquad \overrightarrow{OB}=\vec{a}+\vec{b} \qquad \ldots(i)$

Complete the llgm OABC,

Then $\quad \overrightarrow{OC}=\overrightarrow{AB}=\vec{b}$ and $\overrightarrow{CB}=\overrightarrow{OA}=\vec{a}$

$\therefore \qquad \overrightarrow{OB}=\overrightarrow{OC}+\overrightarrow{CB}=\vec{b}+\vec{a} \qquad \ldots(ii)$

From (*i*) and (*ii*), we have

$$\vec{a}+\vec{b}=\vec{b}+\vec{a}.$$

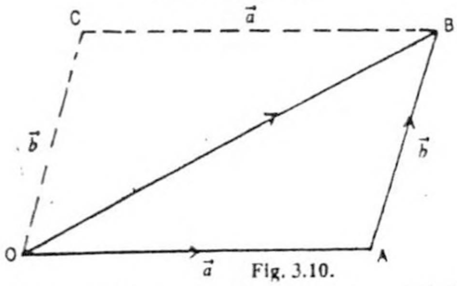

Fig. 3.10.

2. **Vector Addition is Associative :** If $\vec{a}, \vec{b}, \vec{c}$ are any three vectors, then

$$\vec{a}+(\vec{b}+\vec{c})=(\vec{a}+\vec{b})+\vec{c}. \qquad [M.\ Imp.]$$

Proof. Let the vectors $\vec{a}, \vec{b}, \vec{c}$ be represented by the directed segments $\overline{OA}, \overline{AB}, \overline{BC}$ respectively ; so that

$$\vec{a}=\overline{OA}, \quad \vec{b}=\overline{AB}, \quad \vec{c}=\overline{BC}$$

Then $\quad \vec{a}+(\vec{b}+\vec{c})=\overrightarrow{OA}+(\overrightarrow{AB}+\overrightarrow{BC})$

$\qquad =\overrightarrow{OA}+\overrightarrow{AC} \qquad [\Delta$ Law of addition$]$

$\qquad =\overrightarrow{OC} \qquad [\Delta$ Law of addition$]$

$\therefore \qquad \vec{a}+(\vec{b}+\vec{c})=\overrightarrow{OC} \qquad \ldots(i)$

Again, $\quad (\vec{a}+\vec{b})+\vec{c}=(\overrightarrow{OA}+\overrightarrow{AB})+\overrightarrow{BC}$

$\qquad =\overrightarrow{OB}+\overrightarrow{BC} \qquad [\Delta$ Law of addition$]$

$\qquad =\overrightarrow{OC} \qquad [\Delta$ Law of addition$]$

$\therefore \qquad (\vec{a}+\vec{b})+\vec{c}=\overrightarrow{OC} \qquad \ldots(ii)$

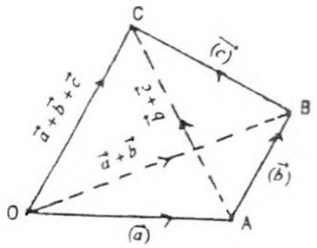

Fig. 3.11.

From (*i*) and (*ii*), we get

$$\vec{a}+(\vec{b}+\vec{c})=(\vec{a}+\vec{b})+\vec{c}.$$

3. **Identity vector for addition :** For every vector $a, \vec{a}+\vec{0}=\vec{a}$, where 0 is the zero vector and is the identity vector for addition.

Proof. Let $\overrightarrow{OA}=\vec{a}$ and $\overrightarrow{AA}=\vec{0}$

Now by addition of two vectors

$$\overrightarrow{OA}=\overrightarrow{OA}+\overrightarrow{AA}=\vec{a}+\vec{0}$$

$\therefore \qquad \vec{a}=\vec{a}+\vec{0}.$

4. **Additive inverse of a vector :** To every vector, **a**, there corresponds the vector **−a** (called its additive inverse) such that **a** + (**−a**) = **0**, where **0** is the zero vector.

Proof. Let $\overrightarrow{OA}=\vec{a}$; then $\overrightarrow{AO}=(-1)\vec{a}$.

Now $\overrightarrow{OA}+\overrightarrow{AO}=\overrightarrow{OO}$ (By definition of addition of two vectors)

$\therefore \qquad \vec{a}+(-\vec{a})=0.$

(*iv*) **Difference of Two Vectors :**

Geometrical Representation of $\vec{a}-\vec{b}$

Let the vector $\vec{a}, \vec{b}$ be represented by the directed segments $\overline{OA}, \overline{AB}$ respectively ; so that

$\vec{a} = \overrightarrow{OA}, \vec{b} = \overrightarrow{AB}$

Produce BA to C, such that

AC = BA

Then $\overrightarrow{AC} = \overrightarrow{BA}$

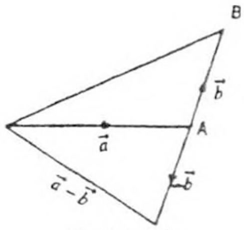

Fig 3.12.

$= -\overrightarrow{AB} = -\vec{b}$

$\therefore \quad \vec{a} - \vec{b} = \vec{a} + (-\vec{b}) = \overrightarrow{OA} + \overrightarrow{AC}$

$= \overrightarrow{OC}$ [By triangle law addition]

Hence $\vec{a} - \vec{b}$ is geometrically represented by the directed segment $\overrightarrow{OC}$.

(*v*) **Multiplication of a Vector by a Scalar :** If $\vec{a}$ be a vector and m a real positive number, then $m\,\vec{a}$ is defined to be a vector having direction as $\vec{a}$ and m times its magnitude.

Geometrical Representation

Let the vector $\vec{a}$ be represented by the directed segment $\overrightarrow{AB}$.

Case I. Let $m > 0$. Choose a point C on AB on the same side of A as B such that

$|\overrightarrow{AC}| = m\,|\overrightarrow{AB}|$

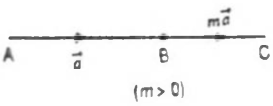

Fig. 3.13.

Then the vector $m\vec{a}$ is represented by $\overrightarrow{AC}$.

Case II. Let $m < 0$. Choose a point C on AB on the side of A opposite to that of B such that

$|\overrightarrow{AC}| = |m|\,|\overrightarrow{AB}|$

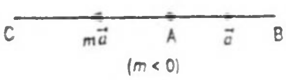

Fig. 3.14.

Then the vector $m\vec{a}$ is represented by $\overrightarrow{AC}$.

(*vi*) **Properties of Scalar Multiplication :**

(*a*) $m\vec{a} = \vec{a}.m$

(*b*) $m(n\vec{A}) = n(m\vec{A})$

(*c*) $(m+n)\vec{A} = m\vec{A} + n\vec{A}$

(*d*) $m\,(\vec{A} + \vec{B}) = m\vec{A} + m\vec{B}$.

6. Section formula :

Statement. If $\vec{a}$ and $\vec{b}$ are the position vectors of two points A and B, then the point C which divides AB in the ratio $m : n$, where m and n are positive real numbers. has the position vector.

$$\vec{c} = \frac{n\vec{a} + m\vec{b}}{m+n}$$

Proof. Let O be the origin of reference and let $\vec{a}$ and $\vec{b}$ be the position vectors of the given points A and B so that

$\overrightarrow{OA} = \vec{a}, \ \overrightarrow{OB} = \vec{b}$

Let C divide AB in the ratio $m : n$.

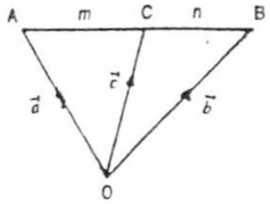

Fig. 3.15.

$$\therefore \quad \frac{AC}{CB} = \frac{m}{n} \qquad \ldots(i)$$

Hence m/n is positive or negative according as C divides AB internally or externally.

We have to express the position vector $\overrightarrow{OC}$ of the point C in terms of those of A and B.

We rewrite (*i*) as, nAC $= m$CB.

And obtain the vector equality, $n\overrightarrow{AC} = m\overrightarrow{CB}$. Expressing the vectors $\overrightarrow{AC}$ and $\overrightarrow{CB}$ in terms of the position vectors of the end points, we obtain

$$n\,(\overrightarrow{OC} - \overrightarrow{OA}) = m\,(\overrightarrow{OB} - \overrightarrow{OC})$$

$$\Rightarrow \quad (m+n)\,\overrightarrow{OC} = n\overrightarrow{OA} + m\overrightarrow{OB}$$

$$\Rightarrow \quad \overrightarrow{OC} = \frac{n\overrightarrow{OA} + m\overrightarrow{OB}}{m+n} = \frac{n\vec{a} + m\vec{b}}{m+n}.$$

Position Vector of the Centroid of a Triangle :

The position vector of the centroid G of a triangle ABC is

$$\frac{\vec{a} + \vec{b} + \vec{c}}{3}$$

where $\vec{a}, \vec{b}, \vec{c}$ are positive vectors of vertices A, B, C respectively.

Proof. $\vec{a}, \vec{b}, \vec{c}$ are position vectors of A, B, C respectively relative to any origin.

If D be the mid-point of BC, then its position vector is

$$\frac{\vec{b} + \vec{c}}{2}$$

[Mid-point formula]

The centroid G divides the median AD in the ratio 2 : 1, *i.e.*, AG : GD = 2 : 1.

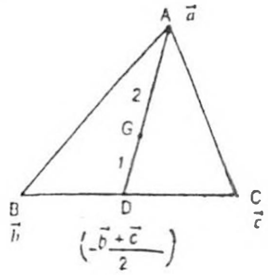

Fig. 3.16.

$\therefore$ Position vector of G

$$= \frac{2\left(\frac{\vec{b}+\vec{c}}{2}\right) + 1.\vec{a}}{2+1}$$

$$= \frac{\vec{b} + \vec{c} + \vec{a}}{3} \qquad \text{[Section formula]}$$

Hence position vector of G

$$= \frac{\vec{a}+\vec{b}+\vec{c}}{3} \quad \ldots(i)$$

7. Definitions :

(*i*) **Linear Combination.** A vector $\vec{r}$ is said to be a linear combination of the vectors $\vec{a}, \vec{b}, \vec{c}, \ldots$, if there exist scalars $x, y, z, \ldots$, such that

$$\vec{r} = x\vec{a} + y\vec{b} + z\vec{c} + \ldots$$

(*ii*) **Linearly Dependent.** A system of vectors $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ is said to be linearly dependent if there exist scalars $x_1, x_2, \ldots, x_n$ (not all zero) such that

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \ldots + x_n\vec{a}_n = \vec{0}.$$

(*iii*) **Linearly Independent.** A system of vectors $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ is said to be linearly independent if there exist scalars $x_1, x_2, \ldots, x_n$ (all zero) such that $x_1\vec{a}_1 + x_2\vec{a}_2 + \ldots + x_n\vec{a}_n = 0$ and a set of vectors $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ is said to be linearly independent if every relation of the type $x_1\vec{a}_1 + x_2\vec{a}_2 + \ldots + x_n\vec{a}_n = \vec{0}$ implies $x_1 = 0, x_2 = 0, \ldots, x_n = 0$.

TEXT-BOOK EXERCISE 3.1
TYPE—I
(*SOLVED EXAMPLES*)

Example 1. Find the component of the vector $\overrightarrow{PQ}$ along the direction OX if P is (x_1, y_1) and Q is (x_2, y_2) with reference to rectangular co-ordinate Ox ; Oy. [*T.B.Q. 1*]

Sol. In this case component of the vector $\overrightarrow{PQ}$ along the direction OX = ON − OM = $x_2 - x_1$. **Ans.**

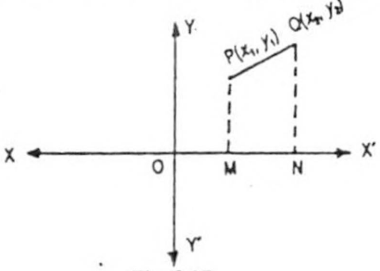

Fig. 3.17.

Example 2. Find the component of the vector $\overrightarrow{AB}$ where A is (1, 0) and B is (5, 0) along the direction $y = -x$ in the increasing direction of x. [*T.B.Q. 2*]

Sol. In this case, the angle made by the vector with the directed line whose equation is $y = -x$ in the anti-clockwise direction is $\theta = 315°$.

Hence, the component has magnitude

$$= |\overrightarrow{AB}|\,|\cos 315°|$$

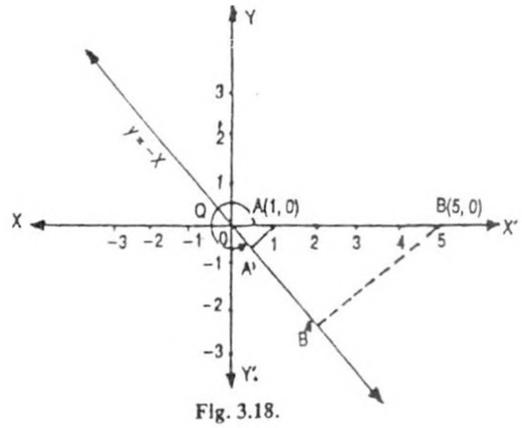

Fig. 3.18.

$$= 4 \times \frac{1}{\sqrt{2}}\hat{i}$$

$$\left[\because \quad \cos 315° = \cos(270° + 45°) = \sin 45° = \frac{1}{\sqrt{2}}\right]$$

$$= 2\sqrt{2}\,\hat{i}$$

where $\hat{i}$ is the unit vector along x-axis.

PRACTICE EXERCISE 3.1 (*i*)

1. Find the component along OX of the position vector of the point P(2, 5).
2. Find the component of the position vector of the point (−2, −3, 5) along the direction OY of the axis of coordinates.
3. Find the component of the vector $\overrightarrow{AB}$ where A has coordinates (1, 0) and B has coordinates (−3, 0) along the line $y = x$ in the increasing direction of X.

TEXT-BOOK EXERCISE 3.2
TYPE—I
(*SOLVED EXAMPLES*)

Example 1. If $\vec{a}, \vec{b}$ are position vectors of the points (1, −1), (−2, m), find the value of m for which $\vec{a}$ and $\vec{b}$ are collinear. [*T.B.Q. 1*]

Sol. Here $\vec{a} = \hat{i} - \hat{j}$ and $\vec{b} = -2\hat{i} + m\hat{j}$

Since $\vec{a}$ and $\vec{b}$ are collinear.

$\therefore \quad \vec{a} = \lambda\vec{b}$, where λ is a scalar

$\Rightarrow \quad \hat{i} - \hat{j} = \lambda(-2\hat{i} + m\hat{j})$

Comparing coefficients of $\hat{i}$ and $\hat{j}$, we get

$$1 = -2\lambda$$

$$\Rightarrow \qquad \lambda = -\frac{1}{2}$$

Also $\qquad \lambda m = -1$

$$\Rightarrow \qquad m = -\frac{1}{\lambda} = -\frac{1}{(-1/2)} = 2$$

Hence $m = 2$. **Ans.**

Example 2. If the position vector $\vec{a}$ of the point $(5, n)$ is such that $|\vec{a}| = 13$, find the value of n. [*T.B.Q.* 2]

Sol. $\qquad |\vec{a}| = \sqrt{5^2 + n^2}$

We are given that

$$|\vec{a}| = 13$$

$$\therefore \qquad \sqrt{5^2 + n^2} = 13$$

$$\Rightarrow \qquad 25 + n^2 = 169$$

$$\Rightarrow \qquad n = \pm 12.$$

Example 3. If A = (0, 1), B = (1, 0), C = (1, 2), D = (2, 1), prove that $\overline{AB} = \overline{CD}$. [*T.B.Q.* 3]

Sol. $\qquad \overrightarrow{AB} = i - j$

and $\qquad \overrightarrow{CD} = i - j$

Now $\qquad |\overrightarrow{AB}| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$

and $\qquad |\overrightarrow{CD}| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$

As $|\overrightarrow{AB}| = |\overrightarrow{CD}|$ and the direction of the two vectors are same.

$\therefore \qquad \overrightarrow{AB} = \overrightarrow{CD}$. **Hence proved.**

Example 4. Find the co-ordinates of the tip of the position vector which is equivalent to $\overline{AB}$, where

(*i*) A = (3, 1), B = (5, 0)

(*ii*) A = (−1, 3), B = (−2, 1). [*T.B.Q.* 4]

Sol. (*i*) $\quad \overrightarrow{AB} = (5-3)\,i + (0-1)\,i$

$\qquad\qquad = 2i - j$

Hence, required co-ordinates are (2, − 1). **Ans.**

(*ii*) $\quad \overrightarrow{AB} = (-2+1)\,i + (1-3)\,j$

$\qquad\qquad = -i - 2j$

Hence, required co-ordinates are (−1, −2). **Ans.**

Example 5. If A, B, C, D are the vertices of a parallelogram and A, B, C have respectively the following co-ordinates,

(*i*) (2, 3), (1, 4), (0, −2)

(*ii*) (−2, −1), (3, 0), (1, −2).

Find the coordinates D. [*T.B.Q.* 5]

Sol. (*i*) $\quad$ A = (2, 3), B = (1, 4), C = (0, 2)

Let $\quad$ D = (*x*, *y*)

$$\overrightarrow{OA} = 2\hat{i} + 3\hat{j}$$

$$\overrightarrow{OB} = \hat{i} + 4\hat{j}$$

$$\overrightarrow{OC} = 0.\hat{i} - 2\hat{j} = -2\hat{j}$$

$$\overrightarrow{OD} = x\hat{i} + y\hat{j}.$$

In a || gm ABCD, diagonals AC and BD bisect each other.

Position vector of P *i.e.*, mid-point of AC is given by

$$\overrightarrow{OP} = \frac{2\hat{i} + 3\hat{j} - 2\hat{j}}{2} = \frac{2\hat{i} + \hat{j}}{2} = \hat{i} + \frac{1}{2}\hat{j}$$

Position vector of P *i.e.*, mid-point of BD is given by

$$\overrightarrow{OP} = \frac{\hat{i} + 4\hat{j} + x\hat{i} + y\hat{j}}{2}$$

$$= \frac{(1+x)\,\hat{i}}{2} + \frac{(4+y)\,\hat{j}}{2}$$

Clearly $\quad \hat{i} + \frac{1}{2}\hat{j} = \left(\frac{1+x}{2}\right)\hat{i} + \left(\frac{4+y}{2}\right)\hat{j}.$

Equating the coefficients of $\hat{i}$ and $\hat{j}$, we get

$$\frac{1+x}{2} = 1 \quad \wedge \quad \frac{4+y}{2} = \frac{1}{2}$$

$1 + x = 2 \qquad$ *i.e.*, $\qquad x = 1$

and $\quad 4 + y = 1 \qquad$ *i.e.*, $\qquad y = -3$

Hence, the co-ordinates of D are (1, −3). **Ans.**

(*ii*) Similar to part (*i*), please try yourself.

Example 6. $\vec{a}$ is a position vector whose tip is (1, −3). Find the co-ordinates of the point B such that $\overline{AB} = \vec{a}$, if A has co-ordinates (−1, 5). [*T.B.Q.* 6]

Sol. $\qquad \vec{a} = \hat{i} - 3\hat{j} \quad$ and $\quad \overrightarrow{AB} = \vec{a}$

$\qquad A = (-1, 5) \quad \therefore \quad \overrightarrow{OA} = -\hat{i} + 5\hat{j}$

Now $\qquad \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$\therefore \qquad \overrightarrow{OB} = \overrightarrow{AB} + \overrightarrow{OA}$

$\qquad \overrightarrow{OB} = (\hat{i} - 3\hat{j}) + (-\hat{i} + 5\hat{j})$

$\qquad\qquad = 0.\hat{i} + 2\hat{j}$

Hence, the co-ordinates of B are (0, 2). **Ans.**

Example 7. If $x\hat{i} + y\hat{j}$ is a vector referred to two rectangular axes in a plane, show that

$$|x\hat{i} + y\hat{h}| = \sqrt{x^2 + y^2}$$

Derive a similar result for a vector in 3-dimensional space. [*T.B.Q.* 7]

Sol. (*a*) Length of the vector $\overrightarrow{OP} = x\hat{i} + y\hat{j}$ is given by

$OP = \sqrt{OQ^2 + QP^2}$ [From Pythagoras Theorem]

$= \sqrt{x^2 + y^2}$ [$\because$ OQ = x and QP = y]

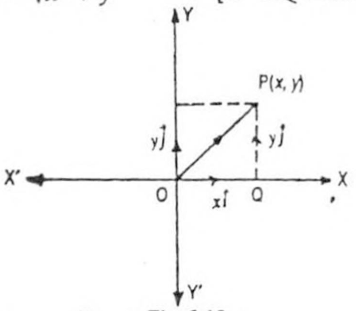

Fig. 3.19.

Thus, if $\vec{r} = x\hat{i} + y\hat{j}$, then $|\vec{r}| = \sqrt{x^2 + y^2}$

(*b*) **Length of the vector $x\hat{i} + y\hat{j} + z\hat{k}$.**

Since OZ $\perp$ plane XOY and NP $\parallel$ OZ

$\therefore$ NP $\perp$ plane XOY [Plane Geometry]

Also ON meets NP in that plane

$\therefore$ NP $\perp$ ON [Plane Geometry]

$\Rightarrow$ $\angle$ONP = 90°

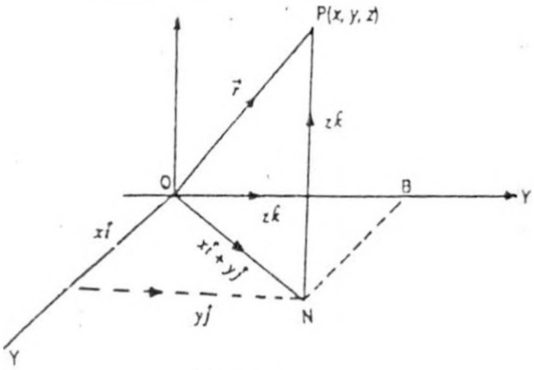

Fig. 3.20.

Now, in rt. $\angle$d ΔONP,

$OP^2 = ON^2 + NP^2$

[By Pythagoras Theorem]

$= (OA^2 + AN^2) + NP^2$

[$\because$ $\angle$OAN = 90°]

$= OA^2 + OB^2 + NP^2$ [AN = OB]

$= x^2 + y^2 + z^2$

[$\because$ OA = x, OB = y, NP = z]

$OP = \sqrt{x^2 + y^2 + z^2}$ ...(*i*)

Thus, if $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then

$|\vec{r}|$ = length of the vector $\vec{r}$ = OP

or $|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$ [From (*i*)]

Hence, the length of the vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is the positive square root of the sum of squares of its components.

Example 8. Find the distance between the points A(2, 3, 1), B(−1, 2, −3). [*T.B.Q. 12*]

Sol. $\because$ A $\equiv$ (2, 3, 1) and B $\equiv$ (−1, 2, −3)

$\therefore$ $\overrightarrow{OA} = 2\hat{i} + 3\hat{j} + \hat{k}$

and $\overrightarrow{OB} = -\hat{i} + 2\hat{j} - 3\hat{k}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (-\hat{i} + 2\hat{j} - 3\hat{k}) - (2\hat{i} + 3\hat{j} + \hat{k})$

$= (-1 - 2)\,\hat{i} + (2 - 3)\,\hat{j} + (-3 - 1)\,\hat{k}$

$= -3\hat{i} - \hat{j} - 4\hat{k}$

Hence $|\overrightarrow{AB}| = \sqrt{(-3)^2 + (-1)^2 + (-4)^2}$

$= \sqrt{9 + 1 + 16} = \sqrt{26}$. Ans.

Example 9. If A, B, C have position vectors (2, 0, 0), (0, 1, 0), (0, 0, 2), show that ΔABC is isosceles. [*T.B.Q. 13*]

Sol. $\because$ A $\equiv$ (2, 0, 0), B $\equiv$ (0, 1, 0), C $\equiv$ (0, 0, 2)

$\therefore$ $\overrightarrow{OA} = 2\hat{i} + 0\hat{j} + 0\hat{k}$,

$\overrightarrow{OB} = 0.\hat{i} + 1.\hat{j} + 0.\hat{k}$

$\overrightarrow{OC} = 0.\hat{i} + 0.\hat{j} + 2\hat{k}$

Now, $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= (0.\hat{i} + 1\hat{j} + 0.\hat{k}) - (2\hat{i} + 0.\hat{j} + 0\hat{k})$

$= (0 - 2)\,\hat{i} + (1 - 0)\,\hat{j} + (0 - 0)\,\hat{k}$

$= -2\hat{i} + \hat{j}$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

$= (0.\hat{i} + 0.\hat{j} + 2\hat{k}) - (0.\hat{i} + 1.\hat{j} + 0.\hat{k})$

$= (0 - 0)\,\hat{i} + (0 - 1)\,\hat{j} + (2 - 0)\,\hat{k}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$

$= (0.\hat{i} + 0.\hat{j} + 2\hat{k}) - (2\hat{i} + 0\hat{j} + 0.\hat{k})$

$= (0 - 2)\,\hat{j} + (0 - 0)\,\hat{j} + (2 - 0)\,\hat{k}$

$= -2\hat{i} + 0.\hat{j} + 2\hat{k} = -2\hat{i} + 2\hat{k}$

$|\overrightarrow{AB}| = \sqrt{(-2)^2 + (1)^2} = \sqrt{4 + 1} = \sqrt{5}$

$|\overrightarrow{BC}| = \sqrt{(-1)^2 + (2)^2} = \sqrt{1 + 4} = \sqrt{5}$

$|\overrightarrow{AC}| = \sqrt{(-2)^2 + (2)^2} = \sqrt{4 + 4} = \sqrt{8}$

Since, $|\overrightarrow{AB}| = |\overrightarrow{BC}|$ *i.e.* AB = BC

Hence, ΔABC is an isosceles triangle.

[$\because$ Its two sides are equal] Ans.

PRACTICE EXERCISE 3.2 (*i*)

1. Prove by vectors that the points P(−2, 1), Q(−5, −1) and R(1, 3) are collinear.
2. If vectors $\vec{a} = 2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$. Find $|\vec{a} + \vec{b}|$.
3. Express $\overline{PQ}$ in terms of unit vectors $\hat{i}$ and $\hat{j}$ when the points are :
(*i*) P(5, −7), Q(−3, 2) (*ii*) P(1, 2), Q(−6, 5).
Find $|\overline{PQ}|$ in each case.
4. If the co-ordinates of the points A, B, C, D are (1, 2), (2, 1), (2, 3) and (3, 2) respectively, show that $\overline{AB} = \overline{CD}$.
5. Find the co-ordinates of the tip of the position vector which is equivalent to $\overline{AB}$, where the co-ordinates of A and B are (−2, 5) and (−3, 2) respectively.
6. If the position vector $\vec{a}$ of the point (*n*, −6) is such that $|\vec{a}| = 10$, find the value of *n*.
7. If $\vec{a}$ be the position vector whose tip is (3, −2), find the co-ordinates of a point B such that $\overline{AB} = \vec{a}$, the co-ordinates of A being (−1, 3).
8. Find the value of *x* so that the points A(*x*, −1) ; B(2, 1) and C(4, 5) are collinear.
9. Show that the vectors $\vec{a} = 2\hat{i}$, $\vec{b} = -\hat{i} + 4\hat{j}$, $\vec{c} = -\hat{i} - 4\hat{j}$ form an isosceles triangle.
10. ABCD is a ‖ gm, if the co-ordinates of A, B, C are (2, 3), (1, 4) and (0, −2) respectively, find the co-ordinates of D.

ADDITIONAL SOLVED EXAMPLES
SECTION—A
[*2 marks questions*]

☞ **Example 1. Show that the three points A(2, −1, 3), B(4, 3, 1) and C(3, 1, 2) are collinear.**

Sol.
$$\overrightarrow{OA} = 2\hat{i} - \hat{j} + 3\hat{k}$$
$$OB = 4\hat{i} + 3\hat{j} + \hat{k}$$
and
$$\overrightarrow{OC} = 3\hat{i} + \hat{j} + 2\hat{k}$$
$\therefore$
$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$
$$= (4\hat{i} + 3\hat{j} + \hat{k}) - (2\hat{i} - \hat{j} + 3\hat{k})$$
$$= 2\hat{i} + 4\hat{j} - 2\hat{k}.$$
$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$$
$$= (3\hat{i} + \hat{j} + 2\hat{k}) - (4\hat{i} + 3\hat{j} + \hat{k})$$
$$= -\hat{i} + 2\hat{j} + \hat{k}$$
Now,
$$\overrightarrow{AB} = 2\hat{i} + 4\hat{j} - 2\hat{k}$$
$$= -2(-\hat{i} - 2\hat{j} + \hat{k}) = -2 \,.\, \overrightarrow{BC}$$

Thus, $\overrightarrow{AB}$ is parallel to $\overrightarrow{BC}$, but one point B is common. Hence, the three given points A, B, C are collinear. **Proved.**

Example 2. Let *a* be a given vector whose initial point is $P(x_1, y_1)$ and terminal point is $Q(x_2, y_2)$. In each of the problems (*i*) to (*v*), find the magnitude and component of the vector along *x* and *y* directions.

(*i*) P(2, 3) ; Q(4, 6) (*ii*) P(−1, 3) ; Q(1, 2)
(*iii*) P(0, 2) ; Q(5, −3) (*iv*) P(−1, −2) ; Q(−5, −6)
(*v*) P(2, 4) ; Q(−5, −3).

Sol. (*i*) In this $x_1 = 2$, $x_2 = 4$, $y_1 = 3$, $y_2 = 6$ and if $\overrightarrow{PQ} = \vec{a}$, then
$$\vec{a} = (4 - 2)\,\hat{i} + (6 - 3)\,\hat{j}$$
$$= 2\hat{i} + 3\hat{j}$$
Its components along *x*-axis and *y*-axis are respectively 2 and 3. **Ans.**

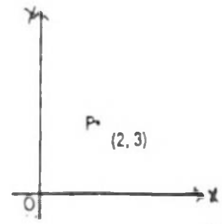

Fig. 3.21.

and $|\vec{a}| = \sqrt{2^2 + 3^2} = \sqrt{4 + 9} = \sqrt{13}$. **Ans.**

(*ii*) In this $x_1 = -1$, $x_2 = 1$, $y_1 = 3$, $y_2 = 2$ and if $\overrightarrow{PQ} = \vec{a}$, then
$$\vec{a} = (x_2 - x_1)\,\hat{i} + (y_2 - y_1)\,\hat{j}$$
$$= (1 + 1)\,\hat{i} + (2 - 3)\,\hat{j} = 2\hat{i} - \hat{j}$$
$$|\vec{a}| = \sqrt{2^2 + 1^2} = \sqrt{5}. \quad \textbf{Ans.}$$
and component of $\vec{a}$ along *x*-axis and *y*-axis are respectively 2 and −1. **Ans.**

(*iii*) In this $x_1 = 0$, $x_2 = 5$, $y_1 = 2$, $y_2 = -3$.

If $\overrightarrow{PR} = \vec{a}$, then
$$\vec{a} = (x_2 - x_1)\,\hat{i} + (y_2 - y_1)\,\hat{j}$$
$$= (5 - 0)\,\hat{i} + (-3 - 2)\,\hat{j}$$
$$= 5\hat{i} - 5\hat{j}$$
$$|\vec{a}| = \sqrt{5^2 + (-5)^2} = \sqrt{25 + 25} = 5\sqrt{2}. \quad \textbf{Ans.}$$

Its component along x-axis and y-axis are 5 and -5 respectively. **Ans.**

(iv) In this question $x_1 = -1$, $x_2 = -5$, $y_1 = -2$, $y_2 = -6$ and if $\overrightarrow{PQ} = \vec{a}$, then

$$\vec{a} = (x_2 - x_1)\,\hat{i} + (y_2 - y_1)\,\hat{j}$$

$$= (-5 + 1)\,\hat{i} + (-6 + 2)\,\hat{j}$$

$$= -4\hat{i} - 4\hat{j}$$

$$|\vec{a}| = \sqrt{(-4)^2 + (-4)^2}$$

$$= \sqrt{16 + 16} = 4\sqrt{2}. \quad \textbf{Ans.}$$

Its components along x-axis and y-axis are -4 and -4 respectively.

(v) In this question $x_1 = 2$, $x_2 = -5$, $y_1 = 4$, $y_2 = -3$ and if $\overrightarrow{PQ} = \vec{a}$, then

$$\vec{a} = (x_2 - x_1)\,\hat{i} + (y_2 - y_1)\,\hat{j}$$

$$= (-5 - 2)\,i + (-3 + 4)\,\hat{j}$$

$$= -7\hat{i} - 7\hat{j}$$

$$|\vec{a}| = \sqrt{(-7)^2 + (-7)^2} = 7.\sqrt{2}. \quad \textbf{Ans.}$$

and its components along x-axis and y-axis are respectively (-7) and (-7). **Ans.**

Example 3. Show that the points (2, −1, 3), (3, −5, 1) and (−1, 11, 9) are collinear.

Sol. Let $A \equiv (2, -1, 3)$, $B \equiv (3, 5, 1)$,
$C \equiv (-1, 11, 9)$.

Clearly, $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$$= (3\hat{i} - 5\hat{j} + \hat{k}) - (2\hat{i} - \hat{j} + 3\hat{k})$$

$$= (3 - 2)\,\hat{i} + (-5 + 1)\,\hat{j} + (1 - 3)\,\hat{k}$$

$$= \hat{i} - 4\hat{j} - \hat{k}$$

and $\overrightarrow{AC} = \overrightarrow{OC} + \overrightarrow{OA}$

$$= (-\hat{i} + 11\hat{j} + 9\hat{k}) - (2\hat{i} - \hat{j} + 3\hat{k})$$

$$= (-1 - 2)\,\hat{i} + (11 + 1)\,\hat{j} + (9 - 3)\,\hat{k}$$

$$= (-3\hat{i} + 12\hat{j} + 6\hat{k}) = -3(\hat{i} - 4\hat{j} - 2\hat{k})$$

$\therefore \quad \overrightarrow{AC} = -3\overrightarrow{AB}$

$\therefore$ The two vectors are parallel but A is common to both. Hence the points A, B, C must be collinear.

Example 4. If $\vec{a}$ is a non-zero vector, find a scalar k such that $|k\vec{a}| = 1$.

Sol. $\quad |k\vec{a}| = 1 \quad$ (Given)

$\Rightarrow \quad |k|\,|\vec{a}| = 1$

$\Rightarrow \quad |k| = \dfrac{1}{|\vec{a}|}$

$[\because \ \vec{a} \neq \vec{0}$ (Given) $\Rightarrow |\vec{a}| = 0]$

$\therefore \quad k = \pm\dfrac{1}{|\vec{a}|} \quad$ **Ans.**

$[\because \ |x| = \lambda$ where $\lambda \geq 0 \Rightarrow x = \pm\lambda]$

Example 5. Find the terminal point of the vector PQ whose initial point is P (2, 3) and components along x and y direction are 1 and 2 respectively.

Sol. Let the coordinates of the terminal point Q be (x, y). Then components of the vector PQ along x and y directions are $(x - 2)$ and $(y - 3)$ respectively.

Therefore, $x - 2 = 1$ or $x = 3$

and $y - 3 = 2$, or $y = 5$

(by the given conditions)

Hence, the terminal point O is (3, 5). **Ans.**

Example 6. Find all the values of λ such that $(x, y, z) \neq 0$ and $(i + j + 3k)\,x + (3i - 3j + k)\,y + (-4i + 5j)\,z = \lambda\,(ix + jy + kz)$ where i, j, k are unit vectors along the co-ordinate axis.

Sol. The given relation can be written as

$$[(1 - \lambda)\,x + 3y - 4z]\,i + [x - (3 + \lambda)\,y + 5z]\,j + [3x + y - \lambda z]\,k = 0$$

$$\Rightarrow \quad (1 - \lambda)\,x + 3y - 4z = 0$$

$$x - (3 + \lambda)\,y + 5z = 0$$

$$3x + y - \lambda z = 0$$

Since $(x, y, z) \neq (0, 0, 0)$ *i.e.* the above equation will have non-trivial solution if

$$\begin{vmatrix} -1\lambda & 3 & -4 \\ 1 & -(3 + \lambda) & 5 \\ 3 & 1 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad \lambda^3 + 2\lambda^2 + \lambda = 0 \quad \text{or} \quad \lambda(\lambda + 1)^2 = 0$$

Hence $\lambda = 0, -1, -1$. **Ans.**

Example 7. Show that the three points A, B, C with position vectors $-2\vec{a} + 3\vec{b} + 5\vec{c}, \vec{a} + 2\vec{b} + 3\vec{c}, 7\vec{a} - \vec{c}$ are collinear.

Sol. Let O be the origin of reference

Then $\quad \overrightarrow{OA} = -2\vec{a} + 3\vec{b} + 5\vec{c},$

$$\overrightarrow{OB} = \vec{a} + 2\vec{b} + 3\vec{c}$$

$$\overrightarrow{OC} = 7\vec{a} - \vec{c}$$

$\therefore \quad \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$$= (\vec{a} + 2\vec{b} + 3\vec{c}) - (-2\vec{a} + 3\vec{b} + 5\vec{c})$$

or $\qquad \overrightarrow{AB} = 3\vec{a} - \vec{b} - 2\vec{c} \qquad \ldots(i)$

Also $\qquad \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$

$$= (7\vec{a} - \vec{c}) - (-2\vec{a} + 3\vec{b} + 5\vec{c})$$

$$= 9\vec{a} - 3\vec{b} - 6\vec{a}$$

or $\qquad \overrightarrow{AC} = 3\,(3\vec{a} - \vec{b} - 2\vec{c})$

$\therefore \qquad \overrightarrow{AC} = 3\overrightarrow{AB} \qquad \ldots(ii)$

(*ii*) Shows that the vectors $\overrightarrow{AC}$ and $\overrightarrow{AB}$ have the same or parallel supports.

But these vectors have a common initial point A, proving thereby that $\overrightarrow{AC}$ and $\overrightarrow{AB}$ have the same support.

$\therefore$ A, B, C are collinear. **Hence proved.**

Example 8. If the mid-points of the consecutive sides of a quadrilateral are joined, then show by vector method that they form a parallelogram.

Sol. Let $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$, be the position vector of the vertices A, B, C, D of the quadrilateral ABCD. Let P, Q, R, S be the mid-points of the sides. Then their position vectors are respectively.

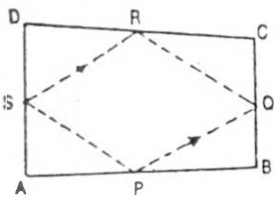

Fig. 3.22.

$$\frac{1}{2}(a+b), \frac{1}{2}(b+c), \frac{1}{2}(c+d), \frac{1}{2}(d+a)$$

Now $\overrightarrow{PQ}$ = position vector of Q – position vector of P

$$= \frac{1}{2}(b+c) - \frac{1}{2}(a+b) = \frac{1}{2}(c-a)$$

Similarly, $\qquad SR = \frac{1}{2}(c-a)$

$\therefore \overrightarrow{PQ} = \overrightarrow{SR} \Rightarrow PQ = SR$ and also PQ $\parallel$ SR.

Since a pair of opposite sides are equal as well as parallel, PQRS is a parallelogram.

Example 9. ABCD is a parallelogram and P is the point of intersection of its diagonals, O is the origin. Prove that

$$\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD} = 4\overrightarrow{OP}.$$

Sol. $\because$ P is the mid- point of AB

$\therefore \qquad \overrightarrow{OP} = \dfrac{\overrightarrow{OA} + \overrightarrow{OC}}{2}$

or $\qquad 2\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{OC} \qquad \ldots(i)$

Again, $\because$ P is the mid-point of BD

$\therefore \qquad \overrightarrow{OP} = \dfrac{\overrightarrow{OB} + \overrightarrow{OD}}{2}$

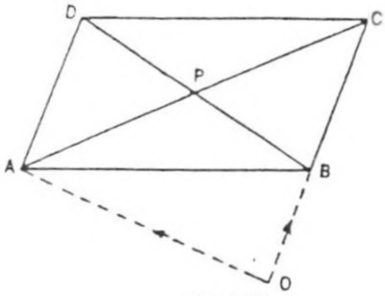

Fig. 3.23.

or $\qquad 2\overrightarrow{OP} = \overrightarrow{OB} + \overrightarrow{OD} \qquad \ldots(ii)$

Adding (*i*) and (*ii*), we get

$$2\overrightarrow{OP} + 2\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{OC} + \overrightarrow{OB} + \overrightarrow{OD}$$

or $\qquad 4\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD}.$

Example 10. **Three vectors of magnitude a, $2a$, $3a$, meet in a point and their directions are along the diagonals of three adjacent faces of a cube. Find their resultant.**

Sol. Let the vectors of magnitudes a, $2a$, $3a$ act along OP, OQ, OR respectively. Then vectors along OP, OQ, OR are

$$a\left(\frac{i+j}{\sqrt{2}}\right), 2a\left(\frac{i+j}{\sqrt{2}}\right), 3a\left(\frac{k+i}{\sqrt{2}}\right) \text{ respectively.}$$

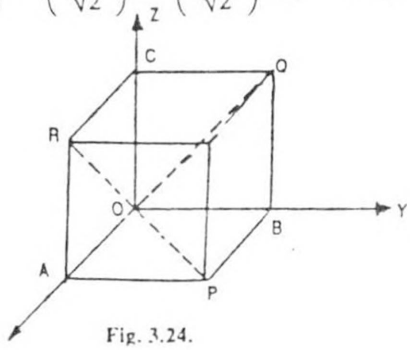

Fig. 3.24.

Their resultant, say $\vec{R}$, is given by

$$\vec{R} = a\left(\frac{i+j}{\sqrt{2}}\right) + 2a\left(\frac{j+k}{2}\right) + 3a\left(\frac{k+i}{\sqrt{2}}\right)$$

$$= \frac{a}{\sqrt{2}}(4i + 3j + 5k)$$

$$\therefore \qquad |\vec{R}| = \sqrt{\frac{a^2}{2}(16+9+25)} = 5a. \quad \text{Ans.}$$

Example 11. Find the position vector of the centroid of the ΔABC when the position vectors of its vertices are $(1, 3, 0), (2, 1, 1), (0, -1, 0)$ respectively.

Sol. The position vectors of A, B and C relative to an origin O are

$$\overrightarrow{OA} = \hat{i} + \hat{j} + 0\hat{k},\ \overrightarrow{OB} = 2\hat{i} + \hat{j} + \hat{k},\ \overrightarrow{OC} = 0\hat{i} - \hat{j} + 0\hat{k}$$

If G be the centroid of the triangle, the position of the centroid is

$$\left(\frac{1+2+0}{3}, \frac{3+1-1}{3}, \frac{0+1+0}{3}\right)$$

or $$\left(1, 1, \frac{1}{3}\right)$$

i.e. $$\overrightarrow{OG} = \hat{i} + \hat{j} + \frac{1}{3}\hat{k}.\quad \text{Ans.}$$

Example 12. Show that the vectors

$$\vec{a} = 3\sqrt{3}\hat{i} - 3\hat{j},\ \vec{b} = 6\hat{j} \text{ and } \vec{c} = 3\sqrt{3}\hat{i} + 3\hat{j}$$

form the sides of an equilateral triangle.

Sol. $$\vec{a} + \vec{b} = (3\sqrt{3}\hat{i} - 3\hat{j}) + 6\hat{j}$$

$$= 3\sqrt{3}\hat{i} + 3\hat{j} = \vec{c}.$$

Since $$\vec{a} + \vec{b} = \vec{c}$$

and $$|\vec{a}| = \sqrt{(3\sqrt{3})^2 + (-3)^2}$$

$$= \sqrt{27+9} = \sqrt{36} = 6$$

$$|\vec{b}| = \sqrt{0+36} = 6$$

$$|\vec{c}| = \sqrt{(3\sqrt{3})^2 + (3)^2}$$

$$= \sqrt{27+9} = \sqrt{36} = 6$$

Hence $\vec{a}, \vec{b}, \vec{c}$ form the sides of an equilateral triangle.

Example 13. Let a, b, c be the position vectors of three points A, B, C. If three numbers α, β, γ (not all zero) can be found such that

$$\alpha a + \beta b + \gamma c = 0$$

and $$\alpha + \beta + \gamma = 0$$

show that the points A, B and C are collinear.

Sol. $$\alpha a + \beta b + \gamma c = 0$$

or $$\alpha a + \beta b = -\gamma c \quad ...(i)$$

Also $$\alpha + \beta = -\gamma \quad (\because \alpha + \beta + \gamma = 0) \quad ...(ii)$$

$\therefore$ $$\alpha a + \beta b = (\alpha + \beta)c$$

[from (*i*) and (*ii*)]

or $$c = \frac{\alpha a + \beta b}{\alpha + \beta}$$

$\Rightarrow$ C divides the join of A and B in the ratio $\beta : \alpha$.

$\therefore$ A, B and C are collinear.

Example 14. Show that the points with position vectors $\hat{i} + 2\hat{j}$, $3\hat{i} - 2\hat{j}$, and $2\hat{i}$ are collinear.

Sol. Let A, B, C be the points with the given position vectors

$\therefore$ $$\overrightarrow{OA} = \hat{i} + 2\hat{j},\ \overrightarrow{OB} = 3\hat{i} - 2\hat{j},\ \overrightarrow{OC} = 2\hat{i}$$

Now $$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= (3\hat{i} - 2\hat{j}) - (\hat{i} + 2\hat{j}) = 2\hat{i} - 4\hat{j}$$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = 2\hat{i} - (\hat{i} + 2\hat{j}) = \hat{i} - 2\hat{j}$$

$\therefore$ $$\overrightarrow{AB} = 2\overrightarrow{AC}$$

Hence, the vectors $\overrightarrow{AB}, \overrightarrow{AC}$ having the same initial point A are parallel. It follows that A, B, C are collinear.

Example 15. If $\vec{a} = (2, 6)$, $\vec{b} = (5, 15)$ and $\vec{c} = (4, 12)$, find λ if $2\vec{a} + 3\vec{b} = \lambda\vec{c}$.

Sol. Two vectors are equal if and only if their corresponding components are equal.

Now $$2\vec{a} + 3\vec{b} = 2(2, 6) + 3(5, 15)$$

$$= (4, 12) + (15, 45) = (19, 57)$$

and $$\lambda\vec{c} = \lambda(4, 12) = (4\lambda, 12\lambda)$$

$\therefore$ $$(4\lambda, 12\lambda) = (19, 57)$$

$\Rightarrow$ $$4\lambda = 19, \text{ or } \lambda = \frac{19}{4}.$$

☞ **Example 16. If the vectors $2\hat{i} + p\hat{j} + \hat{k}$ and $-5\hat{i} + 3\hat{j} + q\hat{k}$ are collinear, find the values of p and q.**

Sol. Since, the given vectors $2\hat{i} + p\hat{j} + \hat{k}$ and $-5\hat{i} + 3\hat{j} + q\hat{k}$ are collinear, we have $2\hat{i} + p\hat{j} + \hat{k} = \alpha(-5\hat{i} + 3\hat{j} + q\hat{k})$ for same values of λ.

$\Rightarrow$ $$(2 + 5\alpha)\hat{i} + (p - 3\alpha)\hat{j} + (1 - q\alpha)\hat{k} = 0$$

$\Rightarrow$ $$2 + 5\alpha = 0,\ p - 3\alpha = 0,\ 1 - q\alpha = 0$$

$\Rightarrow$ $$\alpha = -\frac{2}{5},\ p = 3\alpha = \frac{-6}{5} \text{ and } q = \frac{1}{\alpha} = \frac{-5}{2}$$

Hence, $$p = \frac{-6}{5} \text{ and } q = \frac{-5}{2}.\quad \text{Ans.}$$

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ADDITIONAL PRACTICE EXERCISE 3 (*a*)

1. Show that the three points A(6, −7, −1), B(2, −3, 1) and C(4, −5, 0) and collinear.
2. Show that the three points A(1, −4, −2), B(2, −2, 1) and C(0, 2, −1) are collinear.
3. Show that the three points A(3, −5, 1), B(−1, 0, 8) and C(7, −10, −6) are collinear.
4. Show that the three points A(2, −4, 1), B(4, 4, 3) and C(3, 0, 2) are collinear.
5. Show that the three points A(4, 5, −5), B(0, −11, 3) and C(2, −3, −1) are collinear.
6. Find the lengths of the sides of the triangle ABC whose vertices have position vectors A(3, 4, 5), B(4, 3, 2), C(3, −6, −3).
7. Find a unit vector parallel to the sum of the vectors
$$a = 2i + 4j - 5k, \quad b = \hat{i} + 2\hat{j} + 3\hat{k}$$
8. Define zero vector. What can you say about its direction ? For any three vectors, *a*, *b* and *c*, prove that
$$a + (b + c) = (a + b) + c.$$
9. Show that the points given by $i - 2j + 3k$, $2i + 3j - 4k$, $-7j + 10k$ are collinear.
10. Show that the three points whose position vectors are A(−2, 3, 5), B(1, 2, 3), C(7, 0, −1) are collinear.
11. If A, B, C are points with position vectors $2i + 4j - k$, $4i + 5j + k$ and $3i + 6j - 3k$, show that the ΔACB is right angled.
12. Show that any three vectors in the same plane are linearly dependent *i.e.* if *a*, *b*, *c* are three vectors, show that there exists α, β, γ not all zero such that $\alpha a + \beta b + \gamma c = 0$. What about vectors more than three in number ?
13. Show that in 3 dimensional space, three vectors are linearly independent if and only if they do not lie in a plane.
14. If A, B, C have position vectors (2, 0, 0), (0, 1, 0), (0, 0, 2), show that ΔABC is isosceles.
15. If the vectors *a*, *b*, *c*, *d* represent the consecutive sides of a quadrilateral, show that the necessary and sufficient condition that the quadrilateral be a parallelogram, is that $a + c = 0$ or $b + d = 0$.
16. Find the magnitude of the vector $\vec{a} = 2\hat{i} - 3\hat{j} + \hat{k}$.
17. Find the distance between the points A(2, 3, 1) and B(−1, 2, −3), using vector method.
18. Show that the points A(2, −1, 1), B(1, −3, −5) and C(3, −4, −4) are the vertices of a right-angled triangle.
19. If $\overline{AB} = 2\hat{i} + \hat{j} - 3\hat{k}$ and the co-ordinates of A are (1, 2, −1), find the co-ordinates of B.
20. Find a unit vector parallel to the sum of vectors $\vec{a} = 2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + \hat{k}$.

ADDITIONAL SOLVED EXAMPLES
SECTION—B
[*4 marks questions*]

Example 1. Four points A, B, C, D with position vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ respectively are such that
$$3\vec{a} - \vec{b} + 2\vec{c} - 4\vec{d} = 0.$$

Show that the four points are coplanar. Also, find the position vector of the point of intersection of lines AC and BD.

[*C.B.S.E. 1995, Delhi (Set I, II, III)*]

Sol. It is given that $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are the position vectors of the points, A, B, C, D respectively.

Now, $\overline{OA} = \vec{a}$, $\overline{OB} = \vec{b}$, $\overline{OC} = \vec{c}$, $\overline{OD} = \vec{d}$

$$\therefore \quad \overline{AB} = \overline{OB} - \overline{OA} = \vec{b} - \vec{a}$$
$$\overline{BC} = \overline{OC} - \overline{OB} = \vec{c} - \vec{b}$$
$$\overline{CD} = \overline{OD} - \overline{OC} = \vec{d} - \vec{c}$$

The given points will be coplanar if $\overline{AB}$, BC and $\overline{CD}$ are coplanar.

But $\quad 3\vec{a} - \vec{b} + 2\vec{c} - 4\vec{d} = 0 \quad$ [Given]

$$\Rightarrow \quad 3\vec{a} + 2\vec{c} = \vec{b} + 4\vec{d}$$
$$\Rightarrow \quad \frac{3\vec{a} + 2\vec{c}}{5} = \frac{\vec{b} + 4\vec{d}}{5}$$
$$\Rightarrow \quad \frac{3\vec{a} + 2\vec{c}}{3 + 2} = \frac{\vec{b} + 4\vec{d}}{1 + 4}$$

L.H.S. represents the position vector of a point lying on the join of A and C which divides AC in the ratio 3 : 2.

R.H.S. represents the position vector of a point lying on the join of B and D which divides BD in the ratio 1 : 4.

Since $\quad$ L.H.S. = R.H.S.

$\therefore$ They represent the same point *i.e.*, the point of intersection of AC and BD.

Since intersecting lines are coplanar, therefore the points A, B, C, D are coplanar.

Also, position vector of the points of intersection is
$$\frac{3\vec{a} + 2\vec{c}}{5} \quad \text{or} \quad \frac{\vec{b} + 4\vec{d}}{5}.$$

Example 2. Show that a quadrilateral is a parallelogram if and only if its diagonals bisect each other.

Sol. Let ABCD be the given parallelogram whose diagonals AC and BD intersect at point P.

Let DP : PB = $m_1 : m_2$ and AP = m_3 AC, where m_1, m_2 and m_3 are positive real numbers.

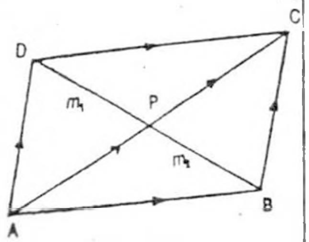

Fig. 3.25.

∵ P lies on DB, therefore, by section formula

$$AP = \frac{m_2 AD + m_1 \cdot AB}{m_2 + m_1} \quad ...(i)$$

Also
$$AP = m_3 AC = m_3 (AB + BC) = m_3 (AB + AD) \quad ...(ii)$$

∴ From (*i*) and (*ii*), we get

$$\frac{m_2 AD + m_1 AB}{m_2 + m_1} = m_3 AB + m_3 AD$$

or
$$\left(\frac{m_2}{m_2+m_1} - m_3\right) Ad + \left(\frac{m_1}{m_2+m_1} - m_3\right) AB = 0$$

As AD and AB are not parallel and zero vectors.

$$\therefore \quad \frac{m_2}{m_2+m_1} - m_3 = 0 \quad \text{and} \quad \frac{m_1}{m_2+m_1} - m_3 = 0$$

or
$$\frac{m_2}{m_2+m_1} = m_3 \quad \text{and} \quad \frac{m_1}{m_2+m_1} = m_3$$

or
$$\frac{m_2}{m_2+m_1} = \frac{m_1}{m_2+m_1}, \quad \therefore m_1 = m_2$$

$$\therefore \quad m_3 = \frac{m_2}{m_2+m_2} = \frac{m_2}{2m_2} = \frac{1}{2}$$

Hence the diagonals of the parallelogram bisect each other.

Conversely : If the diagonals bisect each other, then

AB = AP + PB = PC + DP = DC

and AD = AP + PD = PC + BP = DC

i.e. AB and DC are parallel and equal.

and AD and BC are parallel and equal.

Hence, ABCD is a parallelogram. **Proved.**

Example 3. Prove that the straight line joining the mid-points of the two non-parallel sides of a trapezium is parallel to the parallel sides and equal to half of their sum. [*T.B. Misc. Ex. Q. 10*]

Sol. Let ABCD be a trapezium in which AB ∥ DC. Let E, F be mid-points respectively of the two non-parallel sides BC, AD.

Take A as the origin of reference.

∴ Position vector of A is $\vec{O}$.

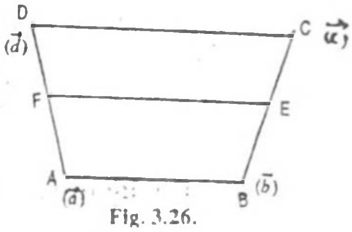

Fig. 3.26.

Let $\vec{b}, \vec{d}$ be the position vectors of B, D respectively with A as the origin of reference, so that

$$\overrightarrow{AB} = \vec{b}, \ \overrightarrow{AD} = \vec{d} \quad ...(i)$$

Since DC ∥ AB there exists a scalar t such that

$$\overrightarrow{DC} = t.\overrightarrow{AC} \ i.e., \ \overrightarrow{DC} = t.\vec{b} \quad ...(ii)$$

∴ Position vector $\overrightarrow{AC}$ of c is given by

$$\overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{DC}$$

[By Triangle law of addition]

or
$$\overrightarrow{AC} = \vec{d} + t.\vec{b} \quad \text{[From (i) and (ii)]}$$

Since E is mid-point of line joining B, C with position vectors $\vec{b}, \vec{d} + t.\vec{b}$ respectively.

∴ By mid-point of formula,

Position vector of

$$E = \frac{\vec{b} + (\vec{d} + t.\vec{b})}{2} = \frac{\vec{d} + (1+t)\vec{b}}{2} \quad ...(iii)$$

Again, since F is mid-point of line joining A, D with position vectors $\vec{0}, \vec{d}$ respectively.

∴ By mid-point formula,

$$\text{Position vector of F} = \frac{0 + \vec{d}}{2} = \frac{\vec{d}}{2} \quad ...(iv)$$

∴ $\overrightarrow{FE}$ = Position vector of E − Position vector of F

$$= \frac{\vec{d} + (1+t)\vec{b}}{2} - \frac{\vec{d}}{2}$$

[From (*iii*) and (*iv*)]

$$\overrightarrow{FE} = \frac{1}{2}(1+t)\vec{b}$$

or
$$\overrightarrow{FE} = \frac{1}{2}(1+t)\overrightarrow{AB} \quad \text{[From (i)]}$$

which shows that FE ∥ AB and hence to DC. **Proved.**

Example 4. "The mid-points of two opposite sides of a quadrilateral and the mid-points of the diagonals are the vertices of a parallelogram"— Prove using vectors. [*T.B. Misc. Ex. Q. 11*]

Sol. Let the position vectors of the vertices B, C, D of the quadrilateral ABCD referred to A as origin be $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$ respectively.

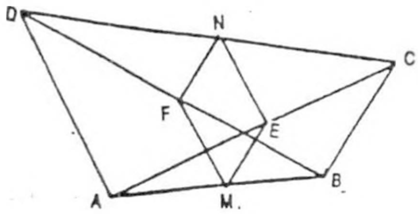

Fig. 3.27.

Let E and F be the mid-points of the diagonals AC and BD. Let M and N be the mid-points of the sides AB and DC. Join NF, NE, ME and MF.

Now p.v. of $M = \frac{1}{2}b$ and p.v. of $N = \frac{1}{2}(c+d)$

Also p.v. of $E = \frac{1}{2}c$ and p.v. of $F = \frac{1}{2}(b+d)$

$$\therefore \quad \overrightarrow{MF} = \text{p.v. of F} - \text{p.v. of M}$$

$$= \frac{1}{2}(b+d) - \frac{1}{2}b = \frac{1}{2}d \qquad \ldots(i)$$

$$\overrightarrow{ME} = \text{p.v. of E} - \text{p.v. of M}$$

$$= \frac{1}{2}c - \frac{1}{2}b = \frac{1}{2}(c-b) \qquad \ldots(ii)$$

$$\overrightarrow{EN} = \text{p.v. of N} - \text{p.v. of E}$$

$$= \frac{1}{2}(c+d) - \frac{1}{2}c = \frac{1}{2}d \qquad \ldots(iii)$$

and $$\overrightarrow{FN} = \text{p.v. of N} - \text{p.v. of F}$$

$$= \frac{1}{2}(c+d) - \frac{1}{2}(b+d)$$

$$= \frac{1}{2}(c-b) \qquad \ldots(iv)$$

From (*i*) and (*iii*), $\overrightarrow{MF} = \overrightarrow{EN}$ *i.e.* MF and EN are equal and parallel.

From (*ii*) and (*iv*), $\overrightarrow{ME} = \overrightarrow{FN}$ *i.e.* ME and FN are equal and parallel.

Hence ENFM is a parallelogram.

☞ **Example 5. If the sum of two unit vectors is a unit vector, prove that the magnitude of their difference is √3.**

Sol. Let $\overline{OA}$ and $\overline{AB}$ be two unit vectors $\vec{a}$ and $\vec{b}$. Then by Triangle Law of addition.

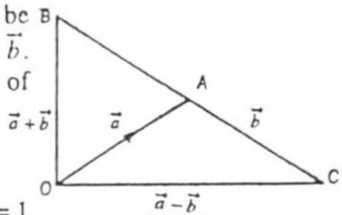

Fig. 3.28.

$$\vec{a} + \vec{b} = \overline{OB}$$

$$\therefore \quad |\vec{a}| = 1, |\vec{b}| = 1$$

and $$|\vec{a} + \vec{b}| = 1 \qquad \text{(Given)}$$

$$\therefore \quad OA = AB = OB = 1$$

Let $\overrightarrow{AC} = -\vec{b}$, then $AC = |\overrightarrow{AC}| = |-\vec{b}| = |\vec{b}| = 1$.

Since OA = AB = AC, then by Geometry ΔBOC is a right triangle, rt. ∠d at O.

$$\vec{a} - \vec{b} = \vec{a} + (-\vec{b}) = \overrightarrow{OA} + \overrightarrow{AC} = \overrightarrow{OC}$$

$$\therefore \quad |\vec{a} - \vec{b}| = |\overrightarrow{OC}| = OC$$

Now $$BC^2 = OB^2 + OC^2$$

$$\therefore \quad OC = \sqrt{BC^2 - OB^2}$$

or $$OC = \sqrt{2^2 - 1^2} = \sqrt{4-1} = \sqrt{3}.$$

Hence the magnitude of their (two unit vectors) difference is √3.

Example 6. D, E, F are the mid-points of the sides of a triangle ABC. For any point O show that

$$\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OD} + \overrightarrow{OE} + \overrightarrow{OF}.$$

[*T.B. Misc. Ex. Q. 23*]

Sol. Let $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ be the position vectors referred to O of the vertices A, B, C of the triangle ABC.

Then $\overrightarrow{OA} = a$, $\overrightarrow{OB} = b$ and $\overrightarrow{OC} = c$

i.e. $$\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = a + b + c. \qquad \ldots(i)$$

Also let D, E, F be the mid-points of the sides AB, BC and CA.

Then $$\overrightarrow{OD} = \text{position vector of D}$$

$$= \frac{1}{2}(\text{p.v. of A} + \text{p.v. of B})$$

$$= \frac{1}{2}(a+b).$$

Similarly $\overrightarrow{OE} = \frac{1}{2}(b+c)$ and $\overrightarrow{OF} = \frac{1}{2}(c+a)$

$$\therefore \quad \overrightarrow{OD} + \overrightarrow{OE} + \overrightarrow{OF} = \frac{1}{2}(a+b) + \frac{1}{2}(b+c) + \frac{1}{2}(c+a)$$

$$= a + b + c = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}, \text{ from } (i)$$

Hence proved.

Example 7. What is the geometric significance of the relation

$$|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}|?$$

Sol. Let $\vec{a} = \overrightarrow{AB}$ and $\vec{b} = \overrightarrow{AD}$. Complete the parallelogram ABCD. Join AC and BD.

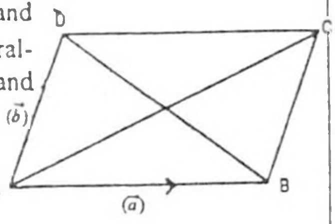

Fig. 3.29.

$$\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BC}$$

$$= \overrightarrow{AB} + \overrightarrow{AD}$$

$$[\because \quad \overrightarrow{BC} = \overrightarrow{AD}]$$

$$= \vec{a} + \vec{b}. \quad \therefore \quad |\vec{a}+\vec{b}| = |\overrightarrow{AC}|$$

Again $\overrightarrow{DB} = \overrightarrow{DA} + \overrightarrow{AB}$

$$= -\overrightarrow{AD} + \overrightarrow{AB} = \overrightarrow{AB} - \overrightarrow{AD} = \vec{a} - \vec{b}$$

$$\therefore \quad |\vec{a}-\vec{b}| = |\overrightarrow{DB}|$$

But $\quad |\vec{a}+\vec{b}| = |\vec{a}-\vec{b}| \quad$ (Given)

$$\therefore \quad |\overrightarrow{AC}| = |\overrightarrow{DB}|$$

or $\quad AC = DB$

i.e., diagonals of the parallelogram are equal.

$\therefore$ The parallelogram is a rectangle (by Geometry) and hence $\vec{a} \perp \vec{b}$.

☞ **Example 8. Prove that the straight line joining the mid-points of the diagonals of a trapezium is parallel to the parallel sides and half of their differences.**

Sol. Let $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$ be the position vectors of the angular points A, B, C, D respectively w.r.t. any origin of the trapezium ABCD whose sides AB and DC are parallel.

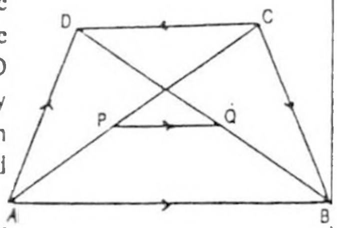

Fig. 3.30.

Let i be the unit vector in the direction of the parallel sides AB and DC.

$$\therefore \quad \overrightarrow{AB} = AB\,i \quad \text{and} \quad \overrightarrow{DC} = DC\,i \quad \ldots(i)$$

where AB and DC are scalars.

Let P be the mid point of diagonal AC and Q be the mid-point of diagonal BD.

$$\text{p.v. of P} = \frac{1}{2}(a+c)$$

$$\text{p.v. of Q} = \frac{1}{2}(b+d) \text{ respectively.}$$

$$\therefore \quad \overrightarrow{PQ} = \text{p.v. of Q} - \text{p.v. of P}$$

$$= \frac{1}{2}(b+d) - \frac{1}{2}(a+c)$$

$$= \frac{1}{2}(b-a) - \frac{1}{2}(c-d)$$

$$= \frac{1}{2}\overrightarrow{AB} - \frac{1}{2}\overrightarrow{DC} = \frac{1}{2}(AB\,i - DC\,i)$$

$$= \frac{1}{2}(AB - DC)\,i$$

This shows that the line PQ is parallel to the unit vector i *i.e.* parallel to the parallel sides AB and DC. Also PQ is half of the difference of AB and DC.

Hence proved.

☞ **Example 9. If $\vec{a}, \vec{b}$ are the position vectors of A, B respectively, find that of a point C in AP produced such that AC = 3AB ; and that of a point D in BA produced such that BD = 2BA.**

Sol. Let O be the origin of reference.

Let $\vec{a}, \vec{b}$ be the position vectors of A, B respectively.

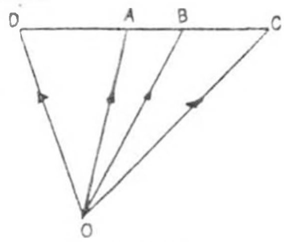

Fig. 3.31.

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= \vec{b} - \vec{a} \quad \ldots(i)$$

(*i*) Since $\quad AC = 3AB \quad$ [Given]

$$\therefore \quad \overrightarrow{AC} = 3\overrightarrow{AB} = 3(\vec{b}-\vec{a}) \quad [\text{Using } (i)]$$

Now in ΔAOC,

$$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC}$$

$$= \vec{a} + 3(\vec{b}-\vec{a}) = 3\vec{b} - 2\vec{a}$$

Hence the position vector of C is $3\vec{b} - 2\vec{a}$.

(*ii*) It is given that BD = 2BA

$$\therefore \quad \overrightarrow{BD} = 2\overrightarrow{BA} = -2\overrightarrow{AB} = -2(\vec{b}-\vec{a}) = 2(\vec{a}-\vec{b})$$

$$[\because \text{ From } (i) \ \overrightarrow{AB} = \vec{b} - \vec{a}]$$

In ΔOBD,

$$\overrightarrow{OD} = \overrightarrow{OB} + \overrightarrow{BD}$$

$$= \vec{b} + 2(\vec{a}-\vec{b}) = 2\vec{a} - \vec{b}$$

Hence the position vector of D is $2\vec{a} - \vec{b}$.

Example 10. The position vectors of four points A, B, C, D are $\vec{a}, \vec{b}, 2\vec{a}+3\vec{b}, \vec{a}-2\vec{b}$ respectively. Express the vectors $\overrightarrow{AC}, \overrightarrow{DB}, \overrightarrow{BC}$ and $\overrightarrow{CA}$ in terms of $\vec{a}$ and $\vec{b}$.

Sol. Let O be the origin of reference.

We are given the position vectors of four points A, B, C and D.

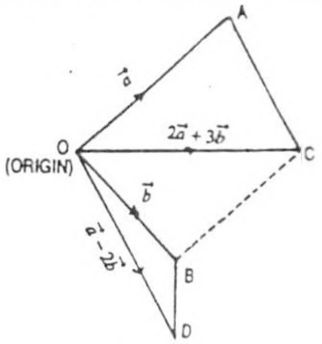

Fig. 3.32.

$$\therefore \quad OA = \vec{a}, \quad \overrightarrow{OB} = \vec{b}$$

$$\overrightarrow{OC} = 2\vec{a}+3\vec{b}, \quad \overrightarrow{OD} = \vec{a}-2\vec{b}$$

(i) $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = 2\vec{a}+3\vec{b}-\vec{a} = \vec{a}+3\vec{b}$

(ii) $\overrightarrow{DB} = \overrightarrow{OB} - \overrightarrow{OD} = \vec{b} - (\vec{a}-2\vec{b}) = \vec{b}-\vec{a}+2\vec{b}$

$$= 3\vec{b}-\vec{a}$$

(iii) $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = 2\vec{a}+3\vec{b}-\vec{b} = 2\vec{a}+2\vec{b}$

(iv) $\overrightarrow{CA} = -\overrightarrow{AC} = -(\vec{a}+3\vec{b})$.

Example 11. Show that the line segment joining the mid-points of two sides of a triangle is parallel to the third side and is half its length.

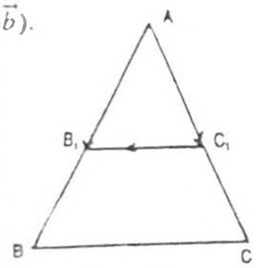

Fig. 3.33.

Sol. Let A of ΔABC be considered as the origin of vectors. Let C_1, B_1 be the mid-points of sides AB, CA respectively.

Then $\quad \frac{1}{2}\overrightarrow{AB} = \overrightarrow{AB_1}, \quad \frac{1}{2}\overrightarrow{AC} = AC_1$,

Now, $\quad \overrightarrow{C_1B_1} = \overrightarrow{C_1A} - \overrightarrow{B_1A}$

$$\overrightarrow{C_1B_1} = \frac{1}{2}\overrightarrow{CA} - \frac{1}{2}\overrightarrow{BA} = \frac{1}{2}(\overrightarrow{CA} - \overrightarrow{BA})$$

$$= \frac{1}{2}(\overrightarrow{CA} + \overrightarrow{AB}) = \frac{1}{2}\overrightarrow{CB}.$$

The above equality shows that $\overrightarrow{C_1B_1}$ and $\overrightarrow{CB}$ have the same direction ; in other words, $C_1B_1 \parallel CB$.

Further, $\quad C_1B_1 = |\overrightarrow{C_1B_1}| = \frac{1}{2}|\overrightarrow{BC}| = \frac{1}{2}BC.$

Example 12. The points D, E, F divide the sides BC, CA, AB of a triangle in the ratio 1 : 4, 3 : 2 and 3 : 7 respectively. Show that sum of the vectors $\overrightarrow{AD}, \overrightarrow{BE}, \overrightarrow{CF}$ is a vector parallel to $\overrightarrow{CK}$ where k divides AB in the ratio 1 : 3.

Sol. Let a, b, c be the position vectors of the points A, B, C respectively.

Then by the question, we have

$$\text{p.v. of D} = \overrightarrow{OD} = \frac{1.\vec{c}+4.\vec{b}}{1+4} = \frac{\vec{c}+4\vec{b}}{5}$$

$$\text{p.v. of E} = \overrightarrow{OE} = \frac{3.\vec{a}+2.\vec{c}}{3+2} = \frac{3\vec{a}+2\vec{c}}{5}$$

$$\text{p.v. of F} = \overrightarrow{OF} = \frac{3.\vec{b}+7.\vec{a}}{3+7} = \frac{3\vec{b}+7\vec{a}}{10}$$

$$\text{p.v. of K} = \overrightarrow{OK} = \frac{1.\vec{b}+3.\vec{a}}{1+3} = \frac{\vec{b}+3\vec{a}}{4}$$

Now $\quad \overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = \frac{\vec{c}+4\vec{b}}{5} - \vec{a} = \frac{\vec{c}+4\vec{b}-5\vec{a}}{5}$

$$\overrightarrow{BE} = \overrightarrow{OE} - \overrightarrow{OB} = \frac{3\vec{a}+2\vec{c}}{5} - \vec{b} = \frac{3\vec{a}+2\vec{c}-5\vec{b}}{5}$$

$$CF = \overrightarrow{OF} - \overrightarrow{OC} = \frac{3\vec{b}+7\vec{a}}{10} - \vec{c} = \frac{3\vec{b}+7\vec{a}-10\vec{c}}{10}$$

$$\therefore \overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF} = \frac{\vec{c}+4\vec{b}-5\vec{a}}{5} + \frac{3\vec{a}+2\vec{c}-5\vec{b}}{5} + \frac{3\vec{b}+7\vec{a}-10\vec{c}}{10}$$

$$= \frac{1}{10}(3\vec{a}+\vec{b}-4\vec{c}) \qquad \ldots(i)$$

and

$$\overrightarrow{CK} = \frac{\vec{b}+3\vec{a}}{4} - \vec{c} = \frac{\vec{b}+3\vec{a}-4\vec{c}}{4}$$

$$= \frac{10}{4}\left(\frac{3\vec{a}+\vec{b}-4\vec{c}}{10}\right)$$

$$= \frac{5}{2}(\overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF}) \quad \text{[Using (i)]}$$

Hence $\quad \overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF}$ is parallel to $\overrightarrow{CK}$.

Example 13. If a and b are two vectors represented by OA and OB, and if C is a point in the line AB such that AC : CB = $m_1 : m_2$, that is m_2 AC = m_1, CB, where m_1 and m_2 are positive real numbers then,

$$c = OC = \frac{m_2 a + m_1 b}{m_1 + m_2}.$$

Proof. From Fig. 3.34

$$OA = OC + CA$$

Therefore, m_2 OA = m_2 OC + m_2 CA, since m_2 is a positive real number. ...(*i*)

Also, $\quad OB = OC + CB$

or, $\quad m_1 OB = m_1 OC + m_1$ CB, since m_1 is a positive real number. ...(*ii*)

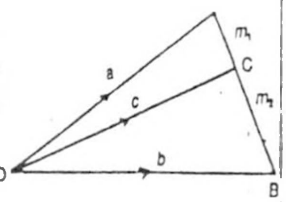

Fig 3.34.

Adding (*i*) and (*ii*), we get

$$m_2 OA + m_1 OB = (m_2 + m_1) OC + m_2 CA + m_1 CB \quad ...(iii)$$

But $\quad m_2 AC = m_1 CB,$

or $\quad m_2 CA + m_1 CB = 0$

Hence, from (*iii*), we get

$$m_2 OA + m_1 OB = (m_2 + m_1) OC$$

or $$c = OC = \frac{m_2 OA + m_1 OB}{m_2 + m_1} = \frac{m_2 a + m_1 b}{m_2 + m_1}$$

Note : If $m_2 = m_1$, that is, if C is the mid-point of AB, then

$$OA + OB = 2OC$$

or $$c = OC = \frac{OA + OB}{2} = \frac{a+b}{2}.$$

Example 14. Show that the medians of a triangle are concurrent.

Sol. Let the position vectors of the vertices A, B, C of a triangle ABC with respect to any origin be *a*, *b*, *c*.

The position vectors of the mid-points D, E, F of the sides are

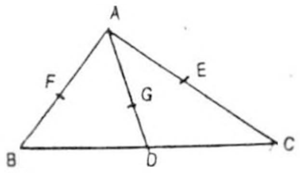

Fig. 3.35.

$$\frac{1}{2}(b+c), \frac{1}{2}(c+a), \frac{1}{2}(a+b) \text{ respectively.}$$

Position vector of the point G dividing AD in the ratio 2 : 1 is

$$\frac{2 \cdot \frac{1}{2}(b+c) + 1.a}{2+1} = \frac{1}{3}(a+b+c) \quad ...(i)$$

By symmetry, we see that this point also lies on the other two medians.

Thus the medians of a triangle are concurrent. Also the position vector of the point of concurrence, is 1/3 (*a*, *b*, *c*) ; *a*, *b*, *c* being the position vectors of the vertices of the triangle.

The point of concurrence of the medians of a triangle is called its centroid.

Example 15. $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are the vectors forming the consecutive sides of a quadrilateral. Show that a necessary and sufficient condition that the figure be a parallelogram is that $\vec{a} + \vec{c} = \vec{0}$ and this implies $\vec{b} + \vec{d} = \vec{0}$.

Sol. ABCD is a quadrilateral.

$$\overrightarrow{AB} = \vec{a}, \overrightarrow{BC} = \vec{b}, \overrightarrow{CD} = \vec{c}$$

and $$\overrightarrow{DA} = \vec{d}$$

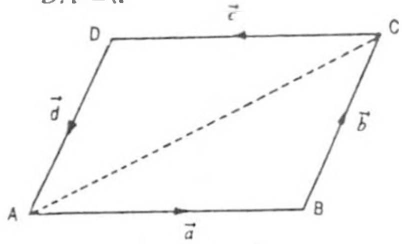

Fig. 3.36.

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC} = \vec{a} + \vec{b}$$

Also $$\overrightarrow{AC} + \overrightarrow{CD} = \overrightarrow{AD}$$

$$\therefore \quad (\vec{a} + \vec{b}) + \vec{c} = -\vec{d}$$

$$\therefore \quad \vec{a} + \vec{b} + \vec{c} + \vec{d} = \vec{0} \quad ...(i)$$

Now if ABCD is a parallelogram, then, AB and DC are parallel and equal.

$$\therefore \overrightarrow{AB} = -\overrightarrow{CD}, \quad \therefore \quad \vec{a} = -\vec{c} \quad \text{or} \quad \vec{a} + \vec{c} = \vec{0}$$

Hence the condition is necessary. Also with the help of (*i*), we get in this case

$$\vec{b} + \vec{d} = \vec{0}.$$

Sufficients. Since $\vec{a} + \vec{c} = \vec{0}$, $\therefore$ $\vec{a} = -\vec{c}$.

$$\therefore \quad \overrightarrow{AB} = -\overrightarrow{CD}$$

$$\therefore \quad \overrightarrow{AB} = \overrightarrow{DC}$$

Thus AB and DC are parallel and equal.

When $\vec{a} + \vec{c} = \vec{0}$, we have from (*i*), $\vec{b} + \vec{d} = \vec{0}$.

$$\therefore \quad \vec{b} = -\vec{d},$$

$$\therefore \quad \overrightarrow{BC} = -\overrightarrow{DA}, \quad \therefore \quad \overrightarrow{BC} = \overrightarrow{AD}.$$

Hence, ABCD is a parallelogram.

Example 16. Show that points

$\bar{a} + 2\bar{b} + 3\bar{c}, -\bar{a} + 2\bar{b} - 4\bar{c}, 3\bar{a} + 3\bar{c}, 2\bar{a} + 3\bar{b} + 10\bar{c}$

are coplanar, $\bar{a}, \bar{b}, \bar{c}$ being any three non-zero, non-coplanar vectors.

Sol. Let the four points in order be A, B, C, D and let O be the origin. Then

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = -2\bar{a} - 7\bar{c}$$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = 2\bar{a} - 2\bar{b}$$

$$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = \bar{a} + \bar{b} + 7\bar{c}.$$

The given points will be coplanar if $\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}$ are coplanar. In that case we can find scalars x and y such that

$$-2\bar{a} - 7\bar{c} = x(2\bar{a} - 2\bar{b}) + y(\bar{a} + \bar{b} + 7\bar{c})$$

$$\Rightarrow \quad -2 = 2x + y, \; 0 = -2x + y, \; -7 = 7y$$

The values $x = -1/2$, $y = -1$ satisfy all the three equations. Thus we can express one vector $\overrightarrow{AB}$ as linear combination of the other two vectors $\overrightarrow{AC}$ and $\overrightarrow{AD}$.

$\therefore$ $\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}$ are coplanar and hence the given points are coplanar.

Example 17. Find by vector method the perimeter of the triangle whsoe vertices are the points (3, 1, 5), (–1, –1, 9) and (0, –5, 1).

Sol. Let the vertices A, B and C of a triangle ABC be the points (3, 1, 5), (–1, –1, 9) and (0, –5, 1) respectively. Then the position vectors of A, B and C referred to (0, 0, 0) as origin are

$$\overrightarrow{OA} = 3\hat{i} + \hat{j} + 5\hat{k}; \quad \overrightarrow{OB} = -\hat{i} - \hat{j} + 9\hat{k}$$

and $\quad \overrightarrow{OC} = -5\hat{j} + \hat{k}$

$\therefore \quad \overrightarrow{AB} = \text{p.v. of B} - \text{p.v. of A} = \overrightarrow{OB} - \overrightarrow{OA}$

$$= (-\hat{i} - \hat{j} + 9\hat{k}) - (3\hat{i} + \hat{j} + 5\hat{k})$$

$$= = -4\hat{i} - 2\hat{j} + 4\hat{k}$$

Similarly, $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = (-5\hat{j} + \hat{k}) + (-\hat{i} - \hat{j} + 9\hat{k})$

$$= \hat{i} - 4\hat{j} - 8\hat{k}$$

and $\quad \overrightarrow{CA} = \overrightarrow{OA} - \overrightarrow{OC} = (3\hat{i} + \hat{j} + 5\hat{k}) - (-5\hat{j} + \hat{k})$

$$= 3\hat{i} + 6\hat{j} + 4\hat{k}$$

$\therefore \quad AB = |\overrightarrow{AB}| = \sqrt{(-4)^2 + (-2)^2 + 4^2}$

$$= \sqrt{16 + 4 + 16} = 6$$

$$BC = |\overrightarrow{BC}| = \sqrt{1^2 + (-4)^2 + (-8)^2} = \sqrt{1 + 16 + 64} = 9$$

$$CA = |\overrightarrow{CA}| = \sqrt{3^2 + 6^2 + 4^2} = \sqrt{9 + 36 + 16} = \sqrt{61}$$

$\therefore$ The required perimeter = AB + BC + CA

$$= 6 + 9 + \sqrt{61} = 15 + \sqrt{61}.$$

Example 18. Write down the equation of the line through the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$.

Sol. The position vectors of P and Q are respectively

$$\overrightarrow{OP} = x_1\vec{i} + y_1\vec{j} + z_1\vec{k}$$

$$OQ = x_2\vec{i} + y_2\vec{j} + z_2\vec{k}$$

If R is any point with position vector

$$\overrightarrow{OR} = x\vec{i} + y\vec{j} + z\vec{k}$$

then there exists a real number λ such that

PR : RQ = λ : 1. Then

$$x\vec{i}' + y\vec{j}' + j\vec{k}' = \overrightarrow{OR}$$

$$= \frac{x_1\vec{i}' + y_1\vec{j}' + z_1\vec{k}' + \lambda(x_2\vec{i}' + y_2\vec{j}' + z_2\vec{k}')}{1 + \lambda}$$

Hence $\quad x = \dfrac{x_1 + \lambda x_2}{1 + \lambda}, y = \dfrac{y_1 + \lambda y_2}{1 + \lambda}, z = \dfrac{z_1 + \lambda z_2}{1 + \lambda}$

i.e. $\quad x - x_1 = \dfrac{\lambda}{1 + \lambda}(x_2 - x_1), \quad y - y_1 = \dfrac{\lambda}{1 + \lambda}(y_2 - y_1),$

$$z - z_1 = \frac{\lambda}{1 + \lambda}(z_2 - z_1)$$

Eliminating λ, we get

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

which is the equation of the line PQ since any $R(x, y, z)$ therein satisfy these equations.

Example 19. Find the unit vector in the direction of the vector $r_1 - r_2$, where $r_1 = \hat{i} + 2\hat{j} - \hat{k}$, $r_2 = 3\hat{i} + \hat{j} - 5\hat{k}$.

Sol. Let $\quad \vec{r} = \vec{r}_1 - \vec{r}_2$

$$= (\hat{i} + 2\hat{j} - \hat{k}) - (3\hat{i} + \hat{j} - 5\hat{k})$$

$$= (1 - 3)\hat{i} + (2 - 1)\hat{j} + (-1 + 5)\hat{k}$$

$\therefore \quad \vec{r} = -2\hat{i} + \hat{j} + 4\hat{k}$

$$|\vec{r}| = \sqrt{(-2)^2 + (1)^2 + 4^2} = \sqrt{4 + 1 + 16} = \sqrt{21}$$

$\therefore$ a unit vector in the direction of vector $\vec{r}$

$$= \frac{\vec{r}}{|\vec{r}|} = \frac{-2\hat{i} + \hat{j} + 4\hat{k}}{\sqrt{21}}$$

$$= \frac{-2}{\sqrt{21}}\hat{i} + \frac{1}{\sqrt{21}}\hat{j} + \frac{4}{\sqrt{21}}\hat{k}.$$

ADDITIONAL PRACTICE EXERCISE 3 (*b*)

1. Show that the four points A, B, C, D with position vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ respectively such that $3\vec{a} - 2\vec{b} + 5\vec{c} - 6\vec{d} = 0$, are coplanar. Also, find the position vector of the point of intersection of the lines AC and BD.
2. Show that the four points P, Q, R, S with position vectors $\vec{p}, \vec{q}, \vec{r}, \vec{s}$ respectively such that $5\vec{p} - 2\vec{q} + 6\vec{r} - 9\vec{s} = 0$, are coplanar. Also find the position vector of the point of intersection of the lines PR and QS.
3. Show that the four points A, B, C, D with position vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ respectively such that $3\vec{a} + 4\vec{b} - 5\vec{c} - 2\vec{d} = 0$ are coplanar. Also, find the position vector of the point of intersection of the lines AB and CD.
4. Show that the four points M, N, R, S with position vectors $\vec{m}, \vec{n}, \vec{r}, \vec{s}$ respectively such that $2\vec{m} + 3\vec{n} - 4\vec{r} - \vec{s} = 0$ are coplanar. Also, find the position vector of the point of intersection of the lines MN and RS.
5. In a regular hexagon ABCDEF, prove that
$$\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD} + \overrightarrow{AE} + \overrightarrow{AF} = 3\overrightarrow{AD}.$$
6. If two concurrent forces are represented by $\lambda\overrightarrow{AO}$ and $\mu\,\overrightarrow{OB}$ prove that their resultant is $(\lambda + \mu) + \overrightarrow{OC}$, where C divides AB such that $\lambda AC = \mu CB$. [*M. Imp.*]
7. Prove that
   (*i*) $|\vec{a} + \vec{b}| \le |\vec{a}| + |\vec{b}|$
   (*ii*) $|\vec{a}| - |\vec{b}| \le |\vec{a} - \vec{b}|$
   (*iii*) $|\vec{a} - \vec{b}| \le |\vec{a}| + |\vec{b}|$.
8. If $\vec{a}$ and $\vec{b}$ are the vectors forming consecutive sides of a regular hexamon ABCDEF express the vectors $\overrightarrow{CD}, \overrightarrow{DE}, \overrightarrow{EF}, \overrightarrow{FA}, \overrightarrow{AC}, \overrightarrow{AD}, \overrightarrow{AE}$ and $\overrightarrow{CF}$ in terms of $\vec{a}$ and $\vec{b}$.
9. ABCDE is a pentagon. Prove that
$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} + \overrightarrow{EA} = \vec{0}.$$
10. Prove that the sum of all the vectors drawn from the centre of a regular octagon to its vertices is the zero vector.
11. Find the coordinates of the point which divides the line joining (x_1, y_1, z_1), (x_2, y_2, z_2) in the ratio $m : l$.
12. Show that the three medians of a triangle meet at a point called the centroid of the triangle which trisects each of the medians.
13. D, E, F are the mid-points of the sides of a triangle ABC. Show that for any point O.
$$\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OD} + \overrightarrow{OE} + \overrightarrow{OF}.$$
14. D, E, F are the middle points of the sides BC, CA, AB respectively of a triangle ABC. Show that
   (*i*) EF is parallel to BC and half of its length.
   (*ii*) The sum of the vectors, $\overrightarrow{AD}, \overrightarrow{BE}, \overrightarrow{CF}$ is zero.
   (*iii*) The medians have a common point of trisection, *i.e.*, they are concurrent.
15. ABCD is a parallelogram and P is the intersection of the diagonals ; O is any point. Show that
$$\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD} = 4\overrightarrow{OP}.$$
16. Prove that the straight line joining the mid-points of he diagonals of a trapezium is parallel to the parallel sides and half their difference.
17. The sides of a parallelogram are $2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\hat{i} + 2\hat{j} + 3\hat{k}$. Find the unit vectors parallel to the diagonals.
18. If the position vectors of P, Q, R, S are
$$2\hat{i} + 4\hat{k},\ 5\hat{i} + 3\sqrt{3}\,\hat{j} + 4\hat{k},\ -2\sqrt{3}\,\hat{j} + \hat{k},\ 2\hat{i} + \hat{k}$$
prove that RS is parallel to PQ and is two-third of PQ.
19. P(2, −1, 3), Q(8, 5, −6) and R(4, 1, 0) are the vertices of a triangle. Show that PQ = 3PR and the direction cosines of QR are
$$\frac{-2}{\sqrt{17}}, \frac{-2}{\sqrt{17}}, \frac{-3}{\sqrt{17}}$$
20. If the position vectors of P and Q are $2\hat{i} + 3\hat{j} + 7\hat{k}$ and $4\hat{i} - 3\hat{j} - 4\hat{k}$ respectively, find $\overrightarrow{PQ}$ and determine its direction cosines.
21. Show that the three points having position vectors $\vec{a} - 2\vec{b} + 3\vec{c}$, $-2\vec{a} + 3\vec{b} + 2\vec{c}$, $-8\vec{a} + 13\vec{b}$ are collinear, whatever be $\vec{a}, \vec{b}, \vec{c}$.
22. Show that the vectors $(\vec{a} - 2\vec{b} + 3\vec{c})$, $(\vec{a} - 3\vec{b} + 5\vec{c})$ and $(-2\vec{a} + 3\vec{b} - 4\vec{c})$ are coplanar where $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar.
23. Show that the vectors $(2\vec{a} - \vec{b} + 3\vec{c})$, $(\vec{a} + \vec{b} - 2\vec{c})$ and $(\vec{a} + \vec{b} - 3\vec{c})$ are not coplanar.
24. Show that the vectors $(2\hat{i} - \hat{j} + \hat{k})$, $(\hat{i} - 3\hat{j} - 5\hat{k})$ and $(3\hat{i} - 4\hat{j} - 4\hat{k})$ are coplanar.
25. Prove that the points with position vectors $(-\hat{j} - \hat{k})$, $(4\hat{i} + 5\hat{j} + \hat{k})$, $(3\hat{i} + 9\hat{j} + 4\hat{k})$ and $(-4\hat{i} + 4\hat{j} + 4\hat{k})$ are coplanar.

MATHEMATICAL MODELING

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A mathematical model is a simplified mathematical representation of a real situation with a mathematical system (a model is something which represents something else). Although a real situation involves a large number of variables and constraints, usually a small fraction of these variables and constraints truly dominate the behaviour of the real system. Thus the simplification of the real system should primarily concentrate on identifying the dominant variables or constraints as well as other data pertinent to problem solving. The assumed real system is abstracted from the real situation by identifying dominant factors (variables, constraints etc) that control the behaviour of the system and such a system always serves as a data for mathematical modeling. A mathematical model is robust if small changes in variables lends to a small change in the behaviour of the model.

The set of natural numbers with usual addition and multiplication form a good mathematical model of a real situations concerned with counting process. Vectors are excellent mathematical models that predict and explain many physical phenomena with perfect accuracy. The concept of direction which is so vague in the physical world is precisely explained by identifying the concept of vector as that of location or coordinate system. (Such an identification is guaranteed by the famous result that every finite dimensional vector space is isomorphic to Euclidean space R^n). We will discuss in greater details some more models in a later section.

Mathematical models are normally thought of as instrument for selecting a good course of action from the set of courses of action that is covered by the model (here a course of action could be a strategy of selecting a content or some such thing). However the models have another very important use: they can be used heuristically (that is an instruments of discovery). They provide an effective tool with which one can explore the structure of a problem and uncover possible course of action that were previously overlooked. For example vectors as models have lead to discovery of several outstanding and useful results in the vectors space theory. The models concerned with drawing of implication diagram (Venn diagram) of given concepts give rise to some very interesting conjectures and their solution later. A good mathematical model presents many features or many predictors of the data; that is, a good mathematical model is one in which many dependent variables are expressed through functions.

Types of models:

There are three types of models which are commonly used: *iconic, analogue, and symbolic.*

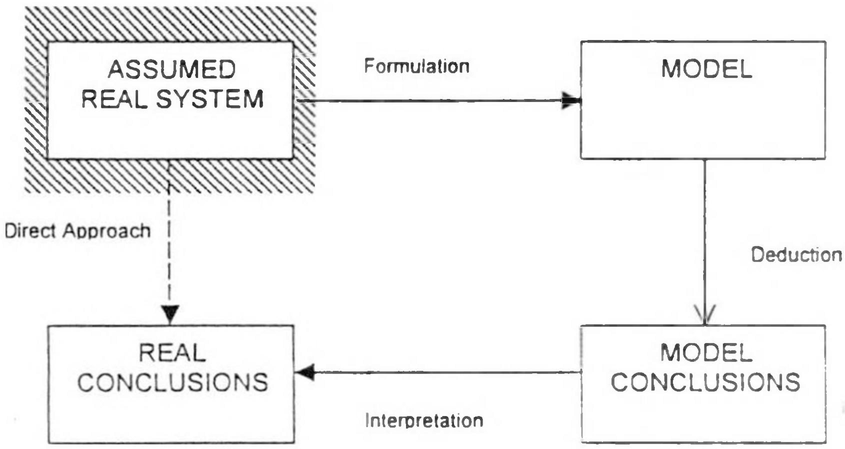
Iconic models are images; they represent the relevant properties of the real situations. For example, Photographs, maps, model aeroplane, drawings of some mathematical objects etc. Iconic model of the sun and its planets in planetorium or model of a field map is scaled down where as a model of atom is scaled up. Iconic models are generally specific, concrete and difficult to manipulate for experimental purposes.

Analogues use one set of properties to represent another set of properties. For example graphs are analogues that use geometric magnitudes and location to represent a wide variety of variables and the relationship between them. Contour lines on a map are analogues of elevation. Bar diagrams are analogues of some statistical information. Flow chart is an analogue of some logical sequence. In general analogues are less specific, less concrete but easier to manipulate than iconic models.

Symbolic models use symbols, numbers to represent variables and relationship between them. Hence they are the most general and abstract type of models. Linear programming model, simple harmonic motion model are some of the examples of symbolic models. Symbolic model are most widely used and result oriented, and the other models (iconic and analogue) are sometimes used as initial approximations which are subsequently refined in to a symbolic model. Symbolic models take the form of mathematical relationships (usually equations or inequations) that reflect the structure of that which they represent.

Process of Modeling

The process of modelling is depicted in the following figure.



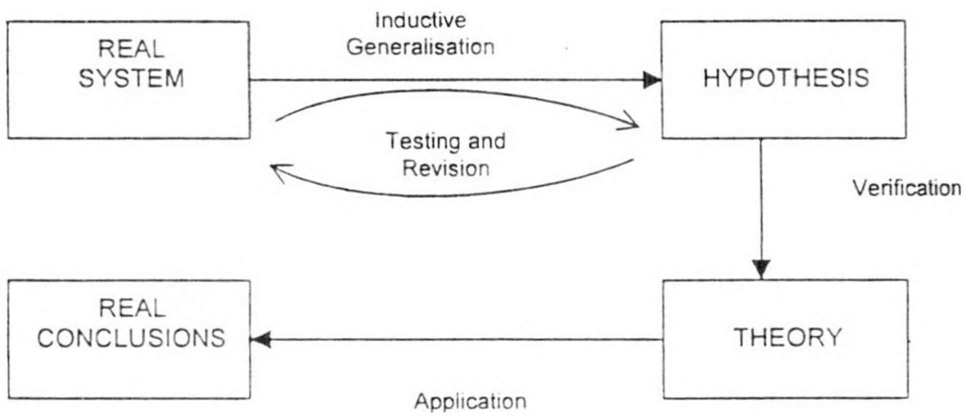
The first step is formulation of the model itself. This step calls for identification of assumptions that can and should be made so that the model conclusions are as accurate as expected. The selection of the essential attributes of the real system and omission of the irrelevant ones require a kind of selective perception which is more an art than a science and which cannot be defined by any precise methodology.

The second step is to analyse the formulated model and deduce its conclusions. It may involve solving equations, finding a good suitable algorithm, running a computer program, expressing a sequence of logical statements - whatever is necessary to solve the problem of interest related to the model.

The final step, interpretation involves human judgement. The model conclusions must be translated to real world conclusions cautiously without discrepancies between the model and its real world referent.

Mathematical Modeling in contrast to experimentally based Scientific Method.

The following figure depicts the process of scientific method.



Here first step is development of a hypothesis which is arrived at generally by induction following a period of informal observation. An experiment is then devised to test the hypothesis of the experiments, if the result contradicts the hypothesis, the hypothesis is revised and retested. The cycle continues until a verified hypothesis or 'theory' is obtained. The first result of the process is Truth, Knowledge or Law of Nature. In contrast to model conclusions theories are independently verifiable statements about factual matters. Models are invented; theories are discovered. Thus modeling is very important but certainly not unique method to deal with complicated real world.

Some mathematical models.

1. ***(a number theoretic model)***

In a party of people with atleast two persons, we are always assured of atleast two persons who know same number of persons in the party.

Here the real situation is the party of people in which a person may have an acquaintance with another person. The conclusion is that there are atleast two persons having the same number of acquaintances in the party We now proceed to model the situation as follows:

Let $P_1, P_2 \ldots\ldots\ldots P_n$ be the persons in the party and let d_i be the number of persons known to P_i in the party. Here once the identification of the variables d_i is done, the rest follows by contrapositive argument.

If the conclusion is wrong then there is a set S of n-1 persons in the party such that each has distinct number of acquaintances and each knows atleast 1 and atmost n-2 members in S. That is, each number d_i corresponding to a member in S in unique, and atleast 1 and atmost n-2. This amounts to getting n-1 distinct integers in the set {1, 2,........, n-2}, a contradiction.

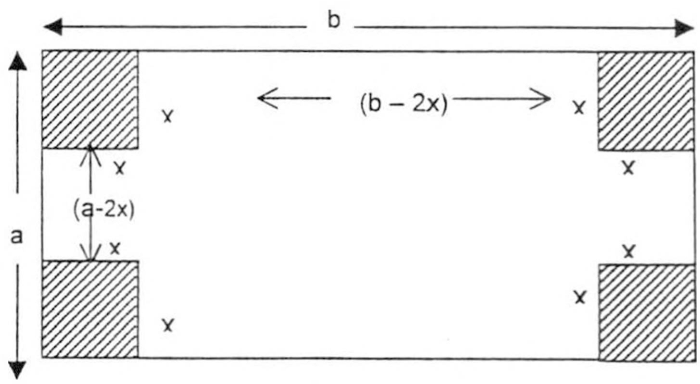
Thus we arrived at the model conclusion by logical sequence of arguments.

2. *(a maxima - minima model)*

Suppose an open box is made from a rectangular piece of tin a sq.mts. by b sq.mts. by cutting out equal squares at each corner and folding up the remaining flaps. What size square should be cut out so that the box will have maximum volume ?

We first draw analogue of the given situation as a prelude to construction of symbolic model (see the following figure).

Analogue of the given situation:



Volume of the open box = $V(x) = x(2-2x).(v-2x)$

Surface area = $ab-4x^2$

First we identify the most significant variable in the given situation. Let x be the length of a side of any of these four squares (all of which are of equal area). The objective is to find a value for x which maximizes $V(x) = x(a-2x)(b-2x)$. $V'=O$ and $V''<O$ imply that the square of dimension $(a+b+\sqrt{a^2+b^2-ab})/6$ be cut out so that the box has maximum volume. Here we note that maximization of surface area $ab-4x^2$ need not imply maximization of volume $V(x)=(a-2x)(b-2x)$.

Applying the same method to the cutout squares, we can make new open boxes with optimal utility. Thus, this model provides a method and solution to make open boxes with optimal use of given rectangular tin sheet.

3. Graphs (Networks) as Mathematical Model:

A graph (or network) is a non-empty set V together with an irreflexive and symmetric relation E on V. The elements of V are marked as vertices and the elements of E are marked as edges (not necessarily straight) joining the vertices in a pair belonging to E. Two vertices are adjacent if they are joined by an edge. For example, if V ={a,b,c,d} and E={(a,b), (a,d), (b,d), (b,c)} the pictorial representation of the graph is as that in the following figure 1.

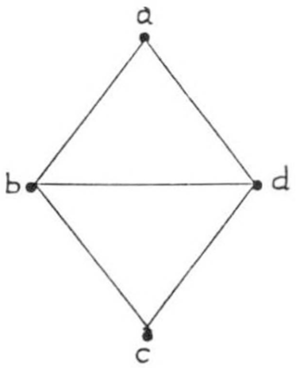


Fig. 1

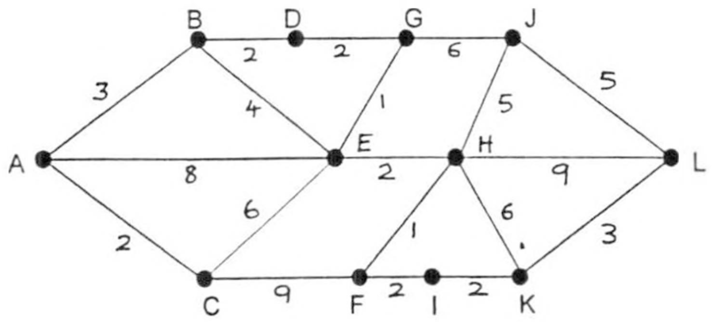


Fig.2

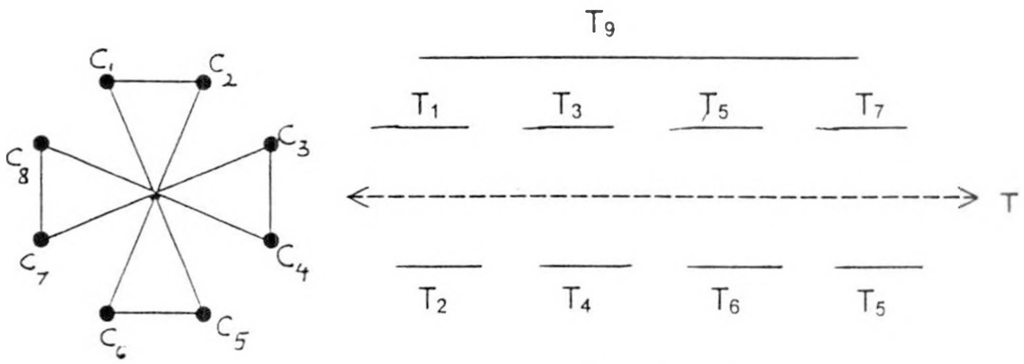
Since the graphs are the most generalized algebraic structure, they often work as excellent models of many real situations. The following examples are just three of those several situations which are easily modeled as graphs.

(i) Shortest path problem: Suppose that we have a map of the form shown in the above figure 2 in which the letters A-L refer to towns which are connected by roads. If the lengths of these roads are as marked in the diagram what is the length of the shortest path from A to L?

There are several methods which can be used to solve this problem. Possibly the simplest of these is to make a model of the graph by knotting together pieces of a string whose lengths are proportional to the lengths of the roads. In order to find the shortest path we hold the knots corresponding to A and L and pull tight and measure the distance corresponding to the tight strings. However there is a more mathematical way of approaching this problem using graph theory.

(ii) Scheduling Problem: Consider a collection $C=\{C_i\}$ of course being offered by a major university. Let T_i be the time interval during which course C_i is to take place. We would like to assign courses to classrooms, so that no two courses meet in the same room at the same time.

We treat C_i as the vertices of the graph G in which C_i and C_j are joined by an edge if and only if T_i and T_j have not empty intersection. We colour the vertices of G such that no two vertices joined by an edge have the same colour. Here each colour corresponds to a classroom. For such graph (called interval graphs) there is an efficient algorithm for colouring its vertices with minimum number of colours. In fact for such graphs the minimum number of colours is equal to the maximum number of mutually adjacent vertices.



Time Intervals

(iii) **Shortage of Chemicals Problem:** Suppose $c_1, c_2, \ldots\ldots\ldots, c_n$ are chemical compounds which need to be refrigerated under closely monitored conditions. If a compound c_i must be kept at a constant temperature between t_i and t_i', the problem is to find minimum number of refrigerators needed to store all the compounds ?

Let G be the interval graph with vertices $c_1, c_2, \ldots\ldots\ldots c_n$ and connect two vertices by a line whenever the temperature intervals of their corresponding compounds intersect. It is not difficult to verify that the intervals (t_i, t_i') satisfy The Helly property (A family of subsets of a set X is said to satisfy the Helly property if pairwise non-empty intersection of members of S imply total non empty intersection of the members of S).

If Q is a clique of G, then the time intervals corresponding to its vertices will have a common point, say t, by Helly property. Therefore a refrigerator set at a temperature t will be suitable for storing the chemicals representing the vertices of Q. Thus a solution to the minimization problem will be obtained by finding minimum clique cover of G. (A clique is a graph in which any two vertices are joined by a line. In fig.1, the sub graphs on {a,b,d} and {b,d,c} will provide a minimum clique cover).

Mathematical modeling plays a great role in teaching Mathematics

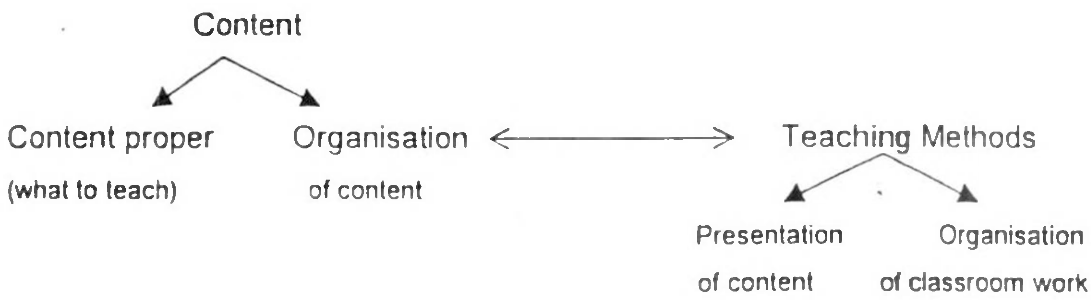
Some of the most important components of teaching a concept in mathematics are:

(i) Motivation for the concept (ii) Simplification of the concept (iii) Problem solving.

Motivation for learning a mathematical concept may be within the mathematics itself or outside the mathematics and a real world situation. For instance, it is very difficult to choose an example of a infinite set from a real world situation; so in such a situation the set of natural numbers can be taken as a motivating factor for the concept of 'infinite sets'. On the other hand a great deal of real world motivate and exemplify several concepts like vector, derivative, integral etc.

By simplication of a concept C we mean breaking of the concept C into simpler sub concepts or more precisely it is identification of meaningful restrictions f on C such that C_f (the restricted C) has a simpler characterization than that of C. Once a concept is simplified into $C_1, C_2, \ldots, C_k$, one is naturally tempted to find various inter-relations among the sub concepts C_i and that is how the concept C in particular and mathematics in general becomes richer.

Content in mathematics can be analysed into content proper (what to teach) and its inner organization, the latter being most closely related to teaching methods. Teaching methods can be analysed into presentation of the subject matter (use of mathematical models etc) and organization of class room work, the former being most closely related to content and mathematical modeling. The analogue model of this para is as follows:



TEACHING MATHEMATICAL CONCEPTS

PROF. K. DORASAMI

The study of mathematics deals with certain objects such as Natural numbers, Circles, Triangles, Functions and Proof.

In learning about these mathematical objects, we are concerned with what these objects are. For example,

1. What an angle is, how to call whether or not something is a rectangle, what is the definition of a parallelogram.
2. What are the relations among mathematical objects.

When we learn what an object is, we are learning a concept of that object.

When we teach students what an object is, and how to identify it, we are teaching a concept of that object.

Concepts are the most basic learnable objects and the first things learned by young children.

By means of concepts, other concepts and other kinds of subject matter are learned.

A concept is the meaning of a term used to designate the concept.

According to Hunt, Marin, and Stone (1966), "A concept is a decision rule which, when applied to the description of an object, specifies whether or not a name can be applied". Thus a student who knows the definition of a circle as the locus of points in a plane from a given point in the plane has a rule that can be used to tell whether any given object is to be called a circle.

Moves in Teaching a Concept

Some concepts are taught, for others the term designating the concepts are used.

For example, a teacher who had deliberately taught a concept of a finite set might not teach a concept of an infinite set but would simply use the term.

1. Defining

Because most concepts in mathematics are precise, definitional moves can be used.

Definition is an elegant move since it employs minimum language. But the very elegance may be a block to learning.

Definitions are often written in the form (1) is a (2) such that (3).

The first space is filled by the term being defined, the second space is filled by a term denoting a superset in which the set of objects denoted by the term defined as included, and third space is filled by one or more conditions that differentiate the set of objects denoted by the term defined from all the other subsets of the superset.

2. Status a Sufficient Condition or Sufficient Condition Move

It is the form in which a characteristic or a property of an object is stated that identifies it as a sufficient condition.

A rhombus is an equilateral parallelogram. Being an equilateral parallelogram is sufficient for being rhombus.

The sufficient condition is more clear in the statements.

"If a quadrilateral is an equilateral parallelogram, it is a rhombus".

"If a parallelogram is a square, it is a rhombus". Other forms are:

A triangle is a right angled triangle provided that it has one right angle.

The logic of the move of sufficient condition enables a student to find examples of objects denoted by a concept, assuming such an example exists.

3. Giving One or More Examples

Examples are objects denoted by the concept i.e. members of the set determined by the concept.

Examples clarify concepts because they are definite, specific, and if well chosen, familiar.

Teachers frequently elicit examples of concepts from students to decide whether the students have acquired the concepts.

Examples cannot be given for every concept. For example, even prime number greater than 2, greatest integer and for self-contradictory concepts like square circle, six-sided pentagon.

4. Giving an Example Accompanied by a Reason Why it is an Example

Accompanying an example with a reason that it is an example is an effective move because the reason is a sufficient condition.

This move is helpful to slow learners, because the logical connection is made explicit by supplying a reason.

$4x^2 + 9y^2 = 36$ is an ellipse because it is of the form $a^2x^2 + b^2y^2 = a^2b^2$

40 is an even number since it is divisible by 2.

5. **Comparing and Contrasting Objects Denoted by the Concept**

By comparing objects of the concept being taught with objects with which students are familiar, a bond of association can be established between familiar and less familiar.

In teaching a concept of parallelogram, the teacher may compare it with non-parallelogram (trapezium).

Comparison points out similarities. But since objects compared are not identical, a contrast identifies some of the differences, if not all.

If a teacher has taught a concept of equal set and then teaches a concept of equivalent set, the next step may be contrast these two concepts in order that the students do not miss the distinction between them.

6. **Giving a Counter Example**

A counter example is an example that disproves a false definition of a concept.

Two kinds of counter examples are possible for an incorrect definition.

1. Give a number (an example) of the set determined by the term defined that is not a member of the set determined by the defining expression.
2. Given a member (an example) of the set determined by the defining expression that is not a member of the set determined by the term defined.

Though this kind of move is effective in sustaining thinking and ultimately facilitating comprehension of the desired concept, students may feel that the teacher was badgering and embarrassing them. Teachers have to exercise good judgement when deciding how frequently to use counter example moves.

7. Stating a Necessary Condition

If two sides are parallel, a quadrilateral is a parallelogram. This statement indicates the absence of a necessary condition for a quadrilateral to be a parallelogram.

One form of the definition of a parallelogram. With the necessary condition is,

If both pairs of opposite sides are parallel, a quadrilateral is a parallelogram.

Another form in which a necessary condition is stated uses only if.

Example: A quadrilateral is a parallelogram only if both pairs of sides are parallel.

A necessary condition move enables a student to identify examples of objects not denoted by a concept.

8. Stating a Necessary and Sufficient Condition

This move is used, if a condition by which objects can be denoted by a concept is both necessary and sufficient condition. One form for this is the explicit use of the terms necessary and sufficient, as

It is both necessary and sufficient that a parallelogram be equilateral for it to be a rhombus. Another form is the use of if and only if. Thus the statement is equivalent to, A parallelogram is a rhombus if and only if it is equilateral.

The definition in terms of necessary and sufficient condition proceeds by subsuming the set of objects to be defined from all other subsets of the superset. Thus, a definition of a rhombus might be;

A parallelogram having pair of adjacent, congruent sides is a rhombus.

The definition implies that there are two conditions necessary for an object to be a rhombus:

1. being a parallelogram and
2. having a pair of adjacent congruent sides. The combination of these two necessary condition is sufficient.

But for some students, the necessary and sufficient conditions in the above statement may not be clear. For them, the teacher can make use of if and only if form.

A sufficient condition move enables a student to identify examples and a necessary condition move enables students to identify non-examples of a concept. A combination of these enables students to discriminate both examples and non-examples of a concept.

An object not in the set determined by a concept is a non-example of the concept.

9. Giving Non-examples

Like the move of giving examples, giving non-examples helps to clarify a concept. Definition of a concept following examples and non-examples of the concept is a common move for a teacher.

10. Giving a Non-example Accompanied by a Reason Why It Is a Non-example

This move is similar to that of giving an example together with a reason that is an example. The reason that accompanies the non-example is the failure to satisfy a necessary condition.

Its logic is that of conditional reasoning,

"If a quadrilateral is not a parallelogram, it is not a rhombus. This quadrilateral is not a parallelogram. Therefore it is not a rhombus".

Strategies of Teaching a Concept

A strategy is defined as a temporal sequence of moves.

So, theoretically, there are thousands of strategies for teaching a concept, of which some are logically impossible.

Examples of Some Strategies of Teaching a Concept

1. Definition – Example – Example with a reason

   Non-example with a reason

2. Example – Non-example – Comparison and Contrast – Characteristic – Definition – Example with a reason – Non-example with a reason.

In such strategies, the definition identifies the necessary and sufficient conditions, examples clarifies them and reasons reinforce necessary and sufficient conditions.

Use of Concepts

1. Knowledge of a concept helps in classifying given objects into examples and non-examples of the concept.

Since we can classify, we can discriminate. For example, a student who has concept of rhombus can pick out rhombus from other quadrilaterals.

2. Knowledge of concepts helps in communication.

Communication breaks when people do not have the knowledge of certain concepts.

A definition of a term tells you both how to use the term and also how to avoid using it.

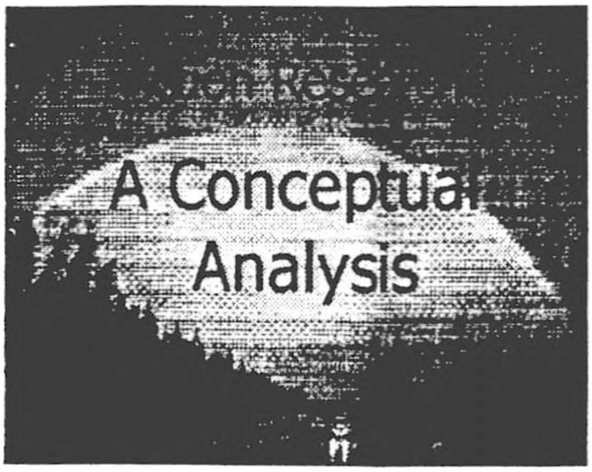
Example: A rhombus is an equilateral parallelogram.

This definition tells that a rhombus means, "an equilateral parallelogram". And if the students do not have the concept of an equilateral parallelogram, the teacher can think of the definition –

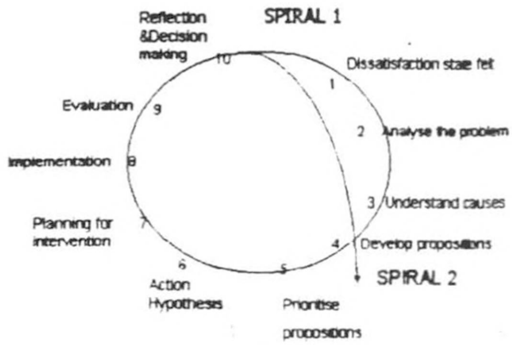
An equilateral parallelogram is a four sided figure whose sides are line segments having the same length.

3. Concepts helps in generalisation.

4. Concepts help in discovery of new knowledge.



Action Research Spiral
McKernan's Time Process Model



3

What is Action Research?

It is a problem-solving approach which helps a practitioner to perceive, understand and assess the situation, and it further facilitates a systematic analysis and working out plausible solutions for the unsatisfactory condition. With this, different alternative solutions can be tried out and finally an intervention can be worked out with which the problem can be solved satisfactorily.

2

STEPS IN THE ACTION RESEARCH

1. Perception of the Problem/dissatisfied state.
2. Analysis of the Problem/dissatisfied state.
3. Understanding the dynamics / causes.
4. Development of propositions (Tentative theory).
5. Prioritizing proposition.

4

6. Development of Action Hypothesis.
7. Planning for Intervention.
8. Execution of Intervention.
9. Evaluation of Intervention.
10. Decision (Reflection, Explanation and Understanding of Action).

Planning for Action Research

What to plan?

- Time
- Human resources and materials
- Collaborators
- Tools and techniques
- Intervention activities
- Collection of evidence

Action Hypothesis

- It includes the proposed intervention stated as capable of minimizing the problem or elevating the situation from dissatisfactory condition to a satisfactory condition.

Why is planning necessary?

Planning is necessary for the following reasons: It

- gives direction to the AR study.
- enables advance preparations.
- ensures optimal efficiency.
- facilitates achieving economy of time and effort.
- minimises ad-hoc decisions, digressions and wastage.
- enables monitoring of the study.
- ensures smooth sailing of the study.

Aspects of Planning

- Concern
- Subjects
- Objectives of the study
- Forming theory/Propositions –Prioritization
- Action Hypothesis
- Intervention strategies
- Scheduling of activities/tasks
- Listing and procuring resources
- Anticipated problems and contingency plan

9

Tools and techniques in AR
What kinds of tools?

- Achievement test
- Diagnostic test
- Psychological tests
- Questionnaires
- Interview schedules
- Checklists

10

What techniques can be used to gather evidence in AR studies?

- Interview
- Video-Recording
- Observation
- Tape Recorder

11

Execution of the Intervention

- Execute the intervention as planned.
- Keep all the precautions in mind.
- Note down/record all intended processes.
- Terminate each session smoothly.

12

What next?

- Collection of evidence/data
- Scoring and tabulation
- Graphical representation of data

EVALUATION OF THE EFFECTIVENESS OF THE INTERVENTION

- What kind of data do we need to evaluate the effectiveness of the intervention?
  - Comprehensive
    (Both qualitative and quantitative)
  - Dependable
  - Relevant
  - Objective
  - Multiple Sources

Graphical representation of data

- Bar Diagram
- Histogram
- Polygon
- Pie Diagram

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Evaluation, Reflection, Decision Making

The data/information that is in descriptive form (word form) are qualitative data.

The data that are expressed in the form of numbers which lend themselves for further manipulation are quantitative data.

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Analysis and Interpretation

In Analysis we organize data and subject it to needed manipulation to elicit meaning. Analysis means categorizing, ordering, manipulating and reading meaning to facilitate discussion and interpretation.

At the Interpretation stage, we draw pertinent inferences in the context of the Action Hypothesis. This leads to decision making.

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Descriptive statistics only!! No Inferential statistics in AR

- Descriptive measures are apt for intact groups studied in Action Research.
- The measure(s) describe the group studied only.
- Inferential statistics has no place in AR as samples are not studied.

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Reflection

- Is the new Practice effective?
- Should I continue with my old practice?
- Is the solution to the impending problem effective?
- Did the intervention bring about improvement to a satisfactory level?
- Is there a scope for enhancing my competence further?

19

Decision Making

- Shall I terminate the intervention?
- Shall I not effect a change in the existing practice?
- Shall I incorporate the new tested intervention in my functioning?
- Shall I try another strategy?
- Are there more effective ways of achieving the goals?
- What changes should I make in the next spiral?

20

Characteristics of AR

- It is a small scale intervention made by a practitioner.
- AR is undertaken in a specific context. The findings are NOT GENERALIZABLE.
- AR is a reflective practice that enhances one's own efficiency.
- AR is practitioner's privilege.
- AR proceeds in a spiral(s).

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Contexts of Action Research

- All Professions and professionals
- In Education,Teachers too.
  - (1) Classroom level
  - (2) HM- School level
  - (3) CRC, BRC, BEO, DEO, ... CPI
  - (4) Teacher-Educators
  - (5) Educational planners, managers, administrators

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WHY ACTION RESEARCH?
It is because it;

- improves one's own professional skills.
- improves the learning environment.
- enhances the quality and/or quantity of desired results.
- solves an immediate problem.
- provides local-specific solutions.
- facilitates overall effectiveness of practice of a profession.

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A change towards higher level of performance is frequently short lived; after a 'shot in the arm', the practice returns to previous level..... A successful change includes, therefore, three aspects:

1. **Unfreezing the present level**
2. **Moving to the new level, and**
3. **Freezing the practice at new level**

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Why should I Conduct Action Research?

- *Are you a professional?*
- *Are you a reflective practitioner?*
- *Do you desire to improve your professional skills?*
- *Are you dissatisfied with what you have been doing?*
- *Do you want to be more effective in your functioning?*
- *Do you want your action to yield better results?*
- *Do you want to work systematically while addressing a problem on hand?*
- *Are you unhappy with the status quo?*
- *Do you want, as a professional, to evaluate your actions objectively?*

25

- Let us all hope that we all become professionals and reflective in our pursuits

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If your answer is 'YES'

- Then you will start seriously thinking about Action Research and you will remain a 'Reflective Practitioner'.

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- Thank you

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SIGNIFICANCE OF VALUE EDUCATION

The problem of value education of the young is assuming increasing prominence in educational discussions during recent times. Parents, teachers and society at large have been concerned about values and value education of children. National policy on Education (NPE) 1986 and revised NPE 1992 has given all importance to the promotion of Value Education in Schools. Education is expected to play a major role in promoting national development in all its ramifications. At the same time, it should bring harmonious development of all the faculties towards adequate preparation for life. The present situation in India demands a system of education, which, apart from strengthening national unity, must strengthen social solidarity through meaningful and constructive value education.

The worldwide resurgence of interest in value education has been explained as the natural response of the modern industrialized societies to the serious erosion of moral values in all aspects of life and the crisis of values experienced in modern times.

It is now commonplace to say that sweeping political, economic and social changes have overtaken human civilization during the past few centuries and these have been largely responsible for the predicament of modern man. The factors such as personal greed, meanness, selfishness, indifference to others' interests and laziness also have brought about large-scale corruption in almost all spheres of life – personal and public, economic and political, moral and religious. We can achieve a better moral standard in our democratic way of national life if we become more industrialized and thus overcome mass poverty and the general feeling of insecurity which gives rise to greed.

We are witnessing a tremendous value crisis throughout the world today. A lackadaisical attitude towards value and its institutions is ubiquitous everywhere around the globe. As the vitality of human belief in values is dying out in every land, the younger generation has started to pooh-pooh the unique religious epics of antiquity and religious institutions, giving room for corrosion of godliness and erosion of spiritual and moral values. As a result, the mind of man has been laciniated and divided into small fractions and fragments which makes the value content of human life a diminishing factor in modern times.

The reappearance of barbaric qualities of selfishness, clashes and conflagration and other destructive forces which are burning the society, give clear indication of the degenerating process of human society. Now, there is an urgent need for a great effort to revive and reform the values of human life and to rejuvenate the foundation of the new civilization.

Concerted efforts and continuous dependence on good books and institutions will give students sterling and inspiring qualities of concentration, infinite love, justice, honesty, purity, selfishness, wisdom, . faithfulness, humility, forgiveness, mercy, trustworthiness, respect for others, obedience, sincerity and a host of other virtues which are *sine qua non* to build the equipment of life. This should be the central theme of value education. Whatever be the cause of the present value crisis, there is no gain – saying the fact that the weakening of moral values in our social life is creating serious social and ethical conflicts. It is this changing context – the declining moral standards in personal and public life on the one hand, and the national ideological commitment to the values of democracy, socialism, secularism and modernization on the other – that constituted the driving force behind the recommendations stressing the importance of value education in educational institutions.

While there is general dissatisfaction with the fall in moral standards of both young and the old and disenchantment with the disregard to moral values witnessed in personal and public life, there has been no concerted attempt on the part of the society to address itself squarely to the problem of value education. Unfortunately, education is becoming day by day more or less materialistic and the value traditions are being slowly given up. A modern Indian is being educated mainly with the bread and butter aim of education; as a result most of our graduates run after money, power and comforts, without caring for any type of value.

The degeneration in the present day life, the demoralization of public and private life, the utter disregard for values, etc. are all traceable due to the fact that moral, religious and spiritual education has not been given due place in our educational system.

The Education Commission of 1964-66 says that "a serious defect in the school curriculum is the absence of provision for education in social, moral and spiritual values". In the life of the majority of Indians, religion is a great motivating force and is intimately bound up with the formation of character and the inculcation of ethical values.

A national system of education that is related to life, needs and inspiration of the people cannot afford to ignore this purposeful force. Value crisis of the present day life is baffling the minds of educators and the educands as well. The effect of the value crisis on present day life is witnessed in the following :

- The democratic ideology that has been accepted by our country is yet to be actualized in the form of social and economic democracy as to realize democratic values guaranteed by the Constitution of India.

- The individual is becoming a prey to the contradictory values and is being converted as a consequence into an extreme radical, a reactionary, a skeptic or cynic.
- The present Indian educational system is reflecting more or less borrowed ideologies and philosophies and the national values are relegated to the back.
- The teacher-educators and teachers are not being clearly oriented to the national values and ideas, ideal and ideologies that they have to inculcate in the students. Hence, they are not in a position to play their role as value educators.
- The student community is drowned in neck-deep poverty, ignorance and unhealthy surroundings. Hence, they are not in a position to comprehend the real values of our contemporary India.
- Our curriculum does not reflect human values and the value system, hence our schools and colleges have become examination centers and not value centers.

The problem with value education, it appears, is that while everybody is convinced of its importance, it is not clear as to what it precisely means and what it involves. In our educational reconstruction, the problem of an integrated perspective on values is pivotal, for its solution alone can provide organic unity for all the multifarious activities of a school or college curriculum programme. An integrated education can provide for integrated growth of personality and integrated education is not possible without integration of values.

In value education, as in any other areas of education, what is asked of the teacher is a total commitment to the development of rational autonomy in both thought and action.

It should be noted that the most important aspect of value education consists not in unwilling adherence to a set of rules and regulations but in the building and strengthening of positive sentiments for people and ideals. Value education should prepare individuals for participation in social life and acceptance of social rules. What is more important in value education is that schools should provide a healthy climate for sharing responsibilities, community life and relationships.

The new National Curriculum Framework for School Education (NCFCE) prepared by NCERT gives uppermost importance to Value Education in schools. NCERT has been contributing richly to the area of Value Education by way of organizing inservice education courses for key level persons, preparation of instructional materials, etc. The RIE, Mysore under the Coordinatorship of Dr Prahallada has brought out a 686 page material titled **"TREASURE TROVE OF VALUES"** which consists of Anecdotes, Fables, Stories, Legends, Biographies and Folk Tales related to values which will be of great use at primary stage.

Also, 115 page Package on Value Education has been brought out by RIEM consisting of importance of Value Education, approaches to Value Education, Lesson Planning in Value Education. The package will be useful for the teachers for the inculcation of values at primary school stage.

Regional Nodal Centre on Value Education at RIEM

The NCERT, New Delhi has been identified by the MHRD (Department of Education). Government of India as the nodal center for strengthening value education in the country at school level. Subsequently, a National Resource Centre for Value Education (NRCVE) has been set up in order to plan and implement programmes on value oriented education. NCERT, New Delhi has launched a National Programme for Strengthening Value Education. This programme has been visualized as a national level initiative to sensitize parents, teachers, teacher educators, educational administrators, policy makers, community agencies etc. about the need for promotion of value oriented education. The focus of the programme is on generating awareness, material development, teachers training, development of school programmes, promotion of research and innovations in the area of education of human values and development of a framework of value education for the school system.

In this context, a Regional Nodal Centre (RNC) has been set up at the RIE, Mysore from September 2002 which will be responsible for linkages networking, monitoring and follow up etc. at the State, District and grassroot level for implementation of value education programmes. The Centre will take up the responsibility of organizing National Consultation and Regional Workshop on Value Education with focus on strategies of awareness generation, material development and teacher's training. The RNC comprises of representatives drawn from SCERTs, IASEs, CTEs, DIETs, NGOs, School Boards, Bureau of Textbooks and eminent professionals/educationists from the southern states.

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THREE DIMENSIONAL GEOMETRY

1. Lines in Space
2. Planes and Sphere

by

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THREE DIMENSIONAL GEOMETRY

The applications of vector algebra in three dimensional geometry are given in these lecture notes. The reader is requested to learn the techniques of vector algebra to see how the coordinate geometry can be made simple with the help of vectors. He is also requested to translate these results to the Cartesian form also.

The derivation of the formula for the shortest distance between two lines in space may be read only by keeping the teaching aid (discussed in the lesson) by the side, so that the concepts may become more clear.

There will be two parts on the applications of vectors; the first will deal with the lines in space while the second with the planes.

I. Lines in Space :

Length of a vector: We have already seen that the position vector of any point P in the 3-dimensional space R^3 is given by

$\vec{r} = xi + yj + zk$ where i, j, k are the unit vectors in three perpendicular directions and x, y, z are the coordinates of the point P. The position vector

$$\vec{r} = xi + yj + zk$$

can also be written as $\vec{r} = (x, y, z)$

The length of the vector $\vec{r}$ is given by

$$|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

Distance Formula :

From the triangle OAB, it is clear that

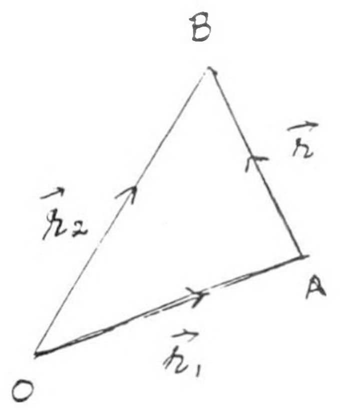
$\vec{AB} = \vec{OB} - \vec{OA}$.

This is the way to express any vector $\vec{AB}$.

If $\vec{OA} = \vec{r}_1$ and $\vec{OB} = \vec{r}_2$

where $r_1 = (x, y, z)$ and $r_2 = (x_2, y_2, z_2)$

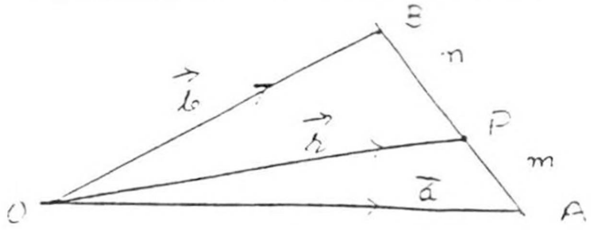
Then, $\vec{r} = \vec{AB} = \vec{r}_2 - \vec{r}_1$



$$|\vec{r}| = |\vec{r}_2 - \vec{r}_1|$$
$$= \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$

This is called the distance formula.

<u>Section Formula</u> : We find the position vector of the point which divides the line joining two given points in the given ratio.



Let A and B be any two points in the 3-dimensional space whose position vectors are $\vec{a}$ and $\vec{b}$ respectively. Let P be the point which divides the line segment AB such that AP : PB = m:n. We wish to find the position vector $\vec{r}$ of the point P. Without loss generality, we can assume that O is the origin.

$$\vec{a} = x_1 i + y_1 j + z_1 k$$
$$\vec{b} = x_2 i + y_2 j + z_2 k$$

Let $\vec{r} = xi + y\,j + zk$

Since P divides AB in the ratio m:n, we have

$$\frac{AP}{PB} = \frac{m}{n}$$

Here m/n is positive or negative according as, P divides AB internally or externally.

From the above, we get n. AP = m. PB

i.e. $n(\vec{r} - \vec{a}) = m.\ (\vec{b} - \vec{r})$

or $(n+m)\,\vec{r} = m\,\vec{b} + n\,\vec{a}$

or $\vec{r} = \dfrac{n\,\vec{a} + m\,\vec{b}}{n + m}$

This is called the section formula in the vector form.

If we substitute the Cartesian coordinates

$$\vec{r} = x\,i + y\,j + z\,k$$
$$\vec{a} = x_1 i + y_1 j + z_1 k$$
$$\vec{b} = x_2 i + y_2 j + z_2 k$$

in the above result, and compare the coefficients of i,j,k, we get

$$x = \frac{mx_2 + nx_1}{m+n}$$

$$y = \frac{my_2 + ny_1}{m+n}$$

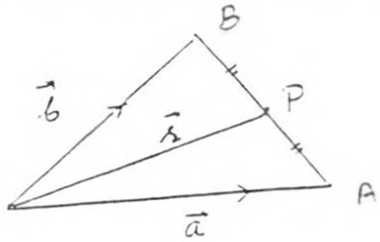
$$z = \frac{mz_2 + nz_1}{m+n}$$

which is the section formula in the Cartesian coordinates.

Middle Point :

From the section formula, it is clear that the position vector of the middle point of the join of two points with position vectors $\vec{a}$ and $\vec{b}$, is given by

$$\vec{r} = \frac{\vec{a} + \vec{b}}{2}$$



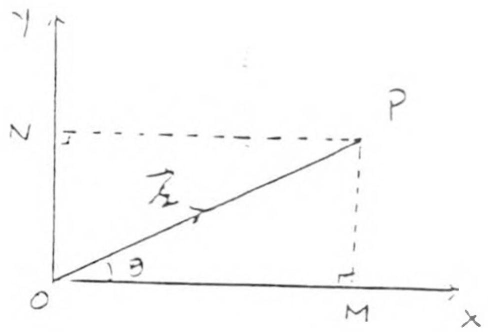
Components of a vector :

In the figure,

$\vec{OM} = \vec{r}\ \cos\theta$

and $\vec{ON} = \vec{r}\ \sin\theta$

where θ is the angle that the vector $\vec{r}$ makes with x-axis.



Direction Ratios of a Vector :

If $\vec{r} = a\ i + b\ j + c\ k$, then a,b,c are called the direction ratios of the vector $\vec{r}$.

Direction cosines : If α is the angle that the vector $\vec{r}$ makes with the x-direction, then

$$\cos\alpha = \frac{\vec{r}\cdot\vec{i}}{|r|\ |i|}$$

$$= \frac{(ai + bj + ck)\cdot i}{|\vec{r}|}$$

$$= \frac{a}{\sqrt{a^2 + b^2 + c^2}} \qquad (i)$$

Similarly if β and γ are the angles that the vector $\vec{r}$ makes with y-direction and z-direction respectively, then

$$\cos \beta = \frac{b}{\sqrt{a^2 + b^2 + c^2}} \qquad \text{(ii)}$$

$$\text{and } \cos \gamma = \frac{e}{a^2 + b^2 + c^2} \qquad \text{(iii)}$$

If $\cos \alpha = l$, $\cos \beta = m$ and $\cos \gamma = n$, then

$\boxed{\text{l,m,n are called direction cosines.}}$

If we add the squares of (i), (ii) and (iii), we get

$$l^2 + m^2 + n^2 = 1$$

Therefore, the relation between the direction ratios and the direction cosines is

$$\boxed{\begin{array}{c} a:b:c = l:m:n \\ \text{with} \\ l^2 + m^2 + n^2 = 1 \end{array}}$$

<u>Parallel vectors have equal direction ratios</u> :

Let $\vec{v}_1 = ai + bj + ck$ and if $\vec{v}_2$ is a vector parallel to $\vec{v}_1$, then

$$\vec{v}_2 = \lambda \vec{v}_1 \text{ for some scalar } \lambda .$$

Then, $\vec{v}_2 = \lambda ai + \lambda bj + \lambda ck$

Hence the direction ratios of $\vec{v}_2$ are $\lambda a , \lambda b , \lambda c$
or a,b,c.

<u>Like parallel vectors have equal direction cosines</u> :

If $\vec{v}_1 = a x + b j + c k$

and $\vec{v}_2$ is a vector parallel to $\vec{v}_1$, then $\vec{v}_2 = \lambda \vec{v}_1$

$= \lambda ai + \lambda bj + \lambda ck$

The direction cosines of the $\vec{v}_1$ are

$$l = \frac{a}{|v_1|}, \quad m = \frac{b}{|v_1|}, \quad n = \frac{c}{|v_1|}$$

Similarly the direction cosines of the $\vec{v}_2$ are

$$\frac{\lambda a}{|\lambda v_1|}, \quad \frac{\lambda b}{|\lambda v_1|}, \quad \frac{\lambda c}{|\lambda v_1|}$$

i.e. $$\frac{a}{|v_1|}, \quad \frac{b}{|v_1|}, \quad \frac{c}{|v_1|}.$$

Also, it is clear that unlike parallel vectors have equal (and opposite sign) direction cosines.

Example:

1. For the vector $\vec{r} = 2i + 2j - k$, the direction ratio is 2:2:-1 and the direction cosines are $\frac{2}{|r|}, \frac{2}{|r|}, \frac{-1}{|r|}$

i.e $\frac{2}{3}, \frac{2}{3}, \frac{-1}{3}$.

It means that the vector $\vec{r} = 2i + 2j - k$ makes the following angles with i direction, j direction and k direction respectively.

$$\cos^{-1}\left(\frac{2}{3}\right), \quad \cos^{-1}\left(\frac{2}{3}\right), \quad \cos^{-1}\left(-\frac{1}{3}\right)$$

2. The vectors $2i + 2j - k$ and $4i + 4j - 2k$ have the same direction ratios and direction cosines. (They are parallel).

3. The vectors $2i + 2j - k$ and $-4i - 4j + 2k$ have the same direction ratios. They have direction cosines equal in magnitude but opposite in sign. (The vectors are unlike parallel vectors).

4. Show that the points A(2,3,4), B(-1,2,-3) and C (-4, 1,-10) are collinear.

There are several ways of answering this question: we can show that the area of the triangle ABC is zero or we can also show that

$$|\vec{AC}| = |\vec{BC}| = 2\,|\vec{AC}|$$

But it is easier to show that the direction ratios of $\vec{AB}$ and $\vec{BC}$ are equal (or proportional).

Direction ratios of $\overrightarrow{AB}$ are (-1,-2): (-2-3) : (-3-4)

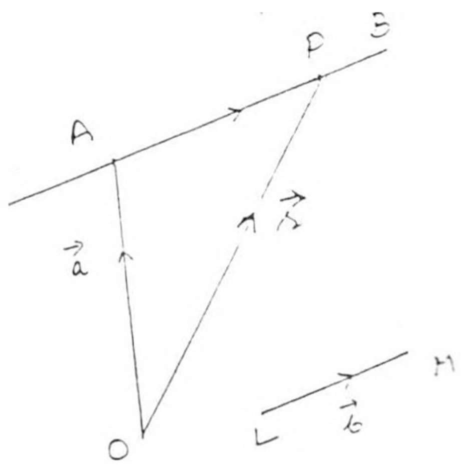
i.e. -3:-1 : -7

Direction ratios of $\overrightarrow{BC}$ are also -3:-1:-7.
Hence $\overrightarrow{AB}$ is parallel to $\overrightarrow{BC}$, showing that A,B,C are collinear.

Angle between the vectors : The angle between the vectors can be found out by applying the formula

$$\vec{a} \cdot \vec{b} = |\vec{a}|\ |\vec{b}|\ \cos$$

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\ |\vec{b}|}$$



Vectorial Equation of a line in Space :

We find the vector equation of the line AB which passes through a given fixed point A and is parallel to a given line LM (vector $\vec{b}$).

Take any point O, as origin of reference. Let $\vec{a}$ be the position vector of the given point A, let $\vec{b}$ be any vector parallel to the given line AB.

Let $\vec{r}$, be the position vector of any point P on the given line. We have

$$\begin{aligned}\vec{r} &= \overrightarrow{OP}\\ &= \overrightarrow{OA} + \overrightarrow{AP}\\ &= \vec{a} + \overrightarrow{AP}\end{aligned}$$

The vector $\overrightarrow{AP}$, being parallel to the vector $\vec{b}$, must be of the form

$\overrightarrow{AP} = t\,\vec{b}$ for some suitable scalar t.

Therefore,

$$\vec{r} = \vec{a} + t\,\vec{b}$$

is the required equation of the straight line.

<u>Cartesian Form</u> : To get the Cartesian form of the above equation, we can substitute the coordinates of the points

or put $\vec{r} = xi + yj + zk$

$$\vec{a} = a_1 i + a_2 j + a_3 k$$

$$\vec{b} = b_1 i + b_2 j + b_3 k$$

Then we get

$$xi + yj + zk = (a_1 i + a_2 j + a_3 k) + t\,(b_1 i + b_2 j + b_3 k)$$

Hence, (comparing coefficients of i,j,k), we get

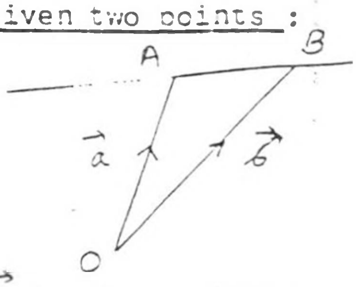
$$t = \frac{x - a_1}{b_1} = \frac{y - a_2}{b_2} = \frac{z - a_3}{b_3}$$

The cartesian equation of the line is

$$\boxed{\frac{x - a_1}{b_1} = \frac{y - a_2}{b_2} = \frac{z - a_3}{b_3}}$$

<u>Equation of the straight line through given two points</u> :

We wish to find the equation of the straight line which passes through the two given points A and B.



Take any point O as origin. Let $\vec{a}$ and $\vec{b}$ be the position vectors of the points A and B respectively.

Then the line AB is parallel to the vector $\vec{b} - \vec{a}$. It passes through A. Hence the equation of the line AB is given by

$$\boxed{\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})}$$ where λ is a parameter.

<u>Cartesian Form</u> : The cartesian form of the above equation is obtained by putting

$$\vec{r} = xi + yj + zk$$

$$\vec{a} = a_1 i + a_2 j + a_3 k$$

$$\vec{b} = b_1 i + b_2 j + b_3 k$$

and comparing the coefficients.

$$xi + yj + zk = a_1 i + a_2 j + a_3 k + \lambda\{(b_1 i + b_2 j + b_3 k) - (a_1 i + a_2 j + a_3 k)\}$$

$$= a_1 i + a_2 j + a_3 k + \lambda\{(b_1 - a_1)i + (b_2 - a_2)j + (b_3 - a_3)k\}$$

Hence we get

$$\lambda = \frac{x-a_1}{b_1-a_1} = \frac{y-a_2}{b_2-a_2} = \frac{z-a_3}{b_3-a_3}$$

The cartesian form of the equation is

$$\boxed{\frac{x-a_1}{b_1-a_1} = \frac{y-a_2}{b_2-a_2} = \frac{z-a_3}{b_3-a_3}}$$

<u>Linearly independent vectors in R^3</u> :

Definition: Three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ in R^3 are said to be linearly independent if $\alpha\vec{a} + \beta\vec{b} + \gamma\vec{c} = 0$, α, β, γ being scalars, implies $\alpha = \beta = \gamma = 0$. The vectors are said to be linearly dependent if they are not linearly independent. In other words, the vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ are said to be linearly dependent if there exists some non zero scalars α, β, γ such that $\alpha\vec{a} + \beta\vec{b} + \gamma\vec{c} = 0$.

For example, the vectors $\vec{a} = (1,2,1)$ $\vec{b} = (2,3,5)$ $\vec{c} = (4,7,7)$ are linearly dependent ($\alpha = 2$, $\beta = 1$, $\gamma = -1$). But the vectors $a = (1,2,1)$ and $b = (2,3,5)$ are linearly independent.

Theorem: A necessary and sufficient condition for three points with position vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ to be collinear is that there exists scalars α, β, γ not all zero, such that

$$\alpha\vec{a} + \beta\vec{b} + \gamma\vec{c} = 0 \quad , \quad \alpha + \beta + \gamma = 0$$

Proof: (Sufficiency part)

Let there be scalars α, β, γ not all zero, such that $\alpha\vec{a} + \beta\vec{b} + \gamma\vec{c} = 0$, $\alpha + \beta + \gamma = 0$

without loss of generality, we take $\gamma \neq 0$.

Then $\alpha + \beta = -\gamma \neq 0$

It is given that $\alpha\vec{a} + \beta\vec{b} + \gamma\vec{c} = 0$.

$$\Rightarrow \alpha\vec{a} + \beta\vec{b} = -\gamma\vec{c}$$

$$\Rightarrow \frac{\alpha \vec{a} + \beta \vec{b}}{\alpha + \beta} = \frac{-\gamma}{\alpha + \beta} \vec{c}$$

$$\Rightarrow \frac{\alpha \vec{a} + \beta \vec{b}}{\alpha + \beta} = \vec{c}$$

Here we have shown that $\vec{c}$ is the position vector of the point C which divides the line joining the points A (with position vector $\vec{a}$) and the point B (with position vector $\vec{b}$) in the ratio $\beta : \alpha$
Thus the points A, B and C are collinear.

<u>Necessary Part</u> : Let the points A, B, C be collinear. The position vectors of A, B, C are $\vec{a}$, $\vec{b}$, $\vec{c}$ respectively.

We can assume that the point C divides the line segment AB in the ratio $\alpha : \beta$

Then $\vec{c} = \dfrac{\alpha \vec{b} + \beta \vec{a}}{\alpha + \beta}$ $\quad \therefore (\alpha + \beta)\vec{c} = \alpha \vec{b} + \beta \vec{a}$

Put $\alpha + \beta = -\gamma$

Then we get $\alpha \vec{b} + \beta \vec{a} + \gamma \vec{c} = 0$ and $\alpha + \beta + \gamma = 0$.

Hence the proof.

Note: 1. We have proved that if the points A, B, C are collinear then, the vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ are linearly dependent. In other words, if the vectors are linearly independent then the points need not be collinear.

2. It is easy to see that the vectors $\vec{a}$ and $\alpha \vec{a}$ are collinear as well as linearly independent.

3. If $\vec{a}$ and $\vec{b}$ are two non zero non collinear vectors, then they are linearly independent. For, if they are linearly dependent, then there exists non zero scalars α, β such that $\alpha \vec{a} + \beta \vec{b} = 0$.

If $\alpha \neq 0$ then $\vec{a} = -\dfrac{\beta \vec{b}}{\alpha}$

which implies that $\vec{a}$ and $\vec{b}$ are collinear, contrary to our assumption.

4. In the same way we can prove that if $\vec{a}$, $\vec{b}$, $\vec{c}$ are three non zero non coplanar vectors, then they are linearly independent.

Angle between any two lines :

Let $\vec{r}_1 = \vec{a}_1 + \lambda \vec{b}_1$

and $\vec{r}_2 = \vec{a}_2 + \mu \vec{b}_2$

be any two straight lines, in space. Then the angle between them can be found out as follows :

The angle between $\vec{r}_1$ and $\vec{r}_2$ is equal to the angle between $\vec{b}_1$ and $\vec{b}_2$

But $\vec{b}_1 . \vec{b}_2 = |b_1| \, |b_2| \cos \theta$

$$\cos \theta = \frac{\vec{b}_1 . \vec{b}_2}{|b_1| \, |b_2|}$$

is the angle between $\vec{r}_1$ and $\vec{r}_2$.

The above method can be applied even if the equations are in the Cartesian form.

Note: The angle θ calculated above does not indicate that the two straight lines intersect. In fact, the angle θ is the angle between the directions of $\vec{b}_1$ and $\vec{b}_2$.

Skew Lines : In the plane, whenever two straight lines are not parallel, then they intersect at some point. But the situation is different in the space. There can be straight lines which are neither parallel nor intersecting. Such lines do not lie in a single plane; and are called skew lines.

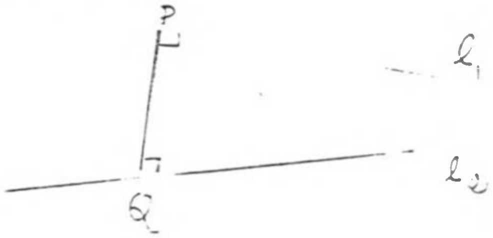
Definition: Two straight lines in R^3 which are not coplanar are called skew lines.

Definition: The length of the common perpendicular to the skew lines is called the shortest distance between the skew lines.

Note: The teacher can make the ideas of skew lines clear with the help of a teaching aid described here: Take two rods AB and CD. Tie one end of a thread to a point P on AB and the other end to a point Q on CD. Hold the rods AB and CD at different levels and make

PQ perpendicular to both AB and CD. (AB and CD need not be parallel). Now $|PQ|$ is shortest distance between the lines.

<u>To find an expression for the shortest distance</u> :



Let the skew lines be

$$\vec{r}_1 = \vec{a}_1 + \lambda \vec{b}_1$$

and $\vec{r}_2 = \vec{a}_2 + \mu \vec{b}_2$

Since PQ is perpendicular to both $\vec{b}_1$ and $\vec{b}_2$, it is clear that PQ is parallel to $\vec{b}_1 \times \vec{b}_2$

The unit vector $\vec{n}$ along $\vec{PQ}$ is given by

$$\vec{n} = \frac{\vec{b}_1 \times \vec{b}_2}{|\, b_1 \times b_2 \,|}$$

Let $\vec{PQ} = d\,\vec{n}$ where d is the shortest distance between the given two skew lines.

Let S and T be any two points with position vectors $\vec{a}_1$ and $\vec{a}_2$ on the lines AB and CD respectively. If θ is the angle between PQ and ST, then PQ = ST cos

This can be realized by taking the projection of ST along the direction of PQ.

Then

$$\cos\theta = \frac{\vec{PQ} \cdot \vec{ST}}{|PQ| \; |ST|}$$

$$= \frac{d\vec{n} \cdot (\vec{a}_2 - \vec{a}_1)}{d \; |ST|}$$

$$= \frac{d\,(\vec{b}_1 \times \vec{b}_2)}{|b_1 \times b_2|} \cdot \frac{(\vec{a}_2 - \vec{a}_1)}{d\;|ST|}$$

$$= \frac{(\vec{b}_1 \times \vec{b}_2)}{|b_1 \times b_2|} \cdot \frac{(\vec{a}_2 - \vec{a}_1)}{|ST|}$$

$$d = PQ = ST \cos$$

$$= \frac{(b_1 \times b_2) \cdot (a_2 - a_1)}{|b_1 \times b_2|} \qquad \ldots (A)$$

(The distance d is to be taken as positive).

Solved Examples:

1. Find the shortest distance between the vectors

$$\vec{r_1} = i + j + \lambda(2i + j + k)$$

and $\vec{r_2} = 2i + j - k + \mu(3i - 5j + 2k)$

Ans: Here in this problem,

$$a_1 = i + j, \quad b_1 = 2i + j + k$$

$$a_2 = 2i + j - k, \quad b_2 = 3i - 5j + 2k$$

Substituting these values in the formula (A)

$$d = \frac{(b_1 \times b_2) \cdot (a_2 - a_1)}{|b_1 \times b_2|}$$

We get $d = \frac{10}{\sqrt{59}}$

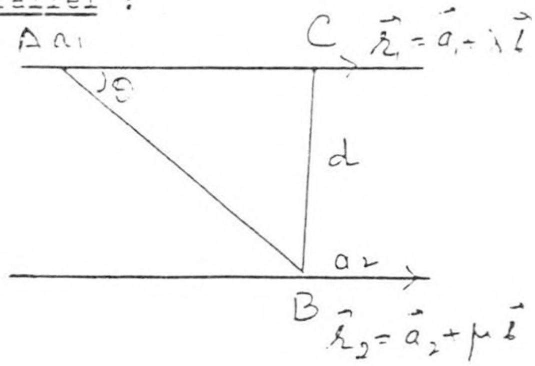
Shortest distance when the lines are parallel :

Let

$\vec{r_1} = \vec{a_1} + \lambda \vec{b}$ and

$\vec{r_2} = \vec{a_2} + \mu \vec{b}$

be the two parallel lines in the space.



Then the two vectors $\vec{r_1}$ and $\vec{r_2}$ can be considered to be in one plane. 'd' as shown in the figure is the shortest distance between the lines.

$d = (\vec{a}_2 - \vec{a}_1) \sin\theta$ ...(1) from the triangle ABC.

But we know that

$$(\vec{a}_2 - \vec{a}_1) \times \frac{\vec{b}}{|\vec{b}|} = |\vec{a}_2 - \vec{a}_1| \left|\frac{\vec{b}}{|b|}\right| \sin\theta . \vec{n}$$

Since d is always considered to be positive, substituting the values of $\sin\theta$ in (1), we get

$$d = (a_2 - a_1) \frac{a_2 - a_1}{(a_2 - a_1)} \times \frac{|\vec{b}|}{|b|}$$

$$= \left| \frac{\vec{b}}{|b|} \times (\vec{a}_2 - \vec{a}_1) \right| \quad \text{———} \quad (2)$$

2. Find the angle between the pair of lines $\vec{r}_1 = 4i - j + \lambda(i + 2j - 2k)$ and $\vec{r}_2 = (i - j + 2k) + \mu(2i + 4j - 4k)$. Also find the shortest distance between them.

Ans: Note that the lines are parallel to the vector $i + 2j - 2k$ and hence the angle between them is zero. Both are of the form

$$\vec{r}_1 = a_1 + \lambda b$$
$$\vec{r}_2 = a_2 + \mu b$$

Hence this problem cannot be solved by the method we adopted for problem 1. Now we use the result (2).

$$d = \left| \frac{\vec{b}}{|b|} \times (a_2 - a_1) \right|$$

$$= \left| \frac{(i + 2j - 2k) \times}{|i + 2j - 2k)|} (i - j + 2k) - (4i - j) \right|$$

$$= \left| \frac{(i + 2j - 2k)}{3} \times (-3i + 2k) \right|$$

$$= \left| 1/3 \; i\,(4) + j\,(6-2) + k(6) \right|$$

$$= \sqrt{\frac{68}{3}}$$

3. Find the shortest distance between the pair of lines

$\vec{r} = i+j - k + (3i-j)$ and

$\vec{r} = 4i - k + (2i + 3k)$

Also find whether they intersect.

Ans: Substituting in the formula (1), we can see that the shortest distance is

$$d = \left| \frac{(3i-j) \times (2i+3k) \cdot (3i-j)}{| b_1 \times b_2 |} \right|$$

$$= \frac{(-3i - 9j + 2k) \cdot (3i-j)}{| b_1 \times b_2 |}$$

$$= \frac{-9 + 9}{94}$$

$$= 0$$

The given lines do intersect.

4. Determine whether the following lines intersect.

$$\frac{x-1}{2} = \frac{y+1}{3} = z;$$

$$\frac{x+1}{5} = \frac{y-2}{1}, \ z = 2$$

Ans : The first set of equations can be written as (when we take the common ratio as λ).

$x = 2\lambda + 1$ $\qquad \vec{r} = xi + yj + zk$

$y = 3\lambda - 1$ $\qquad = (i-j) + \lambda(2i+3j+k)$ ——— (1)

$z = \lambda$

Similarly the second set of equations can be written as

$x = 5\mu - 1$

$y = 1\mu + 2$

$z = 0\mu + 2$ $\qquad \vec{r} = (-i + 2j + 2k) + \mu(5i + j)$ ——— (2)

Now as in exercise (3) above, we can show that the shortest distance d between the lines (1) and (2) is not zero. Hence they do not intersect.

5. Find the angle between the pair of lines with direction ratios 1,1,2 and $\sqrt{3}-1$, $-\sqrt{3}-1$, 4.

Ans: The vector equation of the 1st line is given by

$$\vec{r}_1 = 1i + 1j + 2k$$

and the second line is given by

$$\vec{r}_2 = (\sqrt{3}-1)\,i + (-\sqrt{3}-1)j + 4k$$

The angle between the two lines is given by

$$\cos\theta = \frac{\vec{r}_1 \cdot \vec{r}_2}{|r_1|\;|r_2|}$$

$$= \frac{(i+j+2k)\cdot(\sqrt{3}-1)i + (-\sqrt{3}-1)j + 4k)}{|r_1|\;|r_2|}$$

$$= \frac{\sqrt{3}-1 + -\sqrt{3}-1+8}{\sqrt{6}\sqrt{24}}$$

$$= 1/2$$

$$\theta = 60^{\circ}$$

Assignments Self Test :

1. Find the angle between the pair of lines whose direction ratios are:

i) 1,2,-2; 2,4,-4

ii) 5,-12,13; -3,4,5

iii) 1,2,1; 2,1,-1

2. Determine whether the following pairs of lines intersect :

i) $r_1 = 3i + 2j - 4k + \lambda(i + 2j + 2k)$

and $r_2 = 5j - 2k + \mu(3i + 2j + 6k)$

ii) $\frac{x+4}{3} = \frac{y-1}{5} = \frac{z+3}{4}$

and $\frac{x+1}{1} = \frac{y-4}{1} = \frac{z-5}{2}$

3. Find the angle between the lines

i) $\vec{r}_1 = 3i + 2j - 4k + \lambda(i + 2j + 2k)$

and $\vec{r}_2 = 5j - 2k + \mu(3i + 2j + 6k)$

ii) $\dfrac{x+4}{1} = \dfrac{y-1}{1} = \dfrac{z+3}{2}$ and $\dfrac{x-1}{1} = \dfrac{y-4}{1} = \dfrac{z-5}{2}$

4. Find the shortest distance between the lines whose direction ratios are

1,2,-2 and 2,4,-4.

5. Find the shortest distance between

$\vec{r}_1 = i+j + k + \lambda(3i-j)$

and $\vec{r}_2 = 4i - k + \mu(2i + 3k)$

PLANE :

A plane is completely determined by any one of the following :

i) Three non collinear points.

ii) A line and a point not on the line

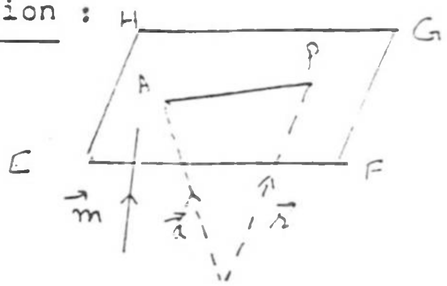
iii) Two intersecting lines

iv) Distance of the plane from the origin and a normal vector to the plane.

v) A point on the plane and a normal vector to the plane.

Here we find vector equation of the plane for some of the above cases.

1. Find the vector equation of the plane through a given point and perpendicular to a given direction :



Let A be the given point with position vector $\vec{a}$, through which the plane EFGH passes. Let $\vec{m}$ be the direction which is perpendicular to the plane EFGH.

We want to find the equation of the plane EFGH.

Let P be any arbitrary point on the plane, whose position vector is $\vec{r}$.

$$\vec{AP} = \vec{r} - \vec{a}$$

The plane is perpendicular to $\vec{m}$

Therefore, $\boxed{(\vec{r} - \vec{a}) \cdot \vec{m} = 0}$

is the vector equation of the plane EFGH.

Example: Equation of the plane passing through the point (a_1, b_1, c_1) and perpendicular to the line with direction ratios A, B, C is given by

$$(\vec{r} - (a_1 i + b_1 j + c_1 k) \cdot (Ai + Bj + ck) = 0$$

If we wish to have the Cartesian equation, then take

$$\vec{r} = xi + yj + zk$$

we get

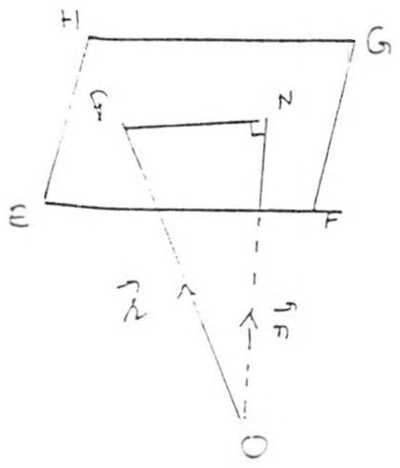
$$(x-a_1)i + (y-b_1)j + (z-c_1)k \ \cdot (Ai + Bj + ck) = 0$$

i.e. $A(x-a_1) + B(y-b_1) + c(z-c_1) = 0$

is the equation of the required plane.

2. Find the vector equation of the plane perpendicular to a given direction and at a given distance from the origin:

Given that the plane EFGH is perpendicular to $\vec{n}$, and the distance ON = d from the origin.



Consider the vector NP.

$\vec{NP} = \vec{r} - d\,\vec{n}$

Also $\vec{NP}$ is perpendicular to $\vec{n}$

Therefore, $\vec{NP} \cdot \vec{n} = 0$

i.e. $(\vec{r} - d\vec{n}) \cdot \vec{n} = 0$

$\boxed{\vec{r} \cdot \vec{n} = d}$ Since n . n = 1

is the required equation.

Cartesian equation : Put $\vec{r} = xi + yj + 3k$ and $\vec{n} = li + mj + nk$, then we have $(xi + yj + 3k).(li + mj + nk) = d$

i.e. $\boxed{lx + my + nz = d}$ is the required equation where l,m,n are the direction cosines of the normal to the plane.

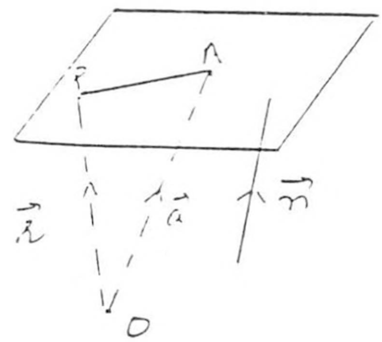
3. Equation of the plane passing through given point and perpendicular to the given direction.

The plane passes through the point A (position vector $\vec{a}$) and perpendicular to the direction $\vec{n}$.

Let P be any arbitrary point on the plane, with position vector $\vec{r}$.

Then AP is perpendicular to $\vec{n}$ given

$\boxed{(\vec{r} - \vec{a}) \cdot \vec{n} = 0}$



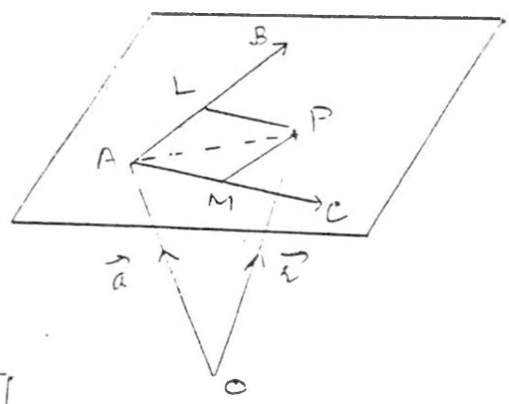
This is the required equation.

4. Equation of the plane passing through given point and parallel to two given lines:

Let $\vec{a}$ be the position vector of the point A, Through which the plane passes. Let $\vec{b}$ and $\vec{c}$ be the vectors parallel to AB and AC.

Let P be any arbitrary point on the plane (position vector $\vec{r}$).

Then, $\vec{OP} = \vec{OA} + \vec{AP}$.

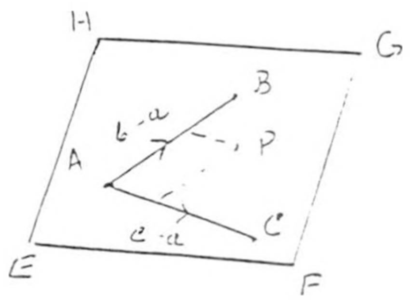


This can be written as

$$\boxed{\vec{r} = \vec{a} + t\,\vec{b} + p\,\vec{c}}$$

where t and p are some scalars.

5. Equation of the plane through three given points :



Let $\vec{a}$, $\vec{b}$, $\vec{c}$ be the position vectors of three given points A,B,C on the plane EFGH.

Then $\vec{AB} = \vec{b} - \vec{a}$

$\vec{AC} = \vec{c} - \vec{a}$

If P is any arbitrary point on the plane, whose position vector is $\vec{r}$, then

$\vec{r} = \vec{a} + t(\vec{b} - \vec{a}) + p(\vec{c} - \vec{a})$

where t and p are some scalars.

$$\boxed{\vec{r} = \vec{a} + t(\vec{b} - \vec{a}) + p(\vec{c} - \vec{a})}$$

is the required equation.

Example: Find the equation of the plane through the points

A (2,2,-1), B(3,4,2), C(7,0,6)

Ans: $\vec{r} = \vec{a} + t(\vec{b} - \vec{a}) + p(\vec{c} - \vec{a})$ is the equation.

To find the scalars t and p we can follow the following method :

$(x,y,z) = (2,2,-1) + t(1,2,3) + p(5,-2,7)$

$t + 5p = x-2$

$2t - 2p = y-2$

$3t + 7p = z+1$

Solving any two equations for t and p and substituting in the third equation, we get

$5x + 2y - 3z - 17 = 0$

which is the required equation of the plane. (See the textbook for an altrnative method).

Angle between two planes :

Let $\vec{r} \cdot \vec{n}_1 = d_1$

Let a plane P and let $\vec{r} \cdot \vec{n}_2 = d_2$

be another plane Q where $\vec{n}_1$ and $\vec{n}_2$ are perpendicular to the planes P and Q.

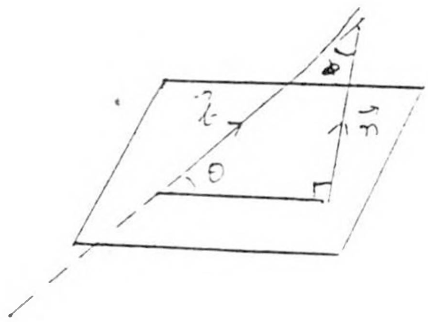
Then the angle between the planes P and Q is the angle between their perpendiculars. If θ is the angle between P and Q, then

$$\boxed{\cos\theta = \vec{n}_1 \cdot \vec{n}_2}$$

The planes are parallel if $\vec{n}_1 = \vec{n}_2$ and perpendicular if $\vec{n}_1 \cdot \vec{n}_2 = 0$.

Angle between a line and a plane :

Let $\vec{r} = \vec{a} + \lambda \vec{b}$
be the line which makes
an angle θ with the plane
$\vec{r} \cdot \vec{n} = c$



From the figure, it is clear that

$$\cos\phi = \frac{\vec{b} \cdot \vec{n}}{|b|}$$

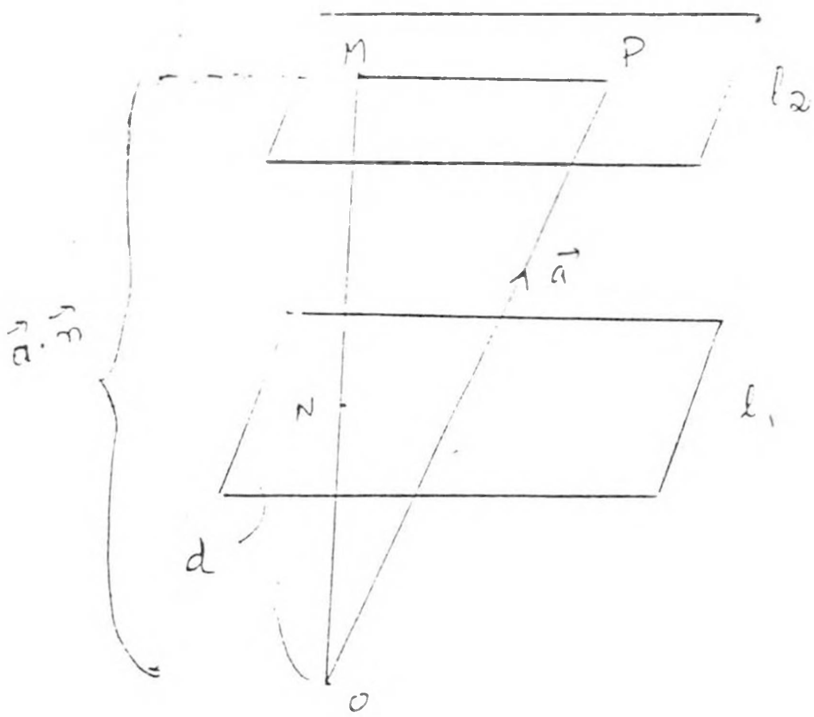
Since $\theta = \frac{\pi}{2} - \phi$

We have $\sin\theta = \cos\phi$

$$\sin\theta = \frac{\vec{b} \cdot \vec{n}}{|b|}$$

where θ is the angle between the line and the plane.

Distance of a point from a Plane :



Let l_1 be the plane and P be the given point. We wish to find the perpendicular distance from P to l_1.

Consider a plane l_2 through the point P and parallel to the plane l_1.

If $\vec{r} \cdot \vec{n} = d$ is the equation of the plane l_1, then

$$(\vec{r} - \vec{a}) \cdot \vec{n} = 0$$ is the equation of the plane l_2

(because the unit vector n is perpendicular to l_2 also). The equation of l_2 can also be written as

$$\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$$

This means that $\vec{a}.\vec{n}$ is the perpendicular distance of the plane l_2 from the point O.

Therefore, the distance from P to the plane l_1

= the distance between the two parallel planes

= OM - ON

= $\vec{a} \cdot \vec{n} - d$

The distance from P to l_1 = $| \vec{a} \cdot \vec{n} - d |$

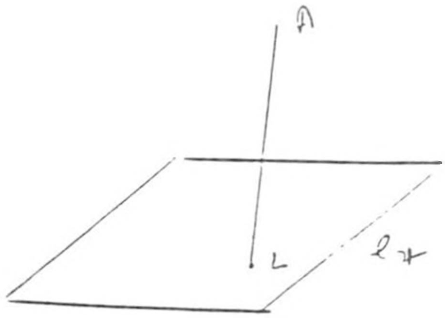
<u>Alternative Method</u> :

Let $\vec{a}$ be the position vector of the given point A and let

$\vec{r} \cdot \vec{n} = q$ .... (1)

be the equation of the plane l_1.

We want to find the distance AL where L is the foot of the perpendicular from A on l_1.



The equation of the line through A and normal the plane l_1 is given by $\vec{r} = \vec{a} + t\,\vec{n}$....(2) where t is scalar.

To find the position vector of the point L, we solve (1) and (2). i.e. At the point of intersection of this line with the plane, we have

$$(\vec{a} + t\,\vec{n}) \cdot \vec{n} = q$$

so that $t = \dfrac{q - \vec{a} \cdot \vec{n}}{n^2}$

$\therefore$ The position vector of L is given by

$$\bar{a} - \frac{c - \bar{a} \cdot \bar{n}}{\bar{n}^2}\,\bar{n}$$

The length AL

$$= |\overrightarrow{AL}|$$

$$= \left| \vec{a} - \frac{c - \bar{a} \cdot \vec{n}}{\bar{n}^2}\,\vec{n} - \vec{a} \right|$$

$$= |c - \bar{a} \cdot \vec{n}|, \text{ for } n^2 = |n|^2 = 1$$

<u>Solved exercises</u> :

1. Show that the line L whose vector equation is

$$\vec{r} = (2i - 2j - 3k) + \lambda(i-j+4k)$$

is parallel to the plane $\vec{r} \cdot (i + 5j + k) = 5$
and find the distance between them.

Ans: If θ is the angle between the line and the plane, then

$$\sin \theta = \frac{\bar{\ell} \cdot \bar{n}}{|\ell|}$$

$$\sin \theta = \frac{(i - j + 4k)}{\sqrt{18}} \cdot \frac{(i + 5j + k)}{\sqrt{27}} = 0$$

$\theta = 0$. They are parallel.

The distance $= |\bar{a} \cdot \vec{n} - c|$

$$= (2i-2j-3k) \cdot \frac{(i+5j+k)}{\sqrt{27}} - \frac{5}{\sqrt{27}}$$

$$= \frac{10}{\sqrt{27}}$$

2. Show that the plane whose vector equation is

$$\vec{r} \cdot (i + 2j - k) = 3$$

contains the line whose vector equation is

$$\vec{r} = i + j + (2j + j + 4k)$$

Ans: $\sin\theta = \frac{(2i+i+4k) \cdot (i+2j-k)}{xxx}$

$= 0$

Hence the Line and Plane are parallel.

The distance $= |\vec{a} \cdot \vec{n} - d|$

$$= (i+j) \cdot \frac{(i+2j-k)}{\sqrt{6}} - \frac{3}{\sqrt{6}}$$

$$= \frac{1+2}{\sqrt{6}} - \frac{3}{\sqrt{6}}$$

$= 0$

Hence the line Lies on the plane.

3. Find the vector equation of the line passing through (3,1,2) and perpendicular to the plane $\vec{r} \cdot (2i - j + k) = 4$.
Find also the point of intersection of this line and the plane.

Ans: The plane is $\vec{r} \cdot (2i - j + k) = 4$.
Hence $\vec{n} = 2i - j + k$ is perpendicular to the plane. The line has to pass through the point (3,1,2).
Hence the equation of the line is $\vec{r} = (3i+j+2k) + \lambda(2i-j+k)$
The point of intersection of the line and the plane will be given by solving $\vec{r} \cdot (2i - j + k) = 4$ ...(1)

$\vec{r} = (3i + j + 2k) + \lambda(2i-j+k)$ ....(2)

Substituting (2) in (1), we get

$4 = 6-1 + 2 + \lambda(4+1+1)$

$\lambda = -1/2$

The point of intersection is

$(3i + j + 2k) + (-1/2)(2i - j + k)$

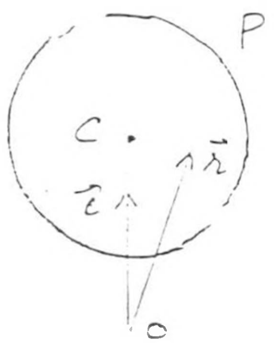
$= (2, \frac{3}{2}, \frac{3}{2})$.

SPHERE

Definition: The set of all points in the space, each of which is at a constant distance $a(>0)$ from a fixed point C is called a sphere.

The fixed point C is called the centre and the constant distance 'a' is called the radius of the sphere.

Central form of a sphere :



Let $\vec{c}$ be the position vector of the centre of the sphere, of radius $a > 0$.
Let $\vec{r}$ be the position vector of any arbitrary point P on the sphere.
Then, $|\vec{CP}| = a$

$\Rightarrow$ |Position vector of P − Position vector of C| = a

$$\Rightarrow \boxed{|\vec{r} - \vec{c}| = a}$$

This is the vector equation of the sphere in the central form.

Cor 1: In particular

$|\vec{r}| = a$ is the equation of the sphere whose centre is the origin and radius is a.

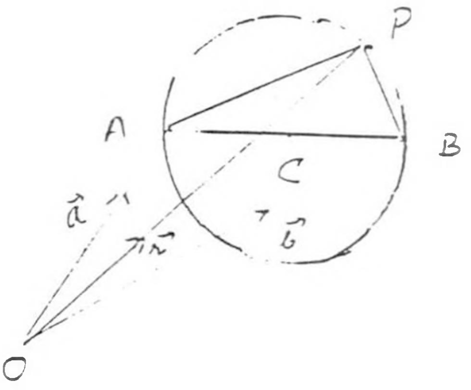
Cor 2: $\vec{r} - \vec{c}$

$$= (x_i + y_j + zk) - (c_1 i + c_2 j + c_3 k)$$

$$= (x-c_1)i + (y-c_2)j + (z-c_3)k$$

$$\boxed{(x-c_1)^2 + (y-c_2)^2 + (z-c_3)^2 = a^2}$$

is the equation of the sphere with centre (c_1, c_2, c_3) and radius a.



Diameter form of the sphere :

Let $\vec{a}$, $\vec{b}$ be the position vectors of the extremities A and B of the diameter AB of the sphere. Let $\vec{r}$ be the position vector of any point P on the surface of the sphere.

Then, $\vec{AP} = \vec{r} - \vec{a}$

$\vec{BP} = \vec{r} - \vec{b}$

It is clear from geometry that

$$\vec{AB} \cdot \vec{BP} = 0$$

$$\boxed{(\vec{r} - \vec{a}) \cdot (\vec{r} \cdot \vec{b}) = 0}$$

which is the equation of the sphere whose diameter is the join of A $(\vec{a})$ and B$(\vec{b})$.

Cartesian Form :

Let A (x_1, y_1, z_1) and B(x_2, y_2, z_2) be the extremities of the diameter AB of the sphere. Let P(x, y, z) be any point on the surface of the sphere. Then,

$$\vec{r} - \vec{a} = (x-x_1)\, i + (y-y_1) j + (z-z_1) k$$

$$\vec{r} - \vec{b} = (x-x_2) i + (y-y_2) j + (z-z_2)\, k$$

$$(\vec{r} - \vec{a}) \cdot (\vec{r} - \vec{b}) = 0$$

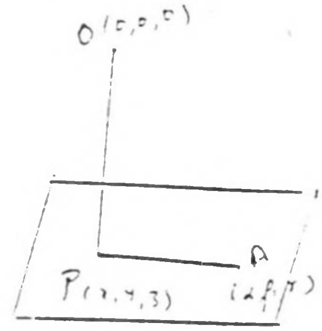
becomes

$$\boxed{(x-x_1)\,(x-x_2) + (y-y_1)\,(y-y_2) + (z-z_1)\,(z-z_2) = 0}$$

which is the Cartesian equation of the sphere whose diameter is the join of the points (x_1, y_1, z_1) and (x_2, y_2, z_2).

Solved Examples :

1. A plane passes through a fixed point A (α, β, γ). Show that the locus of the foot of perpendicular to it from the origin is the sphere $x^2+y^2+z^2 - \alpha x - \beta y - \gamma z = 0$



Ans:

Let $P(x,y,z)$ be the foot of the perpendicular from O on the plane.

$\vec{OP} = (xi + yj + 3k)$

$\vec{PA} = (x-\alpha)i + (y-\beta)j + (z-\gamma)k$

$\vec{OP} \perp \vec{PA}$ can be written as

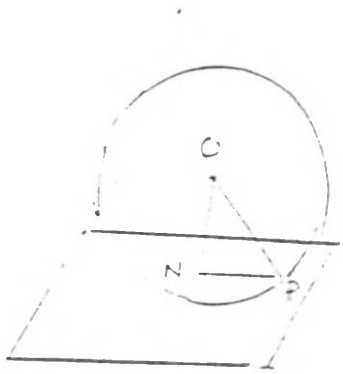
$$(xi + yj + zk)\cdot (x-\alpha)i + (y-\beta)j + (z-\gamma)k = 0$$

i.e. $x^2+y^2+z^2 - x\alpha - y\beta - z\gamma = 0$

2. Prove that the radius of the circular section of the sphere $|\vec{r}| = 5$ cut off by the plane $\vec{r}\cdot(i+j+k) = 3\sqrt{3}$ in 4 units.

Ans: The given sphere is $|\vec{r}| = 5$.

The centre is the origin and the radius is 5.



The given plane can be written as

$$\vec{r}\cdot\frac{(i+j+k)}{\sqrt{3}} = 3$$

Hence the distance of the plane from the centre is p=3.

i.e. $|CN| = 3$

Then $|NP| = \sqrt{CP^2 - CN^2}$

$= \sqrt{5^2-3^2}$

$= 4$ units

3. Prove that the plane $x+2y+2z = 15$ cuts the sphere $x^2+y^2+z^2-2y-4z-11 = 0$ in a circle. Find the centre and radius of the circle.

Ans: The equation of the sphere is $x^2+y^2+z^2-2y-4z-11 = 0$

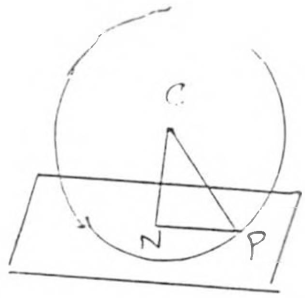
Its centre is (0,1,2) and radius $r = 4$.

The distance of the plane from the centre of the sphere is

$$p = \left|\frac{0+2+4-15}{\sqrt{1+4+4}}\right|$$

$= 3$ $\qquad p < r$

$\therefore$ The plane cuts the sphere in circle.



Then,

radius of the circle is

$= NP$

$= \sqrt{CP^2 - CN^2}$

$= \sqrt{r^2 - p^2}$

$= \sqrt{7}$

Let (α, β, γ) be the coordinates of N. N lies on the plane.

$\therefore \quad \alpha + 2\beta - 2\gamma - 15 = 0$

Also CN is parallel to the normal to the plane.

$$\frac{\alpha}{1} = \frac{\beta-1}{2} = \frac{\gamma-2}{2} = k$$

$\alpha = k, \quad \beta = 2k+1, \quad \gamma = 2k+2$

Substituting these values in the above, we get

$k + 4k + 2 + 4k + 4 - 15 = 0$

$k = 1$

$\alpha = 1, \ \beta = 3, \ \gamma = 4$

Hence the centre C (1,3,4)

and the radius is $\sqrt{7}$.

PROJECTS IN MATHEMATICS

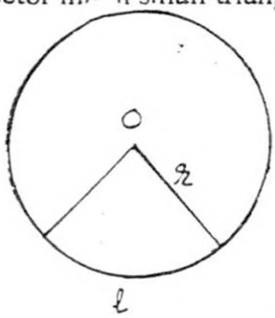
Prof N M Rao

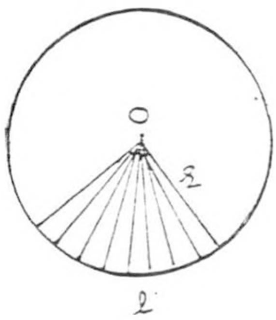
1. Area of the Circle

Objective : To find the area of the circle by using the area of small sectors.

Description :

Take a circle of radius r. Consider a sector of the circle of arc length l and divide the sector into n small triangles as shown in the figure.





The area of each triangle = $\frac{1}{2} rb$ where b = $\frac{l}{n}$.

The total area of the sector of arc length l = n (½ rb)

= ½ r (nb)

= ½ rl

since l = nb

In the same way, the area of the circle = ½ rc, where c is the circumference of the circle.

Area of the circle = ½ rc

= ½ r (2 π r)

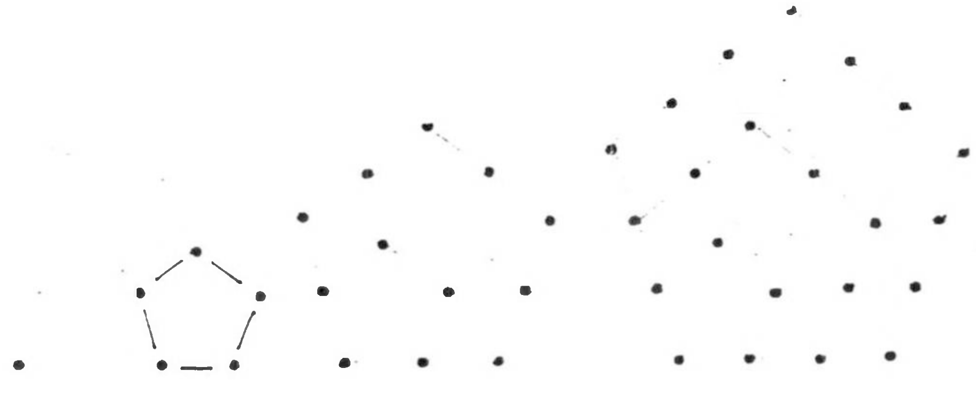
= πr^2

2. Pentagonal Numbers

Objective : To enable the students to acquire the knowledge of pentagonal numbers.

Description:

Numbers can be represented in certain patterns. One of the patterns is by representing the dots. The below shown are the pattern of pentagonal numbers.

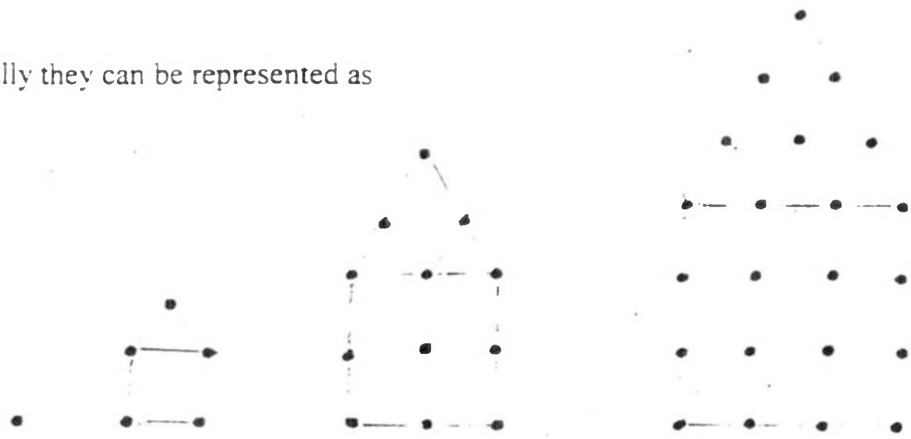


1 5 12 22

1,5,12,22,..... are called pentagonal numbers. These pentagonal numbers are obtained by adding triangular numbers and square numbers. The pattern thus formed with these numbers are

Triangular Numbers	+	Square Numbers	=	Pentagonal Numbers
	+	1	=	1
1	+	4	=	5
3	+	9	=	12
6	+	16	=	22
10	+	25	=	?

Thus pictorially they can be represented as



1 1+4 3+9 6+16

The bindis can be pasted on chart paper and the patterns of the pentagonal numbers can be enjoyed by the students.

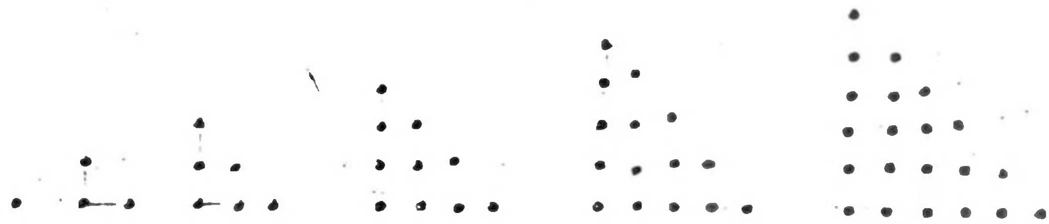
1. The students can be asked to guess the next pentagonal numberand verify it afterwards by adding the corresponding triangular and square numbers.
2. The students can also be asked to find a formula to represent the triangular, square and pentagonal numbers

3. **Tetrahedral Numbers**

Objective : To enable th students to acquire the knowledge of the development of fifth tetrahedral number through Pythagorean, triangular numbers.

Procedure :

The first six Pythagorean numbers are 1,3,6, 10. 15 and 21. They are represented as follows:

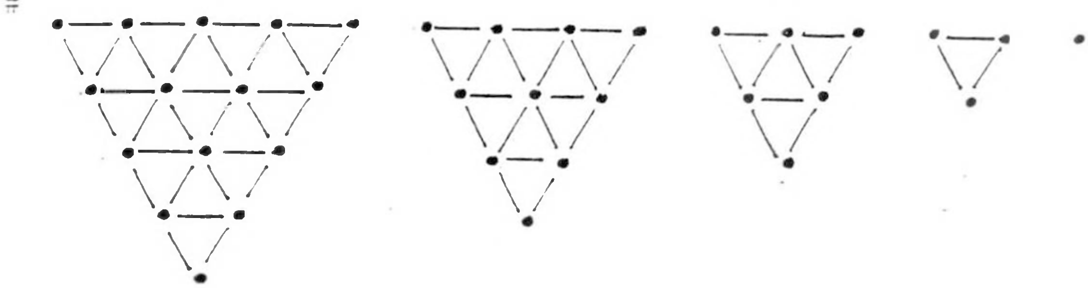


T1	T2	T3	T4	T5	T6
=1	(1+2)	(1+2+3)	(1+2+3+4)	(1+2+3+4+5)	(1+2+3+4+5+6)
=1	=3	=6	=10	=15	=21

Look at the pattern down below and the series in the fourth line :

1	1	1	1	1	1	1	1	
1	2	3	4	5	6	7	8	The Natural Numbers
1	3	6	10	15	21	28	36	The Triangular Numbers
1	4	10	20	35	56	84	120	The Tetrahedral Numbers

The tetrahedral number is built up from Pythagorean, triangular numbers as follows :



15 + 10 + 6 + 3 + 1

1; (1+3)=4; (1+3+6)=10; (1+3+6+10)=20; (1+3+6+10+15)=35

Taking clue from the above table, a model of the tetrahedral numbers is formed by keeping the patterns one upon the other as follows :

1. Keep one ball on the top step.
2. Below that, keep a step having three balls.
3. Next step contains 6 balls.
4. Next lower step contains 10 balls.
5. The fifth step contains 15 balls.

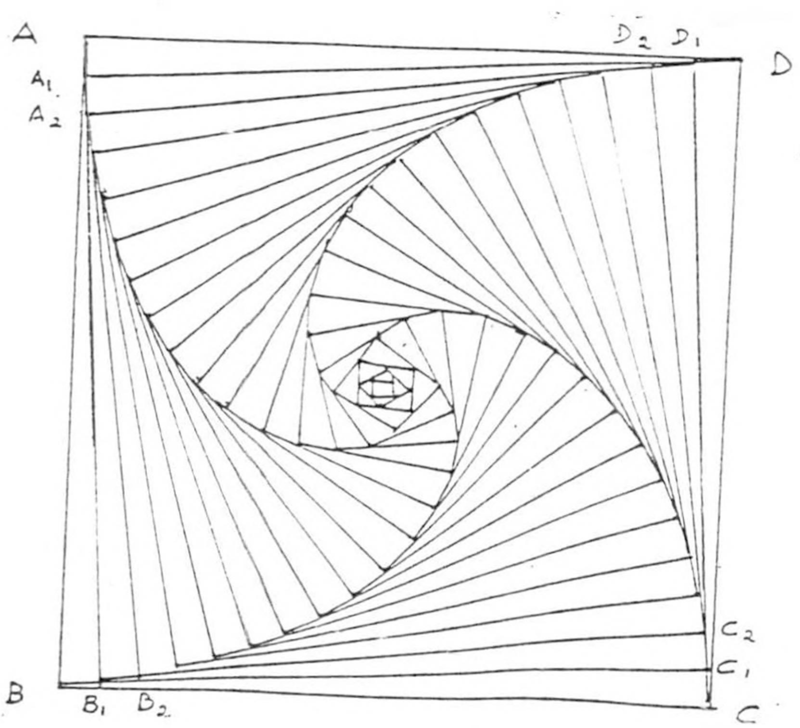
Now the complete model contains 15 + 10 + 6 + 3 + 1 = 35 balls – A tetrahedral number is built up from triangular numbers. Similarly any tetrahedral number can be built up as the sum of triangular numbers.

The students can be asked to prepare a vertical model of the above. They can also be asked to guess a formula to find tetrahedral numbers.

4. Path of Pursuits

Objective: To find the paths of four ants placed at the corners of the square, each one moving in the direction of the ant in front of it. (This path is called the path pursuits).

Take a piece of stiff card board and mark a square ABCD of side 10 cm. Mark the point A_1 on AB at ½ cm distance from A. Similarly mark B_1, C_1 and D_1 at ½ cm from B, C and D respectively. Now mark A_2 at a distance of ½ cm from A_1, on the line A_1B_1, B_2 at ½ cm from B_1 on the line B_1C_1 and so on. Continue in this way until the center of the square is reached. These envelopes are known as curves of pursuit. Since they are the paths which four ants originally placed at the corners of the square, would follow if they were always to walk in the direction of the ant in front of them.

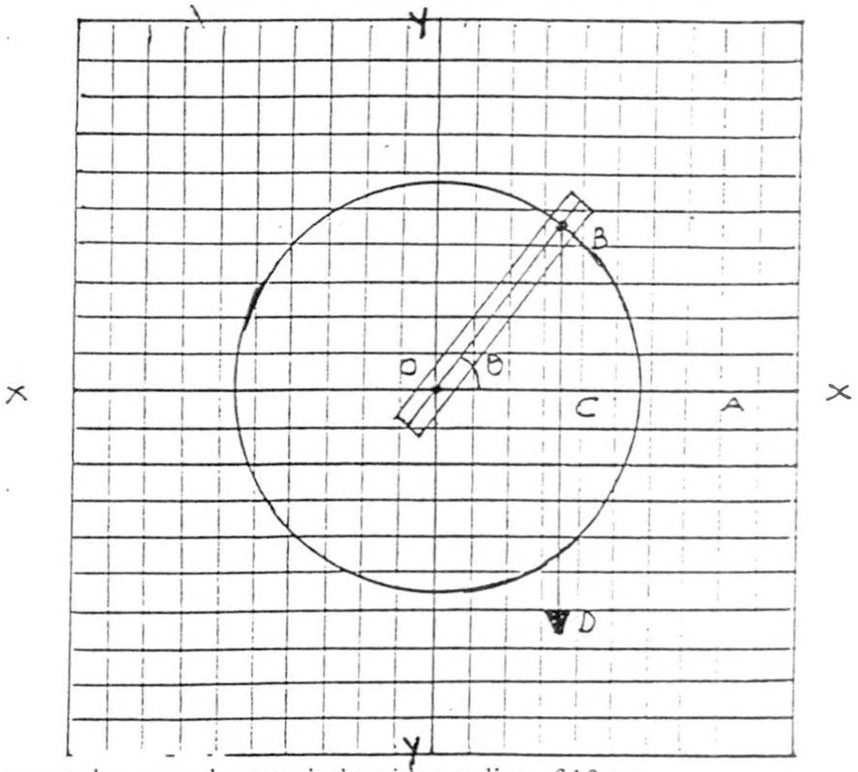


1. Can you stitch the path of pursuits on a black coloured cloth using white thread ?
2. Where is the point at which all four ants meet each other in the end ?
3. Read the chapter on envelopes and evolutes (geometry) to understand the significance of this path.

5. Building Trignometrical Tables

Objective: A simple device can be constructed by the students that will enable them to make their own table of trigonometric ratios for the sine and cosine.

Procedure :



1. On a graph paper, draw a circle with a radius of 10 cm.
2. Cut thin strip of cardboard atleast 12 cm long.
3. Draw a line down the center of the strip.
4. Attach one end of the strip to the center of the circle.
5. At the other end of the strip, 10 cm from the point where it is attached to the circle, make a small hole and attach a piece of thread.
6. At the opposite end of the string, attach a weight to serve as a plumb line.

The strip OB can be rotated around the point O so that OB makes different angles θ with x-axis. The hanging plummet BD cuts the x-axis at the point C. Count the number of spaces of length of the cord BC. Since hypotenuse is fixed at 10 cm, we can easily determine sine ratio. Sin θ = BC/10. As we change the angle by moving the cardboard strip, we can observe the change in the value of sin θ. Similarly the value of cos θ can also be read by counting the number of spaces of horizontal axis OA. Cos θ = OC/10.

From this we can get the value of tan θ, cot θ, sec θ and cosec θ. There may be some error in counting the lengths of BC and OC. Therefore, students are asked to compare these values of sin θ, cos θ, etc. with the standard values given in the trigonometric tables.

6. Solids of Revolution

Objective: To show that various geometrical figures when revolved around a particular axis give various solids.

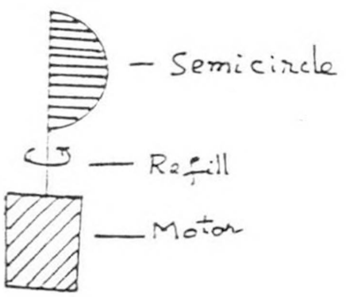
How to use this aid

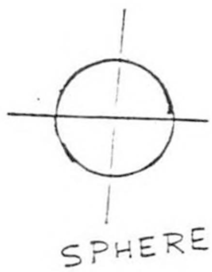
The teaching aid consists of a motor and various objects of following shapes :

a) circular
b) parabolic
c) triangular or angular
d) square or rectangular

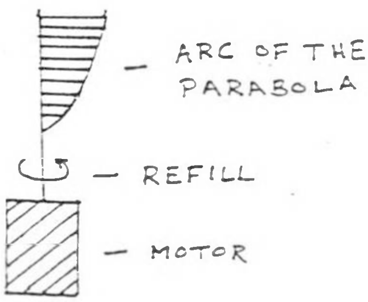
The objects are fixed to a pen refill. that should be attached to the motor which rotates about its axis. We get the following solids of revolution.

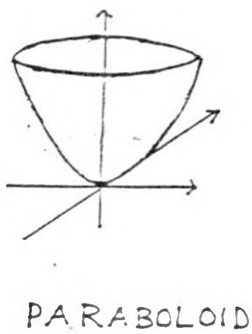
a)





b)





7. Path of the Moving Chord Inside a Circle

Objective : To illustrate that, the path of the moving chord of constant length inside a circle is a circle and to find out the radius of this inner circle.

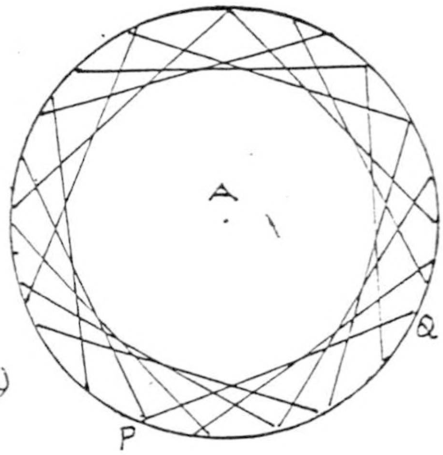


FIG (a)

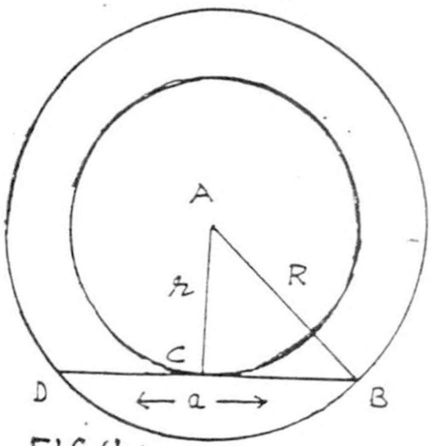


FIG (b)

PQ is a chord of constant length which moves inside the circle of a radius R, centred at the point A. What is the path of PQ ? The students can move the stick PQ inside the circle and convince themselves that the path of the moving chord PQ of constant length inside a circle in a circle. They can repeat the experiment and verify the above fact. It is also clear that the center of the new circle is also A. What is the radius of this inner circle?

To find the radius of the inner circle see Fig. (2).
In which BD = a (length of the chord)
AB = R (radius of the outer circle)
AC = r (radius of the inner circle)

By Pythagoras theorem,

$$AB^2 = AC^2 + BC^2$$

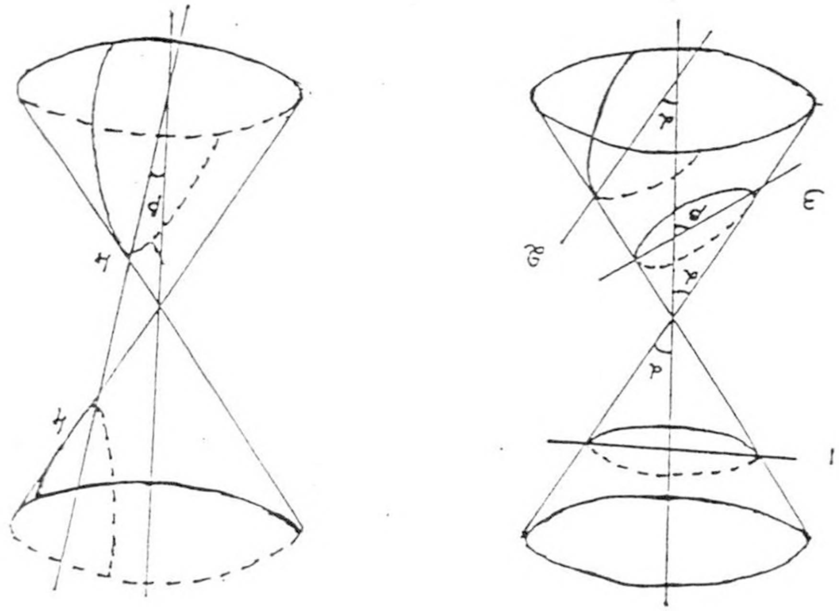
$$R^2 = r^2 + \left(\frac{a}{2}\right)^2$$

$$r = \sqrt{R^2 - \frac{a^2}{4}}$$

1. What happens if the length of the chord PQ is equal to the diameter of the bigger circle ?
2. What happens if the length of the chord PQ is equal to the radius of the bigger chord?

8. Conic Sections

Objective: To show that when a right circular cone is cut in four specific ways we get conic sections namely (1) circle, (2) parabola, (3) ellipse and (4) hyperbola.



How to use

- Hold the model and chart side by side, disjoint the right circular cone at the place marked '1' and see that the edge of the surface is a circle i.e. we get a circle by cutting the right circular cone perpendicular to its axis by a plane.
- Similarly disjoint the cone at the place marked '2' and see that the edge of the surface is a parabola. i.e. when we cut the cone parallel to one of its side we obtain parabola.
- Disjoint the cone at the place marked '3' and see that the edge of the surface is an ellipse. i.e. when we cut the cone at an inclined angle we get ellipse.

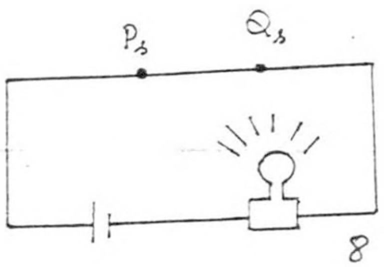
9. Logic Box

Objective: To enable the students to understand the conjunction (^) and Disjunction (v) of two statements and draw their truth tables.

P ^ Q = P and Q (Conjunction)
P v Q = P or Q (Disjunction)

How to use the Teaching Aid :

1. Connect the battery to the circuit. The circuit is now ready to operate.
2. The Ps and Qs switches, are connected in the series circuit. The circuit is given by



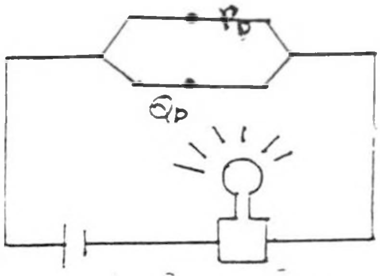
(i) When the switches Ps and Qs are both switched on the light is on (T ^ T = T).
(ii) When either of the switches are off the light is off (T ^ F = F).
(iii) When both the switches are off, the light is off (F ^ F = F).

The truth table for the given "And" circuit is :

Ps	Qs	Ps ^ Qs
T	T	T
T	F	F
F	T	F
F	F	F

This is called the *conjunction.*

3. Now see the disjunction (v) circuit.
Pp and Qp are connected in parallel circuit. The circuit is shown as



(i) When both Pp and Qp are switched on, the light is on (T v T = T).
(ii) When either Pp or Qp are switched on, the light is on (T v F = T).
(iii) When both Pp or Qp are switched off, the light is off (F v F = F).

The truth table is given by

Pp	Qp	Pp v Qp
T	T	T
T	F	T
F	T	T
T	F	F

The 'OR' circuit is off only when both Pp and Qp are off. This is called the disjunction of P, Q (Read as P or Q).

Verify whether the following statements are true or false :

1. (Conjunction) : Either $2 + 3 = 6$ and $4 + 5 = 9$.
2. (Disjunction) : Either $2 + 3 = 6$ or $4 + 5 = 9$

Justify your answer using the logic box.

10. Magic Square

Problem

Prepare a magic square by putting the given numbers between 1 and 20 in the holes of given 3 × 3 box such that sum of columns, rows and diagonals should be 21.

x_1	x_2	x_3
y_1	y_2	y_3
z_1	z_2	z_3

Solution

The least sum from 3 × 3 magic square will be 15, a multiple of 3. Let the sum be "a". To find the numbers in the magic square first subtract 15 from "a" divide by 3 and add 1.

$$\frac{a - 15}{3} + 1 = C \qquad \ldots\ldots \quad (1)$$

C will occupy the position of z_2. The number at x_3 will be $c + 1$. Similarly $y_2 = x_3 + 3$, $z_1 = y_2 + 3$ $(x_3 + 6)$. From these four numbers we get $z_3 = a - (z_1 + z_2)$, $x_2 = a - (z_2 + y_2)$, $z_1 = a - (x_2\ x_3)$, $y_3 = a - (x_3 + z_3 0$.

Here the given sum is 21. From (1) $z_2 = \frac{21 - 15}{3} + 1 = 3$

$x_3 = 3 + 1 = 4$, $y_2 = 4 + 3 = 7$, $z_1 = 7 + 3 = 10$ or $4 + 6 = 10$,
$z_3 = 21 - (10 + 3) = 8$, $x_2 = 21 - (3 + 7) = 11$, $y_1 = 21 - 16 = 5$
$x_1 = 21 - (11 + 4) = 6$, $y_3 = 21 - (4 + 8) = 9$.
Therefore, a magic square of sum 21 is as follows:

6	11	4
5	7	9
10	3	8

This method can be applied for any 3 × 3 magic square.

1. Students are advised to try to form a different magic square in which the sum is 21.
2. Form a magic square of sum 15.

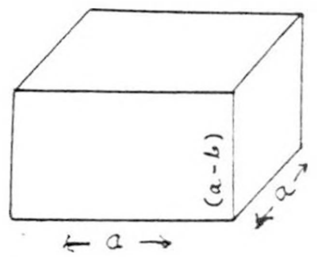
11. Model of $a^3 - b^3$

Objective : This model is to illustrate that

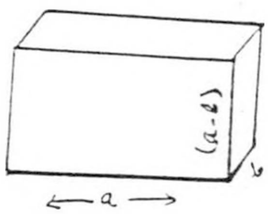
$$a^3 - b^3 = (a - b)\,a^2 + (a - b)\,ab + (a - b)\,b^2$$
$$= (a - b)(a^2 + ab + b^2)$$

Model : There are three wooden blocks of the following dimensions as shown :

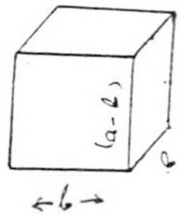
(i) $(a - b) \times a \times a$



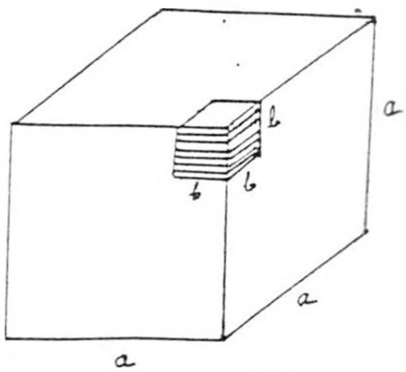
(ii) $(a - b) \times a \times b$



(iii) $(a - b) \times b \times b$



The three wooden blocks can be arranged in such a way that the complete assembly looks like $a^3 - b^3$, i.e. a small cube of volume b^3 has been removed from cube of volume a^3 units.



The students are requested to assemble the wooden blocks and convince themselves about the result :

$$
\begin{aligned}
& (a-b) \times a \times a + (a-b) \times a \times b + (a-b) \times b \times b \\
= \; & (a-b)\, a^2 + (c-b)ab + (a-b)b^2 \\
= \; & (a-b)\,(a^2 + ab + b^2) \\
= \; & a^3 + a^2b + ab^2 - a^2b - ab^2 - b^3 \\
= \; & a^3 - b^3
\end{aligned}
$$

12. Envelopes

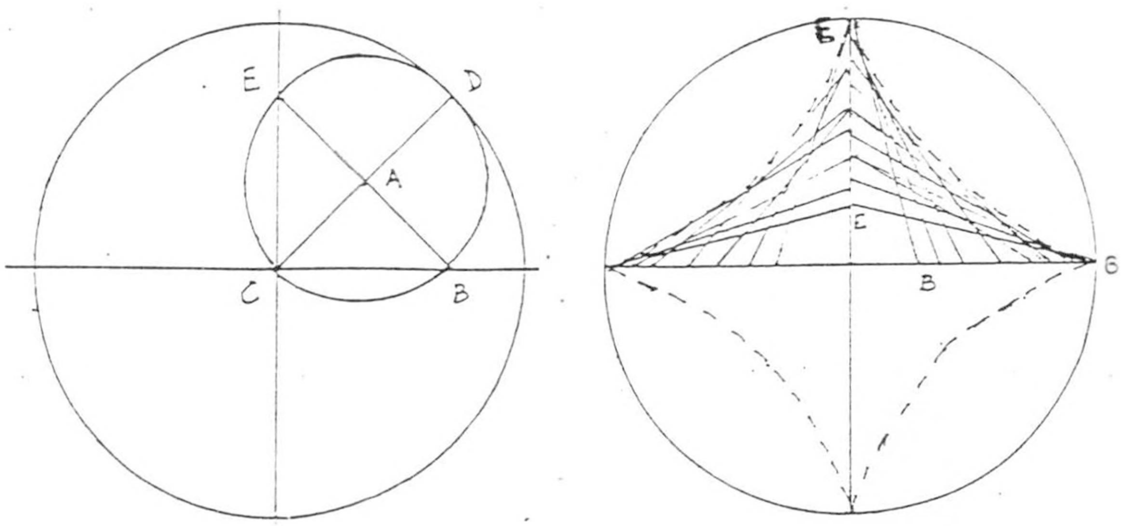
Objective: To enable the stdents to understand the locus of a point and envelope of a set of lines.

Analysis : A set of points obeying a rule is called locus and a set of lines obeying a rule is called an envelope.

Experiment :

Cut a hole whose radius is the diameter of a one rupee coin, in a piece of cardboard. Roll the coin, without slipping, round the hole. What is the locus of

(a) the center of the coin ?
(b) a point on the circumference ?



Answer :

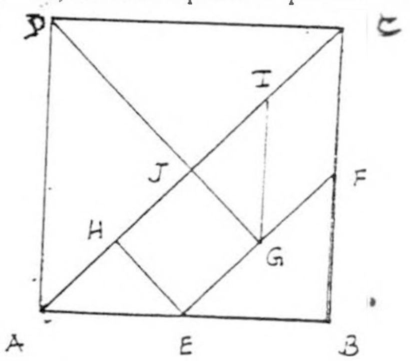
(i) If B is any point on the coin, then the locus of B is the diameter of the hole.
(ii) If BE is the diameter of the coin, the locus of E is the perpendicular diameter of the hole.
(iii) The locus of A, the center of the coin, is a circle.
(iv) The envelope of BE is an astroid.

13. **Tangrans**

Objective : To form the geometrical shapes of squares, rectangles, hexagon, trapezium, etc. from the given pieces and to improve the mental ability of students.

Construction:

1. Take a square cardboard ABCD of side length 20 cms.
2. Draw the diagonal segment AC as shown in the figure.
3. The points E and F are mid points of AB and BC respectively. Draw EF.
4. G is the midpoint of EF. Draw GD.
5. Construct the line segment EH perpendicular on AC from the point E.
6. Draw a line segment GI, from the point G parallel to BC to cut the line AC at the point I.

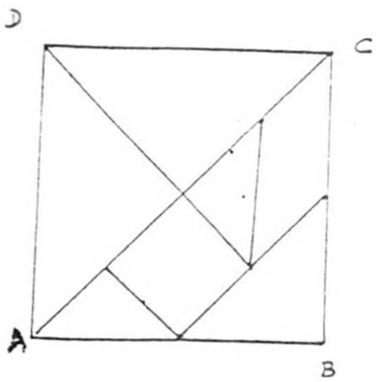


By cutting along the lines as shown in the figure, we get tan gram pieces.

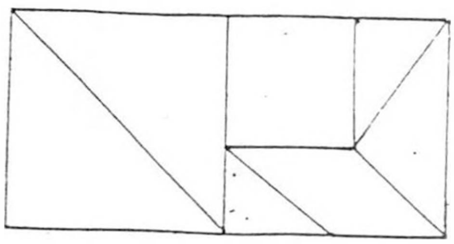
How to use

Take out all the seven tan gram pieces. Ask the learner to arrange the given pieces.

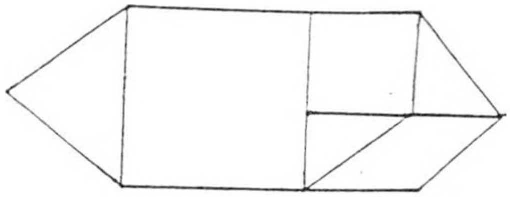
(i) to form a square



(ii) to form a rectangle

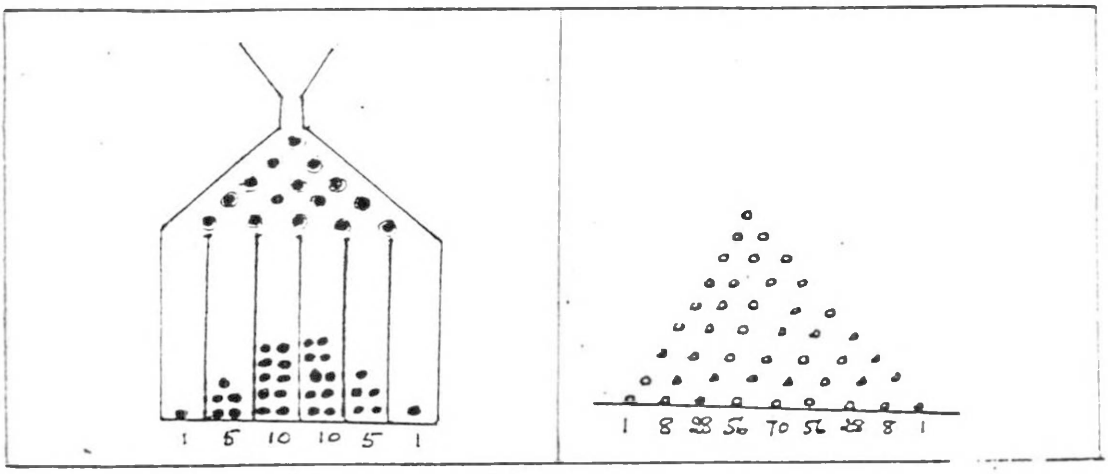


(iii) to form a hexagon



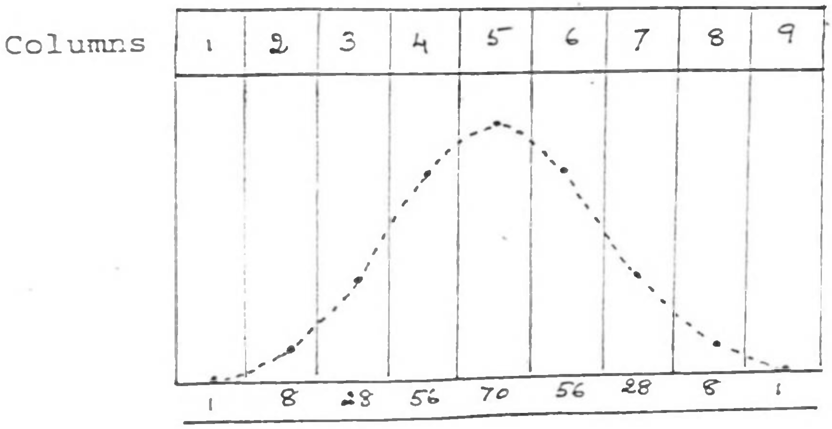
14. Probability Curve Experiment

Objective : This is a wooden model to show that the marbles flowing through a series of nails in the form of Pascal's triangle, will settle down in the shape of a Normal Probability Curve.



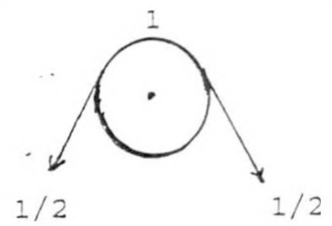
How to use it

The nails are fixed on a wooden board according to the Pascal's Triangle as shown in the figure. Above the nails, a metal box is fitted to pour the marbles uniformly. As we pour the marbles in the metal box, they come and settle in the columns in the form of normal probability curve as shown.



Principle

When we pour the marbles at the top, at the first nail, half of the marbles will flow by the left side of the nail and half the marbles will flow by the right side of the nail as shown :



The marbles coming to left nail in the second row will have two equal possibilities to go to the 3rd row, ¼ to the left and ¼ to the right. Similarly the marbles coming to the right nail in the second row will have two equal chances to go to the 3rd row, ¼ to the left of it and ¼ to the right of it. Hence in the second row, the marbles flow will be as follows : ¼ in the left, ¼ + ¼ = ½ in the center and ¼ in the right. Similarly in the third row, the marbles flow as follows : 1/8, 3/8, 3/8 and 1/8. If we continue in this way, in the eight row marbles flow as follows:

$$\frac{1}{256}, \frac{8}{256}, \frac{28}{256}, \frac{56}{256}, \frac{70}{256}, \frac{56}{256}, \frac{28}{256}, \frac{8}{256}, \frac{1}{256}$$

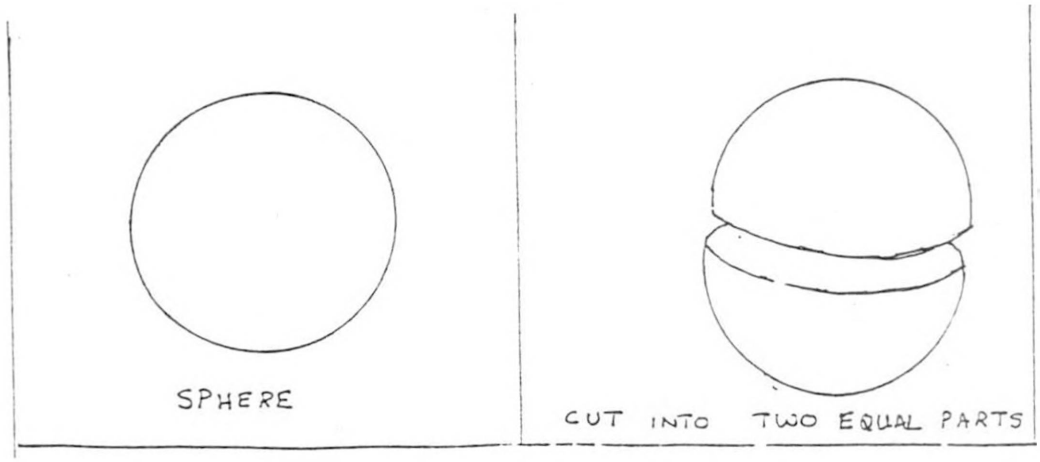
In other words, if we pour 256 marbles from the top, then in a normal case, the number of marbles setting in each column will be as shown below.

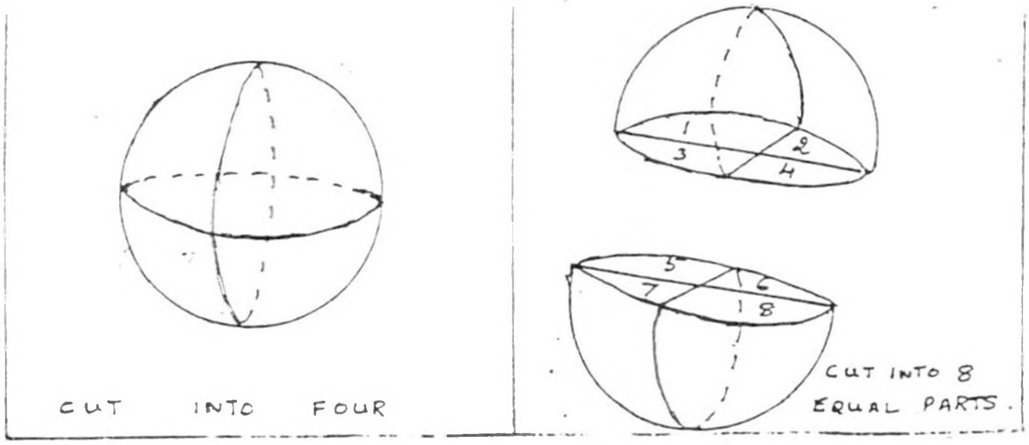
15. Relation between the volume of sphere and volume of cube, constructed from the sphere

Objective: To see the relation between the volume of the original sphere and the volume of the interior of the simple cube constructed from the sphere.

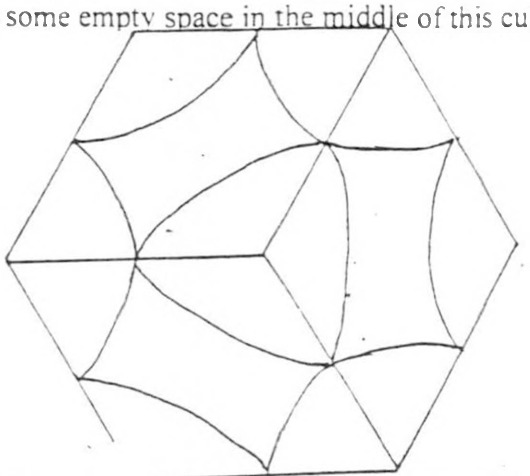
How to use:

Take a sphere of radius 'R' and cut the sphere into eight equal parts as shown below:





Join them from the reverse direction to form an object similar to a cuboid as shown below. Note that there will be some empty space in the middle of this cuboid.



The comparison is between the volume of the sphere and volume of the interior (empty space) of the constructed simple cube.

Volume of sphere = $\frac{4}{3}\pi R^3$

Volume of cube = $(2R)^3 = 8R^3$

Volume of the interior empty space of = $8R^3 - \frac{4}{3}\pi R^3$.

% of empty space in the cube = $\dfrac{8R^3 - (4/3)\pi R^3}{(4/3)\pi R^3} \times 100$

$$= \left(\frac{6-\pi}{\pi}\right) \times 100$$

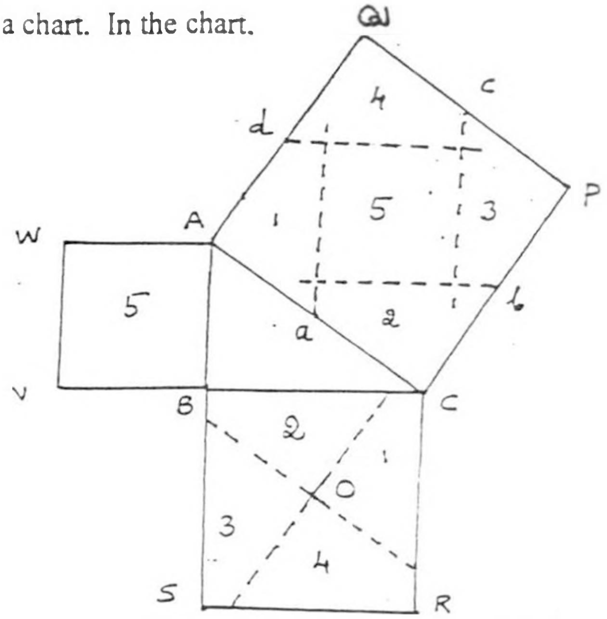
Note that this expression is independent of R.
The percent of empty space remains the same, even if the diameter of the sphere changes.

16. Pythagoras Theorem (Perigal's Dissection Method)

Objectives: To show that in a right angled triangle ABC, $AC^2 = AB^2 + BC^2$ where AC is the hypotenuse, by the Perigal's Dissection Method.

Procedure :

There is a wooden model and a chart. In the chart.



ABC is the given right angled triangle. BCRS is the square on the side BC. O is the point of intersection of the diagonals BR and CS. Draw a line parallel to AC through O. Also draw a line perpendicular to AC through O. They divide the square BCRS to four parts 1,2,3,4 as shown in the figure.

a, b, c, d are mid points of AC, CP, PQ and QA respectively. Draw lines parallel to the line AB through a and c. Draw lines perpendicular to the line AB through b and d. These four lines divide the square ACPQ into five parts 1,2,3,4 and 5 as shown.

There are five plastic cut pieces which are congruent to the shapes 1,2,3,4 and 5.

Place these plastic pieces numbered 1,2,3 and 4 on the square on BC and piece numbered 5 on the square on AB as shown in the figure.

Now place the same five pieces on the square on the hypotenuse AC. The five places exactly fit in the square on the hypotenuse (the areas are equal).

The above method justifies that

$$AC^2 = AB^2 + BC^2$$

Remember that it is not a proof.

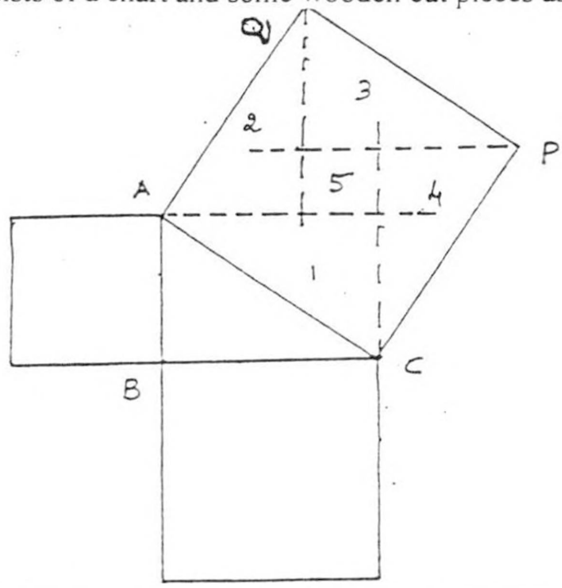
The teachers and the students are welcome to give a mathematical proof for the Perigal's method.

17. Pythagoras theorem (Bhaskaracharya's Dissection Method)

Objectives: To show that in a right angled triangle ABC, $AC^2 = AB^2 + BC^2$ where AC is the hypotenuse, by Bhaskaracharya's Dissection Method.

Procedure

This teaching aid consists of a chart and some wooden cut pieces as shown in the following figure.



ABC is a right angled triangle. ACPQ is the square on the side AC. Draw lines parallel to AB from the vertices Q and C. Also draw lines parallel to BC from the vertices P and A, and hence divide the square ACPQ into four triangles congruent to the triangle ABC and a square in the center whose side length is (BC – AB) as shown in the figure.

Now,

Area of the square ACPQ
$= 4\,(1/2 \times AB \times BC) + (BC - AB)^2$
$= 4\,(1/2 \times AB \times BC) + BC^2 + AB^2 - 2BC\,.\,AB$
$= 2\,AB.\,BC + BC^2 + AB^2 - 2AB\,.\,BC$
$= BC^2 + AB^2$

$\therefore AB^2 + BC^2 = AC^2$

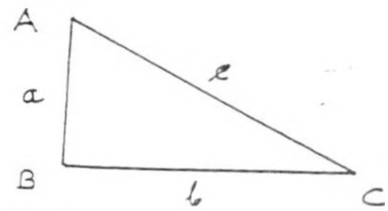
By keeping the wooden pieces in the appropriate places, the students can convince themselves that the result is true.

Now try to give a complete mathematical proof for Bhaskaracharya's method.

18. Pythagoras Theorem (Chau Pei's Dissection Method)

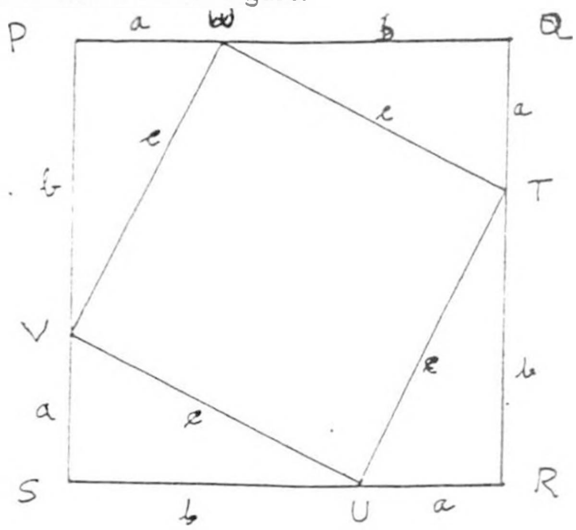
Objectives: To show that in a right angled triangle ABC, $AC^2 = AB^2 + BC^2$ where AC is the hypotenuse, by using the expansion of the expression $(a + b)^2$:

This teaching aid consists of a chart and some wooden cut pieces.



In the right angled triangle ABC, the lengths of the sides are a, b respectively while the length of the hypotenuse is c.

Take a plastic square piece PQRS of side length a + b as shown in the figure.



Then TUVW is a square whose side length is c.

Area of PQRS $= (a + b)^2$
$= a^2 + b^2 + 2ab$ (1)

Area of PQRS = Area of the square TUVW + 4 (Area of the triangle PVW)
$= c^2 + 4 (1/2 \times a \times b)$
$= c^2 + 2ab$ (2)

From (1) and (2)

$$a^2 + b^2 + 2ab = c^2 + 2ab$$
$$a^2 + b^2 = c^2$$
$$AB^2 + BC^2 = AC^2$$

By keeping the plastic pieces in the appropriate places, the students can convince themselves that the result is true.

The students can also be asked to prove mathematically that the four triangles are congruent to each other. Probably this method was adopted by the Chinese Mathematician Chou Pei (AD 40). Please see the book 'History of Mathematics' by Smith.

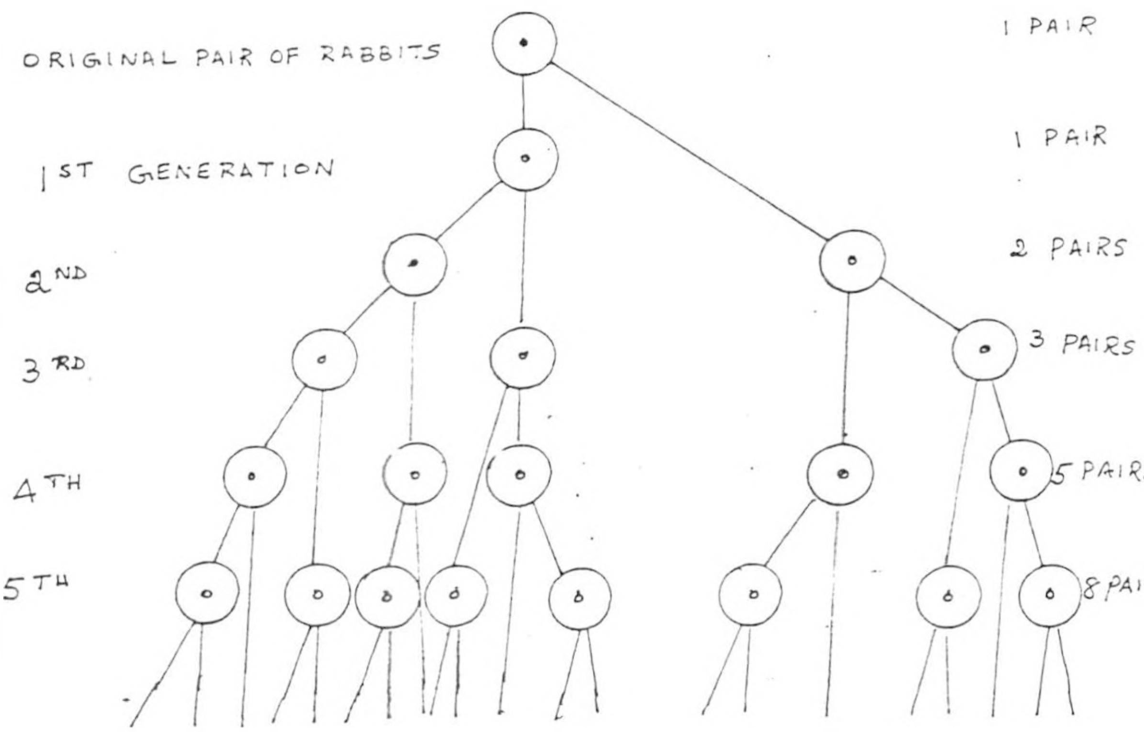
19. Fibonacci Sequence

Objective: This is a model, to show the physical meaning of the 'FIBONACCI SEQUENCE'.

The Fibonacci sequence is 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89......

The Problem :

The problem can be stated as follows. A man brought a pair of rabbits and bred them. The pair produced one pair of offspring after one month and a second pair of offspring after the second month. Then they stopped breeding. Each new pair also produced tow more pairs in the same way and then stopped breeding. How many new pairs of rabbits did he get each month ?



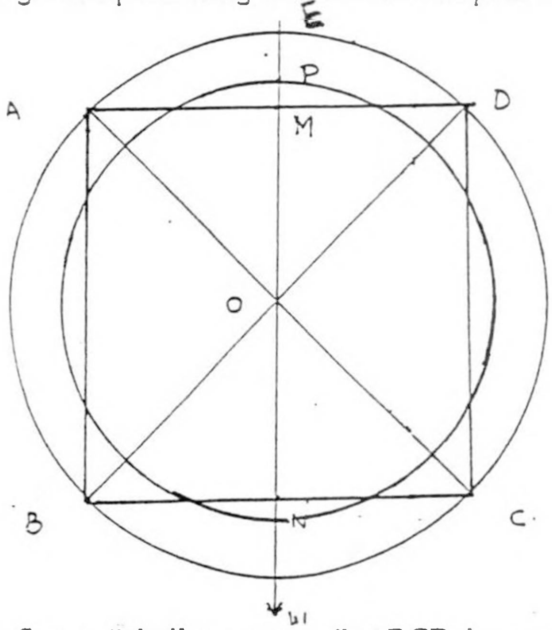
Let us write down in a line, the number of pairs in each generation of rabbits.

1. First we write the number 1 for the single pair we started with (1 new pair).
2. Next we write the number 1 for the pair they produced after a month (1 new pair).
3. The next month, both pairs produced. So the next number is 2 (2 new pairs).
4. Now the original pair stopped producing. The first generation (1 pair) produced 1 pair. The second generation (2 pairs) produced 2 pairs. So the next number we write is 1 + 2 or 3. (Total 3 new pairs).
5. Now the first generation stopped producing. The second generation (2 pairs) produced 2 pairs. The third generation (3 pairs) produced 3 pairs. So, the next number we write is 2 + 3 or 5. (Total 5 new pairs).
6. Each month, only the last two generations produced. So, we can get the next number by adding the last two numbers in the line.
7. The numbers we get in this way are called Fibonacci numbers.

Reference : Land – Language of Mathematics

20. Circling a Square

Objective: This chart can be used to explain "How to construct a circle whose area is equal to the area of the given square using a scale and compass only". (approximately equal).



How to use it

1. In the above figure, "circling a square" ABCD is a square which is to be transformed into a circle so that their areas are equal.
2. AC and BD are the diagonals of the square intersecting at O.

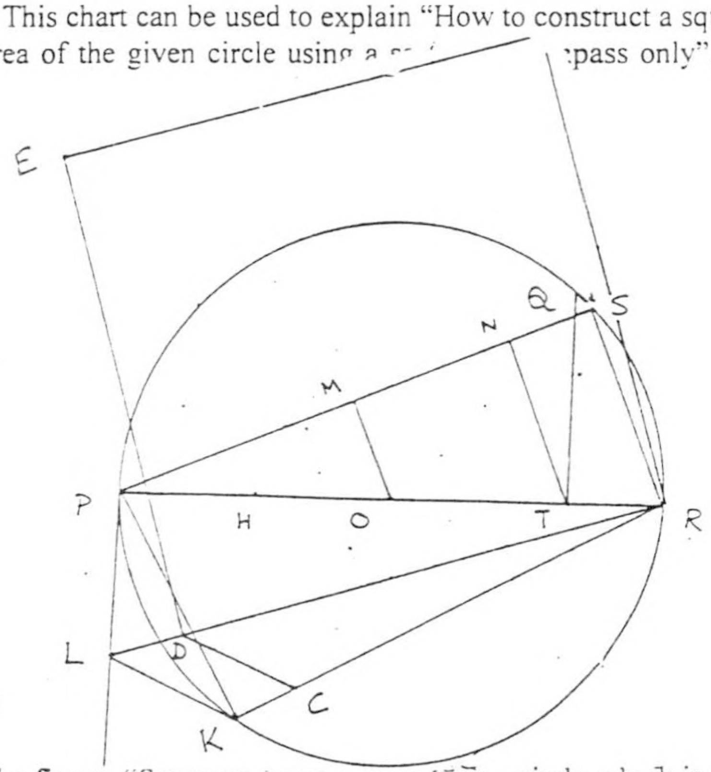
3. EW is a line passing through M, O and N, where M and N are the midpoints of AD and BC respectively.
4. With 'O' as center and OA as radius, a circle is drawn such that it intersects EW at E.
5. EM is divided such that EP = 2PM.
6. With 'O' as center and OP as radius another circle is drawn. The area of this circle is approximately equal in the area of the square ABCD.

Reference : Indian Mathematics and Astronomy by S. Balachandra Rao.

Note : The above problem, "Constructing a circle whose area is equal to the area of the given square" had remained unsolved for centuries in the history of Mathematics. The above method of construction is given by the ancient "Indian Mathematicians" in "Sulva Sutra".

21. Squaring a circle

Objective: This chart can be used to explain "How to construct a square whose area is equal to the area of the given circle usin a ... :pass only". (Approximately equal).



How to use it :

1. In the figure, "Squaring a circle", PQRS is a circle which is to be transformed into a square, so that their areas are equal. O is the center of the circle and PR is the diameter of the circle.
2. PO is bisected at H and OR is trisected at T nearer R.
3. TQ is drawn such that TQ ⊥ PR and a chord RS is placed such that RS = TQ.
4. 'P' and 'S' are joined and OM and TN are drawn parallel to RS.
5. A chord is drawn such as PK = PM and a tangent PL is drawn to the circle at P such that PL = MN. RL, RK and KL are drawn.

6. A point 'C' is marked on RK such that RC = RH and CD is drawn such that CD is parallel to KL, meeting RL at D. Now a square is constructed on RD. Area of this square is equal to the area of the circle PQR approximately.

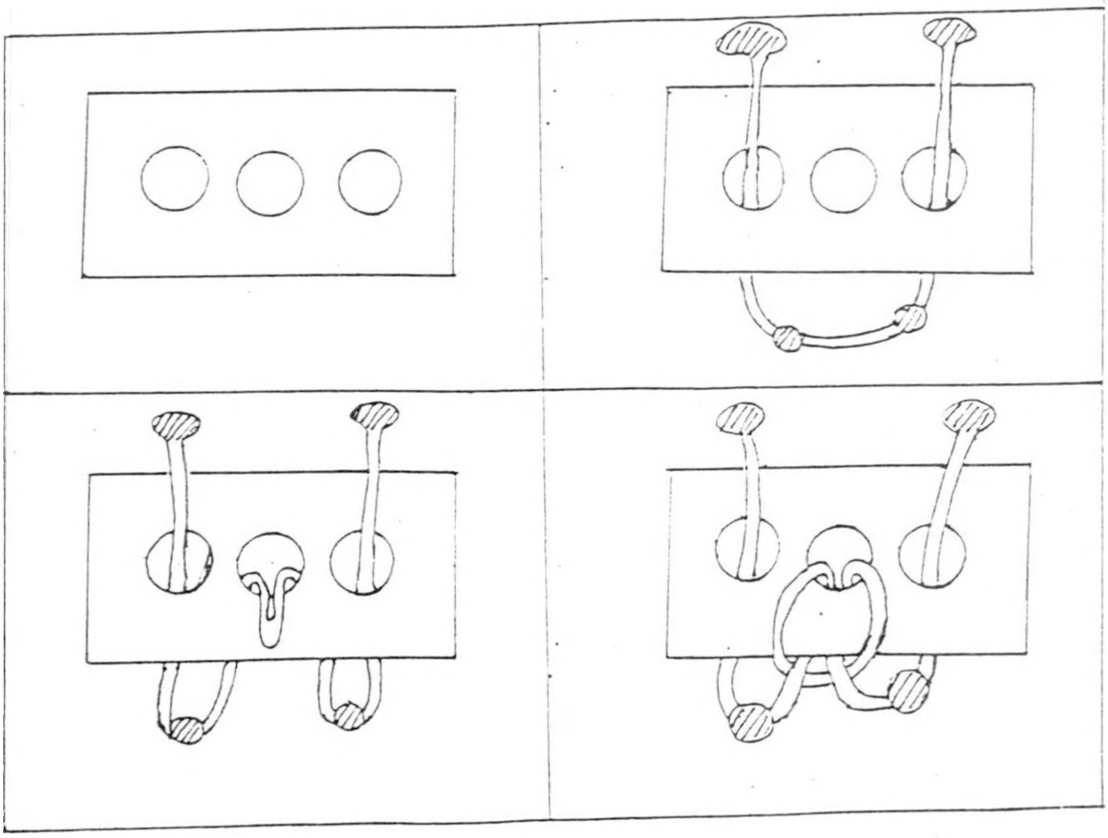
Reference: Indian Mathematics and Astronomy by S. Balachandra Rao.

Note : The above problem "Squaring a circle", i.e. to construct a square whose area is equal to the area of the given square using a scale and compass, had remained unsolved for centuries in the history of Mathematics. But ancient Indian Mathematicians solved the above problem in "Sulva Sutras". The above method of construction is given by "Srinivasa Ramanujan".

22. Buttons and Beads Puzzle

Objective: To improve the mental ability of students
Needed : Cardboard, string, two buttons and two beeds

How to prepare it :



Insert the string through the two beads and insert one end of the string, through hole A and attach a button larger than the hole. In the same direction, thread the other end of the string through hole C and attach a button as in figure 2.

How to use it

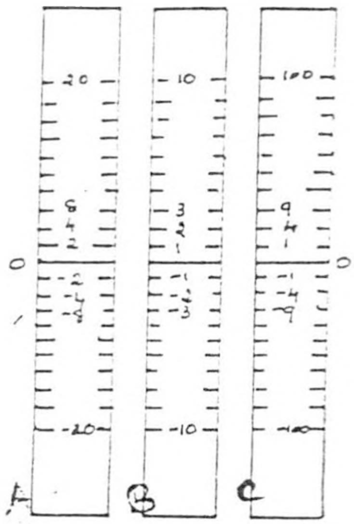
The string is looped through hole B, as in figure 3. Now to loop it back under itself as in figure 4, the loop is first threaded up in hole A and cover the button and then likewise in hole C. Now the puzzle is ready for someone to try to undo the loop and get the beads together.

23. Quadratic Equation Solver

Objective : To enable the user to solve quadratic equations.

How to use it :

The aid consists of three scales namely A, B, C of which the scale B can be moved.



Suppose we have to solve the quadratic equation $x^2 + 6x + 8 = 0$.

Step 1 :	Move sliding scale B so that 0 (zero) on it coincides with 6 (six) (coefficient of x) on scale A.
Step 2:	Note corresponding reading on scale C which is 9 (nine).
Step 3:	Subtract 8 (eight) (constant term or equation) from 9 (nine). The result is 1 (one).
Step 4:	On scale C search the position of number 1 (one). There will be 2 (two) positions on scale C where you find 1 (one).
Step 5 :	Note the corresponding two readings on scale B. They are – 2 and –4. Hence, -2 and –4 are the solutions of the equation $x^2 + 6x + 8 = 0$.

Reference: Teaching of Mathematics by S K Aggarwal.

24. Four Colour Theorem

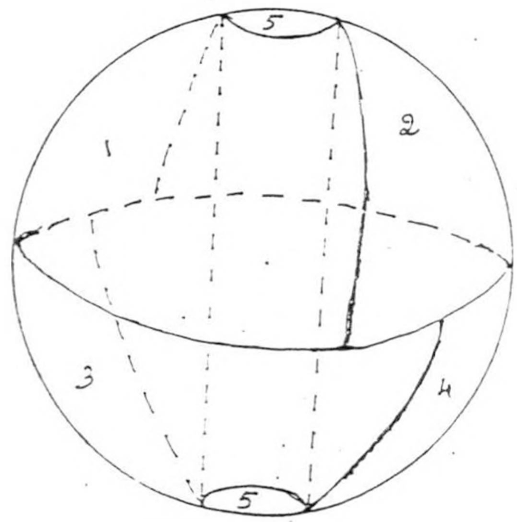
Objective : This is a model to show that the four colour theorem fails to hold in a 3-dimension object.

Analysis :

There is a celebrated theorem called 'four colour theorem' which states that four colours are sufficient to colour any map in the plane in such a way that the neighbouring states do not get the same colour.

Model

A cylindrical hole is constructed in the center of a sphere (football). A horizontal line is drawn around the sphere to make it into two semi-spheres. The horizontal line is connected to the two poles in four different places as shown in the figure. Now the sphere has five regions each having a common boundary with all the remaining four regions.



Therefore, this model requires five colours.

1. Does it disprove the four colour theorem ? (If not, why?).
2. Can you produce a map in the plane, which actually requires four colours ?

25. Euler's Formula V + F = E + 2

Objective: To show that the Euler's formula 'Vertices + Faces = Edges + 2' is satisfied by al the convex polyhedra.

Teaching Aid:

This teaching aid consists of a vertical stand in which all the five regular polyhedra (tetrahedron, hexahedron, octahedron, dodecahedron and icosahedron) made from thermacol and fixed. There are some other convex polyhedra also.

Procedure

The children will have to count the number of vertices, faces and edge of each one of these objects and make a table to find the relation between them.

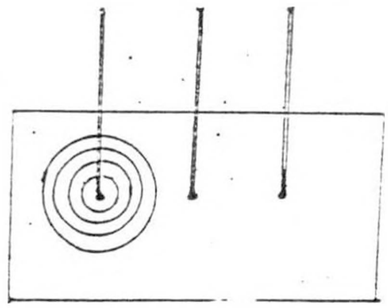
	Name	No. of Faces	No. of Edges	No. of Vertices	V + F	E + 2
1.	Tetrahedron	4	6	4	4 + 4	6 + 2
2.	Hexahedron	6	12	8	8 + 6	12 + 2
3.	Octahedron	8	-	-	-	-
4.	Dodecahedron	12	-	-	-	-
5.	Icosahedron	20	30	12	32	32

Can you produce a polyhedra which does not satisfy the Euler's formula ?

26. Tower of Hanoi

Objective : It is a puzzle called Tower of Hanoi for high school students to develop the inductive reasoning.

Puzzle :



Three vertical rods are fixed on a metallic plate. On one end of the rods, five discs of different sizes have been inserted, the largest disc being at the bottom, in the decreasing order of size.

You will have to put all the discs on any other rod, replacing one at a time and not placing a larger disc on a smaller disc. How many trials are needed to replace all the five discs ?

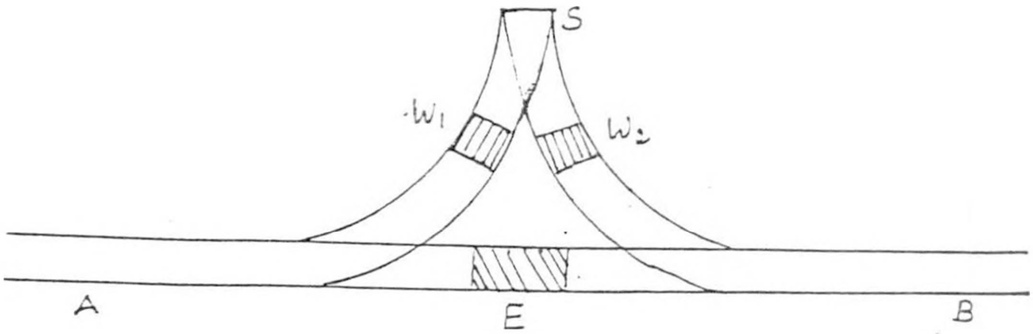
How to do

Children can try by taking two discs first. They will see that the number of trials needed are3. They can repeat this experiment by increasing the number of discs.

At last they can see that if the number of discs are n, then the number of trials required to replace them is $2^n - 1$.

27. Interchanging the Railway Wagons

Problem :



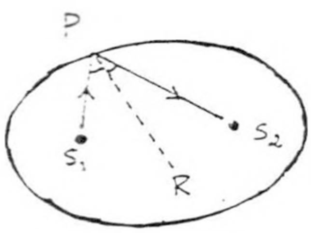
There is a railway line along AB and a slanting line S is connected to AB as shown in the figure. The length of the shunting place S will be sufficient for the wagons W_1 and W_2, but will not be sufficient for the engine E to move. Using the engine E, interchange the positions of the wagons W_1 and W_2.

The children can find the answer to this questions, themselves, by moving wagons W_1 and W_2 through the engine E to different directions. Repeated trials will help to improve their thinking and reasoning powers.

28. Elliptic Carrom Board

Objective: To enable the students to experience a geometrical property of ellipse.

Carrom Board :



An elliptic carom board is prepared, in which the two foci S_1 and S_2 are marked. Keep one carom coin each at S_1 and S_2. The coin at S_1 is pushed to hit any side of the wall of the board. After hitting the wall, the coin S_1 will hit the coin at S_2 and throw it away.

Reason

The perpendicular PR to the wall of the ellipse at any point P divides the angle S_1 PS_2 equally. Hence the angle of incidence S_1PR and the angle of reflexion RPS_2 are equal.

1. What happens if the point P is on the line S_1S_2 ?
2. What happens if the point P is on the perpendicular bisector of the line S_1S_2 ?

BUSINESS MATHEMATICS

Dr B S P Raju

Sinking Fund

Sinking fund is a kind of reserve by which a provision is made to

a) reduce a liability i.e. redemption of debentures or repayment of loan,
b) replace depreciating assets,
c) renew a lease,
d) replace wasting assets i.e. mines.

Let the amount of debt be A; E be the installment amount to credit to sinking fund and 'r' be the interest rate per annum in decimal form that accrues to sinking fund.

Let us consider the case for 3 years.

At the end of the first year, the amount in S.F. (Sinking Fund) is Rs.E.

At the end of II year, this becomes $E(1 + r)$ rupees. (by compound interest formula).

At the beginning of III year, he adds another E rupees so the amount in S.F. is $E(1 + r) + E$.

At the end of III year, this becomes

$$\{ E(1 + r) + E \} \{ 1 + r \}$$

At the beginning of the IV year again he adds E Rupees.

Hence S.F. = $[\{ E(1 + r) + E \} \{ 1 + r \}] + E$.

But this is equal to A.

i.e. $E(1 + r)^2 + E(1 + r) + E = A$.

$E \{ 1 + (1 + r) + (1 + r)^2 = A$.

$$\therefore \; E = \frac{A}{1 + (1 + r) + (1 + r)^2}$$

But $1 + (1 + r) + (1 + r)^2$

$$= 1 \left\{ \frac{(1 + r)^3 - 1}{1 + r - 1} \right\}$$

$$= 1 \left\{ \frac{(1+r)^3 - 1}{r} \right\}$$

$$\therefore \; E = \frac{Ar}{(1+r)^3 - 1}$$

In general for n years,

$$E = \frac{A}{1 + (1+r) + (1+r)^2 + (1+r)^3 + \ldots + (1 + r^n)}$$

$$= \frac{A}{1 \left\{ \frac{(1+r)^n - 1}{1 + r - 1} \right\}} = \frac{A}{\left(\frac{(1+r)^n - 1}{r} \right)}$$

$$= \frac{A\,r}{(1+r)^n - 1}$$

Problem :

A mortgage of Rs.10,000/- is due in 5 years. It calls for interest payments of 8% payable annually to the creditor. What is the annual payment ? The debtor decides to make equal payments at the end of each year for 5 years into a sinking fund investment that earns 4% compounded annually, to accumulate Rs.10,000/- in 5 years. What is the annual payment to the sinking fund, construct a sinking fund schedule.

Interest $\dfrac{10{,}000 \times 1 \times 8}{100} = 800/-$ payable annually.

Installment for payment to sinking fund = $E = \dfrac{Ar}{(1+r)^n - 1}$

$$= \frac{10000 \times 0.04}{(1 + 0.04)^5 - 1} = \frac{400}{0.2166528}$$

$$= 1846.272$$

Period	Interest at 4%	Payment to Sinking Fund	Increase in S.F. Col. 2 + 3	Amount in S.F.	Book Value of Debt
0	--	--	--	--	10,000
1	0	1846.272	1846.272	1846.272	8,153.728
2	73.85088	1846.272	1846.272 + 73.85088 = 1920.1228	1846.272 + 1920.1228 = 3766.3948	6233.606
3	$3766.3948 \times \frac{4}{100} =$ 15065579	1846.272	1846.272 + 150.65 = 1996.9277	5763.3225	4236.678
4	230.5329	1846.272	2076.8049	7840.1274	2159.873
5	313.60509	1846.272	2159.877	10000	0000

1. In order to purchase new carpeting and furniture, the Healys decided to deposit Rs.50/- in a S.B. account at the end of each month for 2 years. How much will they have available at that time, if the interest rate is 5% compounded monthly.

Problems on Partnership

X starts a business on 1st January 1987 with Rs.5000/-.
Y joins on 1st May 1987 with Rs.10,000/-.
On 1st July, Z comes in as a partner with Rs.15,000/-.
And on the same date, X contributes Rs.5000/- and Y contributes Rs.10,000/- as further capital.
The profits for the year ended 31st December 1987 amounted to Rs.16,000/-. The partners agree to share the profits in proportion of their capitals. Find their profits.

X : 5000 for 12 months $5000 \times 12 = 60{,}000$
5000 for 6 months $5000 \times 6 = \underline{30{,}000}$
$\underline{90{,}000}$

Y : 10,000 for 8 months $10000 \times 8 = 80{,}000$
10,000 for 6 months $10000 \times 6 = \underline{60{,}000}$
$\underline{1{,}40{,}000}$

Z : 15,000 for 6 months $15000 \times 6 = 90{,}000$

Their profits should be 90 : 140 : 90 i.e. 9 : 14 : 9

$\therefore$ X's profit is $16{,}000 \times \frac{9}{32} = 4{,}500/-$

Y's profit is $16{,}000 \times \frac{14}{32} = 7{,}000/-$

Z's profit is $16{,}000 \times \frac{9}{32} = 4{,}500/-$

Admission of a Partner

1. Change in the profit sharing ratio.

Ex: If A, B and C are partners sharing in the ratio 6 : 5 : 3 and later they admit D for $\frac{1}{8}$ share. What is the new and sacrificing ratio ?

Solution : Old ratio is 6 : 5 : 3

D's ratio is $\frac{1}{8}$ (given).

A's, B's and C's combined share in the new firm $= 1 - \frac{1}{8} = \frac{7}{8}$.

A will get $\frac{6}{14}$th of the remaining $\frac{6}{14} \times \frac{7}{8} = \frac{6}{16}$.

B will get $\frac{5}{14}$ of the remaining $\frac{5}{14} \times \frac{7}{8} = \frac{5}{16}$.

C will get $\frac{3}{14}$ of the remaining $\frac{3}{14} \times \frac{7}{8} = \frac{3}{16}$.

New profit sharing ratio $\frac{6}{16} : \frac{5}{16} : \frac{3}{16} : \frac{2}{16}$ i.e. 6 : 5 : 3 : 2.

Sacrificing ratio of A is $\frac{6}{14} - \frac{6}{16} = \frac{48 - 42}{112} = \frac{6}{112}$.

Sacrificing ratio of B is $\frac{5}{14} - \frac{5}{16} = \frac{5}{112}$

Sacrificing ratio of C is $\frac{3}{14} - \frac{3}{16} = \frac{24 - 21}{112} = \frac{3}{112}$.

$\therefore$ Sacrificing ratio $= \frac{6}{112} : \frac{5}{112} : \frac{3}{112} = 6 : 5 : 3$.

Goodwill

Goodwill is the attracting force, which attracts the customers towards products of the firm. It is the value of customer's confidence in the business. It is an intangible and invisible asset.

Goodwill = Actual profit earned – Normal profit.
Goodwill = Certain number of times the average profit.

Ex : A and B are equal partners in a firm. Their capitals show credit balances of Rs.18000/- and Rs.12000/- respectively. A new partner C is admitted with $1/5^{th}$ share in the profits. He brings Rs.14000/- for his capital. Find the value of goodwill of the firm at the time of C's admission.

Solution : For $1/5^{th}$ of share C contributes Rs.14000/- (given).
Full capital of the new firm = $14000 \times 5 = 70{,}000$/-.
But combined total capital of the three partners = $18000 + 12000 + 14000 = 44000$.
$\therefore$ Total value of firm's goodwill = $70000 - 44000 = 26000$.

Adjustment of Capital

Ex : A, B and C have been sharing their profit and loss in the ratio of 6: 5 : 3. They admit D to a $1/8^{th}$ share. D brings Rs.16000/- for his share of capital. All the partners decide to make the balance of their capital accounts in the profit sharing ratio, calculate their capital.

Solution : Combined share of A, B and C in the new firm = $1 - \frac{1}{8} = \frac{7}{8}$.

A's new share $\frac{6}{14}$ of $\frac{7}{8} = \frac{6}{14} \times \frac{7}{8} = \frac{6}{16}$.

B's new share $= \frac{5}{14} \times \frac{7}{8} = \frac{5}{16}$.

C's new share $= \frac{3}{14} \times \frac{7}{8} = \frac{3}{16}$.

$\therefore$ New profit sharing ratio among A, B, C and D is $\frac{6}{16} : \frac{5}{16} : \frac{3}{16} : \frac{2}{16} = 6:5:3:2$.

For $1/8^{th}$ share, the new partner D brings Rs.16,000.

$\therefore$ Total capital of new firm will be $8 \times 16{,}000 = 1{,}28{,}000$/-.

$\therefore$ A's capital in new firm = $1{,}28{,}000 \times \frac{6}{16} = 48{,}000$/-.

B's capital in new firm = $1{,}28{,}000 \times \frac{5}{16} = 40{,}000$/-.

C's capital in new firm = $1{,}28{,}000 \times \frac{3}{16} = 24{,}000$/-.

D's capital in new firm = $1,28,000 \times \frac{2}{16} = 16,000$/-.

On the Retirement or Death of a Partner

Ex : If A, B, C and D are partners sharing in the ratio of 6 : 5 : 3 : 2. D retires from the firm. Calculate the new ratio after D's retirement.

Combined share of A, B and C (after excluding D).

$$= 1 - \frac{2}{16} = \frac{14}{16}$$

A's share out of $\frac{14}{16}$ is $\frac{6}{16}$.

$\therefore$ A's share out of 1 is $\dfrac{\frac{6}{16}}{\frac{14}{16}} = \frac{6}{16} \times \frac{16}{14} = \frac{6}{14}$

|||ly B's share is $\frac{5}{14}$.

C's share is $\frac{3}{14}$.

$\therefore$ New ratio is $\frac{6}{14} : \frac{5}{14} : \frac{3}{14}$.

Gain in ratio :

A's gain = New share – old share

$$= \frac{6}{14} - \frac{6}{16} = \frac{6}{112}.$$

|||ly B's gain = $\frac{5}{112}$.

C's gain = $\frac{3}{112}$.

Bills of Exchange

Definition :

A bill of exchange is an instrument, an unconditional order, signed by the maker, directing a certain person to pay a certain sum of money only to or to the order of a certain person or to the bearer of the instrument.

Discounting of the bill :

The drawer may wait for the entire period of the bill to receive its payment. If he is in the immediate need of funds, he can get the bill discounted with the bank. The drawer transfers the possession and also the ownership of the bill. The bank charges certain interest, here known as discount for the period it has advanced the amount. On due date, the bank will present the bill to the drawer and receive the payment.

Discount is always charged for a period between the date of discounting and due date.

Ex : A draws a bill on B for Rs.3,000/- on January 1, 1994 payable after 3 months. The bill is discounted by A, as he is in the immediate need of funds. Calculate the discount in the following cases :

a) The bill has been discounted at 12% on January 4.
b) The bill has been discounted at 12% on February 4.
c) The bill has been discounted at 12 % on March 4.

a) Discount $= 3000 \times \frac{12}{100} \times \frac{3}{12} = 90.$

b) Discount $= 3000 \times \frac{12}{100} \times \frac{2}{12} = 60.$

c) Discount $= 3000 \times \frac{12}{100} \times \frac{1}{12} = 30.$

Retiring a bill under rebate :

Payment of the bill is generally made after the expiry of the specified period. The drawee may make the payment of the bill even before the date of maturity of the bill. In case of receiving payment of the bill even before the due date of the bill, the drawer allows certain discount, here known as "rebate" as a customary trade practice.

Ex : Ansar accepts a bill drawn by Azar for Rs.8000/- on March 15, 1999 payable after 4 months. According to the trade practice in the industry cash rebate at 6% p.a. is allowed. Calculate the amount of rebate in the following cases.

a) Ansar makes payment on April 18, 1999.
b) Ansar makes payment on May 18, 1999.
c) Ansar makes payment on June 18, 1999.

Solution :

a) Rebate $8000 \times \frac{6}{100} \times \frac{3}{12}$ = Rs.120/-

b) Rebate $8000 \times \frac{6}{100} \times \frac{2}{12}$ = Rs.80/-

c) Rebate $8000 \times \frac{6}{100} \times \frac{1}{12}$ = Rs.40/-.

Depreciation :

Depreciation means a fall in the quality, quantity or value of an asset.

I. Factors that cause depreciation

1. Wear and tear due to actual use.
2. Efflux of time – mere passage of time will cause a fall in the value of an asset even if it is not used. Ex. A patent right acquired on lease for 10 years loses 1/10 of its value for every year, even if it is not actually used.
3. Obsolescence – a new invention or a permanent change in demand may render the asset useless.
4. Accidents – when a fixed asset is damaged by an accident, naturally it loses its value.

Except a few cases like land and paintings, all assets depreciate.

Generally, depreciation is used only in respect of fixed assets (are those that are not meant to be sold but are meant to be utilized in the firm's business). Ex. Machinery, Patents, Buildings and goodwill.

II. Need for providing depreciation

1. To assess the profit correctly. Cost of the fixed asset used up in the period should be treated as cost or expense.
2. To estimate the value of the assets possessed by the firm.
3. The amount so kept out of profits for depreciation will be made available for the replacement of the asset when its life is over.

Factors for calculating depreciation :

1. The cost of the asset
2. The estimated residual scrap value at the end of its life.
3. The estimated number of years of its life. (Not the actual but the number of years it is likely to be used by the firm). A machinery may be capable of running for 30 years, but say, due to new inventions, it will be in use only for 10 years; then the estimated life is 10 years and not 30 years).

Methods of calculating depreciations :

a) *Straight line Method or Fixed Percentage on Original cost or Fixed Installment Method*

$$\frac{\text{Cost} - \text{Estimated Scrap Value}}{\text{Estimated Life}}$$

Note : In the case of companies, the scrap value is assumed to be 5% of the original cost of the asset.

This method is useful when the service rendered by the asset is uniform from year to year.

Example 1: A company purchased a lathe machine in the year 1995-96 at a cost of Rs.40,000/-. At the end of its estimated life of 10 years, it is expected to give Rs.5000/- when sold as scrap. Calculate the annual depreciation value.

$$\frac{40{,}000 - 5{,}000}{10} = \frac{35{,}000}{10} = 3{,}500/- \text{ per year}$$

Example 2 : A firm purchased machinery for Rs.22,500/- on 1.1.1998 and spent for its installation Rs.2,500/-. Its life was estimated to be 4 years with a scrap value of Rs.5000/-. Calculate the amount of depreciation.

Purchase cost of the machinery	:	Rs.22,500/-
Installation charges (to be regarded as cost of the machinery)	:	Rs. 2,500/-
		Rs.25,000/-
Scrap vale of the machinery at the end of its life	:	Rs. 5,000/-
Depreciation of the machinery	:	Rs.20,000/-

Depreciation for each year is $\frac{20{,}000}{4}$ = Rs.5,000/-.

b) ***Written down value method :***

In this method, the percentage of deprecation is fixed, but it applies to the value at which the asset in the beginning of the year.

Example : At the rate of 10%, what is the amount of depreciation in the third year, if the cost of the machinery in the beginning is Rs.20,000/-.

Depreciation for the 1^{st} year $20{,}000 \times \frac{10}{100}$ = 2,000/-.

$\therefore$ Cost of the asset in the beginning of 2^{nd} year is $20,000 - 2,000 = 18,000$/-.

Depreciation for the 2^{nd} year $18,000 \times \frac{10}{100} =$ Rs.1,800/-.

$\therefore$ Cost of the asset in the beginning of 3^{rd} year is $18,000 - 1,800 = 16,200$/-.

$\therefore$ Depreciation for the 3^{rd} year $16,200 \times \frac{10}{100} =$ Rs.1,620/-.

Uses : Depreciation in earlier years will be heavy, but will be light as the asset gets old. Repairs on the other hand are light in the earlier years and heavy later.

The total of the two – depreciation and repairs – will be roughly constant.

c) ***Sum of the Digits Method :***

The amount of depreciation for each year is calculated by the formula :

$$\frac{\text{Remaining life of the asset (including the current year)}}{\text{Sum of all the digits of the life of the asset in years}} \times \text{cost of the asset}$$

Example : For an asset costing Rs.50,000/-, Life is estimated for 10 years.

What is the amount to be provided for depreciation in the first year and also in second year ?

Solution : Sum of all the digits of the life of the asset in years is
$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 55$.

$\therefore$ Amount of depreciation for the first year $= \frac{10}{55} \times 50,000 = 9091$.

$\therefore$ Amount of depreciation for 2^{nd} year $= \frac{9}{55} \times 50,000 = 8181$.

d) ***Depletion Method :*** This method is used in the case of mines, quarries, etc.

Depreciation is calculated per tonne of output.

Example : Cost of mine is Rs.20,00,000 and it is estimated that the total quantity of mineral in the mine is Rs.5,00,000 tonnes.

The depreciation per tonne of output is

$$\frac{20,00,000}{5,00,000} = \text{Rs.}4.$$

If the output for the first year is 40,000 times,
then, the depreciation is 40.000×4 = Rs.1.60,000/-.

If the output for 2^{nd} year is 60,000 tonnes.
then the depreciation is 60.000×4 = Rs. 2.40,000/-.

e) *Machine Hour Rate Method*

Effective life of machine may be 20.000 hours.

Example : An asset which costs Rs.45.000/- has a useful life of 24 years and a salvage value (Trade-in-value) of Rs.3000/-. What will be the depreciation expense for the first (1^{st}) year, the 10^{th} year and the 24^{th} year if the sum of year's digit method is used ?

Ans: Rs.3,360/-; Rs.2,100/-; Rs.140/-.

ANNUITY

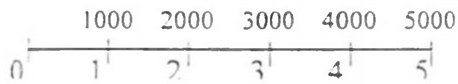
Meaning : An annuity is a series of equal periodic payments or deposits with the interest on each one being compound interest.

TYPES OF ANNUITY (by date of payment)

- ORDINARY ANNUITY : Payments are made at the end of the payment intervals.
- ANNUITY DUE : Payments are made at the beginning of the payment intervals.
- DEFERRED ANNUITY : Payments are made at the end of the payment intervals but do not start until after a designated period of time.

Example for Ordinary Annuity

Find the amount of an ordinary annuity of five deposits of Rs.1000/- each made at the end of each year for 5 years, if the interest rate is 4% compounded annually.



The first deposit of Rs.1000/- is for 4 years;
The second deposit of Rs.1000/- is for 3 years;
The third deposit of Rs.1000/- is for 2 years;
The fourth deposit of Rs.1000/- is for 1 year;
and fifth receives no interest.

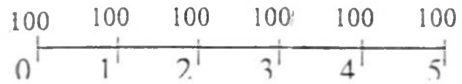
$\therefore$ $S = 1000\,(1.04)^4 + 1000\,(1.04^3 + 1000\,(1.04)^2 + 1000\,(1.04) + 1000$

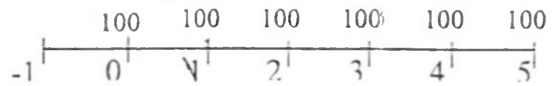
$= 1000\,(1.1698586) + 1000\,(1.1248640) + 1000(1.08160) + 1000(1.04)+1000$

$= 5416.32$

Example for Annuity Due

If a payment of Rs.100 today and a like payment at the end of each year for 5 years, how much will be on deposit at the end of 6 years, if the interest rate is 5% compounded annually.





The first deposit of Rs.100/- is for 5 years.

The second deposit of Rs.100/- is for 4 years.

The third deposit of Rs.100/- is for 3 years.

The fourth deposit of Rs.100/- is for 2 years.

The fifth deposit of Rs.100/- is for 1 year.

The sixth deposit of Rs.100/- is for 0 years.

Example for Deferred Annuity

I deposited a sum of Rs.5000/- in TISCO on 1st January, for secured premium notes. The company agreed to pay me back at the rate of Rs.2000/- every year for about 5 years from the beginning of 1st January 1996.

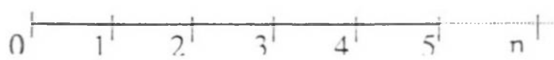
FORMULA TO FIND THE AMOUNT OF ORDINARY ANNUITY

Let P stands for the Principal;

r for interest rate per annum expressed in decimal form

n for number of years the money is left in deposit, and

A for amount or principal plus interest.



The first instalment paid at the end of 1st year will be in deposit for (n-1) years. By using the compound interest formula this amounts to $P(1+r)^{n-1}$.

$|||^{ly}$ the second instalment paid at the end of 2^{nd} year amounts to $P(1 + r)^{n-2}$.

$$\therefore \; A = P(1+r)^{n-1} + P)1+r)^{n-2} + \ldots.. + P(1+r) + P$$

$$= P \{ (1+r)^{n-1} + (1+r)^{n-2} + \ldots. + (1+r) + 1 \}$$

$$= P \{ 1 + (1 + r) + (1 + r)^2 + \ldots\ldots + (1 + r)^{n-2} + (1 + r)^{n-1} \}$$

(writing in reverse order)

$$= P \left\{ \frac{1 \{ (1+r)^n - 1 \}}{1 + r - 1} \right\} \quad = P \left\{ \frac{(1 + r)^n - 1}{r} \right\}$$

(by using summation of G.P. formula)

For annuity done $A = P \left[\frac{(1+r)^{n-1} - 1}{r} \right] - P.$

PRESENT VALUE OF AN ANNUITY

The inverse of finding the amount of an annuity is finding the present value of the annuity.

Present value of an annuity is the amount of money to be deposited in the beginning so that one can withdraw a fixed amount of money at the end of each year for n-years, at which time the original investment and the interest (earned compoundly) exhausted completely.

Example : At age 21 Ram receives an inheritance of 20 equal annual payments of Rs.2000/- each, the first payment coming due at age 22. If money is worth 4% compounded annually, what is the cash inheritance at age 21 ?

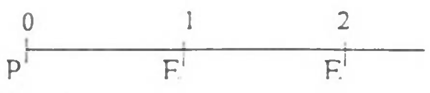
The cash inheritance is the present value of annuity.

FORMULA TO FIND THE PAYMENT VALUE OF ANNUITY

Let p is the present value, it is same as Principal,

r is the interest rate per annum expressed in decimal form,

E is the amount that can be withdrawn at the end of every year.



Consider the case for two years only.

At the end of the 1^{st} year, the amount becomes

$$p\,(1 + r)$$

But $\because$ E rupees is withdrawn, the principal at the beginning of the second year is $p(1 + r) - E$.

So by the end of 2^{nd} year, this becomes $\{p\,(1 + r) - E\}\,\{1 + r\}$

But this is equal to E.

$$\therefore\ \{\,p\,(1 + r) - E\,\}\,\{\,1 + r\} = E$$

$$\text{i.e. } p\,(1 + r) - E = E\,(1 + r)^{-1}$$

$$p(1 + r) = E\,(1 + r)^{-1} + E$$

$$= E\,\{\,1 + r)^{-1} + 1\,\}$$

$$\therefore\ P = \{\,(1 + r)^{-2} + (1 + r)^{-1}\,\}$$

Consider the case for 3 years.

At the end of 1^{st} year, the amount becomes $p(1 + r)$.

$\because$ E rupees is withdrawn, the principal becomes

$$p(1 + r) - E.$$

By the end of 2^{nd} year this becomes

$$\{\,p\,(1 + r) - E\,\}\,\{\,1 + r\,\}$$

$\because$ E rupees is again withdrawn, the principal becomes

$$[\{\,p\,(1 + r) - E\,\}\,\{\,1 + r\,\}\,] - E$$

By the end of 3^{rd} year, this becomes

$$[\,\{\,p\,(1 + r) - E\,\}\,\{\,1 + r\,\} - E\,]\,[\,1 + r\,]$$

But this is equal to E.

$$\therefore\ [\{\,p\,(1 + r) - E\,\}\,\{1 + r\}] - E\,[1 + r] = E$$

$$\Rightarrow\ [\,\{\,p\,(1 + r) - E\,\}\,\{\,1 + r\,\} - E\,] = E(1 + r)^{-1}$$

$$\Rightarrow\ [\,\{\,p\,(1 + r) - E\,\}\,\{\,1 + r\,\}\,] = E\,(1 + r)^{-1} + E$$

$$\Rightarrow\ p\,(1+r) - E = E\,(1+r)^{-2} + E\,(1+r)^{-1}$$

$$\Rightarrow\ p\,(1+r) = E\,(1+r)^{-2} + E\,(1+r)^{-1} + E$$

$$\Rightarrow\ p = E\,\{\,(1+r)^{-3} + (1+r)^{-2} + (1+r)^{-1}\,\}$$

Similarly for n years, we can derive

$$p = E\,\{\,(1+r)^{-1} + (1+r)^{-2} + (1+r)^{-3} + \ldots.. + (1+r)^{-n}\,\}$$

The expression within the brackets is sum of n terms of a G.P. with

$a = (1+r)^{-1}$ and $r = (1+r)^{-1}$

$$\therefore\quad p = E\left\{(1+r)^{-1}\left[\frac{1-\{(1+r)^{-}\}^{n}}{1-(1+r)^{-1}}\right]\right\}$$

$$= E\left\{(1+r)^{-1}\left[\frac{1-(1+r)^{-n}}{1-\dfrac{1}{1+r}}\right]\right\}$$

$$= E\left\{(1+r)^{-1}\left[\frac{1-(1+r)^{-n}}{\dfrac{1+r-1}{1+r}}\right]\right\}$$

$$= E\left\{(1+r)^{-1}\,(1+r)\,\frac{[\,1-(1+r)^{-n}\,]}{r}\right\}$$

$$p = E\left(\frac{1-(1+r)^{-n}}{r}\right)$$

USE OF VENN DIAGRAMS IN TEACHING-LEARNING OF MATHEMATICS

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USE OF VENN DIAGRAMS IN TEACHING-LEARNING OF MATHEMATICS

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ABSTRACT

In this article an attempt has been made to discuss the effectiveness of Venn Diagrammatic Representation Approach (VDRA) in Teaching-Learning of Mathematics

Introduction

Mathematics plays a central role in Science and Technology. The *Numbers*, which are so fundamental to Mathematics, encompass all disciplines of study and all walks of life. The Pythagoreans (around BC 540) literally worshipped the natural numbers and they believed that the entire Universe was made up of these numbers. Mahavira(AD 850) stressed the importance of *Ganita* (Mathematics) in all the three Worlds. But this subject of supreme importance also seems to pose almost insurmountable difficulties for the great majority of the students.

Is Mathematics difficult?

Several studies have observed that many of the students at school level find mathematics a difficult subject and large number of students fail in this subject. Mathematics is difficult not because of abstraction, as has been generally perceived, it is because of precision. Mathematics is difficult because, unlike any other discipline, it demands complete precision (King, 1992)[4]. One of the vast areas of the world of contemplative beauty is mathematics and this alone is sufficient reason for the study of mathematics. However, there is one comment that the inadequacies in teaching of mathematics have created the gap between the scientific community and the rest of the humanity, and thereby hamper the growth of our society. Thus there is a need for overcoming the inadequacies in teaching of

Mathematics. Any model of teaching of mathematics should ensure that mathematics is taught the way mathematics is and mathematics is learnt the way mathematics is.

Teaching is a great art

Taking an active role in the learning process of the child is one of the greatest joys of teaching. Teachers have natural curiosity to observe how children grow and discover the world around them. Teachers feel that they want to be there, to help and play an important role in facilitating their learning. This is a tremendous job. One of the basic teaching functions is to "check for understanding"(Rosenshine, 1983)[6]. Similarly " assess student comprehension " (Good & Grouws, 1979)[2] as one of their instructional behaviours for effective mathematics teaching. Shavelson (1979)[7] argued that it is important for the teacher to estimate the " states of mind" of their students and that these estimates provide essential information for deciding what and how to teach.

The way mathematics is being taught is going through a dramatic change. The introduction to the study of "*Numbers and numerations*" starts when the children begin to gather objects and then form groups or sets. Soon they need a way to describe the numbers of objects in the sets. To begin with, these descriptions are verbal - they count by saying the names of numbers as objects are singled out. Symbols for numbers are eventually introduced and then they learn to write numbers. It is most likely that children learn maths today by beginning with real-life problem or situation that needs to be solved. They are given freedom to use techniques that might be uniquely theirs. The premise is that children as well as teachers are most likely to remember the things that they grapple with and resolve. To-day Maths curriculum is so different from what we were taught. What was emphasised then is not emphasised now. New names have been given to old procedures, and some procedures that took forever to master are no longer taught because a calculator can provide the answer at the push of a button.

The mathematics teacher while developing the problem solving ability among students, often follows two basic steps (Hayes,1981)[3]: *problem representation* and *problem solving*. In the problem representation, a problem is converted from a series of words and numbers into an internal mental representation of the relevant terms. In problem solutions, operations are performed so as to deduce a solution to the problem from the internal mental representation.

Bertrand Russell (1917)[1] rightly points out "Mathematics is the study of assertions of the form 'p implies q', where p and q are each statements about objects that live in mathematical world". Thus mathematics is a study of Sets where a set is identified with a '*precise property*', a property that is true or false, not both. But strangely the *Sets* do not find the right kind of importance in the teaching learning process of mathematics in school education. In the process of making mathematics more functional, the most fundamental element of mathematics -"*Set*" is not used properly as it should have been.

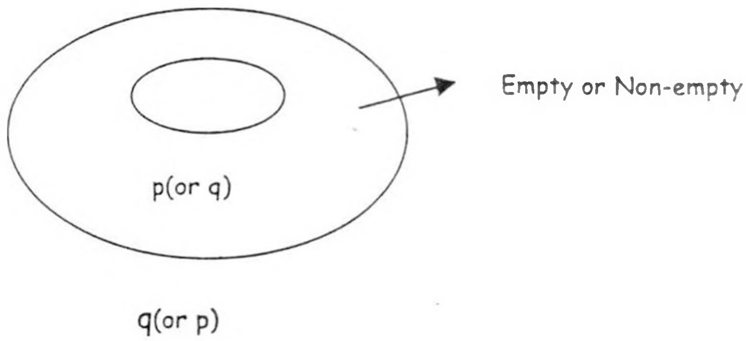
With these in mind, an attempt was made to study the Role of Venn Diagrams in Teaching and Learning of Mathematics

Venn Diagram and its importance

John Venn[6] introduced Venn diagrams in 1880. A Venn diagram represents pictorially interrelations among sets (well defined properties) each of which is denoted by a closed region without holes. Though there are other diagrams like line diagram, directed graph, etc. to illustrate relationships, Venn diagram has an advantage of *space* over the others.

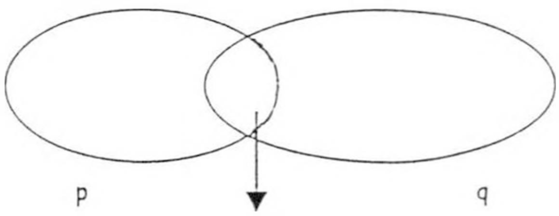
Given two well defined properties p,q, the possible relations between them can be represented by Venn diagram in one of the following ways.

a)



' $p \Rightarrow q$ ' or ' p is q ' or ' p is part of q '

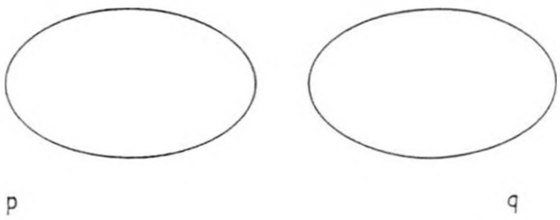
b)



P and q (which in non-empty)

Some of p is q

c)



No p is q

Rationale

The author while doing mathematics, and while teaching mathematics at different levels (graduates and post-graduates) gained adequate experience of

Venn diagrams. This experience of more than two decades made him believe that Venn diagrams could best be employed in:

- *Better understanding of mathematics because of their visual effect;*
- *Providing clarity in teaching-learning of mathematics;*
- *Finding inter-relations among mathematical properties holistically and accurately, and better analysis of the properties;*
- *"Conjecturing" as a consequence of natural creation of some new portion in Venn diagram (For Example - See Annex -1).* This Venn diagram motivated the author[5] to conjecture C_1 and C_2.(See dotted lines in the diagram) and these conjectures are unsolved problems for more than two decades.

Keeping these observations in mind, the author tries to present the salient features of the two simple studies involving the *use of Venn Diagrammatic Approach (VDRA) in teaching -learning of Mathematics.*

Experiment-1

It was the purpose of the study to determine the direct effect of Venn Diagrammatic Representation Approach training on student teachers and to assess the transfer of training to them. The training focussed on different components involved in Venn diagrams - like encoding of information, inferring and mapping the relationships before applying them to specific problem situations establishing the emptiness or non-emptiness of a section of the Venn diagram. The author himself gave training during this period. Students were given frequent opportunities to practice and apply the related components in varied contexts.

The second session of the training programme was for the students to apply successfully the components involved in establishing the relationships. A typical exercise given was - " Draw and Justify Venn Diagram of the ......, ..., ...". At the end, the author asked students " Is Venn diagram useful in teaching learning of mathematics? If 'Yes', why?" A summary of 110 responses received is given below:

All agreed that the Venn diagrams are useful in teaching learning of mathematics. The reasons given by them are -

- *Venn diagrams are better tools for understanding because of their visual effect;*
- *Venn diagrams being pictorial representations convey accurate meaning what words cannot (A picture worth thousand words). The concepts explained through Venn diagrams are better retained for longer duration;*
- *Venn diagrams are self-explanatory; convey the meaning at a glance, have no language bias and easily understood by all types of students;*
- *Through Venn diagrams, abstract ideas are grasped easily and are kept at the concrete level of understanding;*
- *Relations between two or more concepts can be more easily and effectively learnt by Venn diagrams than by using symbols;*
- *Venn diagrams will reduce the verbal explanation of teachers and students and the school students feel happy to draw Venn diagrams;*
- *Venn diagrams help students to solve problems on sets more accurately and with better confidence and thereby reducing the mistakes/errors;*
- *Venn diagrams develop power of reasoning;*
- *With Venn diagrams, one can simultaneously inter-relate many properties;*
- *One can get new results from the basic elements of Venn diagram.*

Experiment -2

The purpose of the experiment-2 was to investigate the questions raised in the experiment-1. Specifically, the investigation was designed to address the issues of influences of age and ability on direct effect of training. The training was administered in two sessions, focussing on the component processes involved in Venn diagram.

Lessons (related to Maths and life - See examples in Annex-2) were taught to three different terminal groups(Grades V, X and XII) separately by author spending one hour per day for two days in case of X and XII and 40 minutes per day for three days for V Standard. V and X grade students did not have any idea of

Venn diagrams earlier and the lessons to them included a brief introduction on Venn diagrams.

Sample

The sample for the studies is students of Regional Institutes of Education, Bhopal and Mysore and Demonstration Multipurpose School (DMS) attached to RIE,Mysore. The samples particularly for Experiment -2 consisted of 33 fifth Grade students, 48 tenth Grade students and 21 twelfth Grade students from DMS and these students were new entrants. The data for the analysis come from the pre-test, post-test information and other information gathered during the intervention programme. These data were collected on separate occasions. The students age ranges from 10 years (Grade- V) to 20 years (undergraduates).

Result

Though the author did not carry out a rigorous research, still felt it is worth sharing these with his fellow professionals. Some of the experiences he had during the processes and different stages of experiments have been very briefly given.

Grade	Pre-test		Post-test		t	df	p
	Mean	SD	Mean	SD			
V N=33	2.61	1.68	5.81	2.44	7.62	32	0.00
X N=48	3.40	2.31	5.71	3.48	5.32	47	0.00
XII N=21	8.24	4.67	16.61	4.96	5.17	20	0.00

The t- values in the above table clearly show that significant difference between the pre-test and post-test scores implying that VDRA has definite influence on the performance of the students at all levels.

These data suggest that VDRA, in teaching of mathematics, offers a more positive view of students' learning than that through mere verbal statements.

Implications

For text-book writers

The findings of these studies give a clear direction for the textbook writers to include Venn Diagrammatic Representation Approach (VDRA) wherever they are appropriate and also to reduce the verbal statements by introducing Venn Diagrammatic Representations.

For Teachers

While transacting mathematics content, VDRA is quite helpful in making their ideas clear and also helps in vivid presentation of the content by reducing the verbal statements. It is necessary for them to focus on different components of VDRA (both separately and collectively) giving adequate opportunities for the students to make mathematical representations and think all plausible relationships while solving any of the mathematical problems.

For Students

The method help in better understanding and also encourages to go for higher order thinking processes like conjecturing, problem solving; decision- making and creativity.

For Evaluators

From the point of view of the evaluator, the study suggests that marking and scoring is much more objective and easier -because of clear visibility of sections of Venn diagram representing relationships.

Conclusion

Considering the vital role that Venn Diagrammatic Representation Approach (VDRA)plays in Mathematics learning, this approach can be effectively used in the classroom instruction.

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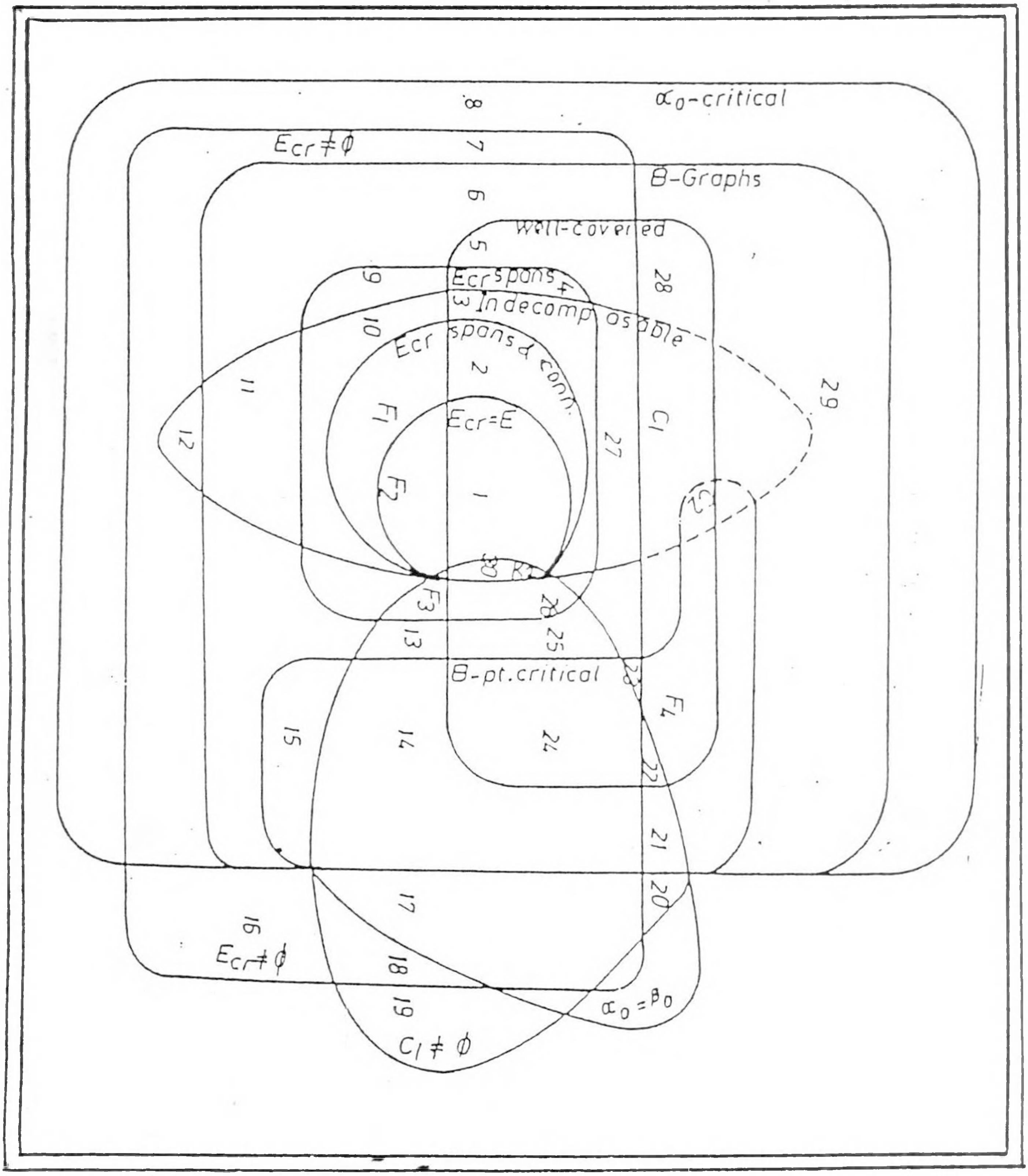

α_0-critical
$E_{cr} \neq \phi$
B-Graphs
well-covered
E_{cr} spans
Indecomposable
E_{cr} spans & conn.
$E_{cr}=E$
C_1
B-pt.critical
$E_{cr} \neq \phi$
$\alpha_0 = \beta_0$
$C_1 \neq \phi$
F_1
F_2
F_3
F_4

1. Examples of the content selected for VDRA:

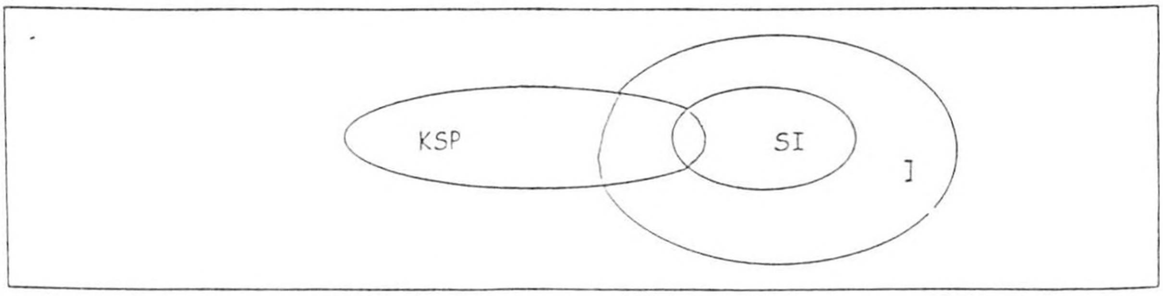
Find inter-relations (mathematical) among the following:

(a) South Indians, Indians and Kannada speaking people

(b) Similar triangles, congruent triangles, equilateral triangles, right angle triangles, quadrilaterals and Polygons

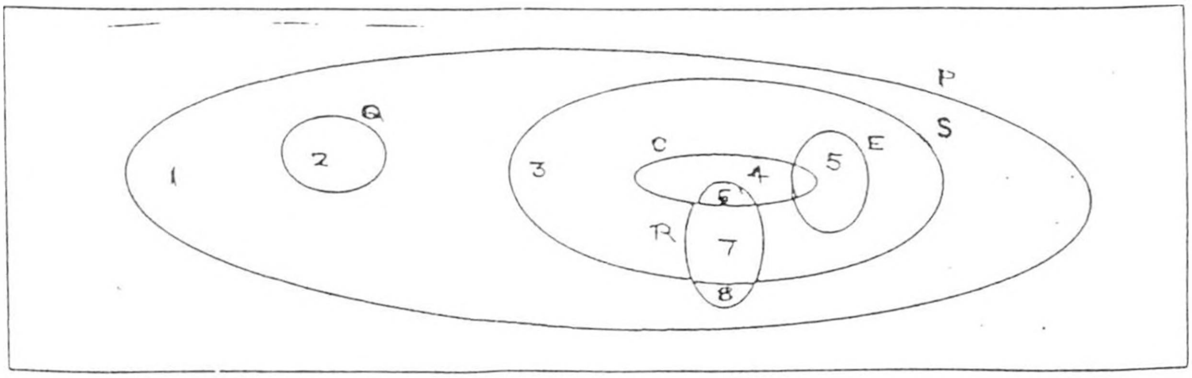
2. VDRA expectations for (a) and (b)

(a)



KSP - Kannada Speaking People   SI - South Indians   I - Indians

(b)

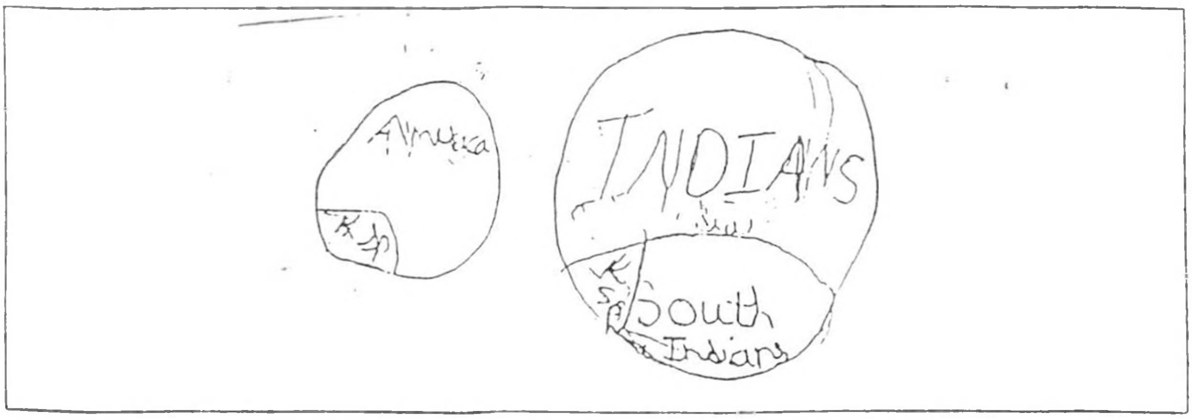


S - Similar triangles   C - Congruent triangles   E - Equilateral triangles   R- Right angle triangles
Q - Quadrilaterals   P - Polygons

Note: Provide, to complete justification, one example for spaces(Gaps) 1,2,3,4,5,6,7,8.

3. An example of creativity shown by the child of V grade during Experiment -2

KSP - Kannada Speaking People   SI - South Indians   I - Indians



APPENDIX 1

By Prof, Shamanna

Session

On

Managerial Skills for Teachers

Date : 20-06-2003

MAIN AREAS:

1. Doing Vs getting things done. The critical role of Teachers in improving quality of education.
2. Achieving Excellence is a function of sharpening Managerial skills.
3. Kaizen-Japanese Tool.
4. Planning - Coordinating - Communicating.
5. Learning as a collaborative process.
6. Team work.
7. Characteristics of Teachers.
8. Changing Role.

IMPROVING QUALITY OF EDUCATION

A. Role of a Teacher

1. Develop team spirit and enthusiasm for learning
2. Promoting discipline and commitment
3. Setting an example
4. Managing change - Change agent
5. Effectiveness and efficiency.

B. Characteristics of a Faculty Member

1. He is hard working and regular in his habits.
2. He has excellent character and gives importance to professional ehtics.
3. He has good knowledge of the subject
4. He is highly motivated and works with team spirit
5. He has genuine concern for his students and has good communication skills.
6. He has an open mind and is flexible in teaching methods in the class room
7. He gives importance to quality, excellence and promoters positive attitudes with students
8. He has good knowledge the problems of students and rules and regulations

DOING THINGS

GETTING THINGS DONE

1. COORDINATION
2. COMMUNICATION
3. PLANNING
4. INSPIRING
5. SOLVING PROBLEMS
6. TEAM WORK SKILLS

PROCESS OBSERVATION

GROUP DYNAMICS

Observers Task
(You are required to observe carefully and make N (TES on the following questions)

1. Observe the process of communication in the Group – What they talk is less important. How they talk is more important.
2. Make brief notes regarding your impression and observations you will have to share your impressions and provide honest feed back to them later.
3. Observed the non-verbal communications – Gestures, facial expression and the posture while they are discussing.
4. You may prepare yourself to answer briefly the questions listed below:
    a. How did the group make the decisions?
       By concensus?
       By majority?
    b. How did it solve the conflict
    c. Do you think the group was effective? Give reasons.
    d. Did the group members were listening to each other?
    e. Did the group have a leader?
    f. Who dominated the discussions?
    g. What suggestions you have to improve their performance? For specific individuals? For group?

Important :-

You will be invited to share your comments and give feed back to group members. Please give brief and frank comments with a view to help members improve their performance.

Notes

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Complete each of the following statements by circling the most appropriate choice.

1. Messages are the most easily understood when:
   (a) you use your full command of the language.
   (b) they are sent in terms the receiver understands.

2. Complex information is more easily understood when you:
   (a) improve clarity by using specific examples and analogies.
   (b) tell the listener to pay careful attention.

3. Key concepts are better remembered when you:
   (a) use repetition to reinforce them.
   (b) express yourself clearly.

4. Organizing a message before transmitting it:
   (a) often takes more time than it is worth.
   (b) makes it easier to understand.

5. The sender can determine the receiver's understanding by:
   (a) asking if he or she understands.
   (b) asking the receiver to report what he or she heard.

6. Listening is more effective when you:
   (a) concentrate on the sender and what is being said.
   (b) anticipate what the speaker is going to say.

7. Understanding is easier when you:
   a) suspend judgment until the sender finishes the message.
   b) assume you know the senders position and judge accordingly.

8. Understanding can be improved by the listener:
   (a) periodically paraphrasing the message back to the sender.
   (b) Interrupting to express feelings and emotions.

9. Good listeners:
   (a) have their response ready when the sender stops talking.
   (b) ask questions when they don't understand.

10. Sending and receiving are both enhanced when:
    (a) the parties maintain good eye contact.
    (b) the parties are defensive and challenge one another.

Encourage team members to review communications skills using this same exercise. Then compare notes and discuss how to improve. This will be another cooperative step in building a stronger team effort.

SEPARATE FACT FROM INFERENCE

Read the narration carefully which follows. Then see how well you can distinguish a FACT from an INFERENCE

Shama, a buyer with the XYZ company, was scheduled for a 10 O' clock meeting in Singh's office to discuss the terms of a large order On the way to that office, the buyer slipped on a freshly waxed floor and as a result received a badly bruised leg. By the time Singh was notified of the accident. Shama was on the way to the hospital for X-ray. Singh called the hospital to enquire but no one there seemed to know anything about Shama. It is possible that Singh called the wrong hospital.

Examine the statements below. Without discussion, put a tick mark against each statement, as to whether it is a fact or an inference (in the Personal Choice Column)

Statements	*Personal Choice*		*Group Choice*	
	Fact	*Inference*	*Fact*	*Inference*
1. Mr. Shama is a buyer				
2. Shama was supposed to meet with Singh.				
3. Shama was scheduled for a 10 O' clock meeting				
4. The accident occured at the XYZ company				
5. Shama was taken to the Hospital for X-Ray.				
6. No one at the hospital which Singh called knew anything about Shama.				
7. Sing had called the wrong hospital.				

Now discuss your personal choices with the group, and enter group choices in the appropriate column.

Appendix 2

A model lesson plan

By Prof. K. Dorasami

REGIONAL INSTITUTE OF EDUCATION, MYSORE 6

Class : IX
Subject: Mathematics
Course: Algebra
Unit : Matrices
Topic : Identity Matrix
Date :
Time :

Name of the Student Teacher:

Name of the Co-operating teacher:

Name of the Co-operating School :

INSTRUCTIONAL OBJECTIVE:

At the end of this lesson, a student will be able to

1. define an identity matrix
2. state the characteristics of an identity matrix
3. cite examples of identity matrix (or) identify identity matrices from the given matrices
4. relate identity matrix with other types of matrices
5. state the condition for a matrix to bd (or not to be) an identity matrix.

TEACHING POINTS

An identity matrix is a square matrix having principle diagonal elements as one and non diagonal elements as zero.

PREVIOUS KNOWLEDGE

1. A square matrix is a matrix with equal number of rows and columns.
2. Symmetric matrix is a sequence matrix whose transpose is the given matrix.

Expected Learning Outcomes	Sequential Learning Activities with inbuilt Evaluation	Black board work
	Introduction	
	1. T: (Surveys the whole class and seeks the class attention) In the last class we learnt about square matrix. What is a square matrix?.. S_5	
	2. S_5: Square matrix is a matrix in which rows and columns are equal.	
	3. T : S_5 said that if a matrix has equal rows and columns, then it is a square matrix. Can someone give a square matrix in which the rows and columns are not equal. (S_2 raises his hand) Yes S_2 ?	
Gives a counter example to show that the defining expression is not the definition of square matrix.	4. S_2: $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$
	5. T : Good. It is a square matrix. When is a matrix said to be a square matrix. S_5 can you?	
	6. S_5: If a matrix has same number of rows and columns then it is a square matrix.	
	7. T : Very good. Now we are going to learn about a special kind of square matrix called an Identity Matrix (writes the concept name on the board). This is an important idea as we will be making use of it later in our study on properties of matrix multiplication.	IDENTITY MATRIX

Expected Learning Outcomes | **Sequential Learning Activities with Inbuilt Evaluation** | **Blackboard Work**

Development of the concept

Let us consider the matrices (writes a set of matrices on the chalk board). The matrices labelled as I are identity matrices, others are square matrices that are not labelled as identity matrices.

What is it that identity matrices have in common that does not occur in matrices that are not identity matrices? Look at the elements. S_{10}?

Blackboard Work:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \quad D = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

$$E = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Expected Learning Outcome: Compares and contrasts the examples and non examples and identifies the characteristics of identity matrix.

8. S_{10}: The principal diagonal elements are one and other are zero.

9. T : Very good! The name identity matrix which we have been using for this group (indicates the identity matrices on chalk board) means a square matrix in which the diagonal elements are zero.
Now consider the matrix (writes a matrix with diagonal elements as three and non diagonal elements as zeros). Why is not this an identity matrix?....S_4?

Blackboard Work:

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Expected Learning Outcomes	Sequential Learning Activities with inbuilt Evaluation	Blackboard work
Gives reason (lack of necessary condition) for the square matrix as the non-example of identity matrix.	10. S_4: The diagonal elements are not one in this matrix. So it is not an identity matrix.	
	11. T: O.K. Are there matrices with diagonal elements as one which are not identity matrices? Why? (a number of pupils raise their hand) You S_4 ?	
Recognises non examples and gives reason for a square matrix to be a non example of identity matrix.	12. S_4: We can have square matrices with diagonal elements as one and non diagonal elements as non zero. Because a matrix to be identity matrix it must also have zero as the non diagonal elements.	
	13. T: Right. We have seen that there are two conditions necessary for a square matrix to be an identity matrix. (Writes them on the board).	1. If a square matrix does not have one as diagonal element, it is not an identity matrix. 2. If it does not have non diagonal elements as zero, it cannot be an identity matrix.

<u>Review and Evaluation</u>

What is an identity matrix?....S_{11}

What attributes will you find in all identity matrices ?

An identity matrix is a kind of __________ matrix ?

What similarities and differences do you find between identity matrix and symmetric matrix ?

Today we learnt that identity matrix is a kind of symmetric matrix (and represents the conceptual hierarchy among the concepts discussed so far).

Matrix

Rectangle Matrix | Matrix | Square Matrix

Row | Column | Symmetric Matrix

Identity Matrix

APPENDIX 3

MATHEMATICS FOR AESTHETIC REASONS

Compiled by

Prof. G. Ravindra

- The famous FOURS: Cognitive (Truth), Metaphysical (Reality), Ethical (Justice), Aesthetical (Beauty).
- "Beauty is truth, Truth is beauty" – that is all ye know on earth, and all ye need to know (John Keats).
- Mathematicians do mathematics for aesthetic reasons.
- One of the vastest areas of world of contemplative beauty is mathematics. This alone is sufficient reason for study of mathematics (King, 1992).
- Mathematics possesses not only truth but supreme beauty – a beauty cold and austere, like that of a sculpture without appeal to any part of weaker nature, sublimely pure, and capable of stern perfection such as only the greatest art can show (Bertrand Russel).
- Mathematicians know beauty when they see, it for that is what motivates them to do mathematics in the first place. And they know where to find truth.
- Despite an objectivity that has no parallel in world of art, the motivation and standards of creative mathematics are more like those of art than of

science. Aesthetic judgements transcend both logic and applicability in the rankings of mathematical theorems: beauty and elegance have more to do with the value of a mathematical idea than does either strict truth or possible utility (Lynn Steen, Ex-President of the Mathematical Association of America).

- To create consists precisely in not making useless combinations and in making those which are useful and which are only a small minority. Invention is discernment, choice . . . The useful combinations are precisely the most beautiful, I mean those best able to charm this special sensibility that all mathematicians know, best of which the profane are so ignorant as often to be tempted to smile at it. (Henri Poincare. The Foundation of Sgemu, 1929).
- Our present system of mathematics instruction which turns on the concept that mathematics is best presented through emphasis on its value as scientific tool. We can overselves do no harm by trying another approach by presenting to our students early on those characteristics of mathematics which in Poicare's words contain "this character of beauty and elegance and which are capable of developing in us a sort of aesthetic emotion".
- The ideas brought fourth from the Unconscious and handed over to the conscious invariably possess the stamp of mathematical beauty (Poincare).

APPENDIX 4

QUESTIONNAIRE TO TEST CREATIVITY IN TEACHING AND LEARNING

By
Dr. (Mrs.) Kalpana Venugopal
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RIE, Mysore

1. I give notes for exercises following the lessons.

   Always/Sometimes/Never

2. I encourage students to take a few topics in the syllabus as seminars/assignments.

   Always/Sometimes/Never

3. The home assignments are mostly the textbook exercises following the lesson.

   Always/Sometimes/Never

4. I try to employ multisensory/direct experience approach in teaching.

   Always/Sometimes/Never

5. I permit students to give their examples/anecdotes/explanation in the course of my teaching in the class.

   Always/Sometimes/Never

6. I do not like to waste time on discussing any issues which come up during the course of the classwork.

   Always/Sometimes/Never

7. I believe in the maximum use of the school laboratories by the students.

   Always/Sometimes/Never

8. While solving problems I wait for students to arrive at the solutions themselves.

   Always/Sometimes/Never

9. The students feel free to ask any question/seek clarification during the course of teaching.

Always/Sometimes/Never

10. I welcome their ides/suggestions regarding any aspect of the subject

Always/Sometimes/Never

11. I make sure that at the end of any discussion/argument, my point of view is final.

Always/Sometimes/Never

12. I do not make a value judgement concerning a student's interpretation of an aspect.

Always/Sometimes/Never

13. I feel humiliated when a student is able to give additional information/solve a problem quicker than me.

Always/Sometimes/Never

14. I do not believe in their imagination going wild.

Always/Sometimes/Never

15. I keep my self updated with the latest information in my area.

Always/Sometimes/Never

16. I cannot supply information beyond that which is available in the textbook.

Always/Sometimes/Never

17. I accept and encourage students to discover something new/different on their own.

Always/Sometimes/Never

18. I give praise frequently.

Always/Sometimes/Never

19. At the end of a lesson I give a few open ended questions as class/home assignment.

Always/Sometimes/Never

20. The test papers I set include questions from the lesson exercises.

Always/Sometimes/Never

21. I set a few analytical/application questions which need not be covered in the syllabus.

Always/Sometimes/Never

22. I expect students to answer the questions in the examination from the notes I have given.

Always/Sometimes/Never

23. I do not set individual goals for the less abled children.

Always/Sometimes/Never

24. I am unable to give students a personal feedback about their performance.

Always/Sometimes/Never

25. I am innovative in engaging my students in some activities in my area.

Always/Sometimes/Never

26. I am able to identify the hidden talents in students.

Always/Sometimes/Never

27. I encourage students to interpret their experiences through various media.

Always/Sometimes/Never

28. I am able to motivate students in various school activities.

Always/Sometimes/Never

29. I allow students to express their feelings.

Always/Sometimes/Never

30. I enable students to apply knowledge gained, to their daily living.

Always/Sometimes/Never

31. I cannot show courtesy to students.

Always/Sometimes/Never

32. I am successful in getting students to assume responsibility.

Always/Sometimes/Never

33. I involve students in planning/consultation in classwork which involves them.

Always/Sometimes/Never

34. I cannot admit a mistake before students when results show I am wrong.

Always/Sometimes/Never

35. I help pupils accept one another.

Always/Sometimes/Never

36. I make each student feel he has a contribution to make.

Always/Sometimes/Never

37. I help students feel that they belong.

Always/Sometimes/Never

38. I let students know that I have confidence in them.

Always/Sometimes/Never

39. I cannot become a part of the group when they work in groups.

Always/Sometimes/Never

40. I let pupils know I like them.

Always/Sometimes/Never

PRE-TEST

1. Draw and justify the Venn diagrams of the following :

   i) one-one function
   ii) onto function
   iii) constant function
   iv) continuous function
   v) polynomial function

2. "No relation is also a relation". How do you justify this statement ?

3. Prove that $\lim_{n \to \infty} \frac{1}{n} = 0$.

4. What logical difficulties do you find in teaching complex number as $x + iy$, $(x, y \in R, i^2 = -1)$? Explain.

5. Does zero vector have direction? Justify your answer.

6. "If 2 is not a prime number, then 2 is an even number". Write the inverse of the proposition.

7. Define f(0) so that $f(x) = \frac{x}{1 - \sqrt{1 - x}}$ becomes continuous at $x = 0$.

8. The sum of two numbers is 48. Find the numbers when their product is maximum.

9. Find $\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx$

10. If a manufacturer's total cost function C is

    $C = \frac{x^2}{25} + 2x$, then find

    i) average cost function
    ii) marginal cost function

11. If P is a probability function then show that $P(\phi) = 0$.

12. Prove or disprove : If P is a probability function and $P(A) = 0$ then $A = \phi$.

******

POST – TEST

1. Draw and justify Venn Diagram of the following : (i) Continuous functions, (ii) Differentiable functions, (iii) one-one and onto functions.

2. If N is the set of all natural numbers, then construct a one-one function from N × N into N by using fundamental theorem of Arithmetic.

3. For any natural number n, prove that $2^n > n$ without using principle of mathematical induction.

4. Prove or disprove :
   If r is rational and x an irrational, then r.x is an irrational number.

5. Explain why no part of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ lies between the lines $x - a = 0$ and $x + a = 0$.

6. Find the present value of an ordinary annuity of 24 payments of Rs.100/- each made monthly and earning interest at 9% per year compounded monthly.

7. Three forces 5p, 5p and 10p act along the sides AB, BC and CA of a given equilateral triangle ABC. Will the system be in equilibrium ? Why ?

8. Find the degree of the differential equation

$$\frac{d^2y}{dx^2} + 3\left(\frac{dy}{dx}\right)^2 = x \log \frac{d^2y}{dx^2}$$

9. A determinant is chosen at random from the set of all determinants of order 2 with elements 0 or 1 only. Find the probability that the determinant chosen is non-zero.

10. There are 10 pairs of shoes in a cupboard from which 4 shoes are picked at random. Find the probability that there is atleast one pair.

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21-DAY TRAINING PROGRAMME FOR MATHEMATICS PGTs OF NVS

02-06-2003 to 22-06-2003

PROVISIONAL TIME TABLE

Day & Date	9.00 am-9.30 am	9.30 am-11.00 am	11.30 am-1.00 pm	2.00 pm-3.15 pm	3.30 pm-5.00 pm	5.00 pm-5.30 pm
Monday 2-6-03	Registration and Inauguration		Identification of difficult areas BCB	Pretest BCB	NMR	Library BSPR
Tuesday 3-6-03	Reporting of previous day's work	GR	BSPR	NBB	BCB	Library BSPR
Wednesday 4-6-03	-do-	GR	NMR	NBB	BCB	Library BSPR
Thursday 5-6-03	-do-	DB	NMR	DB	NBB	Library BSPR
Friday 6-6-03	-do-	NMR	DB	NBB	GR	Library BCB
Saturday 7-6-03	-do-	NMR	DB	NBB	BCB	Library BCB
Sunday 8-6-03	← GROUP WORK → BCB	NMR	BSPR	DB	DB	
Monday 9-6-03	Reporting of previous day's work	GR	NBB	BSPR	NMR	Library BCB
Tuesday 10-6-03	-do-	KD	NMR	NBB	DB	Library BCB
Wednesday 11-6-03	-do-	KD	BSPR	DB	NBB	Library NMR
Thursday 12-6-03	-do-	KD	NMR	NNP	MVG	Library NMR
Friday 13-6-03	-do-	DB	MVG	NNP	NMR	Library NMR
Saturday 14-6-03	-do-	DB	BSPR	MVG	NBB	Library DB
Sunday 15-6-03	← GROUP WORK →					
Monday 16-6-03	Reporting of previous day's work	BSU	CGV	BSPR	MVG	Library BSU
Tuesday 17-6-03	-do-	BCB	DB	MVG	BSU	Library BSU

Wednesday 18-6-03	-do-	DB	BSU	BSPR	MVG	Library BSU
Thursday 19-6-03	-do-	GTN	ASNS	BCB	BSPR	Library BSU
Friday 20-6-03	-do-	BSPR	KV	NMR	KD	Library DB
Saturday 21-6-03	-do-	S	S	NMR	BCB	Library DB
Sunday 22-6-03	-do-	Post-test BCB	BCB	BSPR	BSU	Valedictory

Venue: Chemistry Lecture Theatre
11.00 am to 11.30 am and 3.15 pm to 3.30 pm – Tea Break
1.00 pm to 2.00 pm – Lunch Break

GR	G.Ravindra	Meaning of Mathematics, Logical Thinking. Venn diagrams, Problems on continuous functions and Mathematical Modelling
KD	K. Dorasami	Teaching of concepts in Mathematics, Evaluation in Mathematics
NMR	N.M. Rao	Statics, 3D Geometry, Mathematics Laboratory
DB	D. Basavayya	Probabiliy, Statistics, Computers
NBB	N.B. Badrinarayana	Dynamics
BSPR	B.S.P. Raju	Commercial Mathematics, Vectors
BSU	B.S. Upadhyaya	Mathematical Logic, Boolean Algebra
CGV	C.G. Venkatesha Murthy	Action Research
NNP	N.N. Prahallada	Value Education
ASNS	A.S.N. Rao Sindhe	Higher Order Thinking
KV	Kalpana Venugopal	Creativity in Teaching and Learning, Adolescent Psychology
MVG	M.V. Gopalakrishna	Conic Sections, Advanced Level Problem Solving
BCB	B.C. Basti	Calculus, Differential Equations
GTN	G.T. Narayana Rao	Popular Talk
S	Shamanna	Popular Talk

REGIONAL INSTITUTE OF EDUCATION, MYSORE-570 006

DEPARTMENT OF EXTENSION EDUCATION

Twenty-One Day Training Programme for PGTs in Mathematics for Navodaya Vidyalayas

1.	Rakesh Kumar Sinha PGT Maths JNV Ministry of HRD (Department of Education) Hansdiha, Dumka Jharkhand 814 145
2.	Rani Mathew C. JNV Minicoy Lakshadweep
3.	Bidyadhar Sahu JNV Bagudi Balasore Orissa
4.	Seshanooj Sarkar Alirajpur District Jhabua Madhya Pradesh
5.	Abdhesh Jha PGT (Maths) JNV Rothak West Sikkim PO Naya Bazaar Sikkim 737 121

6.	Pradyumna Kumar Moharana JNV, Kherigadevat Essagash, Guna Madhya Pradesh 473 375
7.	Mahesh M. JNV, Chikmagalur District Karnataka 577 112
8.	Vijaya Naithani JNV, Chara District Udupi Karnataka (Hyderabad Region) 576 112
9.	Ravindra Kumar Rudra PGT (Maths) JNV, Manpur District Indore Madhya Pradesh . 453 661
10.	P.S Rajput JNV Panghata Narwar District Shivpuri Madhya Pradesh 473 865
11.	A.S. Rawat JNV, Shyampur District Sehore Madhya Pradesh . 466 651

12.	Sasi Kumar, D. PGT (Mathematics) JNV, Mahadevpur Lohit District 792 103 (Transfered to JNV, Hassan, Karnataka)
13.	P. Sundara Kumar JNV, Canacona South Goa Goa 403 701
14.	V.B. Vaidya A/P: Navegaon Bandh District Gondia Maharastra 441 702
15.	Asokan N.M. JNV Almatti D S Bijapur District Karnataka 586 201
16.	V. Nagarajan JNV, Selukate Wardha Maharastra 442 001
17.	Y. Ranga Rao JNV Narayanpur Taluk Basavakalyan Bidar Karnataka

18.	V. Srinivas Rao JNV, Yenigadale District Kolar Karnataka 563 156
19.	V. Ramakrishnaiah JNV, Doddaballapur Bangalore (Rural) Karnataka 561 203
20.	Gopinath Meethale Veetil JNV, Tamenglong Manipur 795 141
21.	Prabhash Chandra Jha Tinsukia, JNV Assam 786 126
22.	P. Mary Janet Daisy JNV, Hondrabalu Chamarajanagar District Karnataka
23.	Umadevi Himirika JNV Panchavati Rangat, Middle Andaman Andaman and Nicobar Island
24.	Mary Thomas JNV, Nalbari Bartala P.O. Nalbari District Assam 781 138

25.	K. Parvathi JNV Khurai Sagar district Madhya Pradesh 470 117
26.	Sanjay Kumar Jena JNV, Pailapool Cachar Assam
27.	Manoj Kumar Singh JNV, Rangia, Kamrup P.O. Jamtola District Kamrup Assam
28.	Rakesh Kumar Singh JNV Kharedi District Dahod Gujarat 389 151
29.	Vinod Kumar Tiwari JNV Chandra Kesar Dam District Dewas Madhya Pradesh
30.	Umesh Chandra Jhankar JNV Belpada District Bolangir Orissa 767 026

31.	Krishna Kumar Mishra JNV, Khumbong Imphal, West Manipur 795 113.
32.	Sucy Stanly JNV Mayannur Trichur District Kerala 679 105
33.	Adode Ganesh Mahdavrao (Khedgaon) Taluk Dindori District Nasik Maharastra 422 205.
34.	M. Lokanadham JNV, Chikkajogihally District Bellary Karnataka 583 126
35.	Rajesh Kumar JNV, Akkalkuwa District Nandurbar Maharastra