ENRICHMENT MATERIAL IN MATHEMATICS FOR SENIOR SECONDARY TEACHERS

TOPICS

NUMBER SYSTEM
VECTORS
THREE DIMENSIONAL GEOMETRY
DIFFERENTIAL CALCULUS
INTEGRAL CALCULUS
DIFFERENTIAL EQUATIONS
PROBABILITY
LINEAR PROGRAMMING
NUMERICAL METHODS
ALGORITHMS AND FLOW CHARTS

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N U M B E R   S Y S T E M

1. Binary operations
2. System of Natural Numbers
3. The System of Integers
4. The Rational Number System
5. The Real Number System

by

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BINARY OPERATIONS

Motivation: Children come in contact with algebra as early as in the primary classes when they learn to add and multiply numbers. Let us try to see what really happens when we add or multiply two numbers. In addition, to add, first of all we should be given two numbers, say 4 and 3. When we add what we get is 7, which is again a number of the same type as 4 and 3. So addition is therefore basically a rule which enables us to assign with two given numbers in an order, some number as answer. Multiplication is also a similar rule but a different one, since in multiplication with 4 and 3 we assign the number 12 and not 7. So is the operation of subtraction. The order in which the two numbers are given is important. For let us consider subtraction. In subtraction, the number associated with two given numbers 7 and 3 in that order is 4 whereas the one associated with 3 and 7 in that order is not 4. Thus, addition, subtraction, multiplication are all rules that assign with every ordered pair of elements of a specified set an element of that set - we call such operations or rules as binary operations.

Definition : A binary operation on a set S is a rule that assigns to each ordered pair of elements of the set, a unique element of the set S.

Binary operation on a set S, therefore, is a map from $S \times S \rightarrow S$. Here one may note that the element which is assigned to the ordered pair of elements need not be a third or a new element. For example, in case of addition, the element assigned to the ordered pair of elements (0,2) is 2, which is not a new element. Similarly, in case of multiplication, the element assigned to the ordered pair of elements (3,1) is the number 3 itself.

A binary operation on a set S is normally denoted by '*' and the element the binary operation * assigns to the ordered pair of elements (a,b) is denoted by $a*b$.

Examples: 1. Addition and multiplication are binary operations on the set of natural numbers, whereas subtraction is not a binary operation on this set.

2. Addition, multiplication, subtraction are binary operations on the set of integers, set of even integers, set of rational numbers, set of real numbers, set of complex numbers, but not on the set of odd integers.

3. Multiplication is a binary operation on the set $\{-1,0,1\}$ whereas addition is not a binary operation on this set.

4. On the set of natural numbers, a*b equal to the smaller of the two numbers a and b or the common value if a=b, is a binary operation.

5. On the set of all permutations on n symbols, the multiplication of permutations is a binary operation.

In fact, we do not need any known set or a known operation to define a binary operation. We can define binary operations on any arbitrary abstract set. For example,

6. Let $S = \{a,b,c,d\}$ where a,b,c,d are elements of an abstract set S. Define $* : S \times S \to S$ by the following table

*	a	b	c	d
a	b	a	d	c
b	a	b	(a)	c
c	d	c	c	c
d	c	c	d	a

> Here b * c = a

Then * is clearly a binary operation on S. Note that in order to define a binary operation on S, we need to define 16 entries in our table. The 16 entries should be filled by the elements of S. There

*	a	b	c	d
a				
b				
c				
d				

are 4 choices for filling each of these entries and hence in all the 16 entries can be filled in 4^{16} = 429,49,67,296 ways. Hence there exists 429, 49, 67, 296 binary operations on the set $\{a,b,c,d\}$ But in reality we are not interested in all of them and in fact we are interested only in a very few of them.

<u>Types of binary operations</u> :

A binary operation * on a set S is said to be commutative if $a * b = b * a \;\forall\; a,b \in S$.

Examples: 1. Addition and multiplication are <u>commutative</u> binary operations on the set of natural numbers, set of integers, etc.
2. Subtraction is not a commutative binary operation on the set of integers.

3. The multiplication of permutation is not a commutative binary operation e.g.

$$\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}\cdot\begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}=\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}$$

whereas

$$\begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}\cdot\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}=\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}.$$

4.

*	a	b	c
a	b	c	a
b	c	c	b
c	a	b	a

* is a commutative binary operation on S = { a,b,c}

A binary operation * on a set S is said to be <u>associative</u> if $a * (b * c) = (a * b) * c \quad \forall\ a,b,c \in S$.

<u>Examples</u> :

1. Addition and multiplication are associative binary operations on the set of integers.
2. Subtraction is not an associative binary operation on the set of integers since $2 - (3-4) \neq (2-3) - 4$.
3. Multiplication of permutations is an associative binary operation.
4.

*	a	b	c
a	b	c	a
b	c	c	b
c	a	b	a

* is not an associative binary operation on S = { a,b,c } since $a*(b*c) = a*b = c$ whereas $(a*b)*c = c*c = a$.

<u>Exercises</u> :

1. Show that the multiplication is a binary operation on $S = \{1,-1,i,-i\}$ where $i^2=-1$. Is division a binary operation on S ? Why ?
2. Show that the multiplication is a binary operation on $\{-1,0,1\}$ but not on $\{0,1,2\}$.
3. Show that addition is a binary operation on $S = \{x : x \in I,\ x < 0\}$, but multiplication is not.
4. Let $S = \{A,B,C,D\}$ where $A = \phi$, $B = \{a,b\}$, $C = \{a,c\}$, $D = \{a,b,c\}$. Show that set union ∪ is a binary operations on S whereas set intersection ∩ is not.
5. Is division a binary operation on the set of rationals ? Why ? What is the largest subset of the set of rationals on which the

division is a binary operation ?

6. Which among the above binary operations are commutative and or associative ?

7. Determine whether the following binary operations are commutative and or associative:

*	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	c	b

∘	a	b	c	d
a	d	a	c	b
b	a	c	b	d
c	b	d	a	c
d	c	b	d	a

THE NATURAL NUMBER SYSTEM :

Motivation : We very freely use the terms real numbers, rationals, integers, natural numbers as they are intuitively very clear to us, and there is no doubt in our minds as to what they are. However, if we ask ourselves as to what are real numbers? we may answer that rational numbers and irrational numbers put together are called real numbers. Now, if we ask as to what are rational numbers?, the answer could be that the numbers of the form p/q where p and q are are integers, $q \neq 0$ are rational numbers. Now, if we ask as to what are integers?, the answer could be that the natural numbers, the zero and the negatives of natural numbers put together are integers. But now if we ask as to what are natural numbers?, we may not have any answer except that 1,2,3,.... etc. are called natural numbers or trying to give an explanation in terms of the basic properties of natural numbers. That means, the set of natural numbers is the starting point from which we can build all other sets, the set of integers, the set of rationals, set of reals, set of complex numbers etc. and there is a need therefore, to define the set of natural numbers.

In order to define the set of natural numbers, we have to see what are the most basic properties of natural numbers which cannot be derived as a consequence of other basic properties. Assume these basic properties as postulates or axioms, so that any other property of the set of natural number could be derived as a consequence of this set of basic properties. The postulates which characterise the set of natural numbers are known as Peano's axioms on natural numbers, named after the Italian Mathematician of the same name.

Peano's axioms on natural numbers :

Let there exist a set N such that

1. 1 N
2. For each n N, there exists a unique n^+ N, called the successor of N.

3. There exists no n in N for which $n^+ = 1$ (i.e. 1 is not the successor of any n in N).
4. If m,n are in N with $m^+ = n^+$, then $m = n$. (i.e. no two elements of N have the same successor).
5. Any subset K of N having the properties (i) $1 \in K$, (ii) $k \in K \Rightarrow k^+ \in K$ is equal to N.

The set N is called the set of natural numbers.

Note that Axiom 5 is nothing but the Principle of Mathematical Induction and hence it is referred to as Induction Axiom.

Let $K = \{1\} \cup \{n^+ : n \in N\}$. Then since $n \in N \Rightarrow n^+ \in N$

$K \subset N$. Further $k \in K \Rightarrow k = 1$ or $k = n^+$ for some $n \in N$ $\therefore k^+ = 1^+$ or $(n^+)^+$ $\therefore k^+ \in N$.

Thus, $1 \in K$ and whenever $k \in K$, $k^+ \in K$. Hence by axiom 5 $K = N$. Hence every natural number n other than 1 is the successor of some natural number and more over it can be obtained by taking successors of 1. We denote the successor of 1 by 2, successor of 2 by 3, successor of 3 by 4 etc. so that

$$N = \{1,2,3,4,5,....\}$$

Addition on N : We will now define addition on N as follows :

i) $n + 1 = n^+$

ii) $n + m^+ = (n+m)^+$

As we have already observed, every natural number is a successor of some natural number and we can reach 1 by proceeding backwards in a finite number of steps. Hence the condition (i) and (ii) above define addition for all n, m in N.

The following properties of the operation of addition on N can be proved using axioms 1 to 5.

For all $m,n,p \in N$,

1. $n + m \in N$ (closure)
2. $n + m = m + n$ (commutativity)
3. $m + (n+p) = (m+n) + p$ (Associativity)
4. $m + p = n + p \Rightarrow m = n$ (cancellation)

Proof: 1. Given $n \in N$, let K be the set of all $m \in N$ such that $n + m \in N$. Since $n \in N$, by axiom 2, $n^+ \in N$ $\therefore$ $n+1 \in N$ $\therefore$ $1 \in K$. Let $k \in K$. Then $n + k \in N$ $\therefore$ By axiom 2, again $(n+k)^+ \in N$. But by definition of addition on N, $(n+k)^+ = n+k^+$ $\therefore$ $n+k^+ \in N$. $\therefore$ $k^+ \in K$. Hence by axiom 5, $K = N$ $\therefore$ $n+m \in N$ for all $n,m \in N$.

3. Given natural numbers m and n, let K be the set of all natural numbers p such that $m+(n+p) = (m+n) + p$. Since $m+ (n+1) = m+n^+ = (m+n)^+$ (by definition of addition) $= (m+n) + 1$, it follows that $1 \in K$.

Now, let $k \in K$. Then $m+(n+k) = (m+n) + k$. Then,
$m + (n + k^+) = m + (n + k)^+ = [m + (n+k)]^+ = [(m+n) + k]^+$
$= (m+n) + k^+$. $\therefore$ $k^+ \in K$. Hence by induction axiom $K = N$. Hence $m + (n + p) = (m+n) + p \quad \forall \quad m,n,p \in N$.

2. This is proved in two stages. First, we prove that $n+1 = 1+n$ for every n in N. Let K be the set of all n in N s.t. $n+1 = 1+n$.

Clearly, $1 \in K$. Let $k \in K$. Then $k+1 = 1+k$. Now $k^+ +1 = (k+1)+1 = (1+k)+1 = (1+k)^+ = 1+k^+$ $\therefore$ $k^+ \in K$. Hence by induction axiom, $K = N$. $\therefore$ $n+1 = 1+n$ for all n in N. We will now prove that given n in N, $n+m = m+n$ for all m in N. Let K be the set of all natural numbers m such that $n+m = m+n$. Since we have proved already that $n+1 = 1+n$, $1 \in K$.

Let $k \in K$. Then, $n+k = k+n$. Now $n+k^+ = n+(k+1) = (n+k)+1 = (n+k)^+ = n+k^+$ $\therefore$ $k^+ \in K$. Hence by induction axiom $K = N$.

Hence $n+m = m+n$ for all $n,m \in N$.

4. Given $m,n \in N$, let K be the set of all natural numbers p set. $m+p = n+p \Rightarrow m=n$. Since $m+1 = n+1 \Rightarrow m^+ = n^+ \Rightarrow m=n$ (by axiom 4). Therefore, $1 \in K$. Let $k \in K$. Then $m+k = n+k \Rightarrow m=n$. Now $m+k^+ = n+k^+ \Rightarrow (m+k)^+ = (n+k)^+ \Rightarrow m+k = n+k$ (by axiom 4) $\Rightarrow m=n$. Hence $k^+ \in K$. Hence by induction axiom, $K = N$. Therefore, $m+p = n+p \Rightarrow m=n$ for all $m,n,p \in N$.

Multiplication on N : We define multiplication on N as follows :
For $m,n \in N$

i) $m.1 = m$

ii) $m.n^+ = m.n + m$

As observed in the case of addition, this defines multiplication completely on N.

Following properties of multiplication on N can be proved.

For $m,n,p \in N$

1. $n.m \in N$ (closure)
2. $m.n = n.m.$ (commutativity)
3. $m(n.p) = (m.n).p$ (associativity)
4. $(m.p = n.p \Rightarrow m=n$ (cancellation)

Proof is on the same lines as in the case of addition and therefore, left as an exercise.

Also the addition and multiplication of natural numbers satisfy distributive laws :

a) $m.(n+p) = m.n + m.p$

b) $(n+p).m = n.m + p.m.$

Proof: a) Given m,n in N, let K be the set of all natural numbers p such that $m.(n+p) = m.n + m.p$. Now, $m(n+1) = m.n^{+} = m.n + m = m.n + m.1$. Therefore, $1 \in K$. Let $k \in K$. Then $m(n+k) = m.k + n.k$. Now, $m(n+k^{+}) = m . (n+k)^{+} = m.(n+k) + m = (m.n + m.k) + m$ $= m.n + (m.k + m) = m.n + m.k^{+}$ Hence $k^{+} \in K$. Hence by induction axiom, $K = N$. Hence $m.(n+p) = m.n + m.p$. Proof of b is similar.

<u>Order relation on N</u> : Given $a,b \in N$, define $a < b$ if there exists c in N s.t. $b = a+c$ (we define $a > b$ if $b < a$). Then,

i) for all $n \neq 1$, $1 < n$; $n < n^{+}$

ii) the relation '$<$' is transitive but neither reflexive nor symmetric.

iii) the Trichotomy law holds good. i.e. given n,m in N exactly one of a) $m=n$, b) $m<n$ c) $m > n$ holds good.

Proof: i) If $n \neq 1$, we have proved earlier that $n = m^{+}$ for some $m \in N$. But $m^{+} = m+1 = 1+m$. Hence $n = 1+m$. Therefore, $1 < n$. Also $n^{+} = n+1$. Hence $n < n^{+}$.

(ii) Let $m < n$ and $n < p$. Hence there exist a,b in N set. $n = m+a$ and $p = n+b$. Hence $p = (m+a) + b = m+(a+b)$ and since $a+b \in N$, we have $m < p$. Hence '$<$' is transitive. Let $b \in N$. If possible let $b < b$. Then there exists n in N set. $b = b+n$.

Therefore, $b+1 = (b+n)+1 = (b+n)^{+} = b+n^{+}$. Hence $1 = n^{+}$ by cancellation which is a contradiction to Axiom 3. Hence $b \not< b$. Therefore, '$<$' is not reflexive. Finally, if possible let there exist $a,b \in N$, with $a < b$ and $b < a$. Since '$<$' is transitive, we get $a < a$, a contradiction to what we have proved just. Hence if $a < b$, then $b \not< a$. Hence '$<$' is not symmetric.

(iii) Given $m,n \in N$, if $m=n$, then by what we have seen in (ii) above, neither $m < n$ nor $n < m$. Hence if (a) holds neither (b) nor (c) holds. If (b) holds then $m < n$ and hence $n \not< m$. Hence (c) does not hold. Also $m \neq n$, since $m=n$ would imply $n < n$, a contradiction to what we have seen in (ii). Thus if (b) holds neither (a) nor (c) holds. On very similar lines it follows that if (c) holds neither (a) nor (b) holds. Thus at most one of (a), (b) or (c) holds. We will now show that atleast one of (a), (b) or (c) holds. So assume that (a) and (b) do not hold. Hence $m \neq n$ and $m \not< n$. Let $A = \{a \in N \mid a \leq m\}$ and $B = \{m+a \mid a \in N\}$. Let $K = A \cup B$. Since $1 \in A$, $1 \in K$. Let $k \in K$. Then either $k \leq m$ or $k = m+a$ for some $a \in N$. If $k < m$, then $k^{+} = k+1 \leq m$. Hence $k^{+} \in K$. If $k = m$, then $k^{+} = k+1 = m+1 \in B$. $\therefore$ $k^{+} \in K$. If $k = m+a$, then $k^{+} = (m+a)+1 = m+(a+1) = m + a^{+} \in B$. $k^{+} \in K$. Thus, $1 \in K$, $k \in K \Rightarrow k^{+} \in K$ $\therefore$ $K = N$. Since $n \in N$, $n \in A \cup B$. But since $m \not< n$, $n \neq m+a$ for any $a \in N$. $\therefore$ $n \notin B$ $\therefore$ $n \in A$ $\therefore$ $n \leq m$. But $n \neq m$ $\therefore$ $n < m$. $\therefore$ (c) holds good.

<u>Exercises</u> :

1. Prove that $1.n = n$ for all n in N without using the commutativity of multiplication of natural numbers.
2. Prove for m,n in N

i) $(m+n^{+})^{+} = m^{+} + n^{+}$

ii) $(m.n^{+})^{+} = m.n + m^{+}$

iii) $(m^{+}.n^{+})^{+} = m^{+} + m.n + n^{+}$

iv) $m^{+} . n^{+} = (m.n)^{+} + m+n$

3. If $m,n \in N$ and $m < n$, then show that for all $p \in N$.

i) $m + p < n+p$

ii) $m.p < n.p$

4. Prove that N is a well ordered set.

5. Let $m, n \in N$. Prove (a) If $m=n$, then $k^{+}.m > n$ for every $k \in N$, (b) If $k^{+}.m = n$ for some $k \in N$, then $m < n$.

6. For all $m \in N$, define

$$m^1 = m$$
$$m^{p^+} = m^p.m$$

for all $p \in N$. When $m,n,p,q \in N$ prove that

a) $m^p.m^q = m^{p+q}$ b) $(m^p)^q = m^{p.q}$ c) $(m.n)^p = m^p.n^p$

7. For all $m,n \in N$, show that (a) $m^2 < m.n < n^2$ if $m < n$.
b) $m^2+n^2 > 2m.n$ if $m \neq n$.

THE SYSTEM OF INTEGERS

Motivation : Having already defined the system of natural numbers N, and the operations of addition and multiplication on N, we observe that given two natural numbers m and N, there may not always exist a natural number p such that m + p = n. Or equivalently the equation $m+x = n$, $m,n \in N$ may not always have a solution for x in N. To overcome this inadequacy or defect of the system of natural numbers, we hope to construct a set I which contains N as a subset such that given any two elements, m,n in I there always exists a p in I such that $m+p = n$. We will call this new set I as the set of integers and each element of this set as an integer.

Construction of I : Consider the set N+N of all ordered pairs of natural numbers. We introduce a relation ' $\sim$ ' on N+N as follows :

$$(m,n) \sim (r,s) \text{ iff } m+s = n+r , \quad m,n,r,s \in N.$$

We can show that ' $\sim$ ' is an equivalence relation on the set NXN. The set of all equivalence classes under this equivalence relation on NXN is denoted by I and is called the set of integers. Thus an integer is an equivalence class of ordered pairs of natural numbers under the equivalence relation ' $\sim$ ' defined above. We denote it by $[(m,n)]$.

Imbedding of N in I. For each natural number m in N, we identify it with the integer $[(m^+, 1)]$ In other words, we consider the map

$$\psi : N \longrightarrow I$$

defined by $\psi(m) = [(m^+,1)] = \{(m+1,1) (m+2, 2), (m+3,3),....\}$
Then we can show that ψ is 1-1. Hence ψ is an imbedding of N in I (In fact, when we define + and . on I, they are indeed extensions of addition and multiplication on N).

Addition on I. We define addition on I as follows

$$[(m,n)] + [(r,s)] = [(m+r, n+s)]$$

for all $[(m,n)]$, $[(r,s)]$ in I.
Since $m,n,r,s \in N$, so are m+r and n+s. Hence (m+r, n+s) I.
Hence + is a binary operation on I. Thus, i,j I 2+j I (Closure law of addition).

Consider $[(1,1)] = \{(1,1), (2,2),...., (n,n),\cdots\}$ in I.

Now,

$$[(m,n)] + [(1,1)] = [(m+1,\ n+1)] = [(m,n)] \quad \text{(Why ?)}$$
$$[(1,1)] + [(m,n)] = [(1+m,\ 1+n)] = [(m,n)]$$

for every $[(m,n)]$ in I.

The integer $[(1,1)]$ is called zero and is denoted by 0. Also

$$[(m,n)] + [(r,s)] = [(m,n)]$$

$\Rightarrow [(m+r,\ n+s)] = [(m,n)]$

$\Rightarrow (m+r)+n = (n+s) + m$ by definition of '$\sim$'

$\Rightarrow m(r+n) = n+(s+m)$ by associativity of + in N

$\Rightarrow m +(n+r) = n+(m+s)$ by commutativity of + in N

$\Rightarrow (m+n) + r = (n+m) + s$ by associativity of + in N

$\Rightarrow (m+n) + r = (m+n) + s$ by commutativity of + in N

$\Rightarrow r = s$ by cancellation law of + in N

$\Rightarrow [(r,s)] = [(r,r)] = 0$

Similarly, $[(r,s)] + [(m,n)] = [(m,n)] \Rightarrow [(r,s)] = 0$.

Thus, there exists a unique element 0 in I such that $i+0 = i = 0+i$ for all i in I. 0 is called the <u>identity element</u> with respect to the operation of addition on I.

Also,

$$[(m,n)] + [(n,m)] = [(m+n),\ n+m)] = [(\ m+n,\ m+n)] = 0$$
$$[(n,m)] + [(m,n)] = [(n+m,\ m+n)] = [(n+m,\ n+m)] = 0$$

for all $[(m,n)]$, $[(n,m)]$ in I. Further,

$$[(m,n)] + [(r,s)] = 0$$

$\Rightarrow (m+r,\ n+s) = 0$

$\Rightarrow m+r = n + s$

$\Rightarrow r + m = s + n$

$\Rightarrow (r,s) \sim (n,m)$

$\Rightarrow [(r,s)] = [(n,m)]$

Thus, given any integer i, there exists a unique integer j such that $i + j = 0 = j + i$. This integer j is demoted by $-i$ and is called the additive inverse of i.

Using the associative law of addition of natural numbers, we can prove that for any $[(m,n)]$, $[(r,s)]$, $[(u,v)]$ in I.

$$\{[(m,n)] + [(r,s)]\} + [(u,v)] = [(m,n)] + \{[(r,s)] + [(u,v)]\}.$$

Hence for any $i,j,k \in I$,

$(i + j) + k = i + (j + k)$ (associative law of addition).

Thus, to sum up the addition of integers have the following properties.

1. $i,j \in I \Rightarrow i+j \in I$ (Closure law)
2. $i,j,k \in I \Rightarrow (i+j)+k = i+(j+k)$ (Associative law)
3. There exists a unique element $0 \in I$ such that $i + 0 = i = 0 + i$ for every $i \in I$ (Existence of identity w.r.t. addition).
4. Given $i \in I$ there exists a unique element $j \in I$ such that
   $$i + j = 0 = j + i$$
   (Existence of inverse element w.r.t. addition).

A set S with a binary operation '$\circ$' satisfying closure law, associative law, existence of identity w.r.t. '$\circ$' and Existence of inverse w.r.t. $\circ$ is called a group. Thus the set of integers is a group w.r.t. the operation of addition.

Given, $[(m,n)], [(r,s)] \in I$

$$\begin{aligned}[(m,n)] + [(r,s)] &= [(m+r,\ n+s)] \\ &= [(r+m,\ s+n)] \text{ (addition is commutative on } N) \\ &= [(r,s)]+[(m,n)]\end{aligned}$$

Thus, $i+j = j+i$ for every $i,j \in I$ (commutative law).
Thus addition is commutative on I. A group in which the binary operation is commutative is called a commutative group or an abelian group. Hence the set of integers is an abelian group w.r.t. the operation of addition.

We have already seen that every natural number m is identified with the integer $[(m^+,1)]$. Note here that since m is a natural number $1 < m^+$. Conversely if we consider any integer $[(m,n)]$ with $n < m$, then by definition of order relation on N, there exists r in N such that $m = n+r$. Then $m+1 = (n+r)+1 = n+(r+1) = n+r^+$. Therefore, $(m,n) \sim (r^+,1)$. Hence, $[(m,n)] = (r^+,1)$. Hence the integer $[(m,n)]$ is the natural number r (under the identification mentioned earlier). Hence there is a one-one correspondence between the set of all integers $[(m,n)]$ with $n < m$ and the set of all natural numbers. If we now consider $[(m,n)]$ with $m < n$, then $[(m,n)] = -[(n,m)]$

But $[(n,m)]$ since $m < n$, is identified with a natural number, we can identify $[(m,n)]$ with $-s$ where s is the natural number identified with $[(n,m)]$. If we consider $[(m,n)]$ with $m=n$, we have seen that $[(m,n)]$ is written as 0.

Given any integer (m,n) in I, we know by the Trichotomy law in N that exactly one of

$$m = n,\ m < n,\ n < m$$

holds good. Hence given any integer $[(m,n)]$ it is exactly one of

$$0,\ r,\ -r$$

where r is a natural number. Hence I can be written as

$$\{\ldots, -n, \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots, n, \ldots\}$$

<u>Multiplication on I.</u> Given $[(m,n)]$ $[(r,s)]$ in I, define

$$[(m,n)] \cdot [(r,s)] = [(m.r + n.s,\ m.s + n.r)]$$

Clearly, $[(m,n)] \cdot [(r,s)] \in I$. Hence

$$i, j \in I \Rightarrow i.j\ \ I \text{(Closure law)}$$

Also, one can easily prove that (Prove!)

$$\{[(m,n)] . [(r,s)]\} \cdot [(u,v)] = [(m,n)] \cdot \{[(r,s)] \cdot [(u,v)]\}$$

for all $[(m,n)], [(r,s)], [(u,v)] \in I$. Hence

$$i \cdot (j.k) = (i.j) \cdot k \text{ for all } i,j,k \text{ in } I$$

(Associative law of multiplication).

Further, we see that (Prove!)

$$[(m,n)] . \{[(r,s)] + [(u,v)]\} = [(m,n)] . [(r,s)] + [(m,n)] . [(u,v)]$$

$$\{[(r,s)] + [(u,v)]\} . [(m,n)] = [(r,s)] . [(m,n)] + [(u,v)] . [(m,n)]$$

for all $[(m,n)], [(r,s)], [(u,v)]$ in I. Hence,

$$i. (j + k) = i.j + i.k$$

$$(j + k) . i = j.i + k.i$$

for all $i,j,k \in I$. (Distributive law of multiplication over addition).

A set S with two binary operations denoted by '+' and '.' such that S is an abelian group with respect to '+' and the binary operation '.' satisfies closure law, associative law and distributive law, over '+' is called a Ring. Thus the set of integers is a ring with respect to addition and multiplication. We also see that

$$[(m,n)] \cdot [(r,s)] = [(r,s)] \cdot [(m,n)]$$

for all $[(m,n)]$, $[(r,s)]$ in I. Hence $i.j = j.i$ for every i,j in I (commutative law). A ring in which the operation '.' is commutative is called a commutative ring. Hence the set of integers is a commutative ring. Moreover, for any $[(m,n)]$ in I

$$[(m,n)] \cdot [(2,1)] = [(2m+n, m+2n)]$$
$$= [(m,n)] \text{ (Why ?)}$$

Similarly,

$$[(2,1)] \cdot [(m,n)] = [(m,n)]$$

Note that the integer $[(2,1)]$ is identified with the natural number 1. Hence,

$$i.\,1 = i = 1.i \quad \text{for every } i \text{ in } I.$$

(Existence of identity with respect to multiplication). Hence, the ring of integers has a multiplicative identity.

<u>Subtraction on I</u> : Given two integers i,j in I define

$$i-j = i + (-j)$$

Recall that if $j = [(m,n)]$, then $-j = [(n,m)]$

Then $i-j \in I$. Hence '-' is a binary operation on I. This operation is called subtraction on I.

<u>Positive and Negative Integers</u> : The integer $[(m,n)]$ in I is said to be a positive integer if $n < m$. $[(m,n)]$ in I is said to be a negative integer if $m < n$. Hence, given an integer either it is a positive integer or a negative integer or is the integer 0. By the identification we have already made,

1,2,3,...

are positive integers whereas

-1,-2,-3,..

are negative integers. We denote the set of positive integers by I^+

and the set of negative integers by I^-

Order Relation on I :

Given two integers a,b in I, we define

$a > b$ if and only if $a-b$ is a positive integer.

Let $a = [(m,n)]$ Then $a > 0$ iff $a-0$ is a positive integer.
But $0 = [(1,1)]$. Then $a-0 = [(m,n)] - [(1,1)] = [(m,n)] + [(1,1)]$
(why?) $= [(m+1, n+1)]$ Hence $a-0$ is a positive integer iff
$n+1 < m+1$ i.e. iff $n < m$. Hence

$n > 0$ for $n=1,2,3,4,\ldots$

Hence $a > b$ iff $a-b > 0$
We define $a < b$ iff $b > a$.

The trichotomy law : Given any two integers a,b in I exactly one of
$a = b$, $a > b$, $a < b$
holds good.
Proof is left as an exercise.

Exercise :

1. Prove cancellation law of multiplication of integers viz.
   $x,y,z \in I$, $z \neq 0$, $x.z. = y.z \Rightarrow x=y$
2. Prove that for all a,b,c in I
   a) $a.0 = 0.a = 0$ b) $a.(-b) = -(a.b)$
   c) $a.(b-c) = a.b - a.c$
3. Prove that the operation of subtraction is not associative.
4. Prove that (i) $(-a) + (-b) = -(a+b)$
   (ii) $(-a) . (-b) = a.b$
5. Prove: If $a,b \in I$ and $a.b = 0$, then either $a = 0$ or $b = 0$.
   (A commutative ring with multiplicative identity with the property $a.b = 0 \Rightarrow a = 0$ or $b = 0$ is called an Integral domain. Thus, the set of integers is an Integral domain).
6. If $a,b,c,d, \in I$, prove
   a) $-a > -c$ if $a < c$
   b) $a+c < b+d$ if $a < b$ and $c < d$
   c) $a < b+c \Rightarrow a-b < c$
   d) $a-b = c-d \Leftrightarrow a+d = b+c$

7. Prove that if $a,b \in I$ and $a < b$, then there exists $c \in I^+$ such that $a+c = b$.

8. Prove the following if $a,b,c \in I$.

i) $a+c < b+c \Leftrightarrow a < b$

ii) If $c > 0$, $a.c < b.c \Leftrightarrow a < b$

iii) If $c < 0$, $a.c < b.c \Leftrightarrow a > b$

9. Prove that there exists no integer n such that $0 < n < 1$

(Hint: If possible let there exist an integer n s.t $0 < n < 1$. Let m be the least such integer. Then $m < 1$ $\therefore m.m < m.1$ $(\because m > 0)$ $\therefore m^2 < m$ $\therefore 0 < m^2 < m < 1$).

THE RATIONAL NUMBER SYSTEM

Motivation : Having defined the set of integers I, addition and multiplication on I, we would like to see whether equations like mx = n where m and n are integers have solutions for x in I ? If we consider the equation 2x = -6, it has a solution for x viz. x = -3. However, the equation 3x = 5 has no solution for x in I. Or, in other words, there exists no i in I such that 3.i = 5. The set of integers is therefore inadequate or defective for finding out solutions of equations of the form mx=n, m,n $\in$ I. Hence we try to find a set Q containing I as a subset and in which every equation of the form mx = n, m $\neq$ 0 has a solution. (Note that if m = 0, then 0.x = n forces us to conclude that n = 0).

How do we go about such a task of finding a set Q. Here again we take the help of an equivalence relation. We consider the set of ordered pairs of integers $\{(m,n) \mid m,n \in I,\ n \neq 0\}$ and introduce a relation '$\sim$' as follows :

$$(m,n) \sim (r,s) \text{ iff } ms = nr \quad m,n,r,s, \in I \quad n \neq 0,\ s \neq 0$$

Now we see why we have taken the second entries to be non zero. If we allow the second entry to be zero, then by definition $(1,0)\sim(0,0)$ and $(0,0)\sim(0,1)$ and if we want '$\sim$' to be an equivalence relation, by transitivity, we get $(1,0)\sim(0,1)$ and hence 1.1 = 0.0 i.e. 1=0 which is absurd.

We can easily verify that '$\sim$' is an equivalence relation. Let the set of all equivalence classes under this equivalence relation be denoted by Q. Then Q is called the set of rational numbers and each element of Q is called a rational number. We identify the integer m with the rational number $[(m,1)]$ by considering a map $\eta : I \to Q$ defined by

$$\eta(m) = [(m,1)],\ m \in I.$$

Then,

$$\eta(m) = \eta(n) \Rightarrow [(m,1)] = [(n,1)]$$
$$\Rightarrow m.1 = n.1 \Rightarrow m=n$$

Hence η is a 1-1 map. Thus I can be considered as a subset of Q. We will see later when we define addition and multiplication as Q that these operations are indeed extensions of addition and multiplication of integers.

We write the rational number $[(m,n)]$ as $\frac{m}{n}$ or m/n.

Addition and Multiplication on Q

For $[(m,n)]$, $[(r,s)] \in Q$ we define

$$[(m,n)] + [(r,s)] = [(ms + nr, ns)] \in Q$$

and $[(m,n)] . [(r,s)] = [(mn, rs)] \in Q$

Hence +, . are binary operations on Q. If in particular $[(m,n)]$ and $[(r,s)]$ are integers then we can take n=1 and s=1.

$$[(m,1)] + [(r,1)] = [(m+n, 1)]$$
$$[(m,1)] . [(r,1)] = [(mn, 1)]$$

and hence the addition and multiplication of rational numbers is an extension of the addition and multiplication of integers.

Properties of addition and multiplication of rational numbers :

For $[(m,n)]$, $[(r,s)]$, $[(u,v)] \in Q$

1. $[(m,n)] + [(r,s)] \in Q$

   i.e. $p,q \in Q \Rightarrow p+q \in Q$ (Closure law of addition)

2. $\{[(m,n)] + [(r,s)]\} + [(u,v)]$

   $= [(m,n)] + \{[(r,s)] + [(u,v)]\}$

   i.e. $(p+q) + r$ $p+ (q + r)$ for all $p,q,r \in Q$.

   (Associative law of addition).

3. $[(m,n)] + [(0,1)] = [(m,n)]$

   $[(0,1)] + [(m,n)] = [(m,n)]$

   $[(m,n)] + [(r,s)] = [(m,n)]$

   $\Rightarrow [(m.s + n.r, ns)] = [(m,n)]$

   $\Rightarrow (m.s + n.r, n.s) \sim (m,n)$

   $\Rightarrow (m.s + n.r) \cdot n = m \cdot n \cdot s$

   $\Rightarrow m.sn + n^2 . r = m \cdot n \cdot s$

   $\Rightarrow m \cdot n \cdot s + n^2 . r = m \cdot n \cdot s$

$\Rightarrow n^2 r = 0$

$\Rightarrow r = 0$ since $n^2 \neq 0$ (Recall that we have proved that $a.b = 0$
$\Rightarrow a = 0$ or $b = 0$ in I)

$\therefore [(r,s)] = [(0,s)] = [(0,1)]$

Note that the integer 0 is identified with the rational number $[(0,1)]$.

Hence there exists a unique rational 0 such that
$p + 0 = p = 0+p$ for every p in Q.
(Existence of identity element w.r.t. addition).

4. $[(m,n)] + [(-m, n)] = [(0, n^2)] = [(0,1)] = 0$
$[(-m, n)] + [(m,n)] = [(0, n^2)] = [(0,1)] = 0$

Also,
$[(m,n)] + [(r,s)] = 0$
$\Rightarrow [(m.s + n.r, ns)] = [(0,1)]$
$\Rightarrow (m.s + n.r, n.s) \sim (0,1)$
$\Rightarrow m.s + n.r = 0$
$\Rightarrow n.r = -m.s$
$\Rightarrow (r,s) \sim (-m,n)$
$\Rightarrow [(r,s)] = [(-m,n)]$

Hence given $p \in Q$, there exists a unique $q \in Q$ such that
$p + q = 0 = q + p$
(Existence of additive inverse).

5. $[(m,n)] + [(r,s)] = [(m.s + n.r, n.s)]$
$= [(s.n + r.n, sn)]$
$= [(r.n + s.m, s.n)]$
$= [(r,s)] + [(m,n)]$

Hence, $p+q = q+p$ for all p,q in Q.
(Commutativity of addition).

6. $[(m,n)] . [(r,s)] \in Q$
i.e. $p,q \in Q \Rightarrow p.q \in Q$ (Closure law of multiplication).

7. $\{[(m,n)] . [(r,s)]\} . [(u,v)]$
$= [(m,n)] . \{[(r,s)] . [(u,v)]\}$

i.e. $(p.q).r = p.(q.r)$ for all p,q,r in Q.
(Associative law of multiplication).

8. $[(m,n)] \cdot [(1,1)] = [(m,n)]$
$[(1,1)] \cdot [(m,n)] = [(m,n)]$

Also, $[(m,n)] \cdot [(r,s)] = [(m,n)]$

$\Rightarrow [(mr, ns)] = [(m,n)] \Rightarrow mrn = msm \Rightarrow r = s \Rightarrow [(r,s)] = [(1,1)]$

Observe that the integer 1 is identified with the rational number $[(1,1)]$. Hence there exists a unique rational number 1 s.t.

$p.1 = p = 1.p$ for all p in Q

(Existence of identity w.r.t. multiplication)

9. If $[(m,n)] \neq 0$ (i.e. if $m \neq 0$) then

$[(m,n)] \cdot [(n,m)] = [(mn, nm)] = [(mn, mn)] = [(1,1)] = 1$

and $[(n,m)] \cdot [(m,n)] = [(nm, mn)] = [(mn, mn)] = [(1,1)] = 1$

Also,

$[(m,n)] \cdot [(r,s)] = 1$

$\Rightarrow [(mr, ns)] = [(1,1)] \Rightarrow mr = ns \Rightarrow rm = sn \Rightarrow [(r,s)] = [(n,m)]$.

Thus, given $p \neq 0$ in Q, there exists a unique q in Q s.t.

$p.q = 1 = q.p$

(Existence of multiplicative inverse of non zero elements).

10. $[(m,n)] \cdot \{[(r,s)] + [(u,v)]\}$
$= [(m,n)] \cdot [(r,s)] + [(m,n)] \cdot [(u,v)]$ (Prove!)

$\{[(r,s)] + [(u,v)]\} \cdot [(m,n)]$
$= [(r,s)] \cdot [(m,n)] + [(r,s)] \cdot [(u,v)]$ (Prove!)

i.e. $p.(q+r) = p.q + p.r$

$(q+r).p = q.p + r.p$, for all p,q,r Q

(Distributive law of multiplication over addition).

A commutative ring $(R, +, .)$ with a multiplicative identity, in which every non zero element has a unique inverse w.r.t. the operation '.' is called a <u>Field</u>. Thus the set of rational numbers Q is a field.

<u>Subtraction and Division on Q</u> :

Subtraction '−' and the division '÷' are defined as follows :

$[(m,n)] - [(r,s)] = [(m,n)] + [(-r,s)]$

for all $[(m,n)]$, $[(r,s)]$ in Q

$[(m,n)] \div [(r,s)] = [(m,n)] \cdot [(s,r)]$

for all $[(m,n)]$, $[(r,s)] \neq 0$ in Q.

Order Relation on Q :

We say that a rational number $[(m,n)]$ is <u>positive</u> if m.n is a positive integer. Given two rational numbers $[(m,n)]$ and $[(r,s)]$ we define $[(m,n)] > [(r,s)]$ if $[(m,n)] - [(r,s)]$ is a positive rational. Then we see that a rational $[(m,n)] > 0$ iff $[(m,n)]$ is a positive rational and that given two rationals $p,q \in Q$.

$$p > q \text{ iff } p-q > 0$$

We define $p < q$ iff $q > p$, $\quad p,q \in Q$.

Then we can verify that ' $>$ ' is transitive but neither reflexive nor symmetric. Further given $p,q \in Q$ exactly one of

$$p = q,\ p > q,\ p < q$$

holds good (The Trichotomy law). Hence Q is an ordered field.

Exercises :

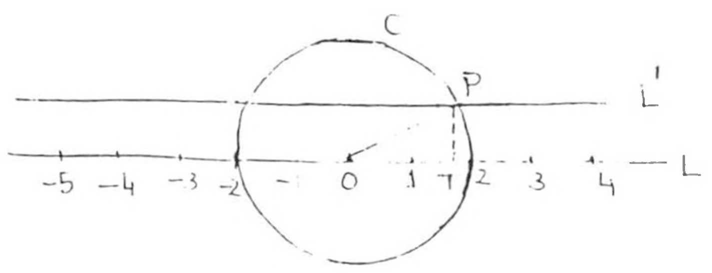
1. Show that addition and multiplication are well defined on Q.
2. Prove Associate laws of addition and multiplication on Q.
3. Prove Distributive law of multiplication over addition on Q.
4. Prove the Trichotomy law on Q.
5. Prove that if x and y are positive rationals with $x < y$ then $\frac{1}{x} > \frac{1}{y}$.
6. Prove that if x and y are rational numbers with $x < y$, then there exists a rational number z s.t. $x < z < y$.
7. Prove that if x and y are positive rationals, there exists a positive integer p such that $px > y$.
8. Prove that if $x,y \in Q$ and $x.y = 0$, then either $x = 0$ or $y = 0$.
9. If $x,y,z \in Q$, prove
   a) $x+z < y+x \Leftrightarrow x < y$
   b) when $z > 0$, $xz < yz \Leftrightarrow x < y$
   c) when $z < 0$, $xz < yz \Leftrightarrow x > y$
10. Prove that if a and b are positive rationals with $a < b$, then $a^2 < a.b < b^2$.

THE REAL NUMBER SYSTEM:

Motivation : Let us recall what we have done so far. Axiomatically, we defined the system of natural numbers N. Then noting that given any two $m,s \in N$, $m+x = s$ may not have a solution in N, we constructed a set I of integers containing N as a subset, in which the equation $m+x = s$, $m,s \in I$ has always a solution in I. Then we observed that the equation $mx = s$, $m,s \in I$, $m \neq 0$ does not always have a solution in I, we constructed a set Q of rationals containing I as a subset such that $mx = s$, $m \neq 0$, $m,s \in Q$ always has a solution in Q. Having constructed the system of rational numbers Q, we ask ourselves whether Q has any inadequacy or defect as observed earlier in case of N and I. In fact, the situation in Q is not about a single defect, but about plenty of defects. For example, just to give a few :

1. The equation $x^2=2$ has no solution in Q. Similarly the equations $x^2=3$, $x^2=5$, $x^2=7,\ldots,x^3=2$, $x^3=3$, $x^3=5,\ldots$ have no solution in Q.
2. The circumference π of a circle with unit diameter is not a rational number. In fact, even $\pi^2 \notin Q$, $\pi^3 \notin Q$, π does not satisfy any polynomial equation over Q.
3. $\lim_{n\to\infty} (1 + \frac{1}{n})^n, n \in I^+$ is not a rational number though each term $(1 + \frac{1}{n})^n$ is a rational number.

Rational Scale : Consider a line L. Fix one of its points and label it as zero. Using a proper scale (i.e. by choosing a unit) attach non zero elements of Q to the points on this line. Call the points of L attached to a rational number as a rational point. This line L is called a rational scale. We will now see that not every point of the line L is a rational point. For this consider a line L' parallel to L at a distance of 1 unit. Choosing 0 as centre, draw a circle C with radius equal to 2. Let the circle C cut the line L' at P as shown.



Drop a perpendicular PT from P on L. Note that OT = $\sqrt{3}$ and hence is not a rational number. Hence T is not a rational point. Hence there are points in L which are not rational points. Or in other words there are some gaps in the rational scale L. The real number system is constructed in such a way that there is no gap on the rational scale. Or in other words, each real number can be represented as a point on the line L and conversely each point on the line L represents a real number. The method used for the purpose is known as method of Dedikind cuts. However, we will not discuss Dedikind cuts here. The set of rationals Q can be imbedded in R, the set of real numbers. Following Dedikind's construction of R one can show that R is a field. Moreover as in case of Q, we can introduce an order relation on R and this ordering satisfies the following two properties in addition to the Trichotomy law.

1. <u>Density property</u> : For each $r,s \in R$, $r < s$, there exists $t \in Q$ such that $r < t < s$.
2. <u>Archimedean property</u> : For each $r,s \in R^+$, with $r < s$, there exists $n \in I^+$ such that $nr > s$.

In addition to the above properties, the set of real numbers also has completeness property viz. "Every non empty subset of R bounded below has a greatest lower bound in R and every non empty subset of R bounded above has a least upper bound in R". Thus, the real number system R is a complete ordered field.

V E C T O R S

1. Vector Algebra
2. Multiplication of Vectors
3. Triple Products

by

Dr.N.B.BADRINARAYAN

VECTOR ALGEBRA

Introduction: The study of vectors is greatly motivated by problems of a physicist, and engineer and a geometor. The vector methods in the hands of an applied scientist is an effective and elegant tool which brings about economy of computation (which is otherwise messy and laborious) and grace. The notion vectors and their methods are being used in a variety of branches of knowledge - mechanics, electricity and magnetism, engineering to mention a few well known ones.

The origin of this branch of mathematics can be traced to early attempts to find geometric representation of imaginary algebraic quantities, which are revealed in the works of a Savilian Professor, John Wallis (1616-1703), a Norwegian surveyor - Casper Wassel (1745-1818) and J.R.Argand. William Rowan Hamilton (1805-1865), an Irish Royal Astronomer of the University of Dublin invented what are called as quaternions whose algebra is comparable to that of vectors, but which is a generalisation of vectors in some sense. Vector algebra, like quaternions , can be considered as a branch of 'Multiple Algebra', Herman Grassman's (1809-1877) - a German mathematician, great work is specially concerned with vector algebra. Professor J.W.Gibb's (1839-1903) vector algebra is based on the fundamental ideas of both Grassman and Hamilton and is found in most texts dealing with this subject.

Basic terminology and notations :

Representing a vector-Notations

A vector has both magnitude and direction. It is represented by a directed segment as shown in Fig.1.

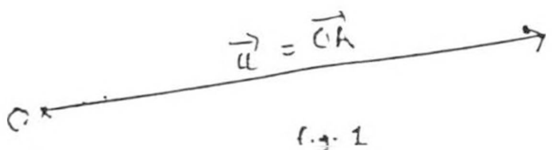


fig. 1

This vector is denoted by $\overrightarrow{OA}$. The arrow shows the direction and the length OA is the magnitude of the vector OA.

O is the initial point (or the tail) and A is the end point (or the terminus) of $\overrightarrow{OA}$. The magnitude of $\overrightarrow{OA}$ is denoted by $|\overrightarrow{OA}|$. By definition, the magnitude of a vector is always a non-negative real number i.e. $|\overrightarrow{OA}| \geqslant 0$ always.

A vector is also represented by the notation $\vec{a}$, a being the magnitude of the vector. The vector $\vec{a}$ is represented by a directed segment of length a drawn in the direction of the vector as in Fig.2. The magnitude of a is denoted by $|\vec{a}|$ or a (without arrow-head).

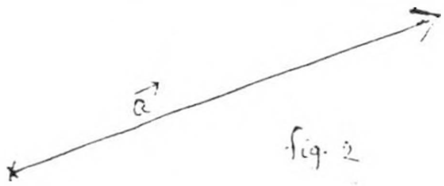


Fig. 2

The magnitude of a vector a is also called its length. Unlike $\vec{a}$ which is a vector, $|\vec{a}|$ is a scalar (or a number),

Note :

1. A directed-segment being a part of a straight line, the straight line itself is called the support of the vector.

2. Any two directed segments of the same length drawn in the same direction represent the same vector.
Such vectors are called free-vectors.
3. In contrast two vectors which are treated as different from each other even if they are represented by two directed segments of equal length, drawn in the same direction, are called bound vectors.
4. In the course vectors are considered are free-vectors.
5. In representing a free-vector, any point of the plane can be taken as the initial point of the directed-segment.
6. Any two vector can be represented by directed segments with the same initial point, without loss of generality.

2. <u>Collinear Vectors</u> : Vector represented by directed segments which are parallel (or along the same line) are called <u>collinear vector</u>. Vectors which are not collinear are called <u>non-collinear vectors</u>.

Consequently, any two collinear vector have either the same direction or opposite directions, and

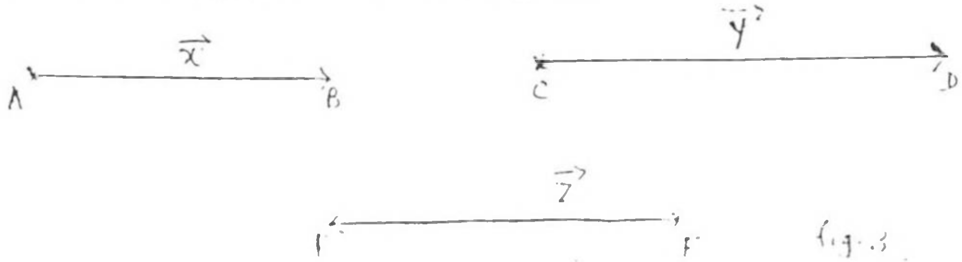


fig.3

In Fig.3, the vectors $\vec{x}$, $\vec{y}$ and $\vec{z}$ are collinear vectors.
(i) $\vec{x}$ and $\vec{y}$ have the same direction. Such vectors are called <u>like-vectors</u>.

(ii) $\vec{x}$ and $\vec{z}$ (or $\vec{y}$ and $\vec{z}$)have opposite directions. Such vectors are called <u>unlike-vectors.</u>

Any two collinear vectors are either like-vectors (i.e. have the same direction) or unlike-vectors (i.e. have opposite directions).

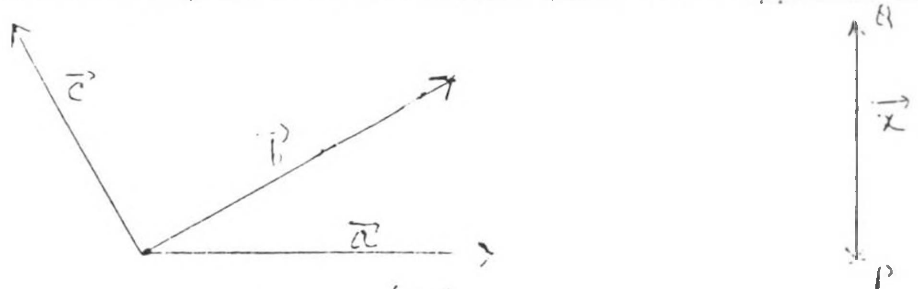


fig.4

Vectors shown in Fig.4 are non collinear vectors. (i.e. no two of them have the same or opposite directions).

Note: 1. Any two like vectors differ in their magnitudes.

2. Equality, zero vector, negative of a vector and unit vector and scalars.

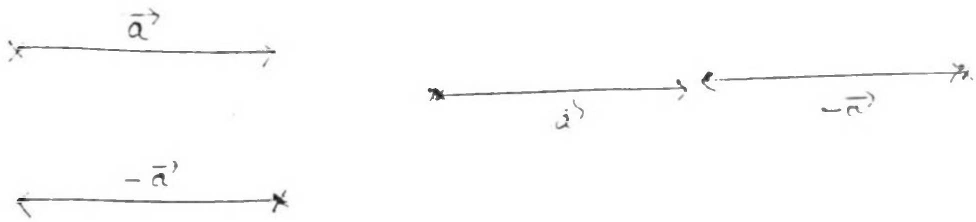
i) Two vectors $\vec{a}$ and $\vec{b}$ having the same magnitude and direction are called equal vectors and we write $\vec{a} = \vec{b}$.

ii) A vector whose magnitude is zero is called the <u>zero vector</u> denoted by 0. Its direction is undefined. Consequently, the zero vector cannot be represented by a directed segment.

A vector $\vec{a}$ which is not $\vec{0}$ is called a proper or non zero vector. However, we use 0 to denote the zero vector as well in these .

iii) Given a vector $\vec{a}$, the vector having the same magnitude as a but opposite direction of a is called the negative of a denoted by $-\vec{a}$.

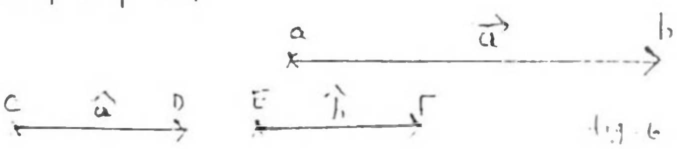
A vector and its negative are represented by two-directed segments of equal length, drawn in opposite direction as shown in Fig.5.



iv) Given a vector $\vec{a}$, a vector which has unit magnitude and is collinear to a is called <u>a unit vector along $\vec{a}$.</u>

<u>A unit vector</u> is essentially one having unit magnitude (i.e. on magnitude is 1). We denote a unit vector by $\hat{a}$.

Then $|\hat{a}| = 1$



$\vec{a}$ and $\vec{b}$ are both unit vectors along a.

Note:

(i) A vector $\vec{a}$ and its negative $-\vec{a}$ are unlike vectors. Therefore, they are collinear vectors.

(ii) A vector a and a unit vector $\vec{a}$ along a are collinear vectors.

(iii) Scalars have magnitude only (but no direction). They are merely real numbers.

<u>Operations on Vectors</u> :

Definition: Let $\vec{a} = \overrightarrow{OA}$ and $\vec{b} = \overrightarrow{AB}$ be two vectors. Then the vector represented by $\overrightarrow{OB}$ is called the sum of $\vec{a}$ and $\vec{b}$.

Then we write $\overrightarrow{CB} = \vec{a} + \vec{b}$.

In other words, $\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{CB}$.

<u>Multiplication of a vector by a Scalar</u> – Scalar multiple of a vector.

Definition: Let $\vec{a}$ be a vector and k a real number. A vector $\vec{b}$ which is collinear with a having the magnitude $|K|\ |a|$ (i.e., $|K|$ times the magnitude of $\vec{a}$) is called a scalar multiple of $\vec{a}$ by K. We then write $\vec{b} = K\,\vec{a}$.

If $K > 0$, a and b are like vectors fig. 7 (i) and if $K < 0$, $\vec{a}$ and $\vec{b}$ are unlike vectors.

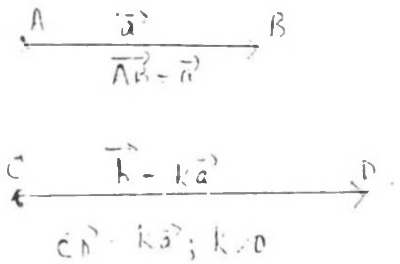


fig 7(i)

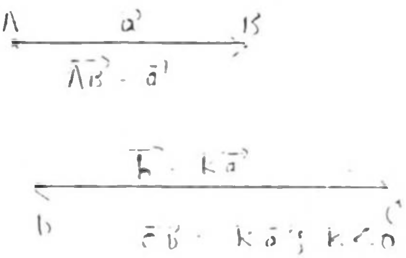


fig 7(ii)

Note:

i) In particular, $\vec{0}$ is a Scalar multiple of any vector a since $\vec{0} = 0\vec{a}$.

ii) If $\vec{a}$ and $\vec{b}$ are proper collinear vectors, then one is a scalar multiple of the other i.e. $\vec{b} = K\,\vec{a}$ for some scalar K.

iii) Conversely, if $\vec{b}$ is a scalar multiple of $\vec{a}$, then $\vec{a}$ and $\vec{b}$ are collinear vectors.

iv) Each vector a is a scalar multiple of itself since we can write $\vec{a} = 1\,\vec{a}$.

We summarise the properties of vector addition and multiplication of a vector by a scalar.

Let V denote the set of all vectors. Then,

1. vector addition is a binary operation on V. i.e. for $\vec{a},\vec{b} \in V$, $\vec{a} + \vec{b} \in V$.

2. $\vec{a} + \vec{b} = \vec{b} + \vec{a}$

3. For $\vec{a},\vec{b},\vec{c} \in V$, $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b}+\vec{c})$

4. For $\vec{a} \in V$, the zero vector $\vec{0}$ satisfies $\vec{a} + \vec{0} = \vec{a} = \vec{0} + \vec{a}$

5. For each $a \in V$, $-a \in V$ has the property $\vec{a} + (-\vec{a}) = 0 = -\vec{a} + \vec{a}$.
   $-\vec{a}$ is called the negative of $\vec{a}$.

6. Multiplication of a vector by a scalar is a binary operation on V. i.e. for any scalar m and $\vec{a} \in V$, $m\,\vec{a} \in V$.

7. For any scalars m, n and vector $\vec{a}$
   $(m + n)\,\vec{a} = m\,\vec{a} + n\,\vec{a}$

8. $m\,(n\vec{a}) = (m\,n)\,\vec{a}$

9. For $\vec{a}$, $\vec{b} \in V$ and any scalar m
   $m\,(\vec{a} + \vec{b}) = m\,\vec{a} + m\,\vec{b}$

10. $1\,\vec{a} = \vec{a}$ for any vector $\vec{a} \in V$.

11. $0\,\vec{a} = \vec{0}$ for any vector $\vec{a} \in V$.

Position vector of a point w.r.t. an origin :

Let P be a point and O be the origin of reference. Then the vector $\vec{OP}$ is called the position vector of P w.r.t. O as the origin.

Given the position vectors of two points P and Q w.r.t. O by $a = \vec{OP}$, $b = \vec{OQ}$, then

$$\vec{PQ} = \vec{OQ} - \vec{OP} = \vec{b} - \vec{a}$$

i.e. $\vec{PQ}$ = Position vector of $\vec{Q}$ - Position vector of P

or $\vec{PQ}$ = The position vector of the end- The position vector of the tail

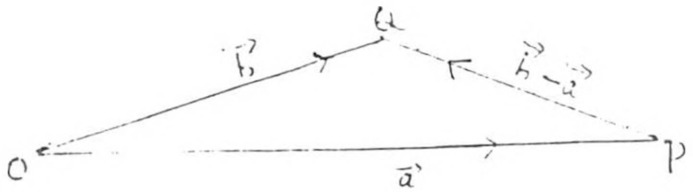


fig. 8.

In particular, the position vector of the origin is 0 always. Equivalently, the position vector of a point w.r.t. itself is $\vec{0}$.

Expressing a vector in terms of two given non collinear vectors - Linear independence and dependence of vectors.

i) Let $\vec{a}$ and $\vec{b}$ be two non collinear non zero vectors and $\vec{c}$ any vector in the plane of $\vec{a}$ and $\vec{b}$.

Taking $\vec{OA} = \vec{a}$ and $\vec{OB} = \vec{b}$.

Let $PA_1 \parallel b$ and $PB_1 \parallel a$ as shown, where $\vec{OP} = \vec{c}$.

Then $\vec{OA_1}$ and $\vec{OA}$ are collinear so that $\vec{OA_1} = x \cdot \vec{OA} = x\vec{a}$ for some scalar x. Similarly, $\vec{OB_1} = y\,\vec{OB} = y\vec{b}$ for some scalar y (because OB_1 and OB are collinear vectors.

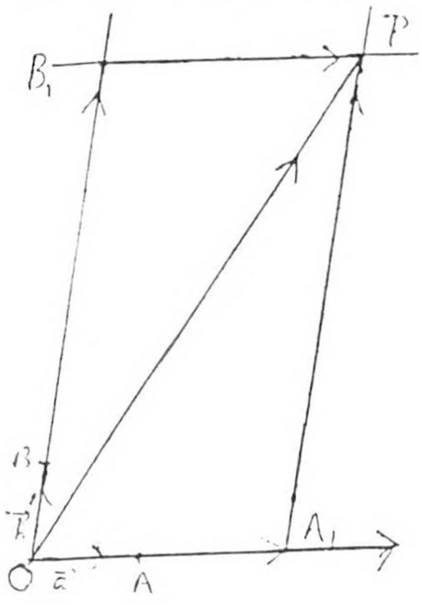


fig. 9.

Now $\vec{c} = \vec{OP} = \vec{OA_1} + \vec{A_1P} = x\,\vec{a} + y\,\vec{b}$
x and y being scalars.

Then we have $\vec{c} = x\,\vec{a} + y\,\vec{b}$.

Definition: (1) $x\,\vec{a} + y\,\vec{b}$ is called a linear combination of $\vec{a}$ and $\vec{b}$, x,y being scalars.

We have proved the following result - Given two non-collinear vectors, any vector in the plane of the given vectors can be expressed as a linear combination of the given vectors.
In particular we can write $\vec{a} = 1\,\vec{a} + o\,\vec{b}$ and $\vec{b} = o\,\vec{a} + 1\,\vec{b}$.

ii) If $\vec{a}$ and $\vec{b}$ are non-collinear vectors and $x\,\vec{a} + y\,\vec{b} = 0$, then $x = y = 0$ and conversely

a) Let $\vec{a}$ and $\vec{b}$ be non collinear vectors and $x\,\vec{a} + y\,\vec{b} = 0$.
We have to prove that $x = y = 0$. Let if possible $x \neq 0$.
Then $x\,\vec{a} + y\,\vec{b} = 0 \implies x\,a = -y\,b$

or $\vec{a} = \left(\frac{-y}{x}\right) b = K\,b,\ K = -\,y/x$

Hence $\vec{a}$ and $\vec{b}$ are collinear contrary to the assumption that $\vec{a}$ and $\vec{b}$ are non collinear.

(b) Conversely, assume that $x\vec{a} + y\vec{b} = 0$ implies $x = y = 0$. We need to prove that, $\vec{a}$ and $\vec{b}$ are non collinear. Contrarily if we assume that $\vec{a}$ and $\vec{b}$ are collinear, then $\vec{a} = K\vec{b}$ for some K.

Then $x\vec{a} + y\vec{b} = 0$ becomes $(xK + y)\, b = 0$

$\Rightarrow xK + y = 0$

Contradicting the assumption that $x\vec{a} + y\vec{b} = 0 \quad x = y = 0$

<u>Definition (2)</u> : Two vectors $\vec{a}$ and $\vec{b}$ are said to be linearly independent
if $x\vec{a} + y\vec{b} = 0$ implies $x = y = 0$

In the light of what is already proved, we have - two (non-zero) vectors are linearly independent if and only if they are non-collinear.

<u>Definition (3)</u> : Two vectors a and b are linearly dependent if they are not linearly independent. Then $x\,a + y\,b = 0 \not\Rightarrow x = y = 0$. Equivalently $x\vec{a} + y\vec{b} = 0$ implies that atleast one of x and y is non zero.

Consequently, two collinear vectors are linearly dependent.

iii) If $\vec{a}$ and $\vec{b}$ are non-collinear vectors and $\vec{c}$ is any vector in the plane of $\vec{a}$ and $\vec{b}$, then $\vec{c} = x\vec{a} + y\vec{b}$ and x,y are unique real numbers.

The first part that $\vec{C} = x\vec{a} + y\vec{b}$ for real x,y in already proved. We prove that x,y are unique. Let if possible,

$$\vec{C} = x\vec{a} + y\vec{b} = x^1\vec{a} + y^1\vec{b}$$

Then, $x\vec{a} + y\vec{b} = x^1\vec{a} + y^1\vec{b}$

i.e. $(x-x^1)\vec{a} + (y-y^1)\vec{b} = 0$

But a and b are non collinear.

Therefore, $x-x^1 = 0 = y-y^1$ or $x=x^1$ and $y=y^1$.

Hence x and y are unique real numbers.

The concepts of linear dependence and independence can be extended to more than two vectors.

Three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ are said to be linearly independent of $x\vec{a} + y\vec{b} + z\vec{c} = 0$ where x,y,z are scalars, implies $x=y=z = 0$.

The vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ are linearly dependent if they are not linearly independent. Consequently, $\vec{a}$, $\vec{b}$, $\vec{c}$ are linearly dependent if $x\vec{a} + y\vec{b} + z\vec{c} = 0$ implies that not all x, y, z are zero. (i.e. $x\vec{a} + y\vec{b} + z\vec{c} = 0 \quad x=y=z = 0$).

In the light of the results, we have proved

1. Any three coplanar vectors are linearly dependent.
2. Any three vectors which do not lie in a single plane are linearly independent.

Vectors in the Cartesian Plane - R^2 and the Cartesian Space-R^3 :

i) Position vectors in R^2

A point P in the Cartesian plane - R^2 is identified by its coordinates (x,y) and we write P = (x,y). Let i denote the unit vector $\vec{OA}$ along the x-axis and j denote the unit vector OB along the y-axis. Drawing PM $\perp$ to ox and PN $\perp$ oy,

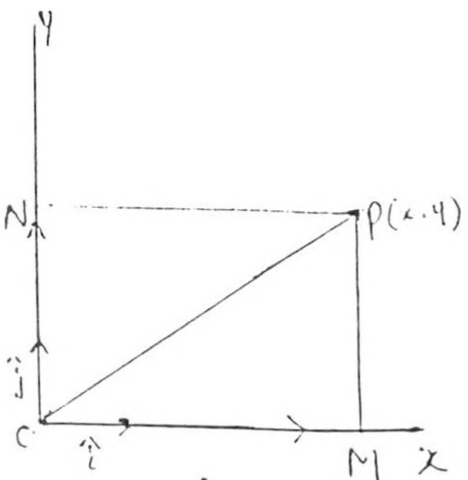


fig. 1c

Since P = (x,y), OM = x, ON = y

Then $\vec{OM} = x\,\hat{i}$, $\vec{ON} = y\,\hat{j}$

$\vec{OP} = \vec{OM} + \vec{MP} = \vec{OM} + \vec{ON} = x\hat{i} + y\,\hat{j}$

$OP = x\,\hat{i} + y\,\hat{j}$

Therefore, the position vector of P = (x,y) is $x\,\hat{i} + y\,\hat{j}$.

$(x,y) \mapsto (x\hat{i}+y\hat{j})$ is a one-one correspondence so that the position vector of P can be taken as (x,y) or $x\hat{i}+ y\hat{j}$.

In particular, $\hat{i} = 1.\hat{i} + 0.\hat{j} = (1,0)$ and $\hat{j} = 0.i + 1.j = (0,1)$.

ii) Position vectors in R^3

A point P in the cartesian space - R^3 is identified by its coordinates (x,y,z). Let i,j and k be the unit vectors along ox, oy and oz respectively.

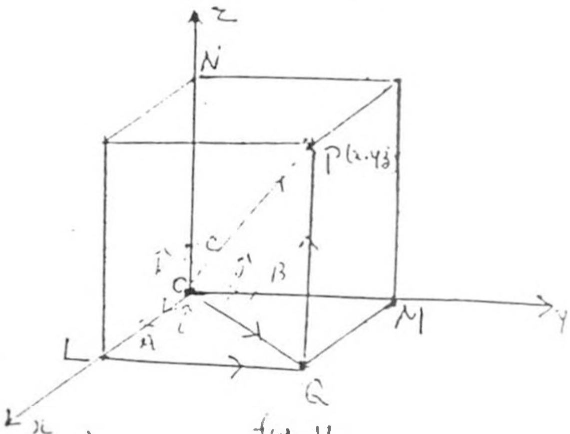


fig. 11

Then taking P = (x,y,z)

OL = x, OM = y and ON = z.

Hence $\vec{OL} = x\hat{i}$, $\vec{OM} = y\hat{j}$, $\vec{ON} = z\hat{k}$.

$\vec{OQ} = \vec{OL} + \vec{LQ} = \vec{OL} + \vec{OM} = xi + yj$

$\vec{OP} = \vec{OQ} + \vec{QP} = xi + yj + \vec{ON} = x\hat{i}+y\hat{j}+z\hat{k}$

Therefore, the position vector of P =(x,y,z) is $\vec{OP} = x\hat{i} + y\hat{j} + z\hat{k}$.

$(x,y,z) \longrightarrow x\hat{i} + y\hat{j} + z\hat{k}$ is a one-one correspondence so that the position vector of P can be taken as (x,y,z) or $x\hat{i} + y\hat{j} + z\hat{k}$.

In particular, $\hat{i} = 1\hat{i} + 0\hat{j} + 0\hat{k} = (1,0,0)$

$\hat{j} = 0\hat{i} + 1\hat{j} + 0\hat{k} = (0,1,0)$

and $\hat{k} = 0\hat{i} + 0\hat{j} + 1\hat{k} = (0,0,1)$

Summary :

1. In R^2, any point P (x,y) has its position vector as $x\hat{i}+ y\hat{j}$. Thus the position vector of any point can be expressed as a linear combination of the unit vectors $\hat{i}$ and $\hat{j}$.

$\hat{i}$ and $\hat{j}$ are called base vectors in R^2. These base vectors are linearly independent.

2. In R^3, any point P (x,y,z) has its position vector $x\hat{i} + y\hat{j} + z\hat{k}$. Thus the position vector of any point can be expressed as a linear combination of the unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$. These unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$ are called base vectors in R^3. These base vectors are linearly independent.

3. Given the position vector $\vec{OP} = x\hat{i} + y\hat{j} + z\hat{k}$, the vectors $x\hat{i}$, $y\hat{j}$ and $z\hat{k}$ are respectively called the x-component, the y-component and the z-component of the vector $\vec{OP}$.

Application of vector methods :

1. Show that the diagonals of a parallelogram bisect each other.

Let the sides $\vec{OA}$ and $\vec{OB}$ of the ▱ gm, represent the vectors $\vec{a}$ and $\vec{b}$ respectively. Let the diagonals intersect at D.

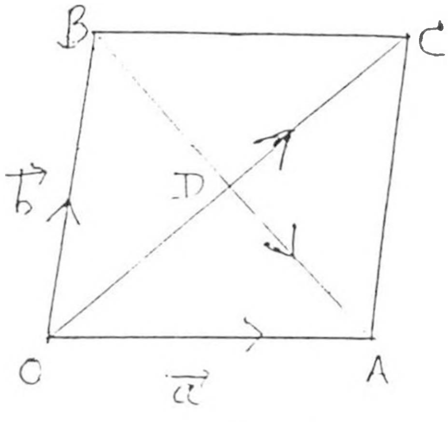


fig 12

Now $\vec{OC} = \vec{a} + \vec{b}$ and $\vec{BA} = \vec{a} - \vec{b}$

$\vec{BD}$ and $\vec{BA}$ are collinear. Likewise $\vec{OD}$ and $\vec{OC}$ are collinear.

Therefore, $\vec{OD} = m\,\vec{OC} = m\,(\vec{a} + \vec{b})$

and $\vec{BD} = \vec{BA} = n\,(\vec{a} - \vec{b})$

for some scalars m and n.

In the triangle OBD, $\vec{OB} + \vec{BD} = \vec{OD}$.

$$\vec{b} + n\,(\vec{a} - \vec{b}) = m\,(\vec{a} + \vec{b})$$

i.e. $(n - m)\,\vec{a} + (1 - n - m)\,\vec{b} = 0$

Since $\vec{a}$ and $\vec{b}$ are non-collinear (and therefore linearly independent) vectors, the equation implies $n-m = 0$, and

$$n = m = 1/2$$

$$\vec{OD} = 1/2\,\vec{OC} \quad \text{and} \quad \vec{BD} = 1/2\,\vec{BA}$$

Taking the magnitudes $OD = 1/2\,OC$ and $BD = 1/2\,BA$. Hence the diagonals bisect each other.

2. Prove that the medians of a triangle are concurrent at a point which trisects each median.

Let $\vec{OA} = \vec{a}$ and $\vec{OB} = \vec{b}$.

Let C, D and E be the mid points of the sides of the Δle OAB as shown in the figure.

Let AD and BC intersect at G.

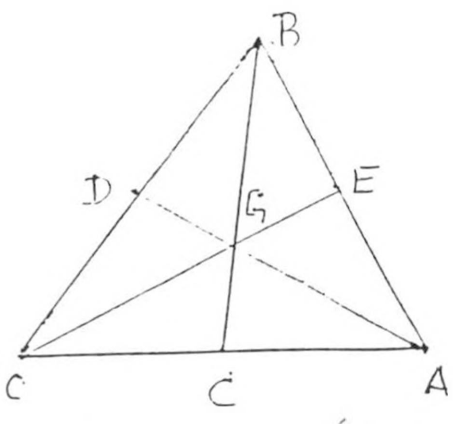


fig 13

Then $\vec{OC} = 1/2\,\vec{OA} = 1/2\,\vec{a}$ and

$\vec{OD} = 1/2\,\vec{OB} = 1/2\,\vec{b}$.

By the mid point formula, $\vec{OE} = 1/2(\vec{a}+\vec{b})$ · (i)

Now $\vec{OC} = \vec{OB} + \vec{BC}$ $\quad \vec{BC} = \vec{OC} - \vec{OB}$

or $\vec{BC} = 1/2(\vec{a} - \vec{b})$, $\vec{BG} = m\,\vec{BC}$ for some scalar m.

i.e. $BG = m\left(\frac{\vec{a}}{2} - b\right)$

$\vec{OG} = \vec{OB} + \vec{BG} = \vec{b} + m \quad (1/2\ \vec{a} - \vec{b})$

or $OG = \frac{m}{2}\vec{a} + (1-m)\vec{b}$ . . (ii)

Again, $\vec{AD} = \vec{OD} - \vec{OA} = \frac{1}{2}\vec{b} - \vec{a}$

Taking $\vec{AG} = n \cdot \vec{AD} = n\left(\frac{1}{2}\vec{b} - \vec{a}\right)$

$$\vec{OG} = \vec{OA} + \vec{AG} = \vec{a} + n\left(\frac{1}{2}\vec{b} - \vec{a}\right)$$

or $\vec{OG} = (1 - n)\ \vec{a} + \frac{n}{2}\ \vec{b}$ - - - (iii)

(ii) and (iii) $\Rightarrow \vec{OG} = \frac{m}{2}\vec{a} + (1-m)\ \vec{b} = (1-n)\ \vec{a} + \frac{n}{2}\ \vec{b}$

$$\Rightarrow \frac{m}{2} = 1 - n \text{ and } 1 - m = \frac{n}{2}$$

$\therefore\ m = n = 2/3 \ \therefore\ \vec{OG} = 1/3\ (\vec{a} + \vec{b}) = 2/3\ \vec{OE}$ ... from (i)

Hence OG and OE are collinear.

Therefore, the medians intersect at G.

Lastly, since $\vec{OG} = 2/3\ \vec{OE}$

$OG = 2/3\ OE$

or $OG/GE = 2/1$ or $OG : GE = 2:1$

Therefore, G is a point of trisection of each median.

3. Prove that the mid points of the sides of any quadrilateral are the vertices of a parallelogram. Let A,B,C,D in the vertices of a quadrilateral with position vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$. Denoting the mid points of the sides by P,Q,R,S, taking some point O as the origin

$\vec{OA} = \vec{a}$, $\vec{OB} = \vec{b}$, $\vec{OC} = \vec{c}$ and $\vec{OD} = \vec{d}$.

$\vec{OP} = \frac{1}{2}(\vec{a}+\vec{b})$, $\vec{OQ} = \frac{1}{2}(\vec{b}+\vec{c})$

$\vec{OR} = \frac{1}{2}(\vec{c}+\vec{d})$, $\vec{OS} = \frac{1}{2}(\vec{a}+\vec{d})$.

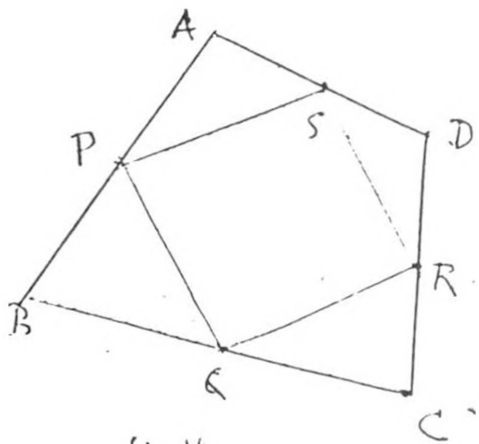


fig 14

$\vec{AC} = \vec{c} - \vec{a}$ (i) $\vec{QP} = \vec{OQ} - \vec{OP} = \frac{1}{2}(\vec{c} - \vec{a})$ ...(ii)

and $\vec{SR} = \vec{OR} - \vec{OS} = \frac{1}{2}(\vec{c} - \vec{a})$ ...(iii)

$\frac{1}{2}\vec{AC} = \vec{SR} = \vec{QP}$. $\vec{PQ} \parallel \vec{SR}$ and $|\vec{PQ}| = |\vec{SR}|$

Hence PQRS is a <u>parallelogram</u>.

4. ABCD is a quadrilateral, E and F are mid points of BC and CD respectively. Show that EF is parallel to and half of BD.

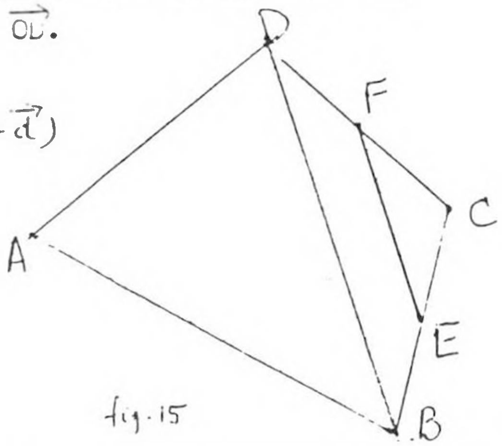


fig. 15

Let $\vec{a} = \vec{OA}$, $\vec{b} = \vec{OB}$, $\vec{c} = \vec{OC}$ and $\vec{d} = \vec{OD}$.

$\vec{BD} = \vec{OD} - \vec{OB} = \vec{d} - \vec{b}$ ---(i)

$\vec{OE} = \frac{1}{2}(\vec{b} + \vec{c})$, $\vec{OF} = \frac{1}{2}(\vec{c} + \vec{d})$

$\vec{EF} = \vec{OF} - \vec{OE}$

or $\vec{EF} = \frac{1}{2}(\vec{d} - \vec{b})$ ---(ii)

(i) and (ii) $\Longrightarrow$ $\vec{EF} = \frac{1}{2}\vec{BD}$

Hence $\vec{EF} \parallel \vec{BD}$ and $\vec{EF} = \frac{1}{2}\vec{BD}$.

5. If $\vec{a}, \vec{b}, \vec{c}$ are the position vectors of A, B, and C, show that the position vector of the centroid of $\triangle$ ABC is $\frac{\vec{a} + \vec{b} + \vec{c}}{3}$

Let E and D be the mid points of AB and BC.

Then $\vec{OE} = \frac{1}{2}(\vec{a} + \vec{b})$

G being the centroid of $\triangle$ ABC

Hence by section formula,

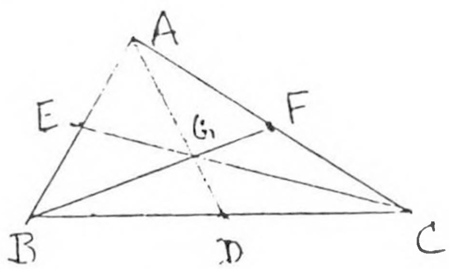


fig. 16

$$\vec{OG} = \frac{\frac{1}{2}\vec{OC} + \frac{1}{2}\vec{OE}}{\frac{1}{2} + 1}$$

$$\vec{OG} = \frac{\vec{a} + \vec{b} + \vec{c}}{3}$$

6. In a quadrilateral, the diagonals bisect each other. Show that the quadrilateral is a parallelogram.

Let $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$ in the position vectors w.r.t. O, of the vertices A, B, C, D of a quadrilateral.

If the diagonals bisect each other at E,

then $\overrightarrow{OE} = \dfrac{\overrightarrow{OA} + \overrightarrow{OC}}{2}$ or $\overrightarrow{OE} = \dfrac{\vec{a} + \vec{c}}{2}$

$\overrightarrow{OE} = \dfrac{\overrightarrow{OB} + \overrightarrow{OD}}{2} = \dfrac{\vec{b} + \vec{d}}{2}$ $\therefore$ $\dfrac{\vec{a} + \vec{c}}{2} = \dfrac{\vec{b} + \vec{d}}{2}$

Hence $\vec{a} + \vec{c} = \vec{b} + \vec{d}$

$\vec{a} - \vec{b} = \vec{d} - \vec{c}$

$\overrightarrow{BA} = \overrightarrow{DC}$. Hence BA = DC and BA||DC.

Therefore, ABCD is a parallelogram.

7. $\vec{a}$ and $\vec{b}$ are the adjacent sides of a regular hexagon ABCDE. Express the remaining sides in terms of $\vec{a}$ and $\vec{b}$.

Let $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{AB} = \vec{b}$.

Then $\overrightarrow{OB} = \vec{a} + \vec{b}$

$\overrightarrow{OC} = 2\,\vec{b}$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

$\overrightarrow{BC} = \vec{a} - \vec{b}, = \vec{b} - \vec{a}$

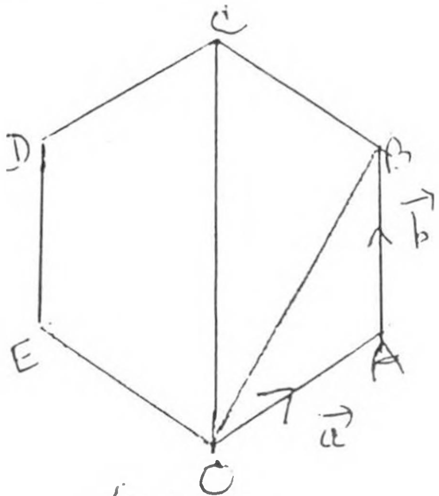


fig 17

Since $\overrightarrow{CD}$ and $\overrightarrow{OA}$ are equal and parallel,

$\overrightarrow{CD} = -\vec{a}$

|||ly $\overrightarrow{DE} = -\vec{b}$

and $\overrightarrow{EO} = -\overrightarrow{BC} = \vec{a} - \vec{b}$.

Hence $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{AB} = \vec{b}$, $\overrightarrow{BC} = \vec{b} - \vec{a}$,
$\overrightarrow{CD} = -\vec{a}$, $\overrightarrow{DE} = -\vec{b}$ and $\overrightarrow{EO} = \vec{a} - \vec{b}$.

8. Show that any three coplanar vector are always linearly dependent.

Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be any three coplanar vector. If any two of these or all of them as collinear, then obviously they are linearly dependent. Suppose $\vec{a}$ and $\vec{b}$ are non collinear.
Then we can write $\vec{c} = m\vec{a} + n\vec{b}$ for some scalars m and n.

$$m\vec{a} + n\vec{b} - \vec{c} = 0$$

Then we have by definition of linear dependence,
$\vec{a}$, $\vec{b}$, $\vec{c}$ are linearly dependent.

9. Show that for any point O, a system of concurrent forces represented by $\vec{OA}$, $\vec{OB}$, $\vec{OC}$ is equivalent to the forces represented by $\vec{OD}$, $\vec{OE}$, $\vec{OC}$, D, E, F being the mid points of $\vec{BC}$, $\vec{CA}$, $\vec{AB}$ respectively.

Let $\vec{OA} = \vec{a}$, $\vec{OB} = \vec{b}$, $\vec{OC} = \vec{c}$

Then $\vec{OD} = \dfrac{\vec{b}+\vec{c}}{2}$, $\vec{OE} = \dfrac{\vec{c}+\vec{a}}{2}$, $\vec{OF} = \dfrac{\vec{a}+\vec{b}}{2}$

$$\vec{OD} + \vec{OE} + \vec{OF} = \vec{a}+\vec{b}+\vec{c} = \vec{OA} + \vec{OB} + \vec{OC}$$

10. D, E, F are the mid points of the sides $\vec{BC}$, $\vec{CA}$ and $\vec{AB}$ of a triangle ABC. Show that the resultant of the three concurrent forces represented by $AD = \frac{2}{3}\vec{BE}$ and $\frac{1}{3}\vec{CF}$ is 1/2 AC.

The medians are concurrent at G(say)

$\frac{2}{3}\vec{BE} = \vec{BG}$ (by the property of the centroid)

Similarly, $\frac{1}{3}\vec{CF} = \vec{GF}$.

Now $\vec{AD} + \frac{2}{3}\vec{BE} + \frac{1}{3}\vec{CF}$

$= \vec{AD} + \vec{BG} + \vec{GF}$

$= \vec{AD} + \vec{BF} = \vec{AD} + \vec{FA}$ (F is the mid pt.of AB)

$= \vec{FD} = \frac{1}{2}\vec{AC}$ since $\vec{FD}$ is || to $\vec{AC}$ and 1/2 of AC.

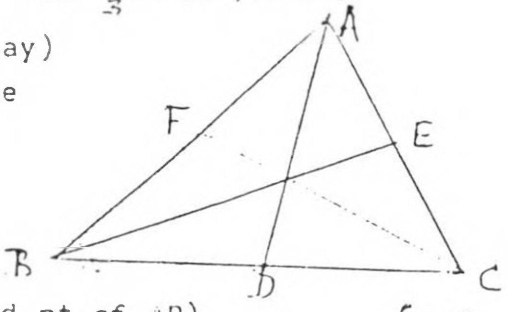


fig.18

Self Test and Assignment :

1. Which of the following are Scalars and which ones are vectors ?
a) weight, b) specific heat, c) density, d) volume
e) speed, f) calorie g) momentum, h) energy
i) density j) magnetic field intensity, k) work done
l) temperature.

2. ABC is a triangle in which $\vec{AB} = \vec{c}$ and $\vec{AC} = \vec{b}$. $\vec{AD}$ is the bisector of A meeting BC at D. Express $\vec{AD}$ in terms of $\vec{b}$ and $\vec{c}$.

3. Show that $-6\vec{a} + 3\vec{b} + 2\vec{c}$, $3\vec{a}-2\vec{b} + 4\vec{c}$, $5\vec{a} + 7\vec{b} + 3\vec{c}$ and $-13\vec{a} + 17\vec{b} - \vec{c}$ are coplanar vectors.

4. ABCDis a quadrilateral in which AB = AD. AX and AY bisect $\hat{BAC}$ and $\hat{DAC}$ respectively, meeting BC and CD at X and Y respectively. By vector method show that XY || BD.

5. If $\vec{a} = (2,3)$, $\vec{b} = (-1,5)$ find (a) $2\vec{a} + 3\vec{b}$, b) $|\vec{a} - 2\vec{b}|$ and c) a unit vector along $2\vec{a} - \vec{b}$.

6. If $\vec{a} = (1,1,-1)$, $\vec{b} = (4,1,2)$ and $\vec{c} = (0,1,-2)$ show that $(\vec{a}-\vec{b}) = 3(\vec{c}-\vec{a})$.

7. $\vec{a}$ and $\vec{b}$ are the adjacent sides of a regular hexagon. Express the remaining sides and the diagonals of the hexagon items on $\vec{a}$ and $\vec{b}$.

8. If $\vec{a}$ and $\vec{b}$ are non collinear vectors such that $x\vec{a} + y\vec{b} = 0$. Show that $x = y = 0$.

9. If $\vec{a}$, $\vec{b}$, $\vec{c}$ are non coplanar vectors such that $x\vec{a} + y\vec{b} + z\vec{c} = 0$, show that $x = y = z = 0$.

10. Given $\vec{a} = (2,-1,1)$, $\vec{b} = (1,3,-2)$, $\vec{c} = (-2,1,-3)$ and $\vec{d} = (3,2,5)$ find x,y,z such that $\vec{d} = x\vec{a} + y\vec{b} + z\vec{c}$.

11. Find a unit vector along the resultant of $\vec{a} = 2i + 4j - 5k$ and $\vec{b} = i + 2j + 3k$.

12. $\vec{a}, \vec{b}, \vec{c}$ are non coplanar vectors find whether the vectors $2\vec{a} - 3\vec{b} + \vec{c}$, $3\vec{a} - 5\vec{b} + 2\vec{c}$ and $4\vec{a} - 5\vec{b} + \vec{c}$ are linearly independent or dependent vectors ?

13. D,E,F are the mid points of the sides of a Δ ABC. Prove that for any point O inside the Δle,
$\vec{OA} + \vec{OB} + \vec{OC} = \vec{OD} + \vec{OE} + \vec{OF}$

Does the result hold if O is outside the Δle ?

14. ABCD is a parallelogram. P and Q are the mid points of BC and CD respectively. Prove by vector method that AP and AQ trisect the diagonal BD.

15. Show that $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3)$ and $\vec{c} = (c_1, c_2, c_3)$ are linearly independent iff

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$

Multiplication of Vectors :

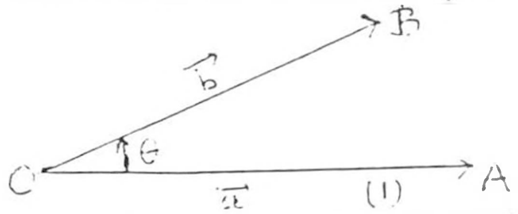
Scalar and Vector Products of two vectors :

1. Scalar Product of two vectors :

Let $\vec{a}$ and $\vec{b}$ be two vectors which include an angle $\theta\ (0 \leq \theta \leq \pi)$

we define the scalar product of $\vec{a}$ and $\vec{b}$ by

$a.b = |a|\ |b| \cos\theta$



(1)

The multiplication defined above is an operation on vectors is called scalar multiplication of vectors.

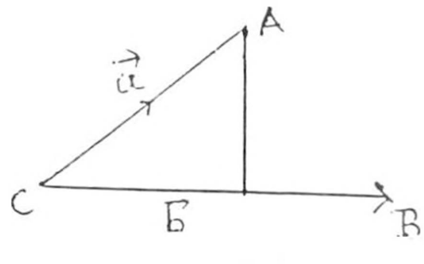
The product is also called the dot product of a and b.

Note :

1. $\vec{a} \cdot \vec{b}$ is a scalar (i.e. a real number) and not a vector.
2. If $\vec{a}$ and $\vec{b}$ are perpendicular vectors, then $\cos\theta = 0$ so that $\vec{a} \cdot \vec{b} = 0$.
3. Conversely, if $\vec{a}$, $\vec{b}$ are non zero vectors such that $\vec{a} \cdot \vec{b} = 0$, then $\vec{a}$ and $\vec{b}$ are at right angles.
4. Since $|\cos\theta| \leq 1$ always, $|\vec{a} \cdot \vec{b}| \leq |\vec{a}|\ |\vec{b}|$.
5. If $\theta = 0$ or π the vectors are collinear. Hence if $\vec{a}$ and $\vec{b}$ are collinear then $|\vec{a} \cdot \vec{b}| = |\vec{a}|\ |\vec{b}|$ and conversely.
6. Since $\cos(-\theta) = \cos\theta$, $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
7. $\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cos\theta$, $\cos\theta = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}|\ |\vec{b}|}$
8. For any vector $\vec{a}$, $\vec{a} \cdot \vec{a} = |\vec{a}|^2$

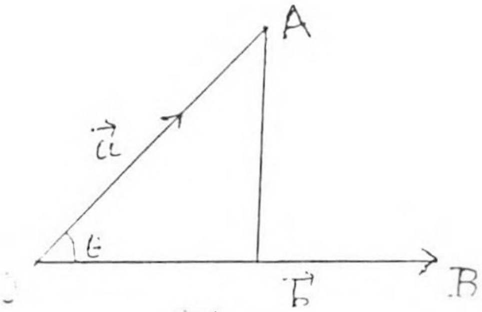
An Interpretation of $\vec{a} \cdot \vec{b}$:

Let $\vec{a}$ be a force under which a particle on which it acts, moves through a displacement $\vec{b}$.



(2)

Then the component of the force $\vec{a}$ in the direction of $\vec{b} = \vec{OK}$ and $\vec{OK} = |\vec{a}| \cos\theta$



(3)

Then the work done by the force $\vec{a}$ in moving the particle through the displacement $\vec{b} = |\vec{a}|\ |\vec{OK}|$

$= |\vec{a}| \cos\theta\ |\vec{b}| = |a|\ |b| \cos\theta = a \cdot b$

ii) If $\vec{OA} = \vec{a}$ and $\vec{OB} = \vec{b}$ drawing $AK \perp OB$, OK is called the projection of $\vec{a}$ on $\vec{b}$. $|OK| = |\vec{a}| \cos\theta = \frac{1}{|\vec{b}|} |\vec{a}|\ |\vec{b}| \cos\theta$

or the projection of $\vec{a}$ on $\vec{b} = \frac{1}{|\vec{b}|} (\vec{a} \cdot \vec{b})$

Similarly, the projection of $\vec{b}$ on $\vec{a} = \frac{1}{|\vec{a}|} (\vec{a} \cdot \vec{b})$

Combining these

The projection of $\vec{a}$ on $\vec{b} \times |\vec{b}| = \vec{a} \cdot \vec{b}$

= (The projection of $\vec{b}$ on $\vec{a}$) $\times |\vec{a}|$

If $\vec{a} \perp \vec{b}$, the projection of one vector on the other is zero.

<u>Scalar products of the unit vectors. i.j,k</u>

We recall that the unit vectors i,j,k are mutually at right angles, being the unit vectors along the x, the y and the z-axes respectively.

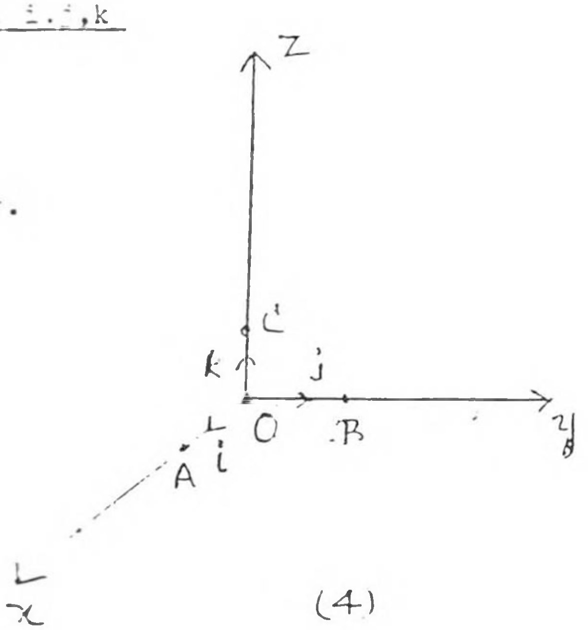
Hence, we find

1. $i \cdot i = i^2 = 1$
2. $j \cdot j = j^2 = 1$
3. $k \cdot k = k^2 = 1$

$i \cdot j = 0$

$j \cdot k = 0$

$k \cdot i = 0$

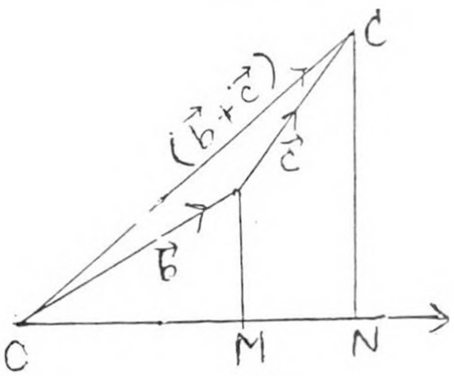


(4)

We put the above information in the multiplication table.

.	i	j	k
i	1	0	0
j	0	1	0
k	0	0	1

(The multiplication table for the dot products of i,j and k).



(5)

Properties of dot products :

i. For any three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$

$\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

Taking $\vec{a}$, $\vec{b}$, $\vec{c}$ as shown in the figure (5).

We observe that

$\overrightarrow{ON}$ = The projections of $\overrightarrow{OC}$ on a

$= \frac{1}{|\vec{a}|} \vec{a} \cdot (\vec{b} + \vec{c})$ - - - (i)

Also $\overrightarrow{ON} = \overrightarrow{OM} + \overrightarrow{MN}$ = Projection of $\vec{b}$ on $\vec{a}$ + projection of $\vec{c}$ on $\vec{a}$.

$= \frac{1}{|\vec{a}|} (\vec{a} \cdot \vec{b}) + \frac{1}{|\vec{a}|} (\vec{a} \cdot \vec{c})$ - - - (ii)

From (i) and (ii),

$$\frac{1}{|\vec{a}|} \vec{a} \cdot (\vec{b} + \vec{c}) = \frac{1}{|\vec{a}|} (\vec{a} \cdot \vec{b}) + \frac{1}{|\vec{a}|} (\vec{a} \cdot \vec{c})$$

$$\Rightarrow \vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$$

This property is called the distributive property.

In general, $\vec{a} \cdot (\vec{b} + \vec{c} + \vec{d} + \vec{e} + \ldots) =$

$\vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} + \vec{a} \cdot \vec{d} + \vec{a} \cdot \vec{e} + \ldots\ldots$

ii) It is already noted that

$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$ for any two vectors $\vec{a}$ and $\vec{b}$. This property is the **commutative property**.

iii) For any scalar m and vectors $\vec{a}$, $\vec{b}$

$m(\vec{a} \cdot \vec{b}) = (m\vec{a}) \cdot \vec{b} = \vec{a} \cdot (m\vec{b})$

iv) $0 \cdot \vec{a} = 0$ for any vector $\vec{a}$.

Summary of the Properties :

1. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$ for any vectors $\vec{a}$ and $\vec{b}$.
2. $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$ for any vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$.
3. $m(\vec{a} \cdot \vec{b}) = (m\vec{a}) \cdot \vec{b} = \vec{a} \cdot (m\vec{b})$
   for any scalar m and vectors $\vec{a}$ and $\vec{b}$.
4. $0 \cdot \vec{a} = 0$ for any vector $\vec{a}$.
5. If $\vec{a}$ and $\vec{b}$ are non zero vectors, then $\vec{a} \cdot \vec{b} = 0$ implies that $\vec{a} \perp \vec{b}$ and conversely,
6. $\vec{a} \cdot \vec{a} = \vec{a}^{\,2} = |\vec{a}|^2$ for any vector $\vec{a}$.
7. If $\vec{a}$ is a unit vector $\vec{a}^{\,2} = \vec{a} \cdot \vec{a} = 1$.
8. For unit vectors i,j,k which are mutually at right angles,

   a) $i^2 = j^2 = k^2 = 1$ b) $i.j = j.k = k.i = 0$

5. The dot product of the position vectors of two parts (a_1, a_2, a_3) and (b_1, b_2, b_3)

$$\vec{a} = a_1 i + a_2 j + a_3 k \text{ and } \vec{b} = b_1 i + b_2 j + b_3 k$$

$$\vec{a} \cdot \vec{b} = (a_1 i + a_2 j + a_3 k) \cdot (b_1 i + b_2 j + b_3 k)$$

$= a_1 b_1 + a_2 b_2 + a_3 b_3$, using the distributive properties and the multiplication table for i,j,k.

Hence, $\vec{a} \cdot \vec{b} = (a_1, a_2, a_3) \cdot (b_1, b_2, b_3) = a_1 b_1 + a_2 b_2 + a_3 b_3$

In particular,

$\vec{a} \cdot \vec{a} = a_1 + a_2 + a_3$

But $|\vec{a}|^2 = \vec{a} \cdot \vec{a}$

$|\vec{a}|^2 = a_1 + a_2 + a_3$

$|\vec{a}|$ = The magnitude of $\vec{a} = + \; a_1 + a_2 + a_3$

It is evident that

1. $|\vec{a}|$ 0 and
2. $|\vec{a}| = 0$ if and only if $a_1 = a_2 = a_3 = 0$ or $\vec{a} = 0$.

ii) The distance formula :

$A = (x_1, y_1, z_1)$ and $B = (x_2, y_2, z_2)$ be any two points of R^3.

Then $\vec{OA} = (x_1, y_1, z_1) = x_1 i + y_1 j + z_1 k$

and $\vec{OB} = (x_2, y_2, z_2) = x_2 i + y_2 j + z_2 k$

$\vec{AB} = \vec{OB} - \vec{OA} = (x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k$

Hence, $|\vec{AB}|$ = the distance between A and B

$$= +\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$

iii) The angle between two vectors (x_1, y_1, z_1) and (x_2, y_2, z_2)

Let $\vec{a} = (x_1, y_1, z_1)$ and $\vec{b} = (x_2, y_2, z_2)$

Then $\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$, θ being the angle between a and b.

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|}$$

$$\cos\theta = \frac{x_1x_2 + y_1y_2 + z_1z_2}{\sqrt{x_1^2 + y_1^2 + z_1^2 \quad (x_2^2 + y_2^2 + z_2^2)}}$$

Gives the angle θ between the vectors $\vec{a}$ and $\vec{b}$.

For vectors in R^2, taking $\vec{a} = (x_1, y_1)$ and

$\vec{b} = (x_2, y_2)$ i) $\vec{a} \cdot \vec{b} = x_1x_2 + y_1y_2$

ii) $|\vec{a}| = + x_1^2 + y_1^2$

iii) Distance between (x_1, y_1) and

$$\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$

iv) $$\cos\theta = \frac{x_1x_2 + y_1y_2}{\sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}}$$

The direction cosines/ratios of a line and angle between two vectors :

Definition 1. Consider the vectors $\overrightarrow{OP} = (x,y,z)$. Let α, β, γ be the angles which $\overrightarrow{OP}$ makes with the x, the y and the z-axes respectively.

Then, $\cos\alpha = \frac{\overrightarrow{OP}\cdot\hat{i}}{|\overrightarrow{OP}|}$, $\cos\beta = \frac{\overrightarrow{OP}\cdot\hat{j}}{|\overrightarrow{OP}|}$, $\cos\gamma = \frac{\overrightarrow{OP}\cdot\hat{k}}{|\overrightarrow{OP}|}$

$\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are called the direction cosines. (d.c's) of $\overrightarrow{OP}$.

Let $\frac{a}{\cos\alpha} = \frac{b}{\cos\beta} = \frac{c}{\cos\gamma}$

Definition 2. $\vec{a}$, $\vec{b}$, $\vec{c}$ are called the direction ratios of $\overrightarrow{OP}$

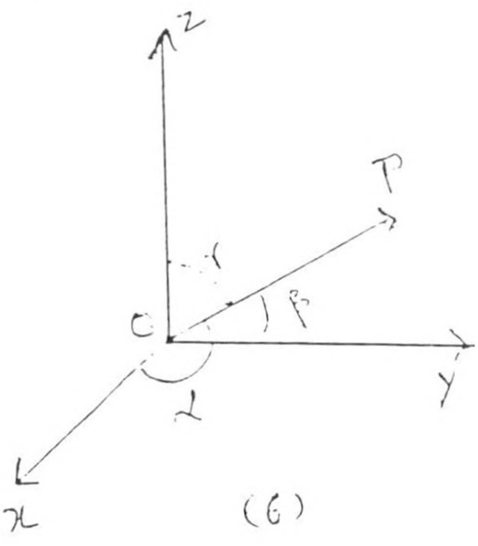
Putting $l = \cos\alpha$, $m = \cos\beta$ and $n = \cos\gamma$

$l = \cos\alpha = \frac{(x\hat{i} + y\hat{j} + z\hat{k})\cdot\hat{i}}{|\overrightarrow{OP}|} = \frac{x}{|\overrightarrow{OP}|}$

Similarly, $m = \frac{y}{|\overrightarrow{OP}|}$ and $n = \frac{z}{|\overrightarrow{OP}|}$

However, $|\overrightarrow{OP}| = x^2+y^2+z^2$

Hence $l^2+m^2+n^2 = \frac{x^2+y^2+z^2}{|OP|^2} = 1$



(6)

We have proved that the direction cosines l,m,n satisfy the equation $l^2 + m^2 + n^2 = 1$

Given the direction ratios a,b,c to find the direction cosines of the vector $\overrightarrow{OP}$.

$a/l = b/m = c/n = K$ (say)

$a^2+b^2+c^2 = K^2 (l^2+m^2+n^2) = K^2.1 = K^2$

$$K = \sqrt{a^2 + b^2 + c^2}$$

The direction cosines $(l,m,n) =$

$$\left(\frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}}\right)$$

Note: If $\vec{CP} = (x,y,z)$, then x,y,z are the direction ratios of $\vec{CP}$.

2. The direction cosines of $\vec{CP}$ are

3. If (l,m,n) are the direction cosines of $\vec{CP}$ any point on CP can be taken as (lt, mt, nt) t being a real number.

<u>Angle of intersection of two vectors :</u>

Let (x_1, y_1, z_1) and (x_2, y_2, z_2) be the vectors $\vec{CP}$ and $\vec{CQ}$ respectively and include an angle θ .

Let (l_1, m_1, n_1) and (l_2, m_2, n_2) be the direction cosine of $\vec{CP}$ and $\vec{CQ}$. Then

$$l_1 = \frac{x_1}{|\vec{CP}|}, \quad m_1 = \frac{y_1}{|\vec{CP}|}, \quad n_1 = \frac{z_1}{|\vec{CP}|}$$

and $$l_2 = \frac{x_2}{|\vec{CQ}|}, \quad m_2 = \frac{y_2}{|\vec{CQ}|}, \quad n_2 = \frac{z_2}{|\vec{CQ}|}$$

Then $$\cos\theta = \frac{\vec{CP}\cdot\vec{CQ}}{|\vec{CP}||\vec{CQ}|}$$

i.e. $$\boxed{\cos\theta = \frac{x_1x_2 + y_1y_2 + z_1z_2}{|\vec{CP}||\vec{CQ}|}}$$

$$\therefore \cos\theta = \frac{x_1}{|\vec{CP}|}\cdot\frac{x_2}{|\vec{CQ}|} + \frac{y_1}{|\vec{CP}|}\cdot\frac{y_2}{|\vec{CQ}|} + \frac{z_1}{|\vec{CP}|}\cdot\frac{z_2}{|\vec{CQ}|}$$

$$\boxed{\cos\theta = l_1l_2 + m_1m_2 + n_1n_2}$$

or $\theta = \cos^{-1}(l_1l_2 + m_1m_2 + n_1n_2)$

In particular, if $CP \perp CQ$, then

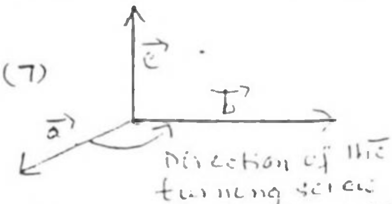
$\cos\theta = 0 = l_1l_2 + m_1m_2 + n_1n_2$

Thus the condition for orthogonality of CP and CQ is

$$l_1l_2 + m_1m_2 + n_1n_2 = 0$$

Vector Product of two vectors :

1. Let $\vec{a}$ and $\vec{b}$ in the given vectors. Let $\vec{c}$ be at right angles to both $\vec{a}$ and $\vec{b}$. The vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ form a right handed system, if $\vec{c}$ is the direction of the movement of a screw as the screw is turned such that $\vec{a}$ rotates towards $\vec{b}$.

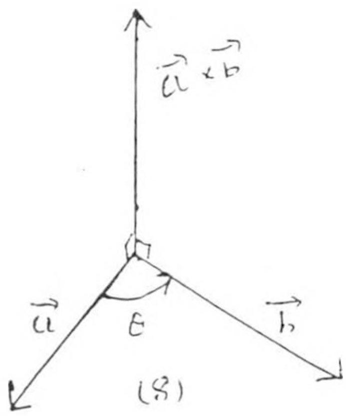


Let $\hat{n}$ be the unit vector so that $\vec{a}$, $\vec{b}$ and $\hat{n}$ form a right handed system.

We define the vector product or the cross product of $\vec{a}$ and $\vec{b}$, DENOTING it by $\vec{a} \times \vec{b}$ by

$$\vec{a} \times \vec{b} = \left(|\vec{a}| \; |\vec{b}| \; \sin\theta \right) \hat{n}$$

$$|\vec{a} \times \vec{b}| = |\vec{a}| \; |\vec{b}| \; |\sin\theta|$$



Note :

1. $\vec{b} \times \vec{a} = -(\vec{a} \times \vec{b})$
2. If $\theta = 0$ or π, a and b are collinear. Then $\vec{a} \times \vec{b} = 0$.
3. In particular $\vec{a} \times \vec{a} = 0$.

2. Properties :

1. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$ (distributive property).
2. $m(\vec{a} \times \vec{b})$ $(m\vec{a}) \times \vec{b} = \vec{a} \times (m\vec{b})$

3. The cross products of i,j,k

Consider the unit vectors i,j,k along the axes. Then $i \times i = j \times j = k \times k = 0$

Since i,j,k form a right handed system,

$i \times j = k$, $j \times k = i$, $k \times i = j$

and $j \times i = -k$, $k \times j = -i$, $i \times k = -j$.

We display this information in the multiplication table.

X	i	j	k
i	0	k	-j
j	-k	0	i
k	j	-i	0

4. The cross product of $\vec{a} = a_1 i + a_2 j + a_3 k$ and $\vec{b} = b_1 i + b_2 j + b_3 k$.

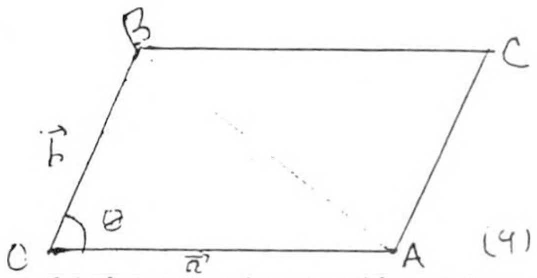
Using the distributive property and the multiplication table, we get

$$(a_1 i + a_2 j + a_3, k) \times (b_1 i + b_2 j + b_3 k)$$

$$= (a_2 b_3 - a_3 b_2)\ i + (a_3 b_1 - a_1 b_3)\ uj + (a_1 b_2 - a_2 b_1)\ k$$

$$\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

5. The meaning of $|\ a \times b\ |$.



(9)

The area of the parallelogram whose adjacent sides are $\vec{a}$ and $\vec{b}$

$= |\ OA\ .\ OB\ .\ \sin\theta\ |$

$= |\ \vec{a}\ |\ \ |\ \vec{b}\ |\ \ |\sin\theta\ |$

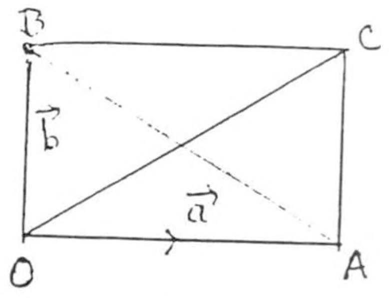
$= |\ \vec{a} \times \vec{b}\ |$

Hence, $|\ \vec{a} \times \vec{b}|$ = The area of the parallelogram whose adj. sides are $\vec{a}$ and $\vec{b}$.

or $1/2\ |\ \vec{a} \times \vec{b}\ |$ = (The area of the triangle formed by $\vec{a}$ and $\vec{b}$).

<u>Problems on dot and cross products</u> .

1. Prove that in a rectangle, the diagonals are equal.



(10)

Let $\vec{OA} = \vec{a}$ and $\vec{OB} = \vec{b}$ be the adj. sides of a rectangle OACB.

Since OACB is a rectangle

$OA \perp OB \therefore a\ .\ b = 0$

$\vec{OC} = \vec{a} + \vec{b}$ and $\vec{BA} = \vec{a} - \vec{b}$

$|\vec{OC}|^2 = (\vec{a}+\vec{b})^2 = |\ \vec{a}\ | + |\ \vec{b}\ | + 2\vec{a}\ .\ \vec{b}$

$= |\vec{a}|^2 + |\vec{b}|^2$

$(\vec{BA})^2 = (\vec{a} - \vec{b})^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a} \cdot \vec{b} = |a|^2 + |b|^2$

Hence, $|\vec{OC}| = |\vec{BA}|$

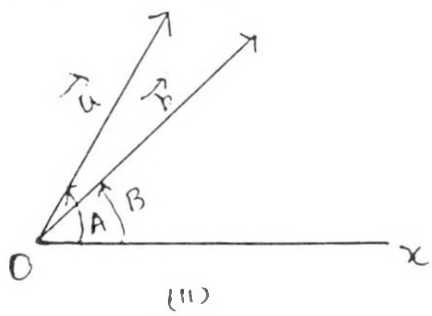
i.e. the diagonals are equal.

2. If the sides of a quadrilateral are equal (so that it is a rhombus), show that the diagonals intersect at right angles.
Let $\vec{a}$, $\vec{b}$ be the sides of the quadrilateral. It is given that
$|\vec{a}| = |\vec{b}|$

The diagonals are given by $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$.

$(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = \vec{a} \cdot \vec{a} - \vec{b} \cdot \vec{b} = |\vec{a}|^2 - |\vec{b}|^2 = 0$
Hence the diagonals cut at right angles.

3. Prove that Cos (A-B) = Cos A Cos B + Sin A Sin B.
Consider the unit vectors
$\vec{OA} = \vec{a} = (\cos A)\, i + (\sin A) j$
and $\vec{OB} = \vec{b} = (\cos B)\, i + (\sin B) j$



$\vec{a} \cdot \vec{b} = |\vec{a}|\, |\vec{b}| \cos (A-B)$ (i)

$= \cos (A-B)$ since $\vec{a}$, $\vec{b}$ are unit vectors and they include an angle (A-B).

Also, $\vec{a} \cdot \vec{b} = \left[(\cos A)i + (\sin A)j\right] \left[(\cos B)i + (\sin B)j\right]$

$= \cos A \cos B + \sin A \sin B$. --- (ii)
From (i) to (ii),
$\cos (A-B) = \cos A \cos B + \sin A \sin B$.

4. Show that A(0,1,1), B(3,1,5) and C(0,3,3) form a right angled triangled with right angle at C.
$\vec{AC} = \vec{OC} - \vec{OA} = (0,3,3) - (0,1,1)$
$= (0,2,2) = 0i + 2j + 2k$.
i.e. $\vec{AC} = 2j + 2k$.

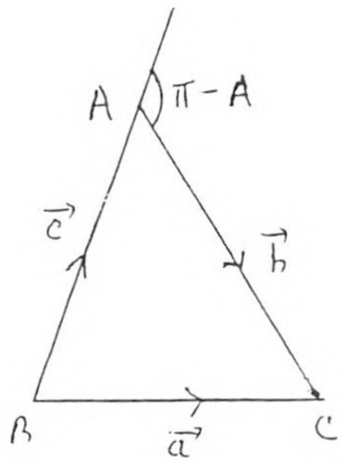
$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = (0,3,3) - (3,1,5) = (-3, 2, -2)$

i.e. $\overrightarrow{BC} = -3i + 2j - 2k$

$\overrightarrow{AC} . \overrightarrow{BC} = -3.0 + 2.2 - 2.2 = 0$

Hence, $\overrightarrow{AC} \perp \overrightarrow{BC}$ and the triangle is right angled at C.

5. Prove the cosine formula.
$a^2 = b^2 + c^2 - 2bc \cos A$ in a triangle ABC.



(12)

Consider the triangle ABC in which $\overrightarrow{BC} = \vec{a}$, $\overrightarrow{BA} = \vec{c}$ and $\overrightarrow{AC} = \vec{b}$.

The angle between $\overrightarrow{BA}$ and $\overrightarrow{AC} = (\pi - A)$

Also in the triangle, $\vec{a} = \vec{b} + \vec{c}$.

$\vec{a} . \vec{a} = (\vec{b} + \vec{c}) . (\vec{b} + \vec{c})$

i.e. $|\vec{a}| = |\vec{b}| + |\vec{c}| + 2\,\vec{b}.\vec{c}$

Since $|\vec{a}| = a$, $|\vec{b}| = b$, $|\vec{c}| = c$,

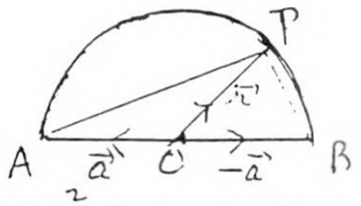
$a^2 = b^2 + c^2 + 2bc \cos(\pi - A)$

$= b^2 + c^2 - 2bc \cos A$.

Hence, $a^2 = b^2 + c^2 - 2bc \cos A$.

6. Show that a diameter of a circle subtends a right angle at its circumference.

Let AB be a diameter and P, a point on the circle.



(13)

Let $\overrightarrow{OA} = \vec{a}$.

then $\overrightarrow{OB} = -\vec{a}$, being equal and opposite to $\vec{a}$.

Let $\overrightarrow{OP} = \vec{r}$, $\overrightarrow{AP} = \vec{r} - \vec{a}$, $\overrightarrow{AP} . \overrightarrow{BP}$
$= |\vec{r}|^2 - |\vec{a}|^2 = 0$
since $|\vec{r}| = |\vec{a}|$

$\overrightarrow{BP} = \vec{r} + \vec{a}$

Hence $AP \perp BP$ or $APB = 90^\circ$.

7. Show that the altitudes of a triangle are concurrent.

Let the altitudes AD and BE intersect at O.

Join CO and let it meet AB at F. We show that CF $\perp$ AB.

Let $\vec{OA} = \vec{a}$, $\vec{OB} = \vec{b}$, $\vec{OC} = \vec{c}$.

$\vec{AD} \perp \vec{BC}$ $\therefore \vec{a} \cdot \vec{BC} = 0$

$\vec{a} \cdot (\vec{c} - \vec{b}) = 0$

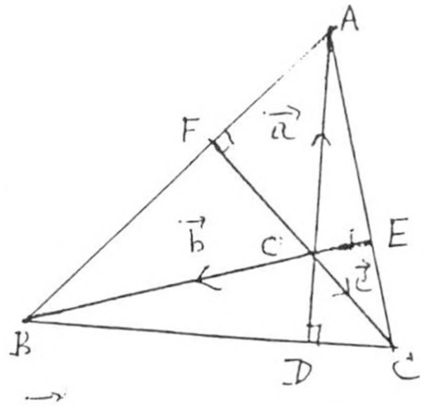
$\vec{a} \cdot \vec{c} = \vec{a} \cdot \vec{b}$ - (i)

BE $\perp$ AC $\therefore \vec{b} \cdot \vec{AC} = 0$

i.e. $\vec{b} \cdot (\vec{c} - \vec{a}) = 0$

i.e. $\vec{b} \cdot \vec{c} = \vec{b} \cdot \vec{a}$

or $\vec{b} \cdot \vec{c} = \vec{a} \cdot \vec{b}$ - - - (ii)



(14)

(i) and (ii) imply, $\vec{a} \cdot \vec{c} = \vec{b} \cdot \vec{c}$

or $(\vec{a} - \vec{b}) \cdot \vec{c} = 0$

or $(\vec{BA} \cdot \vec{C}) = 0$

Hence, CF $\perp$ AB.

8. Derive the formula. Sin (A-B) = Sin A Cos B - Cos A Sin B.

Consider $\vec{a} = \vec{OA} = (\cos A)\, i + (\sin A)\, j$

and $\vec{b} = \vec{OB} = (\cos B)\, i + (\sin B) j$

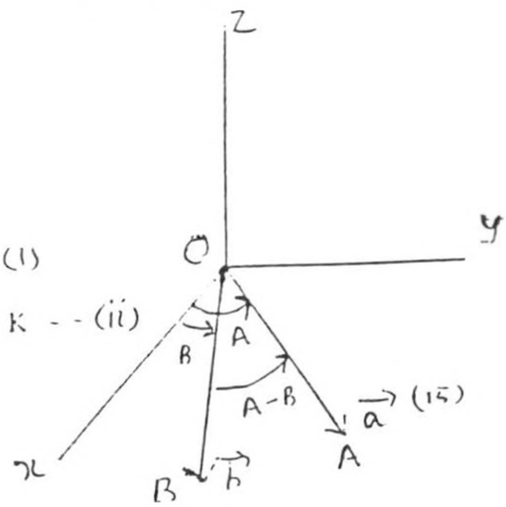
such that $\vec{a} \times \vec{b}$ is in the direction of the z-axis.

Then $\vec{a} \times \vec{b} = |\vec{a} \times \vec{b}|\, k$

$$= \begin{vmatrix} i & j & k \\ \cos B & \sin B & 0 \\ \cos A & \sin A & 0 \end{vmatrix}$$

$= (\sin A \cos B - \cos A \sin B)\, K$ - - (i)

Also, $\vec{a} \times \vec{b} = |\vec{a}|\, |\vec{b}| \sin (A - B)\, K$ - - (ii)



(15)

Hence, from (i) and (ii)

Sin (A-B)

= Sin A Cos B - Cos A Sin B.

9. If $\vec{a}$, $\vec{b}$, $\vec{c}$ are the position vectors of A, B, C, show that the area of $\triangle$ ABC is 1/2 $|\vec{b} \times \vec{c} + \vec{c} \times \vec{a} + \vec{a} \times \vec{b}|$

$\vec{AB} = \vec{b} - \vec{a}$ and $\vec{AC} = \vec{c} - \vec{a}$

The area of $\Delta ABC = 1/2 \; |\vec{AB} \times \vec{AC}|$ --- (i)

$\vec{AB} \times \vec{AC} = (\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})$

$= \vec{b} \times \vec{c} - \vec{b} \times \vec{a} - \vec{a} \times \vec{c} + \vec{a} \times \vec{a}$

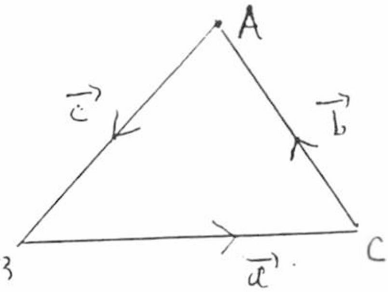
$= \vec{b} \times \vec{c} + \vec{a} \times \vec{b} + \vec{c} \times \vec{a} + 0$

$\vec{AB} \times \vec{AC} = \vec{b} \times \vec{c} + \vec{c} \times \vec{a} + \vec{a} \times \vec{b}$ ...(ii)

From (i) and (ii)

Area of the triangle $ABC = 1/2 \; |b \times c + c \times a + a \times b|$

10. Prove the sine rule:



(16)

For a triangle ABC taking

$\vec{a} = \vec{BC}$, $\vec{b} = \vec{CA}$ and $\vec{c} = \vec{AB}$

$\vec{a} + \vec{b} + \vec{c} = 0$

$\vec{c} \times (\vec{a} + \vec{b} + \vec{c}) = 0$

$\vec{c} \times \vec{a} + \vec{c} \times \vec{b} + \vec{c} \times \vec{c} = 0$

$\vec{c} \times \vec{a} - \vec{b} \times \vec{c} = 0$; $\vec{b} \times \vec{c} = \vec{c} \times \vec{a}$

Similarly, we can show that $\vec{a} \times \vec{b} = \vec{b} \times \vec{c}$

Hence, $\vec{b} \times \vec{c} = \vec{c} \times \vec{a} = \vec{a} \times \vec{b}$

$bc \sin A = \vec{c}\vec{a} \sin B = ab \sin c$

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

or $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

11. Find a unit vector perpendicular to the plane containing the vectors $2i - 6j - 3k$ and $4i + 3j - k$.

Taking $\vec{a} = 2i - 6j - 3k$ and $b = 4i + 3j - k$

$\vec{a} \times \vec{b}$ is $\perp$ to the plane of $\vec{a}$ and $\vec{b}$.

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ 2 & -6 & -3 \\ 4 & 3 & -1 \end{vmatrix} = 15i - 10j + 30k$$

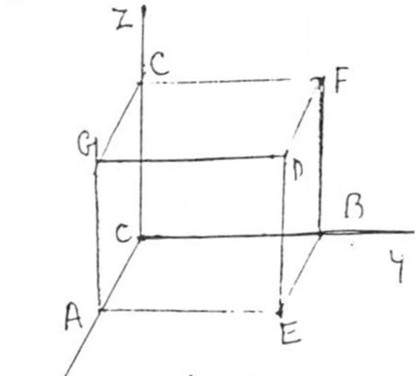
A unit vector along $\vec{a} \times \vec{b}$ is $\dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$

$|\vec{a} \times \vec{b}| = 15^2 + 10^2 + 30^2 = 35.$

A unit vector required is $\left(\frac{3}{7}\right)i - \left(\frac{2}{7}\right)j + \left(\frac{6}{7}\right)k$
Another unit vector is the negative of this vector

12. Find the acute angle between the diagonals of a cube.

Consider the unit cube whose in edges.
OA, OB and OC are of unit length, taken as axes of reference.



(17)

Then D = (1,1,1), A(1,0,0), B=(0,1,0)
C = (0,0,1), E = (1,1,0), F = (0,1,1)
G = (1,0,1).

The d.c.s of CD are proportional to (1-0, 0-0, 0-0) = (1,0,0)
The d.c.s of AF are proportional to (0-1, 1-0, 1-0) = (-1, 1,1)
If θ is the acute angle between the diagonals CD and AF of the cube, then

$$\cos\theta = | l_1 l_2 + m_1 m_2 + n_1 n_2 |$$

where (l_1, m_1, n_1) and (l_2, m_2, n_2) are the d.c.s of OD and AF.

$(l_1, m_1, n_1) = (1/\sqrt{3}, 0, 0)$
and $(l_2, m_2, n_2) = (-1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$

$$\cos\theta = | 1/\sqrt{3} \times -1/\sqrt{3} + 0 \times 1/\sqrt{3} + 0 \times 1/\sqrt{3} | = 1/\sqrt{3}$$

$$\boxed{\theta = \cos^{-1}(1/\sqrt{3})}$$

13. Find the area of the parallelogram whose diagonals are $3i + j - 2k$ and $i - 3j + 4k$.

If $\vec{a}$ and $\vec{b}$ are the adjacent sides of the parallelogram, then
$\vec{a} + \vec{b} = 3i + j - 2k$
$\vec{a} - \vec{b} = i - 3j + 4k$
$\vec{a} = 1/2\,(3i + j - 2k + i - 3j + 4k)$
$= 2i - j + k$
and $\vec{b} = 1/2\,(3i + j - 2k - i + 3j - 4k) = i + 2j - 3k.$
The area of the parallelogram $= | \vec{a} \times \vec{b} |$.

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ 2 & -1 & 1 \\ 1 & 2 & -3 \end{vmatrix} = i + 7j + 5k$$

$$|\vec{a} \times \vec{b}| = 1^2 + 7^2 + 5^2 = 75 = 5\sqrt{3}$$

The area of the parallelogram is $5\sqrt{3}$ sq. units.

14. Show that for any three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$

$(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$

Since $\vec{a} + \vec{b}$ is in the plane of $\vec{a}$ and $\vec{b}$,

$(\vec{a} + \vec{b}) \times \vec{c}$ and $\vec{a} \times \vec{c} + \vec{b} \times \vec{c}$ have the same direction. Therefore, it is enough to show that both these vectors have the same magnitude.

Let $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3)$ and $\vec{c} = (c_1, c_2, c_3)$

Then $\vec{a} + \vec{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$

$$(\vec{a} + \vec{b}) \times \vec{c} = \begin{vmatrix} i & j & k \\ a_1 + b_1 & a_2 + b_2 & a_3 + b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} i & j & k \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$= \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$

Hence the result.

16. Show that $(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) = 2(\vec{a} \times \vec{b})$

Interpret the result geometrically.

$(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b})$

$= \vec{a} \times \vec{a} + \vec{a} \times \vec{b} - \vec{b} \times \vec{a} + \vec{b} \times \vec{b}$

$\quad + \vec{a} \times \vec{b} + \vec{a} \times \vec{b} + 0$

$= 2(\vec{a} \times \vec{b})$.

Equating the magnitudes on both sides of the identity,

$$\left|(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b})\right| = 2 \; |\vec{a} \times \vec{b}|$$

Since $\vec{a} - \vec{b}$ and $\vec{a} + \vec{b}$ are the diagonals of the parallelogram whose adj. sides are $\vec{a}$ and $\vec{b}$, the result means that the area of the parallelogram formed with the diagonals as adj. sides is twice the area of the original parallelogram.

<u>Assignment/Self-Test</u> :

1. Prove : $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
2. Find the angle between $\vec{a} = 2i + 3j + k$ and $\vec{b} = -i + 2j - 6k$.
3. Find the values of p so that $a = (p-1)\, i + 2j + k$ and $\vec{b} = pi + (p-1)\, j + 2k$ are at right angles.
4. Show that 3i-2j+k, i-3j+5k and 2i+j-4k form a right angled triangle.
5. Show by vector method that in a rectangle the diagonals are of equal length.
6. Prove by vector method, that the diagonals of a rhombus are perpendicular bisectors of each other.
7. Find the angle which 3i-6j+2k makes with the coordinate axes.
8. Find the projection of i-2j + k on 4i-j + 2k.
9. Prove the cosine rule : $a^2+b^2 - ab \cos c = c^2$ for any triangle ABC by vector method.
10. Prove by vector method the sine rule for a triangle ABC :

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

11. Find the unit vector perpendicular to the vectors 2i-6j +3k and 4i+3j-k.
12. Find the work done by the force F = i-j+2 in moving a particle through 3i-2j+k.

13. Find the distance of the origin from the plane normal to the vector 2i+j+4k passing through the point (1,-2,3).

14. S.T. $|\vec{a} \times \vec{b}| + |\vec{a} \cdot \vec{b}| = |\vec{a}| \cdot |\vec{b}|$

15. Prove : $\vec{a} \times (\vec{b}+\vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{b}$

16. Given $\vec{a} = 3i-j + 2k$, $\vec{b} = 2i +2j - k$ and $\vec{c} = -i + 3j - 4k$
Form $(\vec{a} \times \vec{b}) \times \vec{c}$ and $\vec{a} \times (\vec{b} \times \vec{c})$.

17. Find the area of the triangle two of whose sides are 2i - 3j + 4k and i + 2j - k.

18. Prove by vector method :
 i) $\cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta$
 ii) $\sin(\alpha - \beta) = \sin\alpha \cos\beta - \cos\alpha \sin\beta$.

19. Evaluate : i) 2i x (3i-2k) ii) (i+2j)
 iii) (4i+j - 2k) x (2i-3j + 4k)
 iv) (i-j+2k) . (-i+2j-4k)
 v) i x (j + k) + j x (k + i) + k x (i + j)

20. Find the area of the triangle whose vertices are A = (3, -1, 2), B = (1, -1, -3) and C = (4, 3, -1).

TRIPLE PRODUCTS

Scalar Triple Product of (three) vectors :

Definition: A scalar product of a vector with the cross products of two other vectors is called a scalar triple product of three vectors.

If $\vec{a}, \vec{b}, \vec{c}$ are any three vectors, then, $\vec{a} \cdot (\vec{b} \times \vec{c})$ is a scalar triple product of $\vec{a}$, $\vec{b}$ and $\vec{c}$. Likewise, $(\vec{a} \times \vec{b}) \cdot \vec{c}$, $\vec{b} \cdot (\vec{c} \times \vec{a})$, $(\vec{b} \times \vec{c}) \cdot \vec{a}$ are also scalar triple products.

A scalar triple product is a scalar.

Geometrical meaning of a scalar triple product :

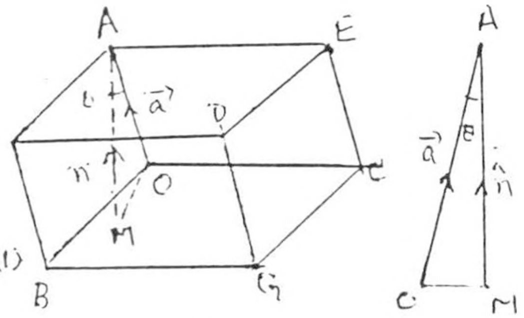
Consider a box whose coterminus edges are $\vec{a} = \vec{OA}$, $\vec{b} = \vec{OB}$ and $\vec{c} = \vec{OC}$. Let n be the unit vector perpendicular to the plane of $\vec{b}$ and $\vec{c}$. Let AM = h = the height of the box it and makes θ with $\vec{a}$.

Then the volume of the box

= Area of OBGC x height

$= |\vec{OB} \times \vec{OC}|\, h = |\vec{b} \times \vec{c}|\, OA \cos\theta$ -(i)

$= |\vec{b} \times \vec{c}|\, |\vec{a}| \cos\theta$



(i)

But $\vec{a} \cdot (\vec{b} \times \vec{c}) = |\vec{a}|\ |\vec{b} \times \vec{c}|\ \cos\theta\ \hat{n}$

so that $|\vec{a} \cdot \vec{b} \times \vec{c}| = |\vec{b} \times \vec{c}|\ |\vec{a}| \cos\theta$ -- (ii)

Hence $|\vec{a} \cdot (\vec{b} \times \vec{c})|$ = volume of the box.

Therefore, the volume of the box whose coterminus edges are $\vec{a}, \vec{b}, \vec{c}$ is $|\vec{a} \cdot \vec{b} \times \vec{c}|$.

Taking the face containing $\vec{a}$ and $\vec{b}$ as the base of the box, the volume the box = $|(\vec{a} \times \vec{b}) \cdot|$

Thus, $|\vec{a} \cdot (\vec{b} \times \vec{c})| = |(\vec{a} \times \vec{b}) \cdot \vec{c}|$ = the volume of the box.

We call this product as the box product and denote it by $[\vec{a}, \vec{b}, \vec{c}]$

Thus $[\vec{a}, \vec{b}, \vec{c}] = \vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$

ii) Properties of Scalar Triple Products :

Already we have seen that

$\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c} = [\vec{a}, \vec{b}, \vec{c}]$ - - (i)

$[\vec{b}, \vec{c}, \vec{a}] = (\vec{b} \times \vec{c}) \cdot \vec{a}$

$= [\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{b}) \times \vec{c} = [\vec{a}, \vec{b}, \vec{c}]$

Thus $[\vec{a}, \vec{b}, \vec{c}] = [\vec{b}, \vec{c}, \vec{a}]$

Similarly, $[\vec{b}, \vec{c}, \vec{a}] = [\vec{c}, \vec{a}, \vec{b}]$

Thus, we have

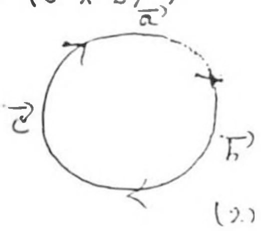
$$[\vec{a}, \vec{b}, \vec{c}] = [\vec{b}, \vec{c}, \vec{a}] = [\vec{c}, \vec{a}, \vec{b}] \quad \cdots (2)$$

In other words, the product is not altered by changing the vectors in cyclic order.

$$[\vec{a}, \vec{b}, \vec{c}] = \vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{a} \cdot (-(\vec{c} \times \vec{b}))$$

$$= -\vec{a} \cdot (\vec{c} \times \vec{b}) = -[\vec{a}, \vec{c}, \vec{b}]$$

$$[\vec{a}, \vec{b}, \vec{c}] = -[\vec{a}, \vec{c}, \vec{b}] \quad --- (3)$$



(2)

Similarly, $[\vec{a}, \vec{b}, \vec{c}] = -[\vec{b}, \vec{a}, \vec{c}]$

and also, $[\vec{a}, \vec{b}, \vec{c}] = -[\vec{c}, \vec{b}, \vec{a}]$

i.e. <u>In other words, the product changes its sign if two vectors are interchanged</u>.

Consequent to (1) we get

$$[\vec{a}, \vec{c}, \vec{b}] = [\vec{c}, \vec{b}, \vec{a}] = [\vec{b}, \vec{a}, \vec{c}] \quad --- (4)$$

$$[m\vec{a}, \vec{b}, \vec{c}] = [\vec{a}, m\vec{b}, \vec{c}] = [\vec{a}, \vec{b}, m\vec{c}] = m[\vec{a}, \vec{b}, \vec{c}] \quad -- (5)$$

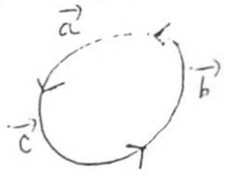
for any scalar m.

Obviously, $[\vec{0}, \vec{b}, \vec{c}] = 0 \quad -- \quad (i)$

$[\vec{a}$, b, c are non zero vectors.

Then, $[\vec{a}, \vec{a}, \vec{b}] = 0 \quad ---- (ii)$

$[\vec{a}, m\vec{a}, \vec{b}] = 0 \quad --- (iii)$



(3)

If $\vec{a}, \vec{b}, \vec{c}$ are coplanar vectors, then $[\vec{a}, \vec{b}, \vec{c}] = 0$

and Conversely--(iv).

<u>The condition for coplanarity of vectors</u>

$\vec{a}, \vec{b}, \vec{c}$ is $[\vec{a}, \vec{b}, \vec{c}] = 0 \quad -- (v)$

iii) Scalar product of $\vec{a} = (a_1, a_2, a_3)$,

$\vec{b} = (b_1, b_2, b_3)$ and $\vec{c} = (c_1, c_2, c_3)$.

$$\vec{b} \times \vec{c} = \begin{vmatrix} i & j & k \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$= (b_2c_3 - b_3c_2)\, i + (b_3c_1 - b_1c_3) j + (b_1c_2 - b_2c_1)\, k$

$\vec{a} \cdot (\vec{b} \times \vec{c})$

$= (a_1 i + a_2 j + a_3 k \cdot (b_2c_3 - b_3c_2)\, i + (b_3c_1 - b_1c_3)\, j$
$+ (b_1c_2 - b_2c_1)\, k$

$= a_1(b_2c_3 - b_3c_2) + a_2 \quad (b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1)$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Hence $[\vec{a}, \vec{b}, \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Vector Triple Product of (three) vectors :

Consider $\vec{a}, \vec{b}, \vec{c}$. Product as $(\vec{a} \times \vec{b}) \times \vec{c}$ and $\vec{a} \times (\vec{b} \times \vec{c})$ are called vector triple products of $\vec{a}, \vec{b}, \vec{c}$.

An expression for $(\vec{a} \times \vec{b}) \times \vec{c}$.

$\vec{d} = \vec{a} \times \vec{b}$ is $\perp$ to $\vec{a}$ and $\vec{b}$

$\vec{d} \times \vec{c}$ is $\perp$ to $\vec{d}$

i.e. $(\vec{a} \times \vec{b}) \times \vec{c}$ is perpendicular to $\vec{a} \times \vec{b}$ (which itself is perpendicular to $\vec{a}$ and $\vec{b}$) as well as to $\vec{c}$.

Therefore, $(\vec{a} \times \vec{b}) \times \vec{c}$ is in the plane of $\vec{a}$ and $\vec{b}$.

Hence, we can write

$\vec{x} = (\vec{a} \times \vec{b}) \times \vec{c} = m\,\vec{a} + n\,\vec{b}$

Now $\vec{x}$ is perpendicular to $\vec{c}$, from (i)

$\vec{x} \cdot \vec{c} = 0$

$(m\,\vec{a} + n\,\vec{b}) \cdot \vec{c} = 0$

$m\,(\vec{a} \cdot \vec{c}) + n\,(\vec{b} \cdot \vec{c}) = 0$

$$\frac{m}{-(\vec{b} \cdot \vec{c})} \qquad \frac{n}{(\vec{a} \cdot \vec{c})} = t \quad \text{(say)}$$

$m = -(\vec{b} \cdot \vec{c})\, t$

and $n = (\vec{a} \cdot \vec{c})\, t$

substituting these values in

$\vec{x} = (\vec{a} \times \vec{b}) \times \vec{c} = t \quad (\vec{a} \cdot \vec{c})\, \vec{b} - (\vec{b} \cdot \vec{c})\, \vec{a}$

$(\vec{a} \times \vec{b}) \times \vec{c} = t \ (\vec{a} \cdot \vec{c})\, \vec{b} - (\vec{b} \cdot \vec{c})\, \vec{a}$

This is an identity which should hold for all choice of $\vec{a}$, $\vec{b}$, $\vec{c}$.

Put $\vec{a} = \vec{c} = i$ and $\vec{B} = j$

$(i \times j) \times i = t \ (i.i)\, j - (j.k)$

$k \times i = t\ (1.j)$

$j = t\, j \qquad t = 1$

Hence $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c})\, \vec{b} - (\vec{b} \cdot \vec{c})\, \vec{a}$ - - (1)

Also, $\vec{a} \times (\vec{b} \times \vec{c}) = -(\vec{B} \times \vec{c}) \times \vec{a}$

$= -\ (\vec{b} \cdot \vec{a})\, \vec{c} - (\vec{c} \cdot \vec{a})\, \vec{b}$

$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{c} \cdot \vec{a})\, \vec{b} - (\vec{b} \cdot \vec{a})\, \vec{c}$ - - (2)

Comparing (1) and (2) we find that

$\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{B}) \times \vec{c}$.

This is obvious even otherwise, because the left hand side vector lies in the plane of $\vec{b}$ and $\vec{c}$ while the right hand side vector lies in the plane of $\vec{a}$ and $\vec{b}$.

An easy method to recall the formula 1 and 2.

Consider $(\vec{a} \times \vec{B}) \times \vec{c}$

| the other vector | the mid vector |

Call $\vec{b}$ as the mid vector and $\vec{a}$ as the other vector noting that both these lie in the brackets.

Then $(\vec{a} \times \vec{B}) \times \vec{c}$ = a scalar times the mid vector - a scalar times the other vector

$= m\, \vec{b} - n\, \vec{a}$

The first scalar on the R.H.S. is formed by the vectors other than the mid vector i.e. $m = (\vec{a} \cdot \vec{c})$.

The second scalar on the R.H.S. is formed by the vectors other than the other vector i.e. $n = (\vec{b} \cdot \vec{c})$

Hence, we can write

$(\vec{a} \times \vec{B}) \times \vec{c} = (\vec{a} \cdot \vec{c})\, \vec{b} - (\vec{b} \cdot \vec{c})\, \vec{a}$

Likewise, in $\vec{a} \times (\vec{b} \times \vec{c})$, the mid vector = b, the other vector = c.

$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\, \vec{B} - (\vec{a} \cdot \vec{b})\, \vec{c}$

Solved problems :

1. Find the volume of the box whose coterminus edges are represented by $\vec{a} = 2i - 3j + k$, $\vec{b} = i - j + 2k$ and $\vec{c} = 2i + j - k$.

The volume is given by

$$V = \begin{vmatrix} 2 & -3 & 1 \\ 1 & -1 & 2 \\ 2 & 1 & -1 \end{vmatrix} = 2\,(1-2) + 3\,(-1-4) + 1(1+2)$$
$$= -2 - 15 + 3 = -14$$

(Ignoring the sign) V = 14 cubic units.

2. Find the constant so that $\vec{a} = 2i - j + k$, $\vec{b} = i + 2j - 3k$, $\vec{c} = 3i + mj + 5k$ are coplanar. For coplanarity of $\vec{a}, \vec{b}, \vec{c}$, $[\vec{a}, \vec{b}, \vec{c}] = 0$

i.e. $\begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & -3 \\ 3 & m & 5 \end{vmatrix} = 0$

$2(10 + 3m) + 1(5 + 9) + 1\,(m-6) = 0$

$7m + 28 = 0$ or $m = -4$.

3. Prove that the four points whose position vectors are A(4,5,1), B (0,-1,-1), C (3,9,4) and D(-4, 4,4) are coplanar.

$\vec{AB} = \vec{OB} - \vec{OA} = (0, -4, -1-5, -1-1) = (-4, -6, -2)$

$\vec{BC} = \vec{OC} - \vec{OB} = (3-0, 9+1, 4+1) = (3,10,5)$

$\vec{CD} = \vec{OD} - \vec{OC} = (-4-3, 4-9, 4-4) = (-7,-5,0)$

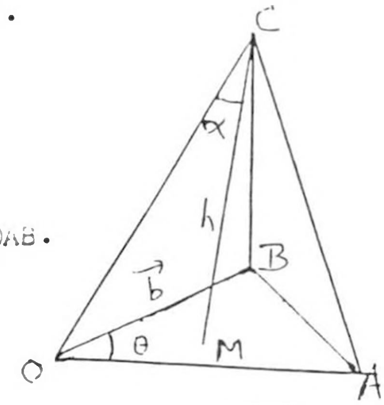
A, B, C, D are coplanar, if $\vec{AB}$, $\vec{BC}$, $\vec{CD}$ are coplanar, for which their scalar triple product must vanish.

$$\begin{vmatrix} -4, & -6 & -2 \\ 3 & 10 & 5 \\ -7 & -5 & 0 \end{vmatrix} = -4 \times 25 + 6 \times 35 - 2(-15 + 70)$$
$$= -100 + 210 - 110 = 0$$

Hence the result.

4. Find the volume of the tetrahedron formed by the vectors $\vec{a} = \vec{OA}$, $\vec{b} = \vec{OB}$ and $\vec{c} = \vec{OC}$.

The volume of the tetrahedron OABC =



(4)

$= 1/3$ (the area of the base) x height.

$= 1/3$ (Area of OAB) x perpendicular from C to $\triangle$ OAB).

$= 1/3$ OAB = x (i) where h = CM = the length of the $\perp$r from C to OAB.

Area $\triangle$ OAB = $1/2$ OA OB sin AOB

$= 1/2\ |a|\ |b|\ \sin\theta$ when $\theta = \widehat{AOB}$

$|a| \cdot |b| \sin\ = 2\ \triangle$ OAB.... (ii)

Let n be the unit vector $\perp$r to the plane OAB as given by the right hand screw rule and h = the length of the perpendicular from C to the plane OAB and α = the angle which the perpendicular makes with OC. (i.e. $\widehat{OCM} = \alpha$)

Then $[\vec{a}, \vec{b}, \vec{c}] = (\vec{a} \times \vec{b}) \cdot \vec{c} = |\vec{a}|\ |\vec{b}|\ \sin\ )\ \hat{n} \cdot \vec{c}$

$= 2\ (\triangle OAB)\ \cdot\ 1 \cdot |\vec{c}| \cos\alpha$

$= 2\ (\triangle OAB)\ h$

$(\triangle OAB) \times h = 1/2\ [\vec{a}, \vec{b}, \vec{c}]$ ...(iii)

From (i) and (iii) $V = 1/3\ \triangle OAB \times h$

$= 1/3 \times 1/2\ [\vec{a}, \vec{b}, \vec{c}] = 1/6\ [\vec{a}, \vec{b}, \vec{c}]$

$V = 1/6\ [\vec{a}, \vec{b}, \vec{c}]$

5. Find the volume of the tetrahedron if the position vectors of its corners are $\vec{\alpha}, \vec{\beta}, \vec{\gamma}$ and $\vec{\delta}$

Putting $a = \vec{\alpha} - \vec{\delta}$ $b = \vec{\beta} - \vec{\delta}$ and $c = \vec{\gamma} - \vec{\delta}$

$\vec{a}, \vec{b}, \vec{c}$ represent the coterminus edges of the tetrahedron.

By the previous problem, we volume is

$V = 1/6\ [\vec{a}, \vec{b}, \vec{c}] = 1/6\ [\vec{\alpha} - \vec{\delta},\ \vec{\beta} - \vec{\delta},\ \vec{\gamma} - \vec{\delta}]$.

$$V = (\vec{\alpha}-\vec{\delta}) \times (\vec{\beta}-\vec{\delta}) \cdot (\vec{\gamma}-\vec{\delta})$$

$$= \left[\vec{\alpha}\times\vec{\beta} - \vec{\alpha}\times\vec{\delta} - \vec{\delta}\times\vec{\beta} + \vec{\delta}\times\vec{\delta}\right]\cdot\left[\vec{\gamma}-\vec{\delta}\right]$$

$$= (\vec{\alpha}\times\vec{\beta})\cdot\vec{\gamma} - (\vec{\alpha}\times\vec{\beta})\cdot\vec{\delta} - (\vec{\alpha}\times\vec{\delta})\cdot\vec{\gamma} + (\vec{\alpha}\times\vec{\delta})\cdot\vec{\delta}$$

$$- (\vec{\delta}\times\vec{\beta})\cdot\vec{\gamma} + (\vec{\delta}\times\vec{\beta})\cdot\vec{\delta} \quad (\because \vec{\delta}\times\vec{\delta} = 0)$$

$$= \left[\vec{\alpha},\vec{\beta},\vec{\gamma}\right] - \left[\vec{\alpha},\vec{\beta},\vec{\delta}\right] - \left[\vec{\alpha},\vec{\delta},\vec{\gamma}\right] - \left[\vec{\delta},\vec{\beta},\vec{\gamma}\right]$$

Since $\left[\vec{\alpha}\times\vec{\delta}\right]\cdot\vec{\delta} = 0 = \left[\vec{\delta}\times\vec{\beta}\right]\cdot\vec{\delta}$

Hence

$$V = \frac{1}{6}\left[\left[\vec{\alpha},\vec{\beta},\vec{\gamma}\right] - \left[\vec{\alpha},\vec{\beta},\vec{\delta}\right] - \left[\vec{\alpha},\vec{\delta},\vec{\gamma}\right] - \left[\vec{\delta},\vec{\beta},\vec{\gamma}\right]\right]$$

6. Prove that A(4,5,1), B(0,-1,-1), C(3,9,4) and D(-4,4,4) are coplanar.

$\vec{AB} = -4i + 6j - 2k$, $BC = 3i + 10j + 5k$ and

$\vec{CD} = -7i - 5j$

$$\left[\vec{AB}, \vec{BC}, \vec{CD}\right] = \begin{vmatrix} -4 & -6 & -2 \\ 3 & 10 & 5 \\ -7 & -5 & 0 \end{vmatrix}$$

= −4 (0+25) + 6 (0+35) − 2(−15+70)

= −100 + 210 − 110 = 0

Hence, A, B, C and D are coplanar.

7. S.T. $\left[\vec{b} + \vec{c}, \vec{c} + \vec{a}, \vec{a} + \vec{b}\right] = 2\left[\vec{a}, \vec{b}, \vec{c}\right]$

L.H.S. $= (\vec{b} + \vec{c}) \times (\vec{c} + \vec{a}) \quad . (\vec{a} + \vec{b})$

$= \vec{b} \times \vec{c} + \vec{b} \times \vec{a} + \vec{c} \times \vec{c} + \vec{c} \times \vec{a}$

$= (\vec{b} \times \vec{c}) . \vec{a} + (\vec{b} \times \vec{a}) . \vec{a} + (\vec{c} \times \vec{c}) . \vec{a} + (\vec{c} \times \vec{a}) . \vec{b}$

$= \left[\vec{b}, \vec{c}, \vec{a}\right] + 0 + 0 + \left[\vec{c}, \vec{a}, \vec{b}\right]$

$= \left[\vec{a}, \vec{b}, \vec{c}\right] + \left[\vec{a}, \vec{b}, \vec{c}\right] = 2\left[\vec{a}, \vec{b}, \vec{c}\right] =$ RHS

8. S.T. $\left[\vec{a} \times \vec{b}, \vec{b} \times \vec{c}, \vec{c} \times \vec{a}\right] = \left[\vec{a}, \vec{b}, \vec{c}\right]^2$

Let $\vec{a} = a_1 i + a_2 j + a_3 k$

$\vec{b} = b_1 i + b_2 j + b_3 k$

$\vec{c} = c_1 i + c_2 j + c_3 k$

$$\left[\vec{a}, \vec{b}, \vec{c}\right] = (\vec{a} \times \vec{b}) . \vec{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = (1) \quad \text{say}$$

We denote the cofactor of each element by the corresponding capital.

Thus, $A_1 = \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} = (b_2c_3 - b_3c_2) =$ The cofactor of a_1

$A_2 = - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} = - (b_1c_3 - b_3c_1) =$ The cofactor of a_2

and so on.

$$\vec{b} \times \vec{c} = \begin{vmatrix} i & j & k \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = A_1 i + A_2 j + A_3 k$$

$$\vec{c} \times \vec{d} = \begin{vmatrix} i & j & k \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = B_1 i + B_2 j + B_3 k$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = C_1 i + C_2 j + C_3 k$$

$$\left[\vec{a} \times \vec{b},\ \vec{b} \times \vec{c},\ \vec{c} \times \vec{a}\right] = \left[\vec{b} \times \vec{c},\ \vec{c} \times \vec{a},\ \vec{a} \times \vec{b}\right]$$

$$= \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

$$\left[\vec{a}, \vec{b}, \vec{c}\right] =$$

$$\left[\vec{a}\times\vec{b},\ \vec{b}\times\vec{c},\ \vec{c}\times\vec{a}\right] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ b_2 & b_2 & b_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1A_1 + b_1B_1 + c_1C_1 & a_1A_2 + b_1B_2 + c_1C_2 & a_1A_3 + b_1B_3 + c_1C_3 \\ a_2A_1 + b_2B_1 + c_2C_1 & a_2A_2 + b_2B_2 + c_2C_2 & a_2A_3 + b_2B_3 + c_2C_3 \\ a_3A_1 + b_3B_1 + c_3C_1 & a_3A_2 + b_3B_2 + c_3C_2 & a_3A_3 + b_3B_3 + c_3C_3 \end{vmatrix}$$

$$= \begin{vmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{vmatrix} = \quad = a, b, c$$

Hence, a x b, b x c, c x a = a, b, c

<u>Second Method</u> :

Putting $\vec{X} = \vec{b} \times \vec{c}$

$(\vec{b} \times \vec{c}) \times (\vec{c} \times \vec{a}) = \vec{X} \times (\vec{c} \times \vec{a})$

$= (\vec{X}.\vec{a})\ \vec{c} - (\vec{X}.\vec{c})\ .\ \vec{a}$ (using the formula for the vector triple product)

$= (\vec{b} \times \vec{c})\ .\ \vec{a})\ \vec{c} - (\vec{b} \times \vec{c})\ \vec{c})\ \vec{a}$

$= \left[\vec{b}, \vec{c}, \vec{a}\right]\ \vec{c}$

$\left[\vec{a} \times \vec{b},\ \vec{b} \times \vec{c},\ \vec{c} \times \vec{a}\right] = (\vec{a} \times \vec{b}),\ (\vec{b} \times \vec{c}) \times (\vec{c} \times \vec{a})$.

$= (\vec{a} \times \vec{b})\ .\left[\vec{b}, \vec{c}, \vec{a}\right]\ \vec{c} = \left[(\vec{a} \times \vec{b})\ .\ \vec{c}\right] \left[\vec{b}, \vec{c}, \vec{a}\right]$.

$= [\vec{a}, \vec{b}, \vec{c}]^2$

9. Simplify $(\vec{a}+\vec{b}) \cdot (\vec{a}+\vec{c}) \times (\vec{b}+\vec{c})$

The given expression $= (\vec{a}+\vec{b}) \cdot \vec{a} \times \vec{b} + \vec{a} \times \vec{c} + \vec{c} \times \vec{b} + \vec{c} \times \vec{c}$

$= (\vec{a}+\vec{b}) \cdot \vec{a} \times \vec{b} + \vec{a} \times \vec{c} + \vec{c} \times \vec{b} + 0$

$= \vec{a} \cdot (\vec{a} \times \vec{b}) + \vec{a} \cdot (\vec{a} \times \vec{c}) + \vec{a} \cdot (\vec{c} \times \vec{b})$

$+ \vec{b} \cdot (\vec{a} \times \vec{b}) + \vec{b} \cdot (\vec{a} \times \vec{c}) + \vec{b} \cdot (\vec{c} \times \vec{b})$

$= 0 + 0 + [\vec{a}, \vec{c}, \vec{b}] + 0 + [\vec{b}, \vec{a}, \vec{c}] + 0$

$= [\vec{a}, \vec{c}, \vec{b}] - [\vec{a}, \vec{b}, \vec{c}]$

$= [\vec{a}, \vec{c}, \vec{b}] + [\vec{a}, \vec{c}, \vec{b}] = 2 [\vec{a}, \vec{c}, \vec{b}]$

10. If $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \times \vec{c}$ S.T. $\vec{a}$ and $\vec{c}$ are collinear vectors and hence show that $(\vec{c} \times \vec{a}) \times \vec{b} = 0$

The data implies (using the Triple Product formulas)

$(\vec{a} \cdot \vec{c}) \vec{b} - (\vec{a} \cdot \vec{b}) \vec{c} = (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{b} \cdot \vec{c}) \vec{a}$

$(\vec{a} \cdot \vec{b}) \vec{c} = (\vec{b} \cdot \vec{c}) \vec{a}$ - (i)

$\vec{c} = \frac{\vec{b} \cdot \vec{c}}{\vec{a} \cdot \vec{b}} \vec{a}$ $\quad \vec{c} = k\vec{a}$, k being a scalar.

Hence, $\vec{a}$ and $\vec{c}$ are collinear vectors.

From (i) $(\vec{b} \cdot \vec{c}) \vec{\ } - (\vec{a} \cdot \vec{b}) \vec{c} = (\vec{c} \times \vec{a}) \times \vec{b} = 0$

11. Find the area of the triangle whose vertices are $A(a_1, a_2, a_3)$, $B(b_1, b_2, b_3)$ and $C(c_1, c_2, c_3)$.

The area of the triangle ABC $= 1/2\ \overrightarrow{AB} \cdot \overrightarrow{AC} \sin A$

$= 1/2\ |\overrightarrow{AB} \times \overrightarrow{AC}|$

$= 1/2\ (\vec{b}-\vec{a}) \times (\vec{c}-\vec{a})$

$= 1/2\ |\vec{b} \times \vec{c} - \vec{b} \times \vec{a} - \vec{a} \times \vec{c} + \vec{a} \times \vec{a}|$

$= 1/2\ |\vec{b} \times \vec{c} + \vec{a} \times \vec{b} + \vec{c} \times \vec{a}|$

$= 1/2\ |a \times b + b \times c + c \times a|$

where $a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\ i$

$$\vec{b} \times \vec{c} = \begin{vmatrix} i & j & k \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = (b_2c_3 - b_3c_2)\, i$$

and

$$\vec{c} \times \vec{a} = \begin{vmatrix} i & j & k \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = (c_2a_3 - c_3a_2)\, i$$

12. Show that $\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) = 0$

$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} . \vec{c}) - (\vec{a} . \vec{B})\, \vec{c}$

$\vec{b} \times (\vec{c} \times \vec{a}) = (\vec{b} . \vec{a})\, \vec{c} - (\vec{b} . \vec{c})\, \vec{a}$

and $\vec{c} \times (\vec{a} \times \vec{b}) = (\vec{c} . \vec{b})\, \vec{a} - (\vec{c} . \vec{a})\, \vec{b}$

Hence, $\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) = 0$

13. If $\vec{a}, \vec{b}, \vec{c}$ are vectors in R^3 such that $[\vec{a}, \vec{b}, \vec{c}] \neq 0$ and

$$\vec{a}' = \frac{\vec{b} \times \vec{c}}{[\vec{a}, \vec{b}, \vec{c}]}, \quad \vec{b}' = \frac{\vec{c} \times \vec{a}}{[\vec{a}, \vec{b}, \vec{c}]}$$

and $\vec{c}' = \dfrac{\vec{a} \times \vec{b}}{[\vec{a}, \vec{b}, \vec{c}]}$

S.T.(1). $\vec{a}' . \vec{a} = \vec{b}' . \vec{b} = \vec{c}' . \vec{c} = 1$

2. $\vec{a}' . \vec{b} = \vec{a}' . \vec{c} = 0;\ \vec{b}' . \vec{a} = \vec{b}' . \vec{c} = 0$ and $\vec{c}' . \vec{a} = \vec{c}' . \vec{b} = 0$

3. $[\vec{a}', \vec{b}', \vec{c}'] = \dfrac{1}{[\vec{a}, \vec{b}, \vec{c}]}$

4. If a, b, c are non co planar, then $\vec{a}', \vec{b}', \vec{c}'$ are also non coplanar.

1. $\vec{a}' . \vec{a} = \dfrac{\vec{b} \times \vec{c}}{[\vec{a}, \vec{b}, \vec{c}]} . \vec{a} = \dfrac{(\vec{b} \times \vec{c}) . \vec{a}}{[\vec{a}, \vec{b}, \vec{c}]}$

$= \dfrac{[\vec{a}, \vec{b}, \vec{c}]}{[\vec{a}, \vec{b}, \vec{c}]} = 1.$ Similarly, $\vec{b}' . \vec{b} = \vec{c}' . \vec{c} = 1$

2. $\vec{a}' \cdot \vec{b} = \dfrac{\vec{b} \times \vec{c}}{[\vec{a}, \vec{b}, \vec{c}]} \cdot \vec{b} = \dfrac{(\vec{b} \times \vec{c}) \cdot \vec{b}}{[\vec{a}, \vec{b}, \vec{c}]}$

$= \dfrac{[\vec{b}, \vec{c}, \vec{b}]}{[\vec{a}, \vec{b}, \vec{c}]} = 0.$

Similarly the other results follow.

3. $[\vec{a}', \vec{b}', \vec{c}'] = \dfrac{\vec{b} \times \vec{c}}{[\vec{a}, \vec{b}, \vec{c}]}, \dfrac{\vec{c} \times \vec{a}}{[\vec{a}, \vec{b}, \vec{c}]}, \dfrac{\vec{a} \times \vec{b}}{[\vec{a}, \vec{b}, \vec{c}]}$

$= \dfrac{1}{[\vec{a}, \vec{b}, \vec{c}]^3} [\vec{b} \times \vec{c}, \vec{c} \times \vec{a}, \vec{a} \times \vec{b}]$

$[\lambda\vec{a}, \lambda\vec{b}, \lambda\vec{c}] = \lambda^3 [\vec{a}, \vec{b}, \vec{c}]$ for a scalar λ.

$\dfrac{[\vec{a}, \vec{b}, \vec{c}]^2}{[\vec{a}, \vec{b}, \vec{c}]^3}$ See Problem (3)

$[\vec{a}', \vec{b}', \vec{c}'] = \dfrac{1}{[\vec{a}, \vec{b}, \vec{c}]}$

Consequently $[\vec{a}, \vec{b}, \vec{c}]\ [\vec{a}', \vec{b}', \vec{c}'] = 1$

4. If $\vec{a}, \vec{b}, \vec{c}$ are non coplanar vectors then $[\vec{a}, \vec{b}, \vec{c}] \neq 0$

Then $[\vec{a}', \vec{b}' = \vec{c}'] = \dfrac{1}{[\vec{a}, \vec{b}, \vec{c}]} \neq 0$

i.e. $[\vec{a}', \vec{b}', \vec{c}'] \neq 0$

Hence, a', b', c' are non coplanar.

The sets of vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{a}', \vec{b}', \vec{c}'$ are called <u>reciprocal</u> vectors.

14. Prove : $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = \begin{vmatrix} (\vec{a} \cdot \vec{c}) & (\vec{a} \cdot \vec{d}) \\ (\vec{b} \cdot \vec{c}) & (\vec{b} \cdot \vec{d}) \end{vmatrix}$

Put $\vec{X} = \vec{c} \times \vec{d}$

Then $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = (\vec{a} \times \vec{b}) \cdot \vec{X}$

$= \vec{a} \cdot (\vec{b} \times \vec{X}) = \vec{a} \cdot (\vec{b} \times (\vec{c} \times \vec{d}))$

$= \vec{a} \cdot [(\vec{b} \cdot \vec{d})\, \vec{c} - (\vec{b} \cdot \vec{c})\, \vec{d}] = (\vec{b} \cdot \vec{d})(\vec{a} \cdot \vec{c}) - (\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c})$

$= \begin{vmatrix} (\vec{a} \cdot \vec{c}) & (\vec{a} \cdot \vec{d}) \\ (\vec{b} \cdot \vec{c}) & (\vec{b} \cdot \vec{d}) \end{vmatrix}.$

15. Prove $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a}, \vec{b}, \vec{d}]\,\vec{c} - [\vec{a}, \vec{b}, \vec{c}]\,\vec{d}$

$= [\vec{a}, \vec{c}, \vec{d}]\,\vec{b} - [\vec{b}, \vec{c}, \vec{d}]\,\vec{a}$.

Put $\vec{X} = \vec{a} \times \vec{b}$

$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = \vec{X} \times (\vec{c} \times \vec{d})$

$= (\vec{X} . \vec{d})\,\vec{c} - (\vec{X} . \vec{c})\,\vec{d}$

$= ((\vec{a} \times \vec{b}) . \vec{d})\,\vec{c} - ((\vec{a} \times \vec{b})) . \vec{c})\,\vec{d}$

$= [\vec{a}, \vec{b}, \vec{d}]\,\vec{c} - [\vec{a}, \vec{b}, \vec{c}]\,\vec{d}$

On the other hand, putting $\vec{X} = \vec{c} \times \vec{d}$

$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = (\vec{a} \times \vec{b}) \times \vec{X}$

$= (\vec{a} \times \vec{X})\,\vec{b} - (\vec{b} . \vec{X})\,\vec{a}$

$= (\vec{a} . (\vec{c} \times \vec{d}))\,\vec{b} - (\vec{b} . (\vec{c} \times \vec{d}))\,\vec{a}$

$= [\vec{a}, \vec{c}, \vec{d}]\,\vec{b} - [\vec{b}, \vec{c}, \vec{d}]\,\vec{a}$

Assignment and Self Test :

1. If $\vec{a} = i - 2j - 3k$, $\vec{b} = 2i + j - k$ and $\vec{c} = i - 3j - 2k$, find

a) $|(\vec{a} \times \vec{b}) \times \vec{c}|$ b) $(\vec{a} \times \vec{b}) . \vec{c}$

c) $\vec{a} \times (\vec{b} \times \vec{c})$ d) $(\vec{a} \times \vec{b}) \times (\vec{b} \times \vec{c})$

e) $(\vec{a} \times \vec{b}) . (\vec{b} \times \vec{c})$

2. Find the area of the triangle having its diagonals,
$\vec{a} = 3i + j - 2k$ and $\vec{b} = i - 3j + 4k$

3. Find the area of the triangle whose vertices are
A (3, -1, 2), B (1, -1, -3) and C (4, -3, 1)

4. If $\vec{a} \neq 0$, $\vec{a} . \vec{b} = \vec{a} . \vec{c}$ and $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$, then show that $\vec{b} = \vec{c}$.

5. Show that $(\vec{a} + \vec{b}) . (\vec{b} + \vec{c}) \times (\vec{c} + \vec{a}) = \vec{a} . \vec{b} \times \vec{c}$

6. Prove $(\vec{a}_1 . \vec{b}_1 \times \vec{c}_1)\ (\vec{a}_2 . \vec{b}_2 \times \vec{c}_2)$

$$= \begin{vmatrix} a_1 . a_2 & a_1 . b_2 & a_1 . c_2 \\ b_1 . a_2 & b_1 . b_2 & b_1 . c_2 \\ c_1 . a_2 & c_1 . b_2 & c_1 . c_2 \end{vmatrix}$$

7. Find a so that $2i - j + k$, $i + 2j - 3k$ and $3i + aj + 5k$ are coplanar.

8. If $\vec{A} = x_1 a + y_1 b + z_1 c$
$\vec{B} = x_2 a + y_2 b + z_2 c$

and $\vec{C} = x_3 a + y_3 b + z_3 c$

Prove $\left[\vec{A}, \vec{B}, \vec{C}\right] = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \left[\vec{a}, \vec{b}, \vec{c}\right]$

9. Prove $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) + (\vec{b} \times \vec{c}) \cdot (\vec{a} \times \vec{d})$

$(\vec{c} \times \vec{a}) \cdot (\vec{b} \times \vec{d}) = 0$

10. Prove $(\vec{a} \times \vec{b}) \cdot (\vec{a} \times \vec{c}) = (\vec{b} \cdot \vec{c})(\vec{a} \cdot \vec{a}) = (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{a})$.

11. Show that the four points $\vec{A}$ (6, -7,0), $\vec{B}$ (16, -19, -4) $\vec{C}$ (3, 0,-6) and D (2,5,10) are coplanar.

12. Prove that the unit vectors i, j, k are self reciprocal vectors.

THREE DIMENSIONAL GEOMETRY

1. Lines in Space
2. Planes and Sphere

by

Dr.N.M.RAO

THREE DIMENSIONAL GEOMETRY

The applications of vector algebra in three dimensional geometry are given in these lecture notes. The reader is requested to learn the techniques of vector algebra to see how the coordinate geometry can be made simple with the help of vectors. He is also requested to translate these results to the Cartesian form also.

The derivation of the formula for the shortest distance between two lines in space may be read only by keeping the teaching aid (discussed in the lesson) by the side, so that the concepts may become more clear.

There will be two parts on the applications of vectors; the first will deal with the lines in space while the second with the planes.

I. Lines in Space :

Length of a vector: We have already seen that the position vector of any point P in the 3-dimensional space R^3 is given by

$\vec{r} = xi + yj + zk$ where i, j, k are the unit vectors in three perpendicular directions and x, y, z are the coordinates of the point P. The position vector

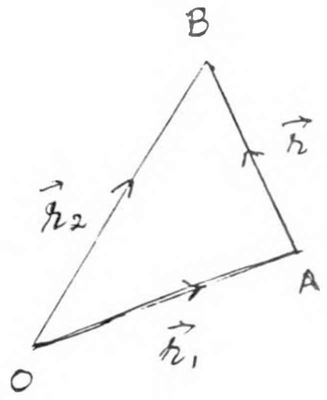
$$\vec{r} = xi + yj + zk$$

can also be written as $\vec{r} = (x, y, z)$

The length of the vector $\vec{r}$ is given by

$$|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

Distance Formula :



From the triangle OAB, it is clear that $\vec{AB} = \vec{OB} - \vec{OA}$.

This is the way to express any vector $\vec{AB}$.

If $\vec{OA} = \vec{r}_1$ and $\vec{OB} = \vec{r}_2$

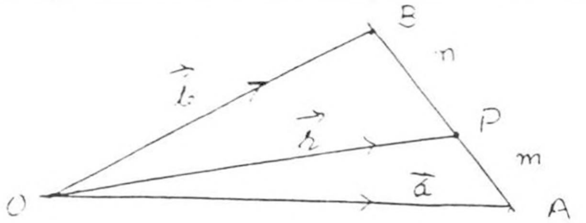
where $r_1 = (x, y, z)$ and $r_2 = (x_2, y_2, z_2)$

Then, $\vec{r} = \vec{AB} = \vec{r}_2 - \vec{r}_1$

$$|\vec{r}| = |\vec{r_2} - \vec{r_1}|$$
$$= \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2 - z_1)^2}$$

This is called the distance formula.

<u>Section Formula</u> : We find the position vector of the point which divides the line joining two given points in the given ratio.



Let A and B be any two points in the 3-dimensional space whose position vectors are $\vec{a}$ and $\vec{b}$ respectively. Let P be the point which divides the Line segment AB such that AP : PB = m:n. We wish to find the position vector $\vec{r}$ of the point P. Without loss generality, we can assume that O is the origin.

$$\vec{a} = x_1 i + y_1 j + z_1 k$$
$$\vec{b} = x_2 i + y_2 j + z_2 k$$

Let $\vec{r} = xi + y\,j + zk$

Since P divides AB in the ratio m:n, we have

$$\frac{AP}{PB} = \frac{m}{n}$$

Here m/n is positive or negative according as, P divides AB internally or externally.

From the above, we get n. AP = m. PB

i.e. $n(\vec{r} - \vec{a}) = m.(\vec{b} - \vec{r})$

or $(n+m)\,\vec{r} = m\,\vec{b} + n\,\vec{a}$

or $$\vec{r} = \frac{n\,\vec{a} + m\,\vec{b}}{n+m}$$

This is called the section formula in the vector form.

If we substitute the Cartesian coordinates

$$\vec{r} = x\,i + y\,j + z\,k$$
$$\vec{a} = x_1 i + y_1 j + z_1 k$$
$$\vec{b} = x_2 i + y_2 j + z_2 k$$

in the above result, and compare the coefficients of i,j,k, we get

$$x = \frac{mx_2 + nx_1}{m+n}$$

$$y = \frac{my_2 + ny_1}{m+n}$$

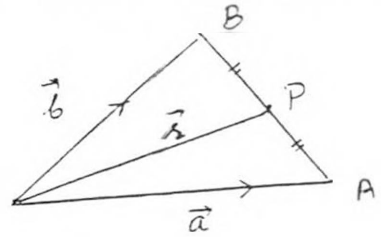
$$z = \frac{mz_2 + nz_1}{m+n}$$

which is the section formula in the Cartesian coordinates.

Middle Point :

From the section formula, it is clear that the position vector of the middle point of the join of two points with position vectors $\vec{a}$ and $\vec{b}$, is given by

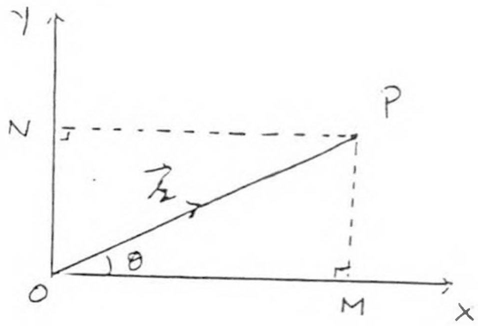
$$\vec{r} = \frac{\vec{a} + \vec{b}}{2}$$



Components of a vector :

In the figure,
$\vec{OM} = \vec{r} \cos\theta$
and $\vec{ON} = \vec{r} \sin\theta$

where θ is the angle that the vector $\vec{r}$ makes with x-axis.



Direction Ratios of a Vector :

If $\vec{r} = a\ i + b\ j + c\ k$, then a,b,c are called the direction ratios of the vector $\vec{r}$.

Direction cosines : If α is the angle that the vector $\vec{r}$ makes with the x-direction, then

$$\cos\alpha = \frac{\vec{r} \cdot \vec{i}}{|r|\ |i|}$$

$$= \frac{(ai + bj + ck) \cdot i}{|\vec{r}|}$$

$$= \frac{a}{\sqrt{a^2 + b^2 + c^2}} \qquad (i)$$

Similarly if β and γ are the angles that the vector $\vec{r}$ makes with y-direction and z-direction respectively, then

$$\cos \beta = \frac{b}{\sqrt{a^2 + b^2 + c^2}} \qquad \text{(ii)}$$

and $$\cos \gamma = \frac{e}{a^2 + b^2 + c^2} \qquad \text{(iii)}$$

If $\cos \alpha = \ell$, $\cos \beta = m$ and $\cos \gamma = n$, then

l,m,n are called direction cosines.

If we add the squares of (i), (ii) and (iii), we get

$$l^2 + m^2 + n^2 = 1$$

Therefore, the relation between the direction ratios and the direction cosines is

$$a:b:c = l:m:n \quad \text{with} \quad l^2 + m^2 + n^2 = 1$$

Parallel vectors have equal direction ratios :

Let $\vec{v}_1$ = ai + bj + ck and if $\vec{v}_2$ is a vector parallel to $\vec{v}_1$, then

$$\vec{v}_2 = \lambda \vec{v}_1 \text{ for some scalar } \lambda .$$

Then, $\vec{v}_2 = \lambda ai + \lambda bj + \lambda ck$

Hence the direction ratios of $\vec{v}_2$ are λa, λb, λc or a,b,c.

Like parallel vectors have equal direction cosines :

If $\vec{v}_1$ = a x + b j + c k

and $\vec{v}_2$ is a vector parallel to $\vec{v}_1$, then $\vec{v}_2 = \lambda \vec{v}_1$

$= \lambda ai + \lambda bj + \lambda ck$

The direction cosines of the $\vec{v}_1$ are

$$l = \frac{a}{|v_1|}, \quad m = \frac{b}{|v_1|}, \quad n = \frac{c}{|v_1|}$$

Similarly the direction cosines of the $\vec{v}_2$ are

$$\frac{\lambda a}{|\lambda v_1|}, \quad \frac{\lambda b}{|\lambda v_1|}, \quad \frac{\lambda c}{|\lambda v_1|}$$

i.e. $\frac{a}{|v_1|}, \quad \frac{b}{|v_1|}, \quad \frac{c}{|v_1|}$.

Also, it is clear that unlike parallel vectors have equal (and opposite sign) direction cosines.

Example:

1. For the vector $\vec{r} = 2i + 2j - k$, the direction ratio is 2:2:-1 and the direction cosines are $\frac{2}{|r|}, \frac{2}{|r|}, \frac{-1}{|r|}$

i.e $\frac{2}{3}, \frac{2}{3}, \frac{-1}{3}$.

It means that the vector $\vec{r} = 2i + 2j - k$ makes the following angles with i direction, j direction and k direction respectively.

$$\cos^{-1}\left(\frac{2}{3}\right), \quad \cos^{-1}\left(\frac{2}{3}\right), \quad \cos^{-1}\left(-\frac{1}{3}\right)$$

2. The vectors 2i + 2j - k and 4i + 4j - 2k have the same direction ratios and direction cosines. (They are parallel).

3. The vectors 2i + 2j - k and -4i -4j + 2k have the same direction ratios. They have direction cosines equal in magnitude but opposite in sign. (The vectors are unlike parallel vectors).

4. Show that the points A(2,3,4), B(-1,2,-3) and C (-4, 1,-10) are collinear.

There are several ways of answering this question: we can show that the area of the triangle ABC is zero or we can also show that

$$|\vec{AC}| = |\vec{BC}| = 2\,|\vec{AC}|$$

But it is easier to show that the direction ratios of $\vec{AB}$ and $\vec{BC}$ are equal (or proportional).

Direction ratios of $\overrightarrow{AB}$ are (-1,-2): (-2-3) : (-3-4)

i.e. -3:-1 : -7

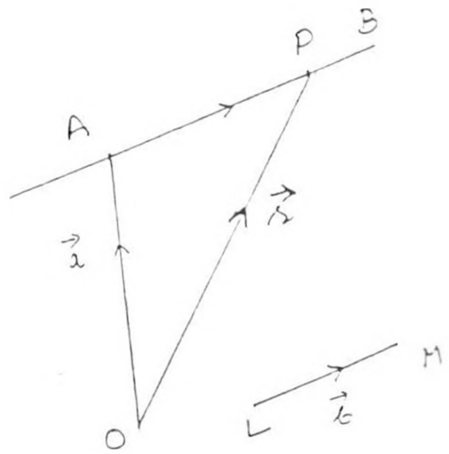
Direction ratios of $\overrightarrow{BC}$ are also -3:-1:-7.
Hence $\overrightarrow{AB}$ is parallel to $\overrightarrow{BC}$, showing that A,B,C are collinear.

<u>Angle between the vectors</u> : The angle between the vectors can be found out by applying the formula

$$\vec{a} \cdot \vec{b} = |\vec{a}|\ |\vec{b}| \cos$$

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\ |\vec{b}|}$$

<u>Vectorial Equation of a line in Space</u> :



We find the vector equation of the line AB which passes through a given fixed point A and is parallel to a given line LM (vector $\vec{b}$).

Take any point O, as origin of reference. Let $\vec{a}$ be the position vector of the given point A, let $\vec{b}$ be any vector parallel to the given line AB.

Let $\vec{r}$, be the position vector of any point P on the given line. We have

$$\begin{aligned} \vec{r} &= \overrightarrow{OP} \\ &= \overrightarrow{OA} + \overrightarrow{AP} \\ &= \vec{a} + \overrightarrow{AP} \end{aligned}$$

The vector $\overrightarrow{AP}$, being parallel to the vector $\vec{b}$, must be of the form

$\overrightarrow{AP} = t\,\vec{b}$ for some suitable scalar t.

Therefore,

$$\vec{r} = \vec{a} + t\,\vec{b}$$

is the required equation of the straight line.

Cartesian Form : To get the Cartesian form of the above equation, we can substitute the coordinates of the points

or put $\vec{r} = xi + yj + zk$

$\vec{a} = a_1 i + a_2 j + a_3 k$

$\vec{b} = b_1 i + b_2 j + b_3 k$

Then we get

$$xi + yj + zk = (a_1 i + a_2 j + a_3 k) + t\,(b_1 i + b_2 j + b_3 k)$$

Hence, (comparing coefficients of i,j,k), we get

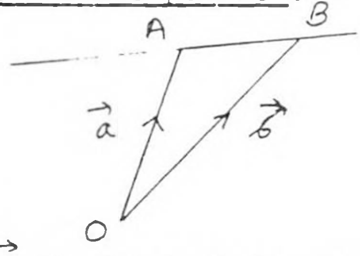
$$t = \frac{x - a_1}{b_1} = \frac{y - a_2}{b_2} = \frac{z - a_3}{b_3}$$

The cartesian equation of the line is

$$\boxed{\frac{x - a_1}{b_1} = \frac{y - a_2}{b_2} = \frac{z - a_3}{b_3}}$$

Equation of the straight line through given two points :

We wish to find the equation of the straight line which passes through the two given points A and B.



Take any point O as origin. Let $\vec{a}$ and $\vec{b}$ be the position vectors of the points A and B respectively.

Then the line AB is parallel to the vector $\vec{b} - \vec{a}$. It passes through A. Hence the equation of the line AB is given by

$$\boxed{\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})}$$ where λ is a parameter.

Cartesian Form : The cartesian form of the above equation is obtained by putting

$\vec{r} = xi + yj + zk$

$\vec{a} = a_1 i + a_2 j + a_3 k$

$\vec{b} = b_1 i + b_2 j + b_3 k$

and comparing the coefficients.

$$xi + yj + zk = a_1 i + a_2 j + a_3 k + \lambda\{(b_1 i + b_2 j + b_3 k) - (a_1 i + a_2 j + a_3 k)\}$$

$$= a_1 i + a_2 j + a_3 k + \lambda\{(b_1 - a_1) i + (b_2 - a_2) j + (b_3 - a_3) k\}$$

Hence we get

$$\lambda = \frac{x-a_1}{b_1-a_1} = \frac{y-a_2}{b_2-a_2} = \frac{z-a_3}{b_3-a_3}$$

The cartesian form of the equation is

$$\boxed{\frac{x-a_1}{b_1-a_1} = \frac{y-a_2}{b_2-a_2} = \frac{z-a_3}{b_3-a_3}}$$

<u>Linearly independent vectors in R^3</u> :

Definition: Three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ in R^3 are said to be linearly independent if $\alpha\vec{a} + \beta\vec{b} + \gamma\vec{c} = 0$, α, β, γ being scalars, implies $\alpha = \beta = \gamma = 0$. The vectors are said to be linearly dependent if they are not linearly independent. In other words, the vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ are said to be linearly dependent if there exists some non zero scalars α, β, γ such that $\alpha\vec{a} + \beta\vec{b} + \gamma\vec{c} = 0$.

For example, the vectors $\vec{a} = (1,2,1)$ $\vec{b} = (2,3,5)$ $\vec{c} = (4,7,7)$ are linearly dependent ($\alpha = 2$, $\beta = 1$, $\gamma = -1$). But the vectors $a = (1,2,1)$ and $b = (2,3,5)$ are linearly independent.

Theorem: A necessary and sufficient condition for three points with position vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ to be collinear is that there exists scalars α, β, γ not all zero, such that

$$\alpha\vec{a} + \beta\vec{b} + \gamma\vec{c} = 0, \quad \alpha + \beta + \gamma = 0$$

Proof: (Sufficiency part)

Let there be scalars α, β, γ not all zero, such that $\alpha\vec{a} + \beta\vec{b} + \gamma\vec{c} = 0$, $\alpha + \beta + \gamma = 0$

without loss of generality, we take $\gamma \neq 0$.

Then $\alpha + \beta = -\gamma \neq 0$

It is given that $\alpha\vec{a} + \beta\vec{b} + \gamma\vec{c} = 0$.

$$\Rightarrow \alpha\vec{a} + \beta\vec{b} = -\gamma\vec{c}$$

$$\Rightarrow \frac{\alpha\vec{a} + \beta\vec{b}}{\alpha + \beta} = \frac{-\gamma}{\alpha+\beta}\vec{c}$$

$$\Rightarrow \frac{\alpha\vec{a} + \beta\vec{b}}{\alpha + \beta} = \vec{c}$$

Here we have shown that $\vec{c}$ is the position vector of the point C which divides the line joining the points A (with position vector $\vec{a}$) and the point B (with position vector $\vec{b}$) in the ratio $\beta : \alpha$
Thus the points A, B and C are collinear.

Necessary Part : Let the points A, B, C be collinear. The position vectors of A, B, C are $\vec{a}$, $\vec{b}$, $\vec{c}$ respectively.

We can assume that the point C divides the line segment AB in the ratio $\alpha : \beta$

Then $\vec{c} = \dfrac{\alpha\vec{b} + \beta\vec{a}}{\alpha + \beta}$ $\quad\therefore (\alpha+\beta)\vec{c} = \alpha\vec{b} + \beta\vec{a}$

Put $\alpha + \beta = -\gamma$

Then we get $\alpha\vec{b} + \beta\vec{a} + \gamma\vec{c} = 0$ and $\alpha + \beta + \gamma = 0$.

Hence the proof.

Note: 1. We have proved that if the points A,B,C are collinear then, the vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ are linearly dependent. In other words, if the vectors are linearly independent then the points need not be collinear.

2. It is easy to see that the vectors $\vec{a}$ and $\alpha\vec{a}$ are collinear as well as linearly independent.

3. If $\vec{a}$ and $\vec{b}$ are two non zero non collinear vectors, then they are linearly independent. For, if they are linearly dependent, then there exists non zero scalars α, β such that $\alpha\vec{a} + \beta\vec{b} = 0$.

If $\alpha \neq 0$ then $\vec{a} = -\dfrac{\beta\vec{b}}{\alpha}$

which implies that $\vec{a}$ and $\vec{b}$ are collinear, contrary to our assumption.

4. In the same way we can prove that if $\vec{a}$, $\vec{b}$, $\vec{c}$ are three non zero non coplanar vectors, then they are linearly independent.

<u>Angle between any two lines</u> :

Let $\vec{r}_1 = \vec{a}_1 + \lambda \vec{b}_1$

and $\vec{r}_2 = \vec{a}_2 + \mu \vec{b}_2$

be any two straight lines, in space. Then the angle between them can be found out as follows :

The angle between $\vec{r}_1$ and $\vec{r}_2$ is equal to the angle between $\vec{b}_1$ and $\vec{b}_2$

But $\vec{b}_1 \cdot \vec{b}_2 = |b_1| \, |b_2| \cos\theta$

$$\cos\theta = \frac{\vec{b}_1 \cdot \vec{b}_2}{|b_1| \, |b_2|}$$

is the angle between $\vec{r}_1$ and $\vec{r}_2$.

The above method can be applied even if the equations are in the Cartesian form.

Note: The angle θ calculated above does not indicate that the two straight lines intersect. In fact, the angle θ is the angle between the directions of $\vec{b}_1$ and $\vec{b}_2$.

<u>Skew Lines</u> : In the plane, whenever two straight lines are not parallel, then they intersect at some point. But the situation is different in the space. There can be straight lines which are neither parallel nor intersecting. Such lines do not lie in a single plane; and are called skew lines.

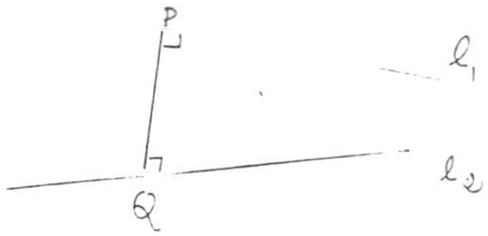
Definition: Two straight lines in R^3 which are not coplanar are called skew lines.

Definition: The length of the common perpendicular to the skew lines is called the <u>shortest distance</u> between the skew lines.

Note: The teacher can make the ideas of skew lines clear with the help of a teaching aid described here: Take two rods AB and CD. Tie one end of a thread to a point P on AB and the other end to a point Q on CD. Hold the rods AB and CD at different levels and make

PQ perpendicular to both AB and CD. (AB and CD need not be parallel). Now $|PQ|$ is shortest distance between the lines.

<u>To find an expression for the shortest distance</u> :



Let the skew lines be

$\vec{r}_1 = \vec{a}_1 + \lambda \vec{b}_1$

and $\vec{r}_2 = \vec{a}_2 + \mu \vec{b}_2$

Since PQ is perpendicular to both $\vec{b}_1$ and $\vec{b}_2$, it is clear that PQ is parallel to $\vec{b}_1 \times \vec{b}_2$

The unit vector $\vec{n}$ along $\vec{PQ}$ is given by

$$\vec{n} = \frac{\vec{b}_1 \times \vec{b}_2}{| b_1 x b_2 |}$$

Let $\vec{PQ} = d\,\vec{n}$ where d is the shortest distance between the given two skew lines.

Let S and T be any two points with position vectors $\vec{a}_1$ and $\vec{a}_2$ on the lines AB and CD respectively. If θ is the angle between PQ and ST, then PQ = ST cos

This can be realized by taking the projection of ST along the direction of PQ.

Then

$$\cos\theta = \frac{\vec{PQ} \cdot \vec{ST}}{|PQ|\ |ST|}$$

$$= \frac{d\vec{n} \cdot (\vec{a}_2 - \vec{a}_1)}{d\ |ST|}$$

$$= \frac{d\,(\vec{b}_1 x \vec{b}_2)}{|b_1 \times b_2|} \cdot \frac{(\vec{a}_2 - \vec{a}_1)}{d\ |ST|}$$

$$= \frac{(\vec{b}_1 x \vec{b}_2)}{|b_1 x b_2|} \cdot \frac{(\vec{a}_2 - \vec{a}_1)}{|ST|}$$

$$d = PQ = ST \cos$$

$$= \frac{(b_1 \times b_2) \cdot (a_2 - a_1)}{|b_1 \times b_2|} \qquad \ldots (A)$$

(The distance d is to be taken as positive).

Solved Examples:

1. Find the shortest distance between the vectors

$\vec{r}_1 = i + j + \lambda(2i + j + k)$

and $\vec{r}_2 = 2i + j - k + \mu(3i - 5j + 2k)$

Ans: Here in this problem,

$a_1 = i + j, \quad b_1 = 2i + j + k$

$a_2 = 2i + j - k, \quad b_2 = 3i - 5j + 2k$

Substituting these values in the formula (A)

$$d = \frac{(b_1 \times b_2) \cdot (a_2 - a_1)}{|b_1 \times b_2|}$$

We get $d = \frac{10}{\sqrt{59}}$

Shortest distance when the lines are parallel :

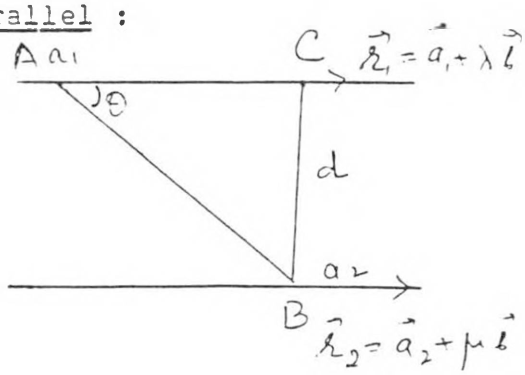
Let

$\vec{r}_1 = \vec{a}_1 + \lambda \vec{b}$ and

$\vec{r}_2 = \vec{a}_2 + \mu \vec{b}$

be the two parallel lines in the space.

Then the two vectors $\vec{r}_1$ and $\vec{r}_2$ can be considered to be in one plane. 'd' as shown in the figure is the shortest distance between the lines.



$d = (\vec{a}_2 - \vec{a}_1) \sin\theta$ ...(1) from the triangle ABC.

But we know that

$$(\vec{a}_2 - \vec{a}_1) \times \frac{\vec{b}}{|\vec{b}|} = |\vec{a}_2 - \vec{a}_1| \left|\frac{\vec{b}}{|b|}\right| \sin\theta . \vec{n}$$

Since d is always considered to be positive, substituting the values of $\sin\theta$ in (1), we get

$$d = (a_2 - a_1) \frac{a_2 - a_1}{(a_2 - a_1)} \times \frac{|\vec{b}|}{|b|}$$

$$= \left|\frac{\vec{b}}{|b|} \times (\vec{a}_2 - \vec{a}_1)\right| \qquad (2)$$

2. Find the angle between the pair of lines $\vec{r}_1 = 4i - j + \lambda(i+2j-2k)$ and $\vec{r}_2 = (i-j+2k) + \mu(2i + 4j - 4k)$. Also find the shortest distance between them.

Ans: Note that the lines are parallel to the vector $i+2j-2k$ and hence the angle between them is zero. Both are of the form

$$\vec{r}_1 = a_1 + \lambda b$$
$$\vec{r}_2 = a_2 + \mu b$$

Hence this problem cannot be solved by the method we adopted for problem 1. Now we use the result (2).

$$d = \left|\frac{\vec{b}}{|b|} \times (a_2 - a_1)\right|$$

$$= \left|\frac{(i+2j-2k) \times}{|i+2j-2k|} (i-j+2k) - (4i-j)\right|$$

$$= \left|\frac{(i+2j-2k)}{3} \times (-3i+2k)\right|$$

$$= \left|1/3\ i\,(4) + j\,(6-2) + k(6)\right|$$

$$= \sqrt{\frac{68}{3}}$$

3. Find the shortest distance between the pair of lines

$\vec{r} = i+j - k + (3i-j)$ and

$\vec{r} = 4i - k + (2i + 3k)$

Also find whether they intersect.

Ans: Substituting in the formula (1), we can see that the shortest distance is

$$d = \left| \frac{(3i-j) \times (2i+3k) \cdot (3i-j)}{|b_1 \times b_2|} \right|$$

$$= \frac{(-3i -9j + 2k) \cdot (3i-j)}{|b_1 \times b_2|}$$

$$= \frac{-9 + 9}{94}$$

$$= 0$$

The given lines do intersect.

4. Determine whether the following lines intersect.

$$\frac{x-1}{2} = \frac{y+1}{3} = z;$$

$$\frac{x+1}{5} = \frac{y-2}{1}, \; z = 2$$

Ans : The first set of equations can be written as (when we take the common ratio as λ).

$x = 2\lambda + 1$ $\quad\quad \vec{r} = xi + yj + zk$

$y = 3\lambda - 1$ $\quad\quad = (i-j) + \lambda(2i+3j+k)$ ——— (1)

$z = \lambda$

Similarly the second set of equations can be written as

$x = 5\mu - 1$

$y = 1\mu + 2$

$z = 0\mu + 2$ $\quad\quad \vec{r} = (-i + 2j + 2k) + \mu(5i + j)$ ——— (2)

Now as in exercise (3) above, we can show that the shortest distance d between the lines (1) and (2) is not zero. Hence they do not intersect.

5. Find the angle between the pair of lines with direction ratios 1,1,2 and $\sqrt{3}-1$, $-\sqrt{3}-1$, 4.

Ans: The vector equation of the 1st line is given by

$$\vec{r_1} = 1i + 1j + 2k$$

and the second line is given by

$$\vec{r_2} = (\sqrt{3}-1)i + (-\sqrt{3}-1)j + 4k$$

The angle between the two lines is given by

$$\cos\theta = \frac{\vec{r_1} \cdot \vec{r_2}}{|r_1|\ |r_2|}$$

$$= \frac{(i+j+2k) \cdot ((\sqrt{3}-1)i + (-\sqrt{3}-1)j + 4k)}{|r_1|\ |r_2|}$$

$$= \frac{\sqrt{3}-1 + -\sqrt{3} - 1 + 8}{\sqrt{6}\sqrt{24}}$$

$$= 1/2$$

$$\theta = 60^{o}$$

<u>Assignments Self Test</u> :

1. Find the angle between the pair of lines whose direction ratios are:

i) 1,2,-2; 2,4,-4

ii) 5,-12,13; -3,4,5

iii) 1,2,1; 2,1,-1

2. Determine whether the following pairs of lines intersect :

i) $r_1 = 3i + 2j - 4k + \lambda(i + 2j + 2k)$

and $r_2 = 5j - 2k + \mu(3i + 2j + 6k)$

ii) $\frac{x+4}{3} = \frac{y-1}{5} = \frac{z+3}{4}$

and $\frac{x+1}{1} = \frac{y-4}{1} = \frac{z-5}{2}$

3. Find the angle between the lines

i) $r_1 = 3i + 2j - 4k + \lambda(i + 2j + 2k)$
and $r_2 = 5j - 2k + \mu(3i + 2j + 6k)$

ii) $\frac{x+4}{1} = \frac{y-1}{1} = \frac{z+3}{2}$ and $\frac{x+1}{1} = \frac{y-4}{1} = \frac{z-5}{2}$

4. Find the shortest distance between the lines whose direction ratios are
1,2,-2 and 2,4,-4.

5. Find the shortest distance between

$r_1 = i+j + k + \lambda(3i-j)$
and $r_2 = 4i - k + \mu(2i + 3k)$

PLANE :

A plane is completely determined by any one of the following :

i) Three non collinear points.

ii) A line and a point not on the line

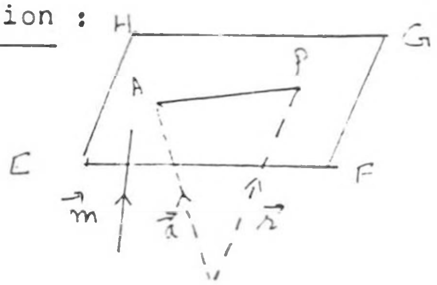
iii) Two intersecting lines

iv) Distance of the plane from the origin and a normal vector to the plane.

v) A point on the plane and a normal vector to the plane.

Here we find vector equation of the plane for some of the above cases.

1. Find the vector equation of the plane through a given point and perpendicular to a given direction :



Let A be the given point with position vector $\vec{a}$, through which the plane EFGH passes. Let $\vec{m}$ be the direction which is perpendicular to the plane EFGH.

We want to find the equation of the plane EFGH.

Let P be any arbitrary point on the plane, whose position vector is $\vec{r}$.

$$\vec{AP} = \vec{r} - \vec{a}$$

The plane is perpendicular to $\vec{m}$

Therefore, $$\boxed{(\vec{r} - \vec{a}) \cdot \vec{m} = 0}$$

is the vector equation of the plane EFGH.

Example: Equation of the plane passing through the point (a_1, b_1, c_1) and perpendicular to the line with direction ratios A, B, C is given by

$$(\vec{r} - (a_1 i + b_1 j + c_1 k) \cdot (Ai + Bj + ck) = 0$$

If we wish to have the Cartesian equation, then take

$$\vec{r} = xi + yj + zk$$

we get

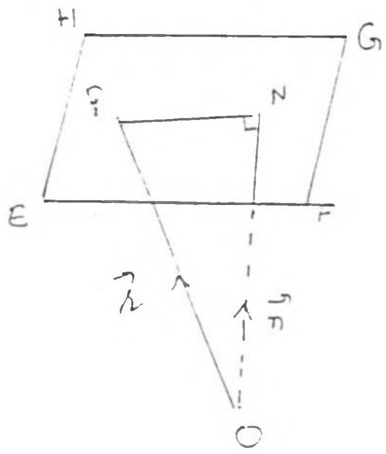
$$(x-a_1)i + (y-b_1)j + (z-c_1)k \cdot (Ai + Bj + ck) = 0$$

i.e. $A(x-a_1) + B(y-b_1) + c(z-c_1) = 0$

is the equation of the required plane.

2. Find the vector equation of the plane perpendicular to a given direction and at a given distance from the origin.

Given that the plane EFGH is perpendicular to $\vec{n}$, and the distance ON = d from the origin.



Consider the vector NP.

$\vec{NP} = \vec{r} - d\,\vec{n}$

Also $\vec{NP}$ is perpendicular to $\vec{n}$

Therefore, $\vec{NP} \cdot \vec{n} = 0$

i.e. $(\vec{r} - d\vec{n}) \cdot \vec{n} = 0$

$\boxed{\vec{r} \cdot \vec{n} = d}$ Since $n \cdot n = 1$

is the required equation.

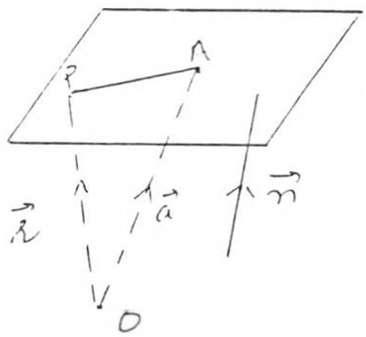
Cartesian equation : Put $\vec{r} = xi + yj + 3k$

and $\vec{n} = li + mj + nk$, then we have $(xi + yj + 3k).(li + mj + nk) = d$

i.e. $\boxed{lx + my + nz = d}$ is the required equation where l,m,n are the direction cosines of the normal to the plane.

3. Equation of the plane passing through given point and perpendicular to the given direction:

The plane passes through the point A (position vector $\vec{a}$) and perpendicular to the direction $\vec{n}$.



Let P be any arbitrary point on the plane, with position vector $\vec{r}$.

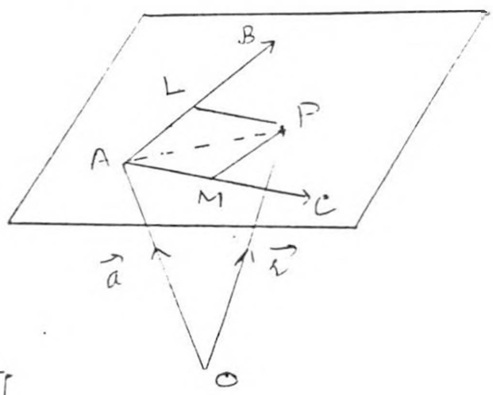
Then AP is perpendicular to $\vec{n}$ given

$\boxed{(\vec{r} - \vec{a}) \cdot \vec{n} = 0}$

This is the required equation.

4. Equation of the plane passing through given point and parallel to two given lines:

Let $\vec{a}$ be the position vector of the point A, through which the plane passes. Let $\vec{b}$ and $\vec{c}$ be the vectors parallel to AB and AC.



Let P be any arbitrary point on the plane (position vector $\vec{r}$).

Then, $\vec{OP} = \vec{OA} + \vec{AP}$.

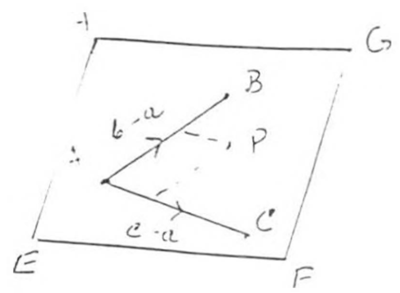
This can be written as

$\boxed{\vec{r} = \vec{a} + t\,\vec{b} + p\,\vec{c}}$

where t and p are some scalars.

5. Equation of the plane through three given points :

Let $\vec{a}$, $\vec{b}$, $\vec{c}$ be the position vectors of three given points A,B,C on the plane EFGH.



Then $\vec{AB} = \vec{b} - \vec{a}$

$\vec{AC} = \vec{c} - \vec{a}$

If P is any arbitrary point on the plane, whose position vector is $\vec{r}$, then

$\vec{r} = \vec{a} + t\ (\vec{b} - \vec{a}) + p\ (\vec{c} - \vec{a})$

where t and p are some scalars.

$$\boxed{\vec{r} = \vec{a} + t\ (\vec{b} - \vec{a}) + p\ (\vec{c} - \vec{a})}$$

is the required equation.

Example: Find the equation of the plane through the points A (2,2,-1), B(3,4,2), C(7,0,6)

Ans: $\vec{r} = \vec{a} + t(\vec{b} - \vec{a}) + p(\vec{c} - \vec{a})$ is the equation.

To find the scalars t and p we can follow the following method :

$(x,y,z) = (2,2,-1) + t(1,2,3) + p(5,-2,7)$

$t + 5p = x-2$

$2t - 2p = y-2$

$3t + 7p = z+1$

Solving any two equations for t and p and substituting in the third equation, we get

$5x + 2y - 3z - 17 = 0$

which is the required equation of the plane. (See the textbook for an alternative method).

Angle between two planes :

Let $\vec{r} \cdot \vec{n}_1 = d_1$

Let a plane P and let $\vec{r} \cdot \vec{n}_2 = d_2$

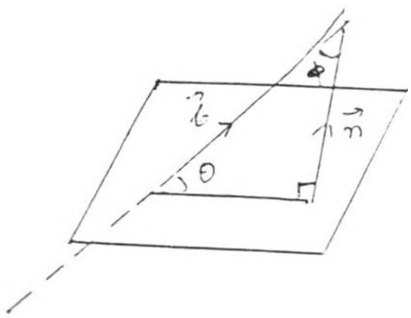
be another plane Q where $\vec{n}_1$ and $\vec{n}_2$ are perpendicular to the planes P and Q.

Then the angle between the planes P and Q is the angle between their perpendiculars. If θ is the angle between P and Q, then

$$\cos\theta = \vec{n}_1 \cdot \vec{n}_2$$

The planes are parallel if $\vec{n}_1 = \vec{n}_2$ and perpendicular if $\vec{n}_1 \cdot \vec{n}_2 = 0$.

Angle between a line and a plane :



Let $\vec{r} = \vec{a} + \lambda \vec{b}$
be the line which makes
an angle θ with the plane
$\vec{r} \cdot \vec{n} = d$

From the figure, it is clear that

$$\cos \phi = \frac{\vec{b} \cdot \vec{n}}{|b|}$$

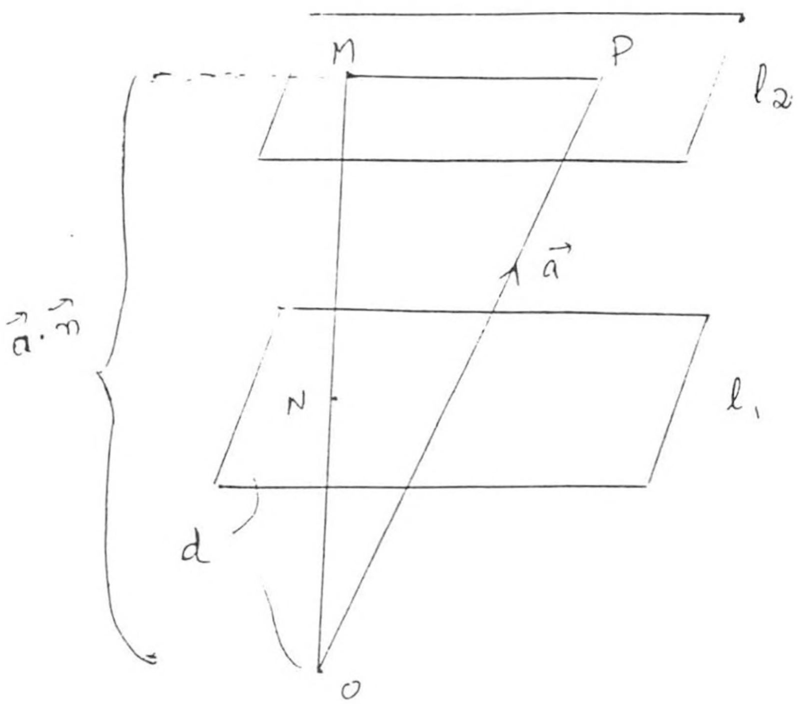
Since $\theta = \frac{\pi}{2} - \phi$

We have $\sin \theta = \cos \phi$

$$\sin \theta = \frac{\vec{b} \cdot \vec{n}}{|b|}$$

where θ is the angle between the line and the plane.

Distance of a point from a Plane :



Let l_1 be the plane and P be the given point. We wish to find the perpendicular distance from P to l_1.

Consider a plane l_2 through the point P and parallel to the plane l_1.

If $\vec{r} \cdot \vec{n} = d$ is the equation of the plane l_1, then

$$(\vec{r} - \vec{a}) \cdot \vec{n} = 0 \text{ is the equation of the plane } l_2$$

(because the unit vector n is perpendicular to l_2 also). The equation of l_2 can also be written as

$$\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$$

This means that $\vec{a}.\vec{n}$ is the perpendicular distance of the plane l_2 from the point O.

Therefore, the distance from P to the plane l_1
= the distance between the two parallel planes
= OM – ON
$= \vec{a} \cdot \vec{n} - d$

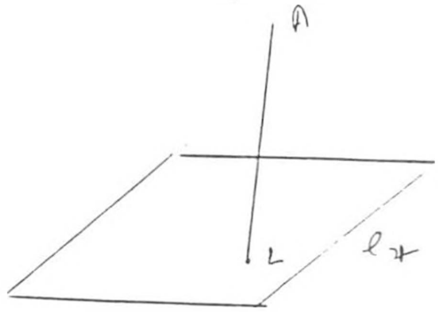
The distance from P to l_1 = $| \vec{a} \cdot \vec{n} - d |$

<u>Alternative Method</u> :

Let $\vec{a}$ be the position vector of the given point A and let

$\vec{r} \cdot \vec{n} = q$ .... (1)

be the equation of the plane l_1. We want to find the distance AL where L is the foot of the perpendicular from A on l_1.



The equation of the line through A and normal the plane l_1 is given by $\vec{r} = \vec{a} + t\,\vec{n}$....(2) where t is scalar.

To find the position vector of the point L, we solve (1) and (2). i.e. At the point of intersection of this line with the plane, we have

$$(\vec{a} + t\,\vec{n}) \cdot \vec{n} = q$$

so that $t = \dfrac{q - \vec{a} \cdot \vec{n}}{n^2}$

∴ The position vector of L is given by

$$\vec{a} + \frac{(\vec{q} - \vec{a}) \cdot \vec{n}}{\vec{n}^2}\, \vec{n}$$

The length AL

$$= |\vec{AL}|$$

$$= \left| \vec{a} + \frac{(\vec{q} - \vec{a}) \cdot \vec{n}}{n^2}\, \vec{n} - \vec{a} \right|$$

$$= |(\vec{q} - \vec{a}) \cdot \vec{n}|, \text{ for } n^2 = |n|^2 = 1$$

Solved Exercises :

1. Show that the line L whose vector equation is

$$\vec{r} = (2i - 2j + 3k) + \lambda(i-j+4k)$$

is parallel to the plane $\vec{r} \cdot (i + 5j + k) = 5$ and find the distance between them.

Ans: If θ is the angle between the line and the plane, then

$$\sin \theta = \frac{\vec{b} \cdot \vec{n}}{|b|}$$

$$\sin \theta = \frac{(i - j + 4k)}{\sqrt{18}} \cdot \frac{(i + 5j + k)}{\sqrt{27}} = 0$$

$\theta = 0$. They are parallel.

The distance $= |\vec{a} \cdot \vec{n} - d|$

$$= (2i-2j+3k) \cdot \frac{(i+5j+k)}{\sqrt{27}} - \frac{5}{\sqrt{27}}$$

$$= \frac{10}{\sqrt{27}}$$

2. Show that the plane whose vector equation is

$$\vec{r} . (i + 2j - k) = 3$$

contains the line whose vector equation is

$$\vec{r} = i + j + (2j + j + 4k)$$

Ans: $\sin \theta = \frac{(2i+j+4k) . (i+2j-k)}{xxx}$

$= 0$

Hence the Line and Plane are parallel.

The distance $= | \vec{a} . \vec{n} - d |$

$= (i+j) . \frac{(i+2j-k)}{\sqrt{6}} - \frac{3}{\sqrt{6}}$

$= \frac{1+2}{\sqrt{6}} - \frac{3}{\sqrt{6}}$

$= 0$

Hence the line Lies on the plane.

3. Find the vector equation of the line passing through (3,1,2) and perpendicular to the plane $\vec{r} . (2i - j + k) = 4$.
Find also the point of intersection of this line and the plane.

Ans: The plane is $\vec{r} . (2i - j + k) = 4$.
Hence $\vec{n} = 2i - j + k$ is perpendicular to the plane. The line has to pass through the point (3,1,2).
Hence the equation of the line is $\vec{r} = (3i+j+2k) + \lambda(2i-j+k)$
The point of intersection of the line and the plane will be given by solving $\vec{r} . (2i - j + k) = 4$ ...(1)

$\vec{r} = (3i + j + 2k) + \lambda (2i-j+k)$ ....(2)

Substituting (2) in (1), we get

$4 = 6-1 + 2 + \lambda (4+1+1)$

$\lambda = -1/2$

The point of intersection is

$(3i + j + 2k) + (-1/2) (2i - j + k)$

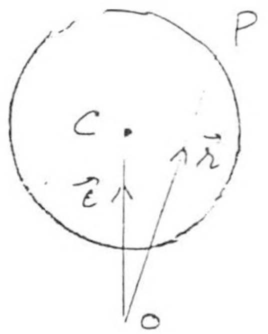
$= (2, \frac{3}{2}, \frac{3}{2})$.

SPHERE

Definition: The set of all points in the space, each of which is at a constant distance $a(>0)$ from a fixed point C is called a sphere.

The fixed point C is called the centre and the constant distance 'a' is called the radius of the sphere.

Central form of a sphere :



Let $\vec{c}$ be the position vector of the centre of the sphere, of radius $a > 0$.
Let $\vec{r}$ be the position vector of any arbitrary point P on the sphere.
Then, $|\vec{CP}| = a$

$\Rightarrow$ |Position vector of P - Position vector of C| = a

$$\Rightarrow \boxed{|\vec{r} - \vec{c}| = a}$$

This is the vector equation of the sphere in the central form.

Cor 1: In particular

$|\vec{r}| = a$ is the equation of the sphere whose centre is the origin and radius is a.

Cor 2: $\vec{r} - \vec{c}$

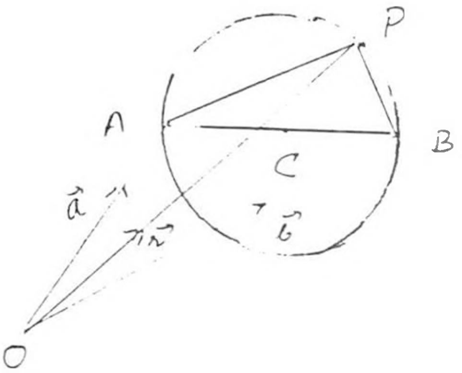
$= (xi + yj + zk) - (c_1 i + c_2 j + c_3 k)$

$= (x-c_1)i + (y-c_2)j + (z-c_3)k$

$$\boxed{(x-c_1)^2 + (y-c_2)^2 + (z-c_3)^2 = a^2}$$

is the equation of the sphere with centre (c_1, c_2, c_3) and radius a.

Diameter form of the sphere :



Let $\vec{a}$, $\vec{b}$ be the position vectors of the extremities A and B of the diameter AB of the sphere. Let $\vec{r}$ be the position vector of any point P on the surface of the sphere.

Then, $\vec{AP} = \vec{r} - \vec{a}$

$\vec{BP} = \vec{r} - \vec{b}$

It is clear from geometry that

$$\vec{AB} \cdot \vec{BP} = 0$$

$$\boxed{(\vec{r} - \vec{a}) \cdot (\vec{r} \cdot \vec{b}) = 0}$$

which is the equation of the sphere whose diameter is the join of A ($\vec{a}$) and B($\vec{b}$).

Cartesian Form :

Let A (x_1, y_1, z_1) and $B(x_2, y_2, z_2)$ be the extremities of the diameter AB of the sphere. Let $P(x, y, z)$ be any point on the surface of the sphere. Then,

$$\vec{r} - \vec{a} = (x-x_1)\, i + (y-y_1) j + (z-z_1) k$$

$$\vec{r} - \vec{b} = (x-x_2) i + (y-y_2) j + (z-z_2)\, k$$

$$(\vec{r} - \vec{a}) \cdot (\vec{r} - \vec{b}) = 0$$

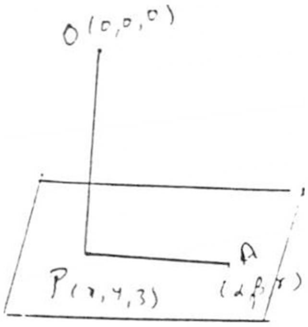
becomes

$$\boxed{(x-x_1)(x-x_2) + (y-y_1)(y-y_2) + (z-z_1)(z-z_2) = 0}$$

which is the Cartesian equation of the sphere whose diameter is the join of the points (x_1, y_1, z_1) and (x_2, y_2, z_2).

Solved Examples :

1. A plane passes through a fixed point A (α, β, γ). Show that the locus of the foot of perpendicular to it from the origin is the sphere $x^2+y^2+z^2 - \alpha x - \beta y - \gamma z = 0$



Ans:

Let $P(x,y,z)$ be the foot of the perpendicular from O on the plane.

$\vec{OP} = (xi + yj + 3k)$

$\vec{PA} = (x-\alpha)i + (y-\beta)j + (z-\gamma)k$

$\vec{OP} \perp \vec{PA}$ can be written as

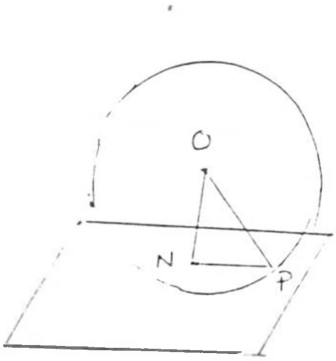
$$(x_1+y_j+zk) . (x-\alpha)i + (y-\beta)j + (z-\gamma)k = 0$$

i.e. $x^2+y^2+z^2 - x\alpha - y\beta - z\gamma = 0$

2. Prove that the radius of the circular section of the sphere $|\vec{r}| = 5$ cut off by the plane $\vec{r} . (i+j+k) = 3\sqrt{3}$ in 4 units.

Ans: The given sphere is $|\vec{r}| = 5$.

The centre is the origin and the radius is 5.



The given plane can be written as

$$\vec{r} . \frac{(i+j+k)}{\sqrt{3}} = 3$$

Hence the distance of the plane from the centre is p=3.

i.e. $|ON| = 3$

$$\text{Then } |NP| = \sqrt{OP^2 - ON^2}$$
$$= \sqrt{5^2-3^2}$$
$$= 4 \text{ units}$$

3. Prove that the plane $x+2y+2z = 15$ cuts the sphere $x^2+y^2+z^2-2y-4z-11 = 0$ in a circle. Find the centre and radius of the circle.

Ans: The equation of the sphere is $x^2+y^2+z^2-2y-4z-11 = 0$

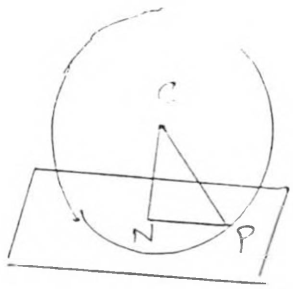
Its centre is (0,1,2) and radius $r = 4$.

The distance of the plane from the centre of the sphere is

$$p = \left|\frac{0+2+4-15}{\sqrt{1+4+4}}\right|$$
$$= 3$$

$p < r$

$\therefore$ The plane cuts the sphere in circle.



Then,
radius of the circle is

$= NP$

$= \sqrt{eP^2 - CN^2}$

$= \sqrt{r^2 - p^2}$

$= \sqrt{7}$

Let (α, β, γ) be the coordinates of N. N lies on the plane.

$\therefore \quad \alpha + 2\beta - 2\gamma - 15 = 0$

Also CN is parallel to the normal to the plane.

$$\frac{\alpha}{1} = \frac{\beta - 1}{2} = \frac{\gamma - 2}{2} = k$$

$\alpha = k, \quad \beta = 2k+1, \quad \gamma = 2k+2$

Substituting these values in the above, we get

$k + 4k + 2 + 4k + 4 - 15 = 0$

$k = 1$

$\alpha = 1, \ \beta = 3, \ \gamma = 4$

Hence the centre C (1,3,4)

and the radius is $\sqrt{7}$.

LIMITS CONTINUITY AND DIFFERENTIATION

by

Mr.B.C.BASTI

1. LIMITS

1.1 Introduction :

We live in a world of change - our values, ideals, hopes and institutions are undergoing constant change. It is interesting to note that certain changes are happening too rapidly, while other changes are not occurring fast enough. This illustrates that, although the topic of change is important, often the concept of rate of change is more relevant. For example, in the study of population growth, it is not sufficient to know that the population changed by doubling. We need to know the rate at which this doubling took place. It is significant that at one time the doubling of the world population took a thousand years, but now the doubling takes only few decades time. The mathematical tool for measuring rates of change is the concept of limits. The concept of limit is needed to pass from the average rate of change to the more useful concept of an instantaneous rate of change. Indeed it is this concept of the limit, that resulted in the invention of Calculus. It may be surprising to discover that Newton did not have a complete understanding of the limit. Many years later Cauchy put the concept of limit on a sound mathematical basis. In this section, the approach to the concept of limit is initially intuitive and later the mathematically elegant Cauchy epsilon-delta approach is given.

There are many topics in school mathematics through which limits can be illustrated. For instance consider the problem of finding circumference of a circle. The circumference of a circle can be taken as the limit of perimeter of inscribed regular polygon as the number of sides tend to infinity. Teachers can also use the action of a bouncing ball. If $\{hn\}_n = 1,2,\ldots$ is a sequence of heights of the bouncing ball, then 0 is the limit of such a sequence.

1.2 Limit of a Function :

Consider the function $f(x) = \frac{x^2-4}{x-2}$ for $x \neq 2$.

f(x) is not defined at 2 because the direct substitution 2 for x results in 0/0 which is an indeterminate form. Let us calculate the values of f(x) for some values x that are very close to but unequal to 2.

From the table it appears that if x is very close to 2, then $f(x) = \frac{x^2-4}{x-2}$ is very near 4. We represent this statement in mathematical shorthand as,

limit of $f(x) = \frac{x^2-4}{x-2}$ as x approaches 2 is 4 or

$$\lim_{x \to 2} f(x) = 4$$

x	$f(x) = \frac{x^2-4}{x-2}$
1.98	3.98
1.99	3.99
2.01	4.01
2.02	4.02

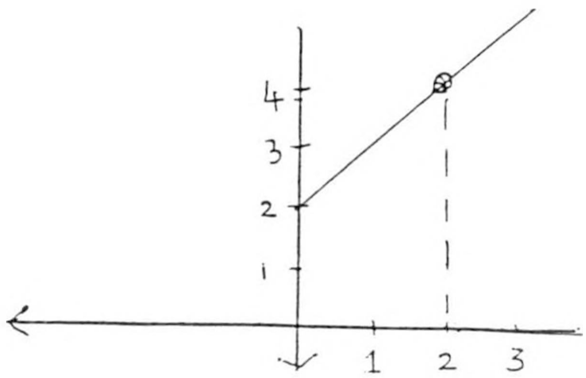


Fig.1

Now we can define f(2) as 4. Here we have used the limit process to define f(2) though originally f(2) was not defined. It is possible to obtain $\lim_{x \to 2} f(x)$ without finding table of values.

Since $f(x) = \frac{x^2-4}{x-2} \quad \frac{(x-2)\ (x-2)}{(x-2)}$ if $x \neq 2$

$= (x+2)$ if $x \neq 2$.

$\lim_{x \to 2} f(x) = \lim_{x \to 2} (x+2) = 2+2 = 4$

Since limit of (x+2) as x tends to 2 can be obtained by substituting x=2 in x+2.

<u>Exercise</u>: Find (i) $\lim_{x \to 3} \frac{x^2-5x+6}{x-3}$

(ii) $\lim_{x \to 3} \frac{x^2-9}{x+3}$

Now we provide intuitive definition of limit of a function.

<u>Definition</u>: If f is a real function defined on a set of real numbers and a in the domain, of f, then we say that limit of f(x) as x a is a real number l if f(x) is very close to l, whenever x is very close to a.

We write this as $\lim_{x \to a} f(x) = l$

If such a l does not exist then we say that $\lim_{x \to a} f(x)$ does not

exi

exist. For instance $\lim_{x \to a} \sqrt{x}$ does not exist.

Next we shall introduce the idea of left hand limit and right hand limit of a function at a point. Let f(x) be a function defined as follows.

$f(x) = \sqrt{2x + 2}$ if $x < 2$.

$\quad = x+4$ if $x \geq 2$

We shall examine whether $\lim_{x \to 2} f(x)$ exists.

First suppose $x \to 2$ from the right side of 2 (or $x \to 2$ and $x > 2$) and symbolically it is written as $x \to 2+$.

Then $\lim_{x \to 2+} f(x) = \lim_{x \to 2} x+4 = 2+4 = 6$

This limit is called as right hand limit of f(x) at 2.

Next suppose $x \to 2$ from the left side of 2 (or $x \to 2$ and $x < 2$) and symbolically it is written as $x \to 2-$.

Then $\lim_{x \to 2-} f(x) = \lim_{x \to 2} \sqrt{2x + 2} = \sqrt{2 \times 2 + 2} = 3$

$\lim_{x \to 2-} f(x)$ is called as left hand limit of f(x) at 2.

Thus $\lim_{x \to 2+} f(x) \neq \lim_{x \to 2-} f(x)$. In this case we say that $\lim_{x \to a} f(x)$ does not exist. Because $\lim_{x \to a} f(x)$ exists if and only if

$\lim_{x \to a+} f(x) = \lim_{x \to a-} f(x)$ when $\lim_{x \to a+} f(x) = \lim_{x \to a-} f(x)$, one of these values is taken as $\lim_{x \to a} f(x)$. Earlier we got $\lim_{x \to 2} \frac{x^2-4}{x-2} = 4$. In this

ase we notice that $\lim_{x \to 2+} \frac{x^2-4}{x-2} = \lim_{x \to 2-} \frac{x^2-4}{x-2} = 4$

The definition of limit given earlier is intuitive and suffers from shortcomings. In the first instance, it lacks mathematical rigour and further it is hardly useful in the development of theory of limits. We can examine more closely the idea of limit so as to arrive at Cauchy's mathematical definition.

Let us begin with $\lim_{x \to 3} (2x+1) = 7$. This means that when x is very close to 3, 2x+1 is very close to 7. Since "close to" is not mathematically defined so far, we have trouble in understanding what we mean by these words. Therefore, our first attempt to explain $\lim_{x \to 3} (2x+1) = 7$ is unsatisfactory. In our second attempt to explain $\lim_{x \to 3} (2x+1) = 7$, we mean that the value of 2x+1 can be made as near 7 as we wish to have it by making x near enough to 3. This leads us to the 'Cauchy definition' for limit of a function.

<u>Definition</u>: $\lim_{x \to a} f(x) = L$ iff for every $\varepsilon > 0$ however small there exists $\delta > 0$ such that $|f(x)-L| < \varepsilon$ whenever x is such that $0 < |x-a| < \delta$

<u>Exercise</u>: Use the above Cauchy definition of limit and show that $\lim_{x \to 3} (2x+1) = 7$

Solution: Let $\varepsilon > 0$ be any given number. Then we have to find δ such that $|(2x+1)-7| < \varepsilon$ whenever $0 < |x-3| < \delta$.

Now $|(2x+1)-7| = 2|x-3|$ iff $0 < |x-3| < \varepsilon/2$

Hence choose $\delta = \varepsilon/2$, so that $|(2x+1)-7| < \varepsilon$ for $0 < |x-3| < \delta = \varepsilon/2$.

$\lim_{x \to 3} (2x+1) = 7$

Exercise: Use the Cauchy definition of Limit and show that

$$\lim_{x \to 2} \left[\tfrac{1}{2}x - 4\right] = -3$$

Solution: Let $\varepsilon > 0$ be any given number.

Then $|(\tfrac{1}{2}x-4) - (-3)| < \varepsilon$ iff $|\tfrac{1}{2}x-1| < \delta$

$|(\tfrac{1}{2}x-4) - (-3)| < \varepsilon$ iff $\tfrac{1}{2}|x-2| < \delta$

$|(\tfrac{1}{2}x-4) - (-3)| < \varepsilon$ iff $0 < |x-2| < 2\varepsilon$

Choose $\delta = 2\varepsilon$, so that $|(\tfrac{1}{2}x-4) - (-3)| < \varepsilon$

whenever $0 < |x-2| < \delta$

Hence $\lim_{x \to 2} \left[\tfrac{1}{2}x-4\right] = -3$

Now we shall illustrate the use of this definition of limit in proving some of the important properties of limits.

Theorem: $\lim_{x \to a} c = c$ (c is any constant)

(i.e. limit of a constant is constant itself).

Proof: Let $\varepsilon > 0$ be given.

Then $|c-c| = 0 \ \forall x$ such that $0 < |x-a| < \delta$ where $\delta > 0$ can be any number. Because $|c-c| = 0$ is always true for any x and so in particular for x such that $0 < |x-a| < \delta$

$$\lim_{x \to a} c = c$$

Theorem: If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$

then $\lim_{x \to a} f(x) + g(x) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = L+M$

(i.e. limit of a sum is sum of limits).

Proof: Let $\varepsilon > 0$ be given. Then $\varepsilon/2 > 0$.

Since $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$. By definition of limit there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

$|f(x) - L| < \varepsilon/2$ for $0 < |x-a| < \delta_1$ and

$|g(x) - M| < \varepsilon/2$ for $0 < |x-a| < \delta_2$

Let δ be the smaller of δ_1, δ_2 then

$|f(x)-L| < \varepsilon/2$ and $|g(x)-M| < \varepsilon/2$ for $0 < |x-a| < \delta$

$$\begin{aligned} \text{Now } |f(x)+g(x) - (L+M)| &= |f(x)-L) + (g(x)-M)| \\ & \quad |f(x)-L| + |g(x)-M| \\ &< \varepsilon/2 + \varepsilon/2 \ \forall\ x \text{ such that } 0 < |x-a| < \delta \end{aligned}$$

$$\lim_{x \to a} f(x) + g(x) = L+M = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

On the same lines as above some more results on the limits may be proved. These results are given at the end as exercises.

Next we shall explain limits at infinity and infinite limits.

Let $f(x) = 1/x$

Let us examine behaviour of $f(x)$ as x approaches zero from right side. The closer x is to zero, the larger $f(x)$ is. In other words, as $x \to 0+$, $f(x)$ goes on increasing without bound. In this case, we write $\lim_{x \to 0} 1/x = +\infty$ (Read ∞ as "plus infinity").

Similarly as $x \to 0-$, $f(x)$ goes on decreasing without bound and we write $\lim_{x \to 0-} f(x) = \lim_{x \to 0-} 1/x = -\infty$

(Read '$-\infty$' minus infinity).

Here, ∞ is a symbol showing the phenomenon of growing larger and larger without bound. Similarly $-\infty$ is a symbol showing the phenomenon of decreasing without bound. Thus ∞ and $-\infty$ are not numbers.

Next let us consider $\lim_{x \to \infty} 1/x$. As x grows larger and larger the values of $1/x$ are close to zero. Therefore, we write $\lim_{x \to \infty} 1/x = 0$.

Also as $x \to -\infty$, $1/x \to 0$ and so we write $\lim_{x \to \infty} 1/x = 0$

However, we shall not attempt formal definitions of the above type of limits.

<u>Exercises</u> :

Use the Cauchy definition of limit to prove the following results.

1. If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$ then, show that

i) $\lim_{x \to a} f(x) - g(x) = L - M$

ii) $\lim_{x \to a} f(x) \cdot g(x) = L \cdot M$

iii) $\lim_{x \to a} f(x)/g(x) = L/M$ provided $M \neq 0$.

2. If $\lim_{x \to a} f(x) = L$ and K a constant, then show that

$\lim_{x \to a} K f(x) = K.L.$

3. Domination Principle

Let $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = L$

Suppose $f(x) \leq h(x) \leq g(x) \quad \forall x$.

Prove that $\lim_{x \to a} h(x) = L$

4. Use $\lim_{n \to \infty} 1/n = 0$ to prove that i) $\lim_{n \to \infty} 1/n^2 = 0$

ii) $\lim_{n \to \infty} 1/n^2+n+1 = 0$

5. Given $h(x) = \dfrac{2x^2-7x+3}{x^2-2x-3}$

Find i) $\lim_{x \to 0} h(x)$ ii) $\lim_{x \to 1} h(x)$ iii) $\lim_{x \to -1} h(x)$

iv) $\lim_{x \to \infty} h(x)$

6. Consider the infinite geometric series $a+ar+ar^2+..+ar^{n-1}+..$

If $S_n = a+ar+...+ar^{n-1}$, define $S = \lim_{n \to \infty} S_n$

If $|r| < 1$, then prove that $S = a/1-r$

7. Consider the circle of radius r. Use the formula for the area $A = \pi r^2$ and show that the circumference C of the circle is given by the formula $C = 2\pi r$.

8. Evaluate the following :

i) $\lim_{x \to 0} \left[\dfrac{(1+x)^3 - (1-x)^3}{x + x^3} \right]$

ii) $\lim_{x \to 0} \left[\dfrac{\sqrt{a+x} - \sqrt{a-x}}{x} \right]$

iii) $\lim_{x \to 3} \left[\dfrac{1}{x^3} - \dfrac{1}{3^3} \Big/ x-3 \right]$

9. Prove that $\lim_{x \to 0} \left(\dfrac{e^x-1}{x} \right) = 1$

10. Show that $\lim_{x \to 0} \dfrac{a^x-1}{x} = \log_e a$

2. CONTINUITY AND DISCONTINUITY OF FUNCTIONS

2.1. Closely related to the limit concept is the concept of continuity. We begin with the assumption that you have some idea of continuity. Ourpurpose is to lead you from an intuitively concept to an appropriate mathematical definition through a discussion that primarily follows the historical development of continuity in mathematics.

Consider first the functions $f(x) = x$, and

$g(x) = \frac{|x|}{x}$ for $x \neq 0$. We observe that the graph of $f(x)$ can be drawn with an uninterrupted stroke of the pencil, whereas the graph of $g(x)$ has a gap at 0.

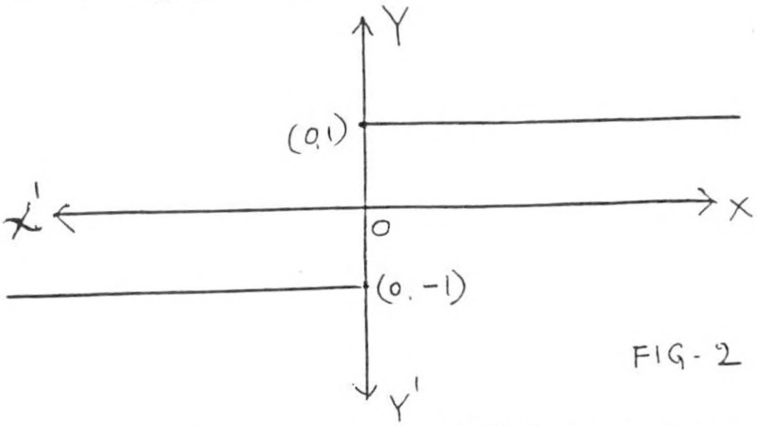


FIG-2

Intuitively we feel that the graph of $f(x)$ is continuous while the graph of $g(x)$ is not continuous as there is a gap in the graph at 0. In fact $g(0)$ is not defined. Even if we define $g(0) = 0$ still the graph of $g(x)$ is not continuous. The reason is that $\lim_{x \to 0} g(x)$ does not exist. Hence one requirement for continuity of a function say $h(x)$ at a point 'b' is that $\lim_{x \to 6} h(x)$ must exist.

Now consider another function defined as follows :

$f(x) = x$ if $x \neq 0$

$= 2$ if $x = 0$

Here $\lim_{x \to 0} f(x) = 0$. Even though $\lim_{x \to 0} f(x)$ exists the graph of $f(x)$ is not continuous at 0. The reason is that $\lim_{x \to 0} f(x) \neq 2 = f(0)$. If we alter the definition of f at 0 and define $f(0) = 0$, then $f(x)$ becomes continuous at 0. From these illustrations we conclude that a function $f(x)$ is continuous at a point c if

i) $\lim_{x \to c} f(x)$ exists, ii) $f(c)$ is defined and

iii) $\lim_{x \to c} f(x) = f(c)$

Now we are in a position to give the mathematical definition of continuity of function at a point.

Definition : Let $f(x)$ be a function defined in an interval containing the point x_1. Then f is said to be continuous at x_1 iff i) $f(x_1)$ exists, ii) $\lim_{x \to x_1} f(x)$ exists iii) $\lim_{x \to x_1} f(x) = f(x_1)$.

If any one of these three criteria is not met, then f is said to be discontinuous at x_1. Earlier we gave Cauchy definition for limit of a function. Now we shall use this to give another definition of (usually called epsilon delta definition) of continuity.

Definition : Let $f(x)$ be a function defined in an interval containing 'a'. If $f(x)$ exists then f is said to be continuous at a iff given $\varepsilon > 0$ $\exists$ $\delta > 0$ such that

$|f(x) - f(a)| < \varepsilon$ $\forall$ x with $0 < |x-a| < \delta$

2.2 Continuity of a function on an interval

Let $f : I \longrightarrow R$ (R being set of all real numbers) be a function defined on an interval I. Then f is said to be continuous on I iff f is continuous at every point of I. Thus f is not continuous on I iff $\exists\, x \in I$ such that f is not continuous at x.

For instance consider the identity function $f(x) = x$ defined on any interval I, then f is continuous on I. Because if a is any point of I, then $f(a) = a$ and so $f(a)$ exists. Also

$$\lim_{x \to a} f(x) = \lim_{x \to a} x = a.$$

$$\lim_{x \to a} f(x) = a = f(a)$$

f is continuous at a. But a is an arbitrary point of I. Hence f is continuous at every point of I and so f is continuous on I.

Now we shall prove an important result on limits which is quite useful in deciding whether or not a given function is continuous at a point.

Let $f(x)$ be a function defined in an open interval containing a point 'a'. Then when $x \longrightarrow a$, x may approach 'a' through left side of a (or through those values of x for which $x \longrightarrow a$) or x may approach a through right side of a. If x approaches a from left side we write $x \longrightarrow a -$ similarly $x \longrightarrow a +$ means that x approaches a from right side.

Theorem : $\lim_{x \to a} f(x) = L$ (L is a real number)

if and only if $\lim_{x \to a+} f(x) = L = \lim_{x \to a-} f(x)$

Proof: First suppose $\lim_{x \to a} f(x) = L$

Let $\varepsilon > 0$ be given. Then $\exists$ $\delta > 0$ such that

$|f(x) - L| < \varepsilon$ whenever $0 < |x-a| < \delta$

If $a < x < a+\delta$, then $0 < |x-a| < \delta$ and so

$|f(x) - L| < \varepsilon$. Hence $\lim_{x \to a+} f(x) = L$

Similarly, $\lim_{x \to a-} f(x) = L$

Conversely suppose $\lim_{x \to a+} f(x) = \lim_{x \to a-} f(x) = L$

Let $\varepsilon > 0$. There exists $\delta_1 > 0$ such that if $a < x < a+\delta_1$

then $|f(x) - L| < \varepsilon$. Also $\exists$ a δ_2 such that if $a - \delta_2 < x < a$

then $|f(x) - L| < \varepsilon$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then if $|x-a| < \delta$

either $a < x < a+\delta_1$ or $a-\delta_2 < x < a$ so that $|f(x)-L| < \varepsilon$

$\lim_{x \to a} f(x) = L$

2.3 Discontinuous functions

Definition : A function y = f(x) is said to be discontinuous at x = a iff f(x) is not continuous at a.

The discontinuity of f(x) at x = a can occur in any one of the following ways.

1. $\lim_{x \to a} f(x)$ does not exist.

2. $\lim_{x \to a} f(x)$ exists but is not equal to f(a).

3. $\lim_{x \to a} f(x)$ is infinite.

Now we shall illustrate these possibilities by means of some examples.

Illustration 1 : Let f(x) be a function defined on $[0,2]$ as follows:

$$f(x) = x \quad \forall\, x \in [0,1)$$
$$= x+1 \quad \forall\, x \in (1,2]$$
$$f(1) = 3/2$$

As x approaches 1 from the left side (i.e. $x \to 1-$) we have

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} x = 1$$

As x approaches 1 from right side, we have, $\lim_{x \to 1+} f(x) = \lim_{x \to 1} x+1 = 2$

Thus $\lim_{x \to 1-} f(x) \neq \lim_{x \to 1+} f(x)$

In this case $\lim_{x \to 1} f(x)$ does not exist because if it exists then $\lim_{x \to 1-} f(x) = \lim_{x \to 1+} f(x) = \lim_{x \to 1} f(x)$

Such a discontinuity is called as ordinary discontinuity or discontinuity of first kind of f(x) at x = 1.

Illustration 2

Let $f(x) \; x = x \quad \forall x \in [0,2]$ and $x \neq 1$

$= 2$ if $x = 1$.

Then $\lim_{x \to 1+} f(x) = \lim_{x \; 1-} f(x) = \lim_{x \to 1} f(x) = 1$

But $f(1) = 2$.

Hence $\lim_{x \to 1} f(x) \neq f(1)$

Hence f is disconti nuous at $x = 1$.

But this discontinuity of f at $x = 1$ can be removed by altering the value of $f(1)$.

Instead of defining $f(1) = 2$ if we define $f(1) = 1$, then f becomes continuous at $x = 1$.

Hence this type of discontinuity of f is called as removable discontinuity.

Illustration 3

If neither $\lim_{x \to a+} f(x)$ nor $\lim_{x \to a-} f(x)$ exist then $f(x)$ is said to have a discontinuity of second kind at $x = a$.

For instance define a function f on $[0,1]$ by,

$f(x) = +1$ if x is rational

$= -1$ if x is irrational.

Then both $\lim_{x \to \frac{1}{2}+} f(x)$ and $\lim_{x \to \frac{1}{2}-} f(x)$ do not exist.

Hence f has second kind discontinuity at $x = \frac{1}{2}$.

Illustration 4

If one of the two limits $\lim_{x \to a+} f(x)$, $\lim_{x \to a-} f(x)$ exists while the other does not exist then the point x = a is called a point of mixed discontinuity for f.

For instance define a function f(x) on [1,2] as follows :

$f(x) = x$ for $0 \leq x < 1$

$f(x) = 0$ if x is rational $\}$ $\forall x \in [1,2]$

$= 1$ if x is irrational

Then $\lim_{x \to 1} f(x) = 1$ but $\lim_{x \to 1+} f(x)$ does not exist.

Hence f has mixed discontinuity at x = 1.

Illustration 5 If either of the limits $\lim_{x \to a+} f(x)$, $\lim_{x \to a-} f(x)$ is infinite then f(x) is said to have an infinite discontinuity at x =a.

Consider $f(x) = 1/x \quad \forall x \in (0,1]$

$= 0$ if $x = 0$

Then $\lim_{x \to 0+} f(x) = \infty$. Therefore, f has an infinite discontinuity at x = 0.

EXERCISES :

1. Let $f(x) = \dfrac{2x^4-6x^3+x^2+3}{x-1} = x \neq 1$.

   Is f continuous at x = 1?

   Explain the type of discontinuity f has at x = 1 if f is discontinuous at x = 1.

2. Let $f(x) = \frac{x}{x^2-1}$

Then find out the values of x at which f(x) is continuous.

3. Let $f(x) = \frac{x-|x|}{x}$ for $x \neq 0$, $f(0) = 1$.

Examine the continuity of f(x) at x = 0.

4. Find the points of discontinuity of the function

$$f(x) = \frac{x}{(x-2)(x-4)}$$

5. If f(x) is continuous at 'c', then show that there exists $\delta > 0$, such that f is bounded on $(c-\delta, c+\delta)$.

6. Give an example of a function defined on a closed interval such that the function is discontinuous at every point of that interval.

7. If f(x) is a continuous function on $[a,b]$ then show that f is bounded on $[a,b]$.

8. If f(x) is continuous on $[a,b]$ and $f(a) > 0$, $f(b) < 0$ then show that f(x) = 0 for some $x \in (a,b)$.

9. Let $f(x) = 2x+1$ when $x < 1$

$= 3$ when $x = 1$.

$= x+2$ when $x > 1$.

Show that f(x) is continuous at x = 1.

10. Let $f(x) = x$ when $0 \leq x < 1$

$= 3$ when $x = 1$

$= 2x+1$ when $x > 1$

Examine the continuity of f(x) at x = 1.

3. DERIVATIVES

3.1 Introduction :

Newton and Leibnitz had been able to solve independently the two basic problems viz. finding the tangent line to a curve at any given point and finding the area under a curve. The tools that Newton and Leibnitz indepencently invented to solve these two basic problems are now called the 'derivative' and the 'integral'. Moreover, one of the great bonanzas of history is that the derivative and integral **which** were invented to solve two particular problems, have applications to a great number of different problems in diverse academic fields.

The power of calculus is derived from two sources. First, the derivative and the integral can be used to solve a multitude of problems in many different academic disciplines. The second source of power is found in the relevancy of the calculus to the problems facing mankind. Among the present day, applications of the calculus are the building of abstract models for the study of the ecology of populations, management practices, economics and medicine.

3.2 Gradient of a curve :

The gradient of a curve at any point is defined as the gradient (or slope) of the tangent to the curve at this point. An approximate value for the gradient of a curve at a point can be found by plotting the curve, drawing the tangent by eye and measuring its slope. This method has to be used for a curve when the coordinates of a finite number of points are known, but its equation is not known. When the equation of a curve is known, an accurate method for determining gradients is necessary so that we can further our analysis of curves and functions.

Consider first the problem of finding the gradient of a curve at a given point A. If B is another point on the curve (not too far from A), then the slope of the chord AB gives us an approximate value for the slope of the tangent at A. The closer B is to A, the better is the approximation. In other words, as $B \rightarrow A$, slope of chord $AB \rightarrow$ slope of the tangent at A. Let us now consider an example where we can use this definition to find the gradient of a curve at a particular point of the curve.

For this purpose, we introduce the following symbolism. A variable quantity, prefixed by δ, means a small increase in that quantity,

δx is a small increase in x,

δy is a small increase in y.

Here δ is only a prefix and it cannot be treated as a factor.

Now consider the curve $y = x\,(2x-1)$ and the problem of finding gradient at the point on the curve where $x = 1$.

If $x = 1$, $y = 1$, let A be the point (1,1). Let B be a point on the curve very close to A. Then x coordinate of B is $1 + \delta x$ (where δx is very small or close to zero).

$$y \text{ coordinate of } B = (1+\delta x)\left[2\,(1+\delta x) - 1\right]$$
$$= (1+\delta x)\quad(2\delta x + 1)$$

Slope of AB = increase in y/increase in x.

$$= \frac{(1 + \delta x)\ (2\delta x + 1) - 1}{(1 + \delta x) - 1}$$

$$= \frac{2\ (\delta x)^2 + 3\delta x}{}$$

$$= 2\delta x + 3$$

As B approaches A, $\delta x \rightarrow 0$

Hence gradient of the curve at A $= \lim_{B \rightarrow A} \left[\text{slope of AB}\right]$

$$= \lim_{\delta x \rightarrow 0} \left[2\delta x + 3\right]$$

$$= 3$$

Now we found that the gradient of the curve $y = x\ (2x-1)$ is 3 at the point on the curve where $x = 1$. We will now derive a function for the gradient at any point on the curve. Then we can find the gradient at a particular point by substitution into this derived function. Instead of taking a fixed point on the curve, we shall take A as any point (x,y) on the curve. Let B be another point on the curve whose x coordinate is $x + \delta x$.

Then B is the point $(x + \delta x, \left[x + \delta x\right]\left[2x + 2\delta x - 1\right]$

The slope of chord AB $= \frac{(x + \delta x)\ (2x + 2\delta x - 1) - x\ (2x-1)}{x}$

$$= \frac{2x^2 + 4x\delta x + 2\ (\delta x)^2 - dx - x - 2x^2 + x}{\delta x}$$

$$= \frac{4x\delta x - \delta x + 2\ (-\delta x)^2}{dx}$$

$$= \left[4x - 1 + 2\delta x\right]$$

Then the gradient at any point A on the curve =

$$\lim_{\delta x \rightarrow 0} 4x - 1 + 2\delta x$$

$$= 4x - 1.$$

So the function $4x-1$ gives the gradient at any point on the curve $y = x(2x-1)$.
We can now find the gradient of the curve at a particular point on $y = x(2x-1)$ by substituting the x coordinate of that point into the function $4x-1$. Thus the gradient of the curve at $x = 1$ is $4.1 - 1 = 3$ which we obtained earlier.

The function $4x-1$ is called the gradient function of $y = x(2x-1)$ and the process of deriving is called differentiation with respect to x. Since $4x-1$ was derived from the function $x(2x-1)$, it is called the derivative or derived function of $x(2x-1)$. Symbolically we write, $d/dx \left[x \ (2x-1)\right] = 4x-1$ where d/dx stands for "derivative w.r.t. x of". We also write $dy/dx = 4x-1$. Sometimes, we call dy/dx as "differential coefficient of y w.r.t. x". The above method of finding derivatives is called as "finding derivatives from first principles".

3.3 Equations of Tangents and Normals :

Now that we know how to find the gradient of a curve at a given point on the curve, we can find the equation of the tangent or normal to the curve at that point.

Illustration 1

Find the equation of the tangent to the curve $y = x^2-3x+2$ at the point where it cuts the y-axis
$y = x^2-3x+2$ cuts the y-axis where $x = 0$ and $y = 2$.
The slope of the tangent at $(0,2)$ = the value of dy/dx when $x = 0$.

$$= \left[d/dx \left[x^2-3x+2 \right] \right]_{x=0} = \left[2x-3 \right]_{x=0} = -3$$

Thus the tangent is a line with slope -3 and passing through (0,2).
So its equation is $y-2 = -3\ (x-0)$.
Hence the desired equation is $y = -3x+2$.

Illustration 2

Find the equation of the normal to the curve $y = x^2 + 3x-2$ at the point where the curve cuts the y-axis.

As shown in the illustration 1, the slope of the tangent to the curve at (0,2) is -3.
Hence the slope of normal to the curve at (0,2) is 1/3.
Hence the equation of normal to the curve at (0,2) is given by $y-2 = 1/3x$ or $3y = x + 6$.

Exercises :

1. Differentiate the following functions w.r.t. x from first principles.

i) $y = x^2$ ii) $y = 3x^2$, iii) $y = 1/x^2$ iv) $y = x^3+3$

v) $y = x^2 - 2x + 1$

2. Find the equation of the tangent to the curve $y = x^2+5x-2$ at the point where this curve cuts the line $x = 4$.

3. Find the equations of the normals to the curve $y = x^2-5x+6$ at the points where the curve cuts the x-axis.

4. Find the coordinates of the point on $y = x^2$ at which the gradient is 2. Hence find the equation of the tangent to $y = x^2$ whose slope is 2.

5. Find the value of K for which $y = 2x + K$ is a normal to $y = 2x^2-3$.

6. Find the equation of the normal to $y = x^2-3x+2$ whose slope is 2.

7. Find the equation of the tangent to $y = 2x^2-3x$ whose slope is 1.

8. Find the equation of the tangent to $y=(x-5)(2x+1)$ which is parallel to the x-axis.

A P P L I C A T I O N S   O F   D E R I V A T I V E S

1. Mean Value Theorem
2. Derivative as Rate Measurer
3. Differentials and Approximations

by

Mrs.S.VASANTHA

APPLICATIONS OF MEAN VALUE THEOREM

The Mean Value Theorem for derivative is of great importance in Calculus because, many useful properties of functions can be deduced from it. A special case of this result known as Rolle's theorem was first proved by Michael Rolle, a French Mathematician in 1691. A formal statement of the Mean Value Theorem is given here for convenience.

(Ref: Th. 4.10 of the Textbook)

Statement : Let f be a real function, continuous on the closed interval $[a,b]$ and differentiable in the open interval (a,b), then, there is a point $C \in (a,b)$ such that

$$\frac{f(b) - f(a)}{b-a} = f^1(c) \tag{1}$$

C is called a mean value.

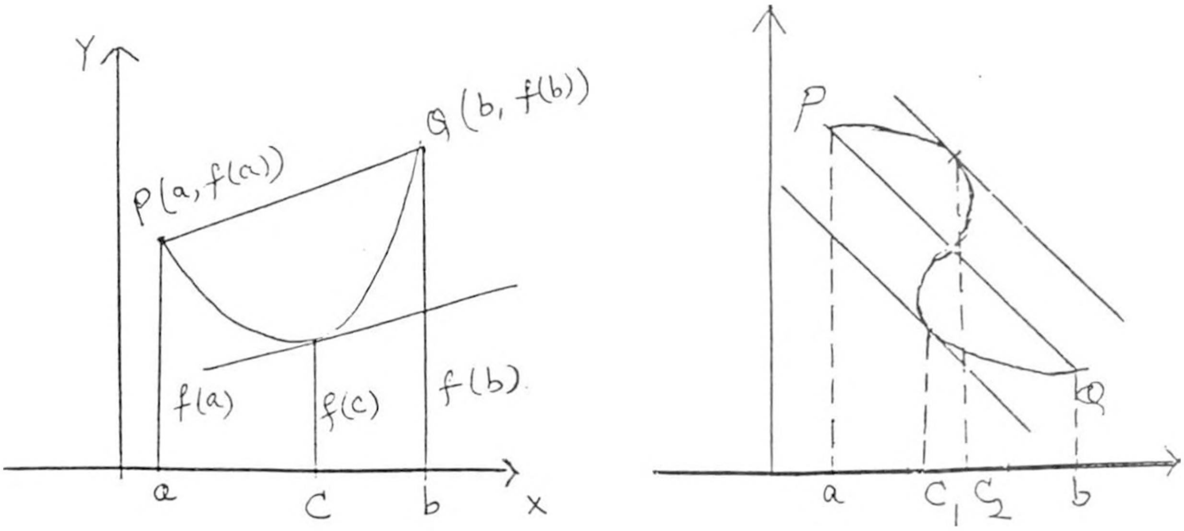


fig 1

Intuitively (1) can be interpreted thus - If we assume f(t) to be the distance travelled by a movingparticle at time t. Then the lefthand side of (1) represents the mean or average speed in the time interval a,b and the derivative $f^1(t)$ on Rhs represents the instantaneous speed at time t. (1) asserts that at some instant C during the motion of the particle, the average speed is equal to the instantaneous speed.

Geometrically, (1) implies that the slope of the tangent at (C. f(c)) in fig.1. [$(C_1, f(C_1))$ and $(C_2, f(C_2)$ in fig.2], is equal to the slope of the chord PQ.

This is seen in the figure by the fact that the chord PQ is parallel to the tangent line at C (in fig.1) (and at C_1 and C_2 in figure 2).

There may be two or more mean values also on a given interval, depending on the graph of f.

Although the M.V. Theorem guarantees that there will be atleast one mean value for a function whose graph is a smooth curve on a given interval, the theorem gives no information about the exact location of these mean values. We just know that the point C lies somewhere between a and b. Generally, an accurate location of C is difficult. Many useful conclusions can be drawn by simply knowing about the existence of atleast one mean value.

Some Consequences of Mean Value Theorem :

1. A generalization of M.V.Theorem can be obtained by considering the parametric representation of a function whose graph is a smooth curve on $[a,b]$

Let $x = g(t)$, $y = f(t)$; $a \leq t \leq b$ .... (2)

be the parametric form of the given function.

Slope of the chord joining the end points $(g(a).f(a))$ and $(g(b), f(b))$ of the curve $= \dfrac{f(b) - f(a)}{g(b) - g(a)}$ ... (3)

The slope of the tangent to the curve the point C

$$= \frac{f^1(c)}{g^1(c)} \quad \ldots. \quad (4)$$

The Mean Value Theorem asserts that there always exists a mean value C in (a.b) for which

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f^1(c)}{g^1(c)} - \quad a < c < b \quad --- (A)$$

$$g^1(c) \neq 0$$

(A) is referred to as Cauchy's M.V. Theorem.

2. <u>Algebraic sign of the first derivative of a function</u> gives useful information about the behaviour of its graph. Using Mean Value Theorem, the algebraic sign of the derivative of a given function can be determined.

Theorem: Let f be continuous on $[a,b]$ and derivable in (a,b), then,

a) If $f^1(x) > 0 \ \forall x \in (a,b)$, then f is strictly increasing on $[a, b]$

b) If $f^1(x) < 0 \ \forall x \in (a,b)$ then f is strictly decreasing on $[a, b]$

c) If $f^1(x) = 0 \ \forall x \in (a,b)$, then f is a constant.

Proof : (a) For any points x_1 and x_2 with $a \leq x_1 < x_2 \leq b$, the Mean Value Theorem applied to $[x_1, x_2]$ gives

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f^1(c); \quad x_1 < c < x_2 \quad --- (5)$$

Since $f^1(c)$ is given to be > 0 and $x_2 - x_1 > 0$, wesee that $f(x_2) - f(x_1) > 0$ implying that $f(x_1) < f(x_2)$ or f is strictly increasing on $[a,b]$.

Proof of (b) is left as an exercise.

Proof of (c) : Put $x_1 = a$ in (5).

We get $$\frac{f(x_2) - f(a)}{x_2 - a} = f^1(c) \quad \ldots \quad (6)$$

Since $f^1(c) = 0$, $(6) \Rightarrow f(x_2) = f(a) \quad \forall x_2 \in [a, b]$

Hence f is a constant on $[a,b]$

Using this result, it is possible to determine the intervals of increase and decrease of functions.

The well-known sufficient condition for the existence of an extrema for a function also follows from the above theorem.

3. The Mean Value Theorem can be used to show that : Any two integrals of the same derived function can differ atmost by a constant.

<u>Proof</u> : Suppose $F(x)$ and $G(x)$ have the same derivative $f(x)$ over some interval $a \leq x \leq b$.

Consider $H(x) = F(x) - G(x)$ ... (1) (1)

Apply Mean Value Theorem to $H(x)$ on $[a,c]$

where C is : $a \leq c \leq b$ to obtain

$H(c) - H(a) = H^1(\xi)(c-a), \quad a \leq \xi \leq c.$

Since $H^1(x) = F^1(x) - G^1(x) = 0$ by hypothesis, $x \in [a,b]$

$H(c) - H(a) = 0$ and so $H(c) = H(a)$

$\Rightarrow F(c) - G(c) = F(a) - G(a)$ where

$F(a) - G(a)$ is a fixed quantity. Let $F(a) - G(a) = C$

Since c is any value of x in $[a,b]$,

we have $F(c) - G(c) = C, \forall c \in [a,b]$

$\therefore$ $F(x)$ and $G(x)$ can differ by a constant C.

Now, $F(x)$ and $G(x)$ which are any two integrals of $f(x)$ can differ only by a constant C.

<u>Differentials and Mean Value Theorem</u>

Recall that the differential dy of a function $y = f(x)$ is defined by the equation

$$dy = f^1(x) \cdot \Delta x = f^1(x)\,dx \text{ for small } \Delta x.$$

Here, dy is an approximate value of Δy, we know that,

$$\Delta y = f(x + \Delta x) - f(x) \quad \ldots \quad (2)$$

Can we improve this approximation ?

Mean Value Theorem helps us to answer this question.

Now, instead of considering x and $x + \Delta x$ let us consider any two values of x say, a and b.

Then we get $\Delta y = f(b) - f(a)$.... (3)

and $dy = f^1(a)\ (b-a)$ .... (4)

Since $dy \approx \Delta y$, $f(b) - f(a) \approx f^1(a)\ (b-a)$ (5)

Can we now find an approximation to $f(b) - f(a)$, which is better than $f^1(a)\ (b-a)$?

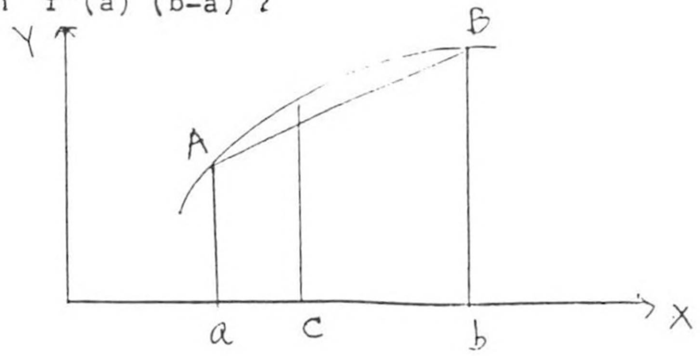


fig.2.

From the figure, $\frac{f(b) - f(a)}{b-a}$, is slope of chord AB. But by Mean Value Theorem there exists a C; $a < c < b$, such that slope of AB = Slope of tangent at $(C, f(c)) = f^1(c)$.

$$\therefore \frac{f(b) - f(a)}{b-a} = f^1(c) \implies f(b) - f(a) = (b-a)\ f^1(c) \quad \ldots (6)$$
$$a < c < b$$

Comparing (5) and (6) we see that (6) results from (5) when we replace a by c in $f^1(a)$, c being the mean value. Also, (6) is an estimate of $\Delta y = f(b) - f(a)$. In fact (6) gives an exact expression for Δy or $f(b) - f(a)$, whereas (5) gives a mere approximation to

$f(b) - f(a)$ or Δy. Hence we have proved that the approximation of Δy by the differential dy can be bettered by using the Mean Value Theorem. For such an improved approximation of Δy, Δx need not be very small.

(5) If for a given function $y = f(x)$ derivable on (a,b) and continuous on $[a,b]$ we further assume that $f^1(x)$ is continuous on $[a,b]$, then f^1 ought to attain its maximum and minimum values (bounds) atleast once on $[a,b]$. By Mean Value Theorem, we have

$$\frac{f(b) - f(a)}{b - a} = f^1(c), a < c < b \quad .....(*)$$

(*) now implies that $f^1(c)$ cannot exceed max. f^1 nor can it be less than min. f^1 on $[a,b]$. So, we obtain

$$\text{Least value of } f^1 \text{ on } [a,b] \leq \frac{f(b) - f(a)}{b - a} \leq \text{greatest value of } f^1 \text{ on } [a,b]$$

or

$$\min_{x \in [a,b]} f^1(x) \leq \frac{f(b) - f(a)}{b - a} \leq \max_{x \in [a.b]} f^1(x) \quad \ldots \quad (1)$$

(1) can now be used to restate the Mean Value Theorem as follows : The mean value of a continuous function on a closed interval must actually be a value attained by the function.

(1) can also be used to estimate the value of a function at a given point when a and f^1 are known.

Assignment Problems

1. Use Mean Value Theorem to deduce the following inequalities :

a) $| \sin x - \sin y | \leq |x-y|$

b) $ny^{n-1}(x-y) \leq x^n - y^n \leq nx^{n-1}(x-y)$

if $0 < y \leq x$, $n = 1,2,3,\ldots$

2. The function $y = |4-x^2|$, $-3 \leq x \leq 3$ has a horizontal tangent at $x = 0$ even though the function is not differentiable at $x=-2$ and $x=2$. Does this contradict Mean Value Theorem? Explain.

3. A motorist drove 30 miles during a one hour trip. Show that the Car's speed was equal to 30 miles/hour atleast once during the trip.

4. Show that

$$\frac{d}{dx}\left(\frac{x}{x+1}\right) = \frac{d}{dx}\left(-\frac{1}{x+1}\right)$$

even though $\frac{x}{x+1} \neq \frac{-1}{x+1}$

Explain.

5. Show that the Mean Value Theorem can be given by the equation.

$$\frac{f(x+h) - f(x)}{h} = f^1(x + \theta h), \quad 0 < \theta < 1.$$

Determine θ as a function of x and h when

a) $f(x) = x^2$ (b) $f(x) = e^x$

c) $f(x) = \log x$, $x > 0$

DERIVATIVE AS A RATE MEASURER

Consider a particle P moving in a straight line. Its motion can be described by the function

$S = f(t)$, where S is the position of P at any time instant t. Let V be the Velocity of the moving particle P, at the time instant t. We wish to obtain V as the derivative $f^1(t)$.

Recall that the average velocity of P in a time interval Δt is the difference quotient $\frac{\Delta s}{\Delta t}$ and

$$\frac{\Delta s}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{(t + \Delta t) - t} = \frac{(S + \Delta S) - S}{(t + \Delta t) - t} \quad \text{- - -} \quad (1)$$

V, the Instantaneous velocity of P at time t is now computed from the values of $\frac{\Delta s}{\Delta t}$ for progressively smaller values of Δt.

This leads to V as $\lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}$

$$\text{or } V = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{t} = f^1(t) \quad \text{- - -} \quad (2)$$

(2) Implies that when the position function $S = f(t)$ of a moving particle is known, the rate of motion of the particle w.r.t. time can be given by the derived function $f^1(t)$.

When the motion of P is uniform, the average velocity itself represents the instantaneous velocity, as the velocity of motion remains constant at all instants of time.

If P moves with variable velocity, then average velocity $\frac{\Delta s}{\Delta t}$ differs with differing values of Δt. By taking an instant 't' as a time-interval of length zero, (an instant is at a point of time) $\frac{\Delta s}{\Delta t}$ reduces to $\frac{0}{0}$ for a given instant of time 't', which is meaningless. However, for small values of Δt, $\frac{\Delta s}{\Delta t}$ gives approximate values of instantaneous velocity V. Hence it is reasonable to define V with the aid of the limit concept. Thus,

$$V = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = f^1(t)$$

Note: V is independent of the increment Δt, but depends on the value of t and the type of function f(t).

Variable Physical magnitudes as derivatives: More examples.

1. Acceleration : When the velocity function $v = f(t)$ of a particle performing non-uniform motion is known, the instantaneous rate of change of its velocity (acceleration) is computed by

$$\text{Acceleration} = \frac{dv}{dt} = f^1(t) = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}$$ when the

quotient $\frac{\Delta v}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t}$ is the average acceleration.

2. Heat Capacity: as a derivative

Let $q = H(t)$ give the quantity of heat q, absorbed by a physical body when heated to the temperature t. Heat capacity C is the rate of change of the quantity of heat absorbed w.r.t. temperature. C is expressed as a derivative. If Average Heat Capacity C_{av} is the quotient $\Delta q/\Delta t$,

then, $$C = \lim_{\Delta t \to 0} C_{av} = \lim_{\Delta t \to 0} \frac{\Delta q}{\Delta t} = \lim_{\Delta t \to 0} \frac{H(t + \Delta t) - H(t)}{\Delta t}$$

$$= H^1(t).$$

3. Reaction rate of a chemical reaction

Let the function $m = \phi(t)$ represents the mass of a chemical substance entering into a chemical reaction during time t.

The rate of change of mass of the substance w.r.t. the time t is called the reaction rate. This can be expressed as a derivative.

If Average reaction rate R_{av} for the time interval Δt is given by the quotient

$$\frac{\Delta m}{\Delta t} = \frac{\phi(t + \Delta t) - \phi(t)}{\Delta t}$$

Then the reaction rate R for a given amount of substance at time t is

$$R = \lim_{\Delta t \to 0} \frac{\Delta m}{\Delta t} = \lim_{\Delta t \to 0} R_{av} = \phi^1(t)$$

The above examples show how derivatives are used to express certain variable physical magnitudes as rates of change w.r.t. some other physical magnitudes.

In general, the derivative of a function estimates the rate of change of a given function. Hence

$$f^1(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

gives the measure of the rate at which f(x) changes with respect to x, at a given point x.

Related rates - Problems :

Before attempting to solve some problems we recall the chain rule, as it is often tailor-made in solving the related rates problems.

If $Z = f(y)$ and $y = g(x)$

then $\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$ ... (1)

where $\frac{dz}{dy} = f^1(y)$ and $\frac{dy}{dx} = g^1(x)$

(1) Tells us that the rate of change of Z w.r.t. X is the product of the rate of change of Z w.r.t. y and the rate of change of y w.r.t. x.

Problem 1. A variable right triangle ABC in the xy-plane has its right angle at the vertex B, a fixed vertex A at the origin and the third vertex C restricted to lie on the parabola $y = 1 + \frac{7}{36} x^2$. The point B starts at (0,1) at time t = 0 and moves upward along the y axis at a constant velocity of 2 cm/sec. How fast is the area of the triangle increasing when t = 7/2 sec ?

Solution: Clearly the moving vertex C of the expanding triangle has for its coordinates C (x,y) where x is the base and y the height of the triangle, x and y are both variables. C(x,y) satisfies the equation $y = 1 + \frac{7}{36} x^2$. Note that the triangle remains right angled while varying in its size.

The velocity of the moving vertex B along y axis (= dy/dt) is a constant (=2 cm/sec). ∴ The equations relating the variables x, y and t are

(1) ... Area $A = 1/2\, xy$

(2) $y = 1 + \frac{7}{36} x^2$

(3) $y = 1 + 2t$

(4) $7x^2 = 7\ 2t$

We must find $\frac{dA}{dt}$ at $t = 7/2$ sec.

$\frac{dA}{dt} = \frac{1}{2}\,(x.dy/dt + y.dx/dt)....$ (5)

Substituting $x = 6$ and $y = 8$, (found from (3) and (4) for $t = 7/2$)

and Using the values $\frac{dx}{dt} = \frac{6}{7}$ and $\frac{dy}{dt} = 2$ in the equation (5)

We obtain $\frac{dA}{dt} = \frac{66}{7}$ cm^2/sec at $7/2 = t$

∴ The triangle is increasing its area at the rate of 66/7 cm^2/sec.

Problem 2. A stone is dropped into a quiet pond and waves move in circles outward from the place where it strikes, at a speed of 3" per second. At the instant when radius of one of the wave rings is three feet, how fast is its enclosed area increasing ?

Solution: Radius r and area A are the variables. The equation relating the variables are

$A = \pi r^2$ so that $\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$

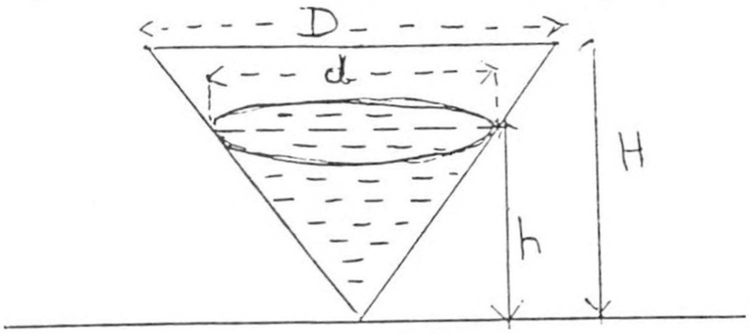
The speed of the wave outward from the center is the rate at which the radius increases = $\frac{dr}{dt}$

$$\frac{dr}{dt} = 3''\ /\text{sec.}$$ At $r = 3$ the rate of

increase in area = $\frac{dA}{dt} = 2\ \ 3 . \frac{1}{4} = \frac{3}{2} = 4.71$ sq. ft/sec.

Problem 3 : Water runs into conical paraffin paper cup five inches high and 3 inches across the top at the rate of one cubic inch per sec. When it just half filled, how rapidly is the surface of the water rising ?

Solution: The height (H) and the diameter (D) of the conical cup are the given constants. Let h be the height of the surface of water in the conical cup, when the volume of the water already in the cup is V. h and d (the diameter of water in the cup) are both variables.



The rate of increase in the volume of water = **rate** of inflow of water into the cup = dv/dt = 1 cubic inch per second. The rate of rise in the surface of water in the cone = rate of increase of height h = dh/dt

$$V = \text{Volume of the conical cup} = \frac{1}{3}\pi \frac{D^2H}{12} = 11.7$$

$\frac{1}{2} V = \frac{11.7}{2} = 5.85$ cu. inc. is the volume of water in the cup when first half filled.

We must find dh/dt when v = 5.85 and dv/dt = 1.

V, Volume of water in the cup =

$$V = \frac{\pi d^2 h}{12} \Rightarrow hd^2 = \frac{12}{\pi} V \qquad (1)$$

(1) relates the variables h and V, but also contains 'd'. We must express d in terms of h or V.

We have $\frac{d}{h} = \frac{D}{H} = \frac{3}{5} = .6$

$$d = .6h \qquad (2)$$

Using (2) in (1)

$$h.\ (.36\ h^2) = \frac{12\ V}{\pi} = .36\ h^3$$

$$h = \sqrt[3]{\frac{12}{.36\pi}} \cdot \sqrt[3]{V}$$

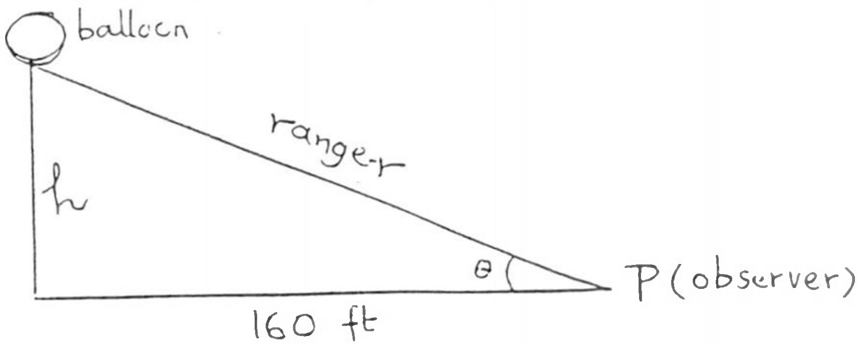
After computing cube roots, we can write $h = 2.2\ V^{1/3}$

$$\frac{dh}{dt} = 2.2\ \frac{1}{3}\ V^{-2/3}\ \frac{dV}{dt}$$

$= \frac{.74}{\sqrt[3]{V^2}} \cdot \frac{dV}{dt}$ when V = 5.85 (half filled) and $\frac{dV}{dt} = 1$

$$\frac{dh}{dt} = \frac{.74}{\sqrt[3]{(5.85)^2}}\ .1 = \frac{.74}{\sqrt[3]{35.2}} = .23 \text{ in/sec.}$$

Problem 4: A balloon is rising vertically from the ground at a constant rate of 15 ft/sec. An observer situated at a point P 160 ft away from the point of lift-off tracks it. Find the rate at which the angle at P and the range r are changing when the balloon is 160 ft. above the ground.



Solution: Variables are angle θ and the range r,

From the figure, $\tan\theta = \frac{h}{160}$ (1)

Differentiating (1) on both sides w.r.t. t

$$\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{1}{160} \cdot \frac{dh}{dt} \quad \ldots \quad (2)$$

At h = 160 (1) gives $\tan\theta = 1$ $\theta = \pi/4$.

$\sec^2\theta = (\sqrt{2})^2 = 2$; $\frac{dh}{dt} = 15$ ft/sec. (given). ∴ (2) becomes

2. $\frac{d\theta}{dt} = \frac{1}{160} \times 15$ $\frac{d\theta}{dt} = \frac{15}{320}$ rad/sec. $= \frac{3}{64}$ radians/sec.

Angle P is increasing at the rate of $\frac{3}{64}$ radians/sec when h = 160 ft.

<u>Now to find the rate of change of the range r</u>

From the figure, $h^2 + 160^2 = r^2$ (3)

(Note h and r are variables)

differentiating (3) w.r.t. t.

$$2h \cdot dh/dt = 2r \cdot \frac{dr}{dt} \quad (4)$$

when h = 160, $r = \sqrt{160^2+160^2} = 160\sqrt{2}$

$$\frac{dh}{dt} = 15 \text{ ft/sec.}$$

$$\frac{dr}{dt} = \frac{160}{160\sqrt{2}} \cdot 15 = \frac{15}{\sqrt{2}} \quad \frac{15\sqrt{2}}{2} \text{ ft/sec.}$$

Range r is varying at the rate of $\frac{15\sqrt{2}}{2}$ ft/sec.

A step by step guide to solve related rates problems :

1. Draw a figure. Name the variable and constant magnitudes. Label these in the figure.
2. Mark the variable/variables whose rate/rates of change you must find.
3. Form equations relating variable and constants.
4. Substitute known values (if necessary) and differentiate. Obtain a single equation expressing the rate that you want in terms of the rates and quantities already known.

Problems for Assignment :

1. Suppose a rain drop is a perfect sphere. Assume that through condensation, the rain drop accumulates moisture at rate proportional to the surface area. Show that the radius increases at a constant rate.
2. A balloon 200 ft off the ground and rising vertically at the constant rate of 15 ft/s. An automobile passes beneath it travelling along a straight road at the constant rate of 45m/hour. How fast is the distance between them changing one second later ? (Ans. 33.7 ft/sec).
3. A light is at the top of pole 50 ft high. A ball is dropped from the same height from a point 30 ft away from the height. How fast is the shadow of the ball moving along the ground 1/2 second later ? (Ans. 1500 ft/sec.).

4. Two ships A and B are sailing straight away from the point D along routes such that the angle AOB = 120°. How fast is the distance between them changing, if at a certain instant DA=8 miles? Ship A is sailing at the rate of 20 miles/hr and ship B at the rate of 30 miles/hr ? (Hint: Use law of Cosines) 260/37 miles/hr.

5. A particle is moving in the circular orbit $x^2+y^2=25$. As it passes through the point (3,4), its Y-coordinate is decreasing at the rate of 2 units per second. How is the X-coordinate changing ? (Ans: 8/3 units/sec).

Additional Problems for Assignment :

1. Find the height of a right cone with least volume circumscribed about a given sphere of radius R. (Ans.4R)

2. It is required to make a cylinder, open at the top the walls and the bottom of which have a given thickness. What should be the dimensions of the cylinder so thatfor given storage capacity, it will require the least material ? (Ans. R = 3 V/R is the inner radius of the base, V = inner volume).

3. Out of sheet metal having the shape of a circle of radius R, cut a sector such that it may be bent into a funnel of maximum storage capacity. (Ans. The central angle of the sector = $2\pi\sqrt{2/3}$

4. Of all circular cylinders inscribed in a given cube with side a so that their axis coincide with the diagonal of the cube and the circumferences of the base touch its planes. Find the cylinder with maximum volume.

$$h = \frac{a\sqrt{3}}{3}$$

$$r = \frac{a}{\sqrt{6}}$$

5. In a rectangular coordinate system a point (X_o, Y_o) is lying in the first quadrant. Draw a straight line through this point so that it forms a triangle of least area with the positive directions of the axis.
(Ans. $X/2X_o + Y/2Y_o = 1$).

6. Given a point in the axis of the parabola $Y^2 = 2px$ at a distance of a from the vertex, find the abscissa of the point of the curve closest to it. (Ans. X = a-p).

7. Assuming that the strength of a beam of rectangular cross-section is directly proportional to the width and to the cube of the altitude, find the width of a beam of maximum strength that may be cut out of a log of diameter 16 cms. (Ans. width = 8 cm).

8. A torpedo boat is standing at anchor 9 km from the closest point of the shore. A messenger has to be sent to a camp 15 km (along the shore) from the point of the shore closest to the boat. Where should the messenger land so as to get to the camp in the shortest possible time ? (if he does 5 kms/hr walking and 4 km/hr rowing). (Ans. at a point 5 km from the camp).

9. Show that the volume of the largest right circular cylinder which can be inscribed in a given right circular cone is 4/9 the volume of the cone.

10. If sum of the surface areas of cube and a sphere is constant, what is the ratio of an edge of the cube to the diameter of the sphere when a) the sum of their volumes is a minimum? b) the sum of their volumes is a maximum ?

11. A lamp 50 ft above the horizontal ground and a stone is dropped from the same height from a point 12 ft away from the lamp. Find the speed of the shadow of the stone on the ground when the stone has fallen 10 ft.

12. The volume of a certain mass of a gas under pressure P lbs wt/sq inch is v cu.inches where PV = 1200. If the volume increases at the rate of 40 cubic inches/min. find the rate of change of pressure when vol = 20 c.inches,

(Ans. 120 lbs/min).

13. A circular blot of ink on a blotting paper expands in such a way that the radius r cms at t secs is given by

$$r = t - \frac{1}{8t^2}.$$

Find the rate at which the blot is increasing at the end of 2 seconds. (Ans. $\frac{2079\pi}{512}$).

DIFFERENTIALS AND APPROXIMATIONS.

In this section, we attempt to define derivative as a quotient of two quantities called differentials and see how this definition is useful in carrying out approximate calculations.

Recall that, derivative $f^1(x)$ of a given function $y = f(x)$ is defined as the limit of a quotient,

i.e. $f^1(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = dy/dx$

Note that $f^1(x)$ itself is not a quotient.

It is wrong to interpret that dy/dx is obtained by dividing dy by dx, Where $dy = \lim_{\Delta x \to 0} \Delta y$ and $dx = \lim_{\Delta x \to 0} \Delta x$

This interpretation leads to the result 0/0.

However, using the notion of derivative as a limit, it is possible to define a new quantity 'dy' called the differential of y so that the quotient dy/dx will indeed become equal to the derivative $f^1(x)$.

Meaning of differential :

Consider $y = f(x)$, derivable at x.

Then, $f^1(x) = dy/dx = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ (1)

(1) implies that $\frac{\Delta y}{\Delta x}$ differs from $\frac{dy}{dx}$ (or $f^1(x)$) by an infinitesimally small quantity ε.

(Here Δx, ε, are examples of infinitesimals).

$\therefore \frac{\Delta y}{\Delta x} = f^1(x) + \mathcal{E}$ or

$$\Delta y = f^1(x) . \Delta x + \mathcal{E} . \Delta x \qquad (1)$$

The term $\mathcal{E} . \Delta x$ in (1) being the product of the infinitesimals, is much smaller when compared with the term $f^1(x) . \Delta x$. For Δx sufficiently small, we see that $f^1(x) . \Delta x$ is a good approximation of Δy if we neglect the term $\mathcal{E} . \Delta x$.

Now let us define the differential dy of y by $dy = f^1(x) . \Delta x$ Denoting dx, the differential of x as Δx itself (why ?) We obtain $dy = f^1(x) . dx$ and $dx = \Delta x$ so that,

The derivative $f^1(x)$ = the quotient $\frac{dy}{dx}$

$f^1(x)$ = the quotient of the differentials dy and dx.

<u>Illustration</u>: (1) Consider a square of side x units. An error of .01 has crept into the measurement of its side. Estimate the error in its area.

Let us take x = 12 units.

Error in the measurement of x = .01

$\therefore \Delta x = .01$

If the function in question is $y = x^2$

then, $y = 2x . \Delta x + (\Delta x)^2 = 2 \times 12 \times (.01) + (.01)^2$

whereas $dy = f^1(x) . dx = f^1(x) . \Delta x = 24x . 01$

neglecting $(.01)^2$; $\Delta y \approx dy$.

Error in this estimation is .0001.

(2) To see the advantage gained by approximating Δy by dy, consider $f(x) = x^4$

$$\Delta y = 4x^3 \cdot \Delta x + 6x^2 (\Delta x)^2 + 4x \cdot (\Delta x)^3 + (\Delta x)^4$$

For small Δx, the powers of Δx get progressively smaller.

Replacing Δy by dy,

$dy = f^1(x) . \Delta x = 4x^3 . \Delta x$ is a good approximation to Δy.

It is worth noting here, how much simpler it is to compute dy as compared to Δy.

When the functions under investigation get more complex, the usefulness of approximating Δy by dy becomes even more pronounced.

The geometric meaning of differential.

Refer to the figure 4.22 given in the text book.
A variation of the same figure is supplied here.

Geometrically, the approximation by the differential is the tangent line approximation to the curve y = f(x) at a given point P(x,y). Note that the tangent to a differentiable curve always runs close to the curve near the point of tangency.

From the figure it is clear that Δy and dy are not the same. While Δy gives the actual change in the function y = f(x) as x changes to x+ Δx, dy gives the increment in the function represented by the tangent line to the curve y = f(x) at P(x,y). In other words, if the function y = f(x) were replaced by its tangent line at P, dy would be the increment in the function representing the tangent line corresponding to the increment dx in x. The slope of this tangent line is $f^1(x)$ at $P(x,y)$. The difference in $\triangle y$ and dy is the vertical portion of $\triangle y$ between the tangent line and the graph of $f(x)$. The less the graph curves, nearer is it to the tangent line and better, the approximation is dy to $\triangle y$.

. Errors and approximate calculations :

1. Differentials are used to estimate the square roots, cube roots, fourth roots and so on. (Ref. text).

2. Estimation of small errors: Physical measurments using instruments are subject to small errors. Differentials are used to estimate the accuracy and the error involved in measurements.

For example, when the diameter (d) of a small steel ball is measured by a vernier and if the reading is correct to $\frac{1}{100}$ of an inch. The true measurement differs from the vernier reading by $\frac{1}{100}$ th of an inch.

If Δx is the error in the measurement of a magnitude x, the corresponding error which results in $y = f(x)$ is approximately $\Delta y = f^1(x). \Delta x = dy$. This error is called the absolute error.

The ratio of this error Δy to the magnitude y is $\frac{\Delta y}{y}$, and is called relative error.

$100.\frac{\Delta y}{y}$ is called the percentage error in y.

Now, going back to the problem of steel balls, the actual measurement gives the diameter as $d + \Delta x$. The relative error here is $-\frac{\Delta x}{d}$. Now we want to find the corresponding error in the volume of the sphere.

Volume of the sphere $= V(d) = \frac{1}{6}\pi d^3$

$\Delta V \approx dV = \frac{1}{2}\pi d^2. \Delta x.$

Hence the relative error in the volume is

$$\frac{\Delta V}{V(d)} \approx \frac{dV}{V(d)} = \frac{\frac{1}{2}\pi d^2 \cdot \Delta x}{\frac{1}{6}\pi d^3} = \frac{3}{d}\Delta x = 3 \cdot \frac{\Delta x}{d}$$

= 3 times the relative error in the diameter.

Example 1 : If $f(x) = x^4 - 4x^2 + 7x - 5$
find f(2.99).

Here we take x = 3, and Δx .01

$f^1(x) = 4x^3 - 8x + 7$

$f^1(3) = 91$, $f^1(x).\ \Delta x = f^1(3) \cdot \Delta x = -0.91$

$f(2.99) = f(3) + f^1(3).\ \Delta x$

$= 61 + (-0.91) = 60.09 = 60.09$

Example 2 : Find the linear approximation to

$f(x) = \sqrt{1 + 2x}$ near x = 2.

We must evaluate $f(2) + f^1(2)\ (x-2)$

taking $\Delta x = (x-2)$

$$f^1(x) = \frac{1}{2}(1+2x)^{-1/2} \cdot 2 = \frac{1}{\sqrt{1+2x}}$$

Its value at x = 2 is

$$f^1(2) = \frac{1}{\sqrt{1+2.2}} = \frac{1}{\sqrt{5}}$$

$f(2) = \sqrt{5}$

$$\therefore\ f(2) + f^1(2)\ (x-2) = 5 + \frac{1}{\sqrt{5}}(x-2)$$

We have $f(x) \approx \sqrt{5} + \frac{1}{\sqrt{5}}(x-2) = 5 + \frac{x}{\sqrt{5}} - \frac{2}{\sqrt{5}}$

$$= \frac{x}{\sqrt{5}} + \sqrt{5} - \frac{2}{\sqrt{5}} = \frac{x}{\sqrt{5}} + \frac{3}{\sqrt{5}}$$

Linear approximation of $\sqrt{1+2x} = f(x)$ near 2 is

$$f(2) + f^1(2)\,(x-2) = (x/\sqrt{5}) + \frac{3}{\sqrt{5}}.$$

> If $y = f(x)$ is differentiable at x_0
> then $f(x) \approx f(x_0) + f^1(x_0)\ (x-x_0)$ for x near x_0

3) How accurately should we measure the edge x of a cube to compute the volume $v = x^3$ within 1% of its true value.

Solution : We want inaccuracy Δx in our measurement to be small enough to make corresponding increment ΔV in volume to satisfy the inequality

$$|\Delta v| \leq \frac{1}{100} \times V = \frac{1}{100} \cdot x^3$$

Using differentials, $dV = 3x^2 .\quad x$

$$V \quad 3x^2 . \quad x$$

$$|3x^2 . \quad x| \quad \frac{x^3}{100}$$

$$|\quad x| \quad \frac{x}{3.100} = \frac{x}{3}\, x.\,01 = \frac{1}{3} \cdot \frac{x}{100}$$

Hence we must measure edge x with an error that is no more than one third of one percent of the true value.

Using differentials $dV = 3x^2 . \Delta x$

$$\Delta V \approx 3x^2 . \Delta x$$

$$\therefore \quad |3x^2 . \Delta x| \leq \frac{x^3}{100} \Rightarrow |\Delta x| \leq \frac{x}{3 . 100} = \frac{1}{3} . \frac{x}{100}$$

$\therefore$ error in the measurement of x (edge) should not exceed a third of one percent of the true value.

Assignments :

1. Estimate $\sqrt[4]{17}$.
2. Calculate Sin 59° approximately, knowing that Sin 60° = $\sqrt{3}/2$ Remember that in calculus formulae presuppose radian measure for angles.
3. The width of a river is calculated by measuring the angle of elevation from a point on one bank of the top of a tree 50 feet high and directly across on the opposite bank. The angle is 45° with a possible error of 20′. Find the possible error in the calculated width of the river.
4. A given quantity of metal is to be cast in the form of a solid right circular cylinder of radius 5" and height 10". If the radius is made 1/20th of an inch too large, what is the error in the height ?
5. The edge of a cube is measured as 10 cm with a possible error of one per cent. The cube's volume is to be calculated from this measurement. About how much error is possible in the volume calculation ?
6. About how accurately must the interior diameter of a 10 meter high storage tank of cylindrical shape be measured to calculate the tank's volume to within an error of one percent of its true value.
7. The radius of a circle is increased from 2.00 to 2.02 meters
   a) estimate the change in area
   b) calculate the error in the estimate in (a) as a percent of the original area.

8. If $f(x) = x^4-2x+3$ and given $f(8) = 4083$ find the value of $f(8.001)$.

9. If $f(x) = x^3+x^2+x-3$, find $f(1.09)$ approximately.

10. Show that the relative error in the volume of a sphere is three times the relative error in the radius.

I N T E G R A T I O N

1. Definite Integral and Properties of Definite Integral

2. Volumes of Solids by Definite Integrals

by

Dr.V.SHANKARAM

DEFINITION OF DEFINITE INTEGRAL

Introduction :

Historically, the basic problem of integrals is to find the areas and volumes by certain approximation methods. The first abstract proofs of rules for finding some areas and volumes are said to have been developed by Eudoxus between 400 B.C. and 350 B.C. Later his method of approximation was developed and exploited by Archimedes. This method, called method of exhaustion is at the root of all modern developments in the theory of measure and integral. In the 19th century, this method culminated in the theory of Riemann integration, defined by means of Riemann sums.

In modern times, the method of exhaustion can be stated as follows: Let S be a surface of known area s. Also suppose that S' is a surface of known area s' contained in s and s" is a surface of known area s" containing s. Then $s' \le s < s''$. The approximating surfaces s' and s" are taken as polygons or sums of slices, mainly trapezoidal or rectangular according to the particular figure s under the method of Eudoxus and Archimedes. In fact, the definition of area as a sum of rectangular areas is in vogue from 16th century A.D.

Calculus (both differential and integral) was invented by both Newton and Leibnitz — independent of each other. Newton, influenced by his teacher Barron used calculus to solve the problems of dynamics. Thus he conceived all functions as functions of a universal independent variable known as time (t). So he had no concept of functions of several variables and partial derivatives.

For Newton, the primary concept was that of fluxion (derivative) and arose from kinematical considerations. Newton did not isolate the concept of integral; nor he introduced one symbol for integration. His first basic problem was to find fluxion (derivatives). Integration was used in a geometric form to find fluents (anti-derivatives or indefinite integrals), functions when fluxions (derivatives) are given. Newton based his theory mainly on the fact: The derivative of a variable area F(x) under a curve is the ordinate f(x) of this curve. For Newton, integration was the inverse process of differentiation, as he was mainly interested in the following problem — Given an equality relation containing fluxions, find the relation for fluents, which is the basic problem to solve ordinary differential equations. He solved these by use of series.

On the other hand, Leibnitz thought of the derivative as the slope of a tangent, and the integral as summa omnimum linae. The main purpose of all his work was to devise a universal language, that is, a general formalism for systematisation and organisation of knowledge. To a great extent, he succeeded in creating such a formalism for calculus. In fact, the present formalism in calculus is mainly his including the integral symbol (a stylised form of the letter S standing for summa omnimum). The terms constant, variable, function and integral used in calculus are due to Leibnitz.

G.F.B.Riemann in 1854 (published in 1867) gave necessary and sufficient condition for the existence of integrals called Riemann integral and showed that continuous functions satisfy his condition. The definition of integral as a limit of sum of areas as given in the text books is due to him.

Resume of the key concepts :

Two important ideas underlie the treatment of definite integrals in the text: 1. Definite integral $\int_a^b f(x).\,dx$ as a limit of the sum of areas and 2. Fundamental Theorem of Integral Calculus.

Here, we give an alternative treatment of Fundamental Theorem of Integral Calculus.

Statement Fundamental Theorem of Integral Calculus

If f(x) is integrable in (a,b), $a < b$, and if there exists a function F(x), such that $F'(x) = f(x)$ in (a,b), then

$$\int_a^b f(x)\,.\,dx = F(b) - F(a)$$

Proof : Let $a = x_0 < x_1 < x_2 < \cdots < x_n = b$

Then, by the Mean Value Theorem of Differential Calculus,

$F(x_r) - F(x_{r-1}) = (x_r - x_{r-1})\,F'(\xi_r)$, $x_{r-1} < \xi_r < x_r$

Taking the sum of the respective sides of the above equations, we have

$$\sum_{r=1}^{n} F'(\xi_r)\,\delta_r = \sum_{r=1}^{n} \left[F(x_r - x_{r-1})\right]$$

[where $\delta_r = x_r - x_{r-1}$]

$$= F(b) - F(a) \qquad (1)$$

Suppose that δ is the length of the largest of the sub-intervals (x_{r-1}, x_r). Then as $\delta \to 0$, all the ξ_r's will also tend to 0. So we have

$$\underset{\delta \to 0}{Lt} \sum F'(\xi_r)\,\delta_r = F(b) - F(a)$$

Now f(x) and so F'(x) is integrable in (a,b).

Hence

$$\underset{\delta \to 0}{Lt} \sum F'(\xi_r)\,\delta_r = \int_a^b F'(x)\,.\,dx = \int_a^b f(x).dx \qquad (2)$$

From (1) and (2) we have

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

The following points are to be noted regarding the above theorem.

1. This theorem is very useful and important as it gives us an easy method of evaluating the definite integral <u>without calculating</u> the limit of the sum by establishing a connection between the integration as a limit of a sum and the integration as inverse operation of differentiation.

2. $\int_a^b f(x)\,dx$ is a function of lower limit a and upper limit b, and <u>not</u> a function of the variable x.

3. In $\int_a^x f(x).dx$ the upper limit is the variable x. So $\int_a^x f(x).dx$ is not a definite integral, but another form of the indefinite integral. For example,

$$\int f(x).dx = F(x). \text{ Then}$$

$$\int_a^x f(x).dx = F(x) - F(a) = F(x) + \text{a constant} = \int f(x).dx.$$

Extended Definition of $\int_a^b f(x).dx$

The following definition of $\int_a^b f(x).dx$ is an extension of the definition given in the text.

Let f(x) be a bounded function defined in the interval (a,b); and let the interval (a,b) be divided in any manner into n sub-intervals

$(a, x_1), (x_1,x_2),\ldots, (x_{r-1},x_r),\ldots, (x_{n-1},b)$ of lengths $\delta_1, \delta_2, \ldots, \delta_n$ respectively where $a < x_1 < x_2 < \ldots < x_{r-1} < x_r < \ldots < x_{n-1} < b$.

In each of these sub-intervals select an arbitrary point and let these points be such that

$\xi_1 \in (a,x_1),\quad \xi_2 \in (x_1,x_2),\ldots, \xi_r \in (x_{r-1},x_r), \ldots, \xi_n \in (x_{n-1},b)$

Now let $S_n = \sum_{r=1}^{n} \delta_r \cdot f(\xi_r)$.

Now let n increase indefinitely so that the longest of the lengths $\delta_1, \delta_2, \ldots, \delta_n$ tends to 0. In such a case clearly each of $\delta_1, \delta_2, \ldots, \delta_n$ tends to 0. Now, if in such a situation (i.e. max.$(\delta_1, \delta_2, \ldots \delta_n) \to 0$), S_n tends to a finite limit which does not depend on the manner in which (a,b) is divided into sub-intervals and the points $\xi_1, \xi_2, \ldots, \xi_n$ are selected; then this limit (if it exists) is defined as the definite integral of f(x) from a to b and symbolically denoted by $\int_a^b f(x).dx$.

In the textbook, for the sake of simplicity, the sub-intervals are supposed to be equal and the points $\xi_1, \xi_2, \ldots, \xi_n$ are taken to be the end-points of the sub-intervals.

Areas of difficulty :

Here are solved some problems the types of which are not discussed in the text.

Problem 1. Evaluate $\int_a^b x^m . dx$ where m is any real number $\neq 1$ and $0 \leq a < b$.

Solution: Consider the sub-intervals

$(a, ar), (ar, ar^2), (ar^2, ar^3), \ldots, (ar^{n-1}, ar^n)$ of (a,b) where $ar^n = b$ i.e. $r = (b/a)^{1/n}$.

Clearly as $n \to \infty$, $r = \left(\frac{b}{a}\right)^{1/n} \to 1$ so that each of the lengths of the sub-intervals

$ar - a, ar^2 - ar, \ldots, (ar^n - ar^{n-1})$

i.e. $a(r-1), ar(r-1), \ldots, ar^{n-1}(r-1)$ tends to 0.

Now by the extended definition of $\int_a^b f(x) . dx$,

$$\int_a^b x^m . dx = \lim_{n \to \infty} \left[a^m . a(r-1) + (ar)^m . ar(r-1) + (ar^2)^m (ar^2)(r-1) + \ldots \text{ to } n \text{ terms} \right]$$

$$= \lim_{r \to 1} a^{m+1}(r-1)\left\{1 + r^{m+1} + r^{2(m+1)} + \ldots \text{ to } n \text{ terms}\right\}$$

$$= \lim_{r \to 1} \frac{a^{m+1}\left((r^{m+1})^n - 1\right)}{r^{m+1} - 1}$$

as the series in the flower bracket is a G.P. with common ratio r^{m+1} with $m+1 \neq 0$.

Simplifying the last expression we have

$$\int_a^b x^m \, dx$$

$$= \underset{r \to 1}{Lt} \; a^{m+1} \left(\frac{r-1}{r^{m+1}-1} \right) \left\{ (r^n)^{m+1} - 1 \right\}$$

$$= \underset{r \to 1}{Lt} \; a^{m+1} \cdot \frac{1}{m+1} \left\{ \left(\frac{b}{a} \right)^{m+1} - 1 \right\}, \quad \text{as } \underset{r \to 1}{Lt} \frac{r-1}{r^{m+1}-1} = \frac{1}{m+1}$$

with $m+1 \neq 0$

$$= \underset{r \to 1}{Lt} \; \frac{b^{m+1} - a^{m+1}}{m-1}$$

$$= \frac{b^{m+1} - a^{m+1}}{m+1}$$, the expression under the limit being independent of r.

Series represented by Definite Integrals

The definition of the definite integral can be used with profit to evaluate easily the limits of the sums of certain series, when the number of terms in the series tends to infinity. The method lies in identifying a definite integral equal to series. In fact,

$$\int_a^b f(x).dx = \underset{h \to 0}{Lim} \; h \sum f(a + rh) \text{ where } nh = b-a$$

$$\text{or } \underset{n \to \infty}{lt} \; \frac{b-a}{n} \sum \left(f(a) + r \, \frac{(b-a)}{n} \right) = \int_a^b f(x).dx$$

If a = 0, b = 1, we have

$$\underset{n \to \infty}{lt} \; 1/n \sum f(r/n) = \int_0^1 f(x) \, . \, dx$$

In the above discussion, r takes the values either $0,1,2,\ldots,n-1$ or $1,2,3,\ldots,n$. These two sets of numbers represent the left and right extremities of the elementary vertical rectangles (columns) in the calculation of area represented by $\int_a^b f(x).dx$. (Refer to the definition of $\int_a^b f(x).dx$ in the text).

The following are illustrative examples.

Problem 2. Evaluate

$$\lim_{n \to \infty} \left[\frac{1}{n+m} + \frac{1}{n+2m} + \ldots\ldots\ldots + \frac{1}{n+nm} \right]$$

Solution: The given expression

$$= \lim_{n \to \infty} \left[\frac{1}{n} \left(\frac{1}{1+\frac{m}{n}} + \frac{1}{1+\frac{2m}{n}} + \ldots + \frac{1}{1+\frac{(n-1)m}{n}} + \frac{1}{1+\frac{nm}{n}} \right) \right]$$

$$= \int_0^1 \frac{dx}{1+mx}$$ by definition of the definite integral $\int_a^b f(x).dx$

$= 1/m \log (1 + mx)$

$= 1/m \log (1 + m) - \log 1$

$1\ 1/m \log (1 + m)$

Problem 3. Evaluate

$$\lim_{n \to \infty} \left\{ \left(1 + \frac{1}{n}\right) \left(1 + \frac{2}{n}\right) \ldots\ldots \left(1 + \frac{n}{n}\right) \right\}^{\frac{1}{n}}$$

Solution :

Let $A = \left\{ \left(1 + \frac{1}{n}\right) \left(1 + \frac{2}{n}\right) \ldots\ldots \left(1 + \frac{n}{n}\right) \right\}^{\frac{1}{n}}$

Then $\lim_{n \to \infty} \log A$

$$= \lim_{n \to \infty} \frac{1}{n} \sum \log \left(1 + \frac{r}{n}\right)$$

$$= \int_0^1 \log (1 + x)$$

Now put $z = 1 + x$

Then $x = 0$ implies $z = 1$ and $x = 1$ implies $z = 2$.

So $\lim\limits_{n \to \infty} \log A$

$= \int_1^2 \log z . dz$

$= \Big[z \log z - z\Big]_1^2$

$= 2 \log 2 - 2 - 1 \log 1 + 1$

$= 2 \log 2 - 1 = 2 \log 2 - \log e$

$= \log 4/e$

So $\lim\limits_{n} A = \dfrac{4}{e}$

<u>Assignments</u> :

Using the definition of $\int_a^b f(x).dx$ as a limit of a sum, evaluate the following definite integrals (1 to 10) :

1. $\int_0^1 e^{-x}.dx$
2. $\int_0^1 x^2.dx$
3. $\int (ax+b)\, dx$
4. $\int_0^{\pi/2} \sin x .\, dx$
5. $\int_a^b \cos \theta \, . \, d\theta$
6. $\int_0^1 \sqrt{x} \, . \, dx$
7. $\int_0^1 \frac{1}{\sqrt{x}} \, . \, dx$
8. $\int_2^4 \frac{1}{x^2} \, dx$
9. $\int_1^3 e^x.dx$
10. $\int_0^{\pi/4} \sec^2 x.dx$

Evaluate the following limits using definite integrals.

11. $\lim\limits_{n \to \infty} \left\{ \frac{1}{n+1} + \frac{1}{n+2} + \ldots\ldots\ldots + \frac{1}{n+n} \right\}$

12. $\lim\limits_{n \to \infty} \left[\frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \ldots\ldots + \frac{n}{n^2+n^2} \right]$

13. $\lim\limits_{n \to \infty} \left[\dfrac{1^2}{n^3+1^3} + \dfrac{2^2}{n^3+2^3} + \ldots\ldots + \dfrac{n^2}{2n^3} \right]$

14. $\lim\limits_{n \to \infty} \sum\limits_{r=1}^{\infty} \dfrac{n+r}{n^2+r^2}$

15. $\lim\limits_{n \to \infty} \left[\dfrac{\sqrt{(n+1)} + \sqrt{(n+2)} + \ldots\ldots + \sqrt{(2n)}}{n\sqrt{n}} \right]$

16. $\lim\limits_{n \to \infty} \sum\limits_{r=1}^{n} \dfrac{n}{(n+r)\sqrt{\{r(2n+r)\}}}$

17. $\lim\limits_{n \to \infty} \left\{ \left(1 + \dfrac{1^2}{n^2}\right)\left(1 + \dfrac{2^2}{n^2}\right) \ldots\ldots \left(1 + \dfrac{n^2}{2n^2}\right) \right\}^{1/n}$

18. $\lim\limits_{n \to \infty} \left\{ \left(1 + \dfrac{1}{n^2}\right)^{2/n^2} \left(1 + \dfrac{2^2}{n^2}\right)^{4/n^2} \left(1 + \dfrac{3^2}{n^2}\right)^{6/n^2} \ldots\ldots \left(1 + \dfrac{n^2}{n^2}\right)^{2n/n^2} \right\}$

Answers :

1. $e^{-b} - e^{-a}$
2. $1/3$
3. $a/2 + b$
4. 1
5. Sinb - sin a
6. 2/3
7. 2
8. $1/4$
9. $e^3 - e$
10. 1
11. log 2
12. $\pi/4$
13. $(1/3) \log 2$
14. $\frac{\pi}{4} + (1/2) \log 2$
15. $(4/3)\sqrt{2} - 2/3$
16. /3
17. $2e^{(1/2)} (\pi - 4)$
18. 4/e

PROPERTIES OF DEFINITE INTEGRALS

Here we will discuss and clarify certain important properties of definite integrals which have not been discussed in the text.

1. $\int_a^b f(x).dx = \int_a^b f(z).\, dz$

Proof :

Suppose that $\int f(x).dx = \phi\,(x)$

Then, we have by Fundamental Theorem of Integral Calculus

$$\int_a^b f(x).dx = \phi(b) - \phi(a) \qquad (1)$$

Also, $\int_a^b f(z).dz = \phi(z)$ and by the Fundamental Theorem of Integral Calculus,

$$f(z).dz = \phi(b) - \phi(a) \qquad (2)$$

From (1) and (2), we have the result.

This property states that a definite integral is independent of the variables with respect to which the integration is performed.

2. $\int_a^{na} f(x).dx = n \int_0^a f(x).dx$ if $f(x) = f(a+x)$

Proof :

$$\int_0^{na} f(x).dx = \int_0^a f(x).dx + \int_a^{2a} f(x).dx + \ldots. + \int_{(n-1)a}^{na} f(x).dx$$

Set $z + a = x$. Then $dx = dz$

Also, $x = a$ implies $z = 0$ and $x = 2a$ implies $z = a$

So, $\int_a^{2a} f(x).dx = \int_0^a f(z+a).dz = \int_0^a f(a+x)dx$

$= \int_0^a f(x).dx$

Again with the same substitution, z+a = x, we can see that

$$\int_{2a}^{3a} f(x).dx = \int_{a}^{2a} f(z+a).dz = \int_{a}^{2a} f(x).dx = \int_{0}^{a} f(x).dx$$

Similarly, we can show that

$$\int_{(n-1)a}^{na} f(x).dx = \int_{(n-2)a}^{(n-1)a} f(x).dx = \ldots = \int_{a}^{2a} f(x).dx = \int_{0}^{a} f(x)dx$$

Hence we get the result.

Illustration: Since cosx = cos (x+ π)

we have

$$\int_{0}^{6\pi} \cos x.\, dx = 6 \int_{0}^{\pi} \cos x.\, dx$$

3. $\int_{0}^{2a} f(x).dx = \int_{0}^{a} f(x).dx + \int_{0}^{a} f(2a-x).dx$

Proof :

By formula 7.2 of the textbook

$$\int_{0}^{2a} f(x).dx = \int_{0}^{a} f(x).dx + \int_{a}^{2a} f(x).dx$$

Substitute 2a - z for x. Then dx = -dz.

Moreover, when x = a, z = a, and when x = 2a, z = 0; so

$\int_{a}^{2a} f(x).dx = -\int_{a}^{0} f(2a-z) = \int_{0}^{a} f(2a-z)$ by formula 7.1 of the textbook $= \int_{0}^{a} f(2a-x)$

Hence, $\int_{0}^{2a} f(x).dx = \int_{0}^{a} f(x).dx + \int_{0}^{a} f(2a-x)$

4. i) $\int_{0}^{2a} f(x).dx = 2\int_{0}^{a} f(x).dx$ if $f(2a-x) = f(x)$ and

ii) $\int_{0}^{2a} f(x) = 0$, if $f(2a-x) = -f(x)$

Proof :

i) $\int_0^{2a} f(x).dx = \int_0^a f(x).dx + \int_0^a f(2a-x).dx$ by the previous result.

$= \int_0^a f(x).dx + \int_0^a f(x).dx$

$= 2\int_0^a f(x).dx$

ii) The proof can be written as in 4(i).

5. If f(x) is integrable in the closed interval a,b and if f(x) 0 for all x in [a,b], then $\int_a^b f(x).dx \geqslant 0$ $(b > a)$.

Proof :

Since f(x) is integrable in [a,b], f(x).dx exists. Since $f(x) \geqslant 0$ in [a,b] in the sub-interval (x_{r-1}, x_r) of [a,b] the lower bound $m_r \geqslant 0$, and so the lower sum s for the partition of $[a,b] = \sum m_r \delta_r \geqslant 0$.

So l, which is the exact upper bound of the set of numbers s, is $\geqslant 0$.

Now, since $\int_a^b f(x).dx$ exists, $l = \int_a^b f(x).dx$

Hence, $\int_a^b f(x).dx$ exists.

6. If f(x) and g(x) are integrable in a,b) and $f(x) \geqslant g(x)$ for all x in [a,b] then $\int_a^b f(x).dx \geqslant \int_a^b g(x).dx$

Proof:

Let h(x) = f(x) - g(x)

Then as f(x) and g(x) are integrable in [0,1] , h(x) is so.

Also, as $f(x) \geqslant g(x)$ in [a,b] , $h(x) \geqslant 0$ in [a,b].

Applying the previous result, we find that

$$\int_a^b h(x).\,dx \geqslant 0$$

i.e. $\int_a^b (f(x) - g(x))\ dx \geqslant 0$

i.e. $\int_a^b f(x).dx - \int_a^b g(x).dx \quad 0$

i.e. $\int_a^b f(x).dx \geqslant \int_a^b g(x).dx$

7. If $f(x)$ is integrable in (a,b), then

$$\int_a^b |f(x)|.dx \geqslant \left|\int_a^b f(x).dx\right|$$

Proof:

Let $\{a = x_0, x_1, x_2, \ldots, x_{n-1}, x_n = b\}$ be a partition of $[a,b]$ and let $\delta_r = x_r - x_{r-1}$.

Then we have

$$|f(\xi_1)\delta_1 + f(\xi_2)\cdot\delta_2 + \cdots + f(\xi_n)\cdot\delta_n|$$

$$\leqslant |f(\xi_1)\cdot\delta_1| + |f(\xi_2)\cdot\delta_2| + \cdots + |f(\xi_n)\cdot\delta_n|$$

$$= |f(\xi_1)||\delta_1| + |f(\xi_2)||\delta_2| + \cdots + |f(\xi_n)||\delta_n|$$

$$= |f(\xi_1)|\cdot\delta_1 + |f(\xi_2)|\cdot\delta_2 + \cdots + |f(\xi_n)|\cdot\delta_n$$

where $\xi_r \in [x_{r-1}, x_r]$ and each δ_r is clearly positive.
Now, let $n \longrightarrow \infty$ so that max. $(\delta_1, \delta_2, \ldots, \delta_n)$ $\longrightarrow 0$ i.e. each $\delta_r \longrightarrow 0$

Then clearly

$$\text{lt} \left| \sum f(\xi_r) \cdot \delta_r \right| \leq \text{lt} \sum |f(\xi_r)| \, \delta_r$$

i.e. $$\left| \int_a^b f(x) \cdot dx \right| \leq \int_a^b |f(x)| \cdot dx$$

Solved Examples :

The following examples will illustrate the use of the properties of the definite integrals in solving problems.

Example 1 :

Show that

$$\int_0^{\pi/2} \log \sin x. \, dx = \int_0^{\pi/2} \log \cos . \, dx = (\pi/2) \log \sqrt{2}$$

$$\int_0^{\pi/2} \log \sin x. \, dx$$

$$= \int_0^{\pi/2} \log \sin (\pi/2 - x) . \, dx$$

$= \int_0^{\pi/2} \log \cos x. \, dx$ by Formula 7.4 of textbook

Now if each of the definite integrals $\int_0^{\pi/2} \log \sin x . \, dx$ and $\int_0^{\pi/2} \log \cos x. \, dx$ is taken to be I, then

$$2I = \int_0^{\pi/2} \log \sin x. \, dx + \int_0^{\pi/2} \log \cos x . \, dx$$

$$= \int_0^{\pi/2} (\log \sin x + \log \cos x) \, dx = \int_0^{\pi/2} \log (\sin x. \cos x) \, dx$$

$$= \int_0^{\pi/2} \log \frac{\sin 2x}{2} . \, dx = \int_0^{\pi/2} (\log \sin 2x - \log 2) . \, dx$$

$$= \int_0^{\pi/2} \log \sin 2x . \, dx - (\pi/2) \log 2$$

Set $2x = u$. Then $dx = du/2$.

We have

$$\int_0^{\pi/2} \log \sin 2x.\, dx = \tfrac{1}{2} \int_0^{\pi} \log \sin u \,.\, du$$

$$= \tfrac{1}{2} \int_0^{\pi} \log \sin x.\, dx = \int_0^{\pi/2} \log \sin x \,.\, dx \text{ by result 4(i)}$$

$= I$

So, $2I = I - \pi/2 \log 2$

i.e., $I = -(\pi/2) \log 2 = (\pi/2) \log (\tfrac{1}{2})$

Example 2 :

Show that $\displaystyle\int_0^1 \frac{\log (1+x)}{1 + x^2} dx = (\pi/8) \log 2$

Set $x = \tan u$

Then $dx = \sec^2 u \; du$ Moreover, $x = 0 \Rightarrow u = 0$

and $x = 1 \Rightarrow u = \pi/4$

So $I = \displaystyle\int_0^{\pi/4} \log (1 + \tan u)\, du$

$$= \int_0^{\pi/4} \log (1 + \tan (\pi/4 - u))\, du = \int_0^{\pi/4} \log \left(1 + \frac{1 - \tan u}{1 + \tan u}\right) du$$

$$= \int_0^{\pi/4} \log \frac{2}{1 + \tan u} \,.\, du$$

$$= \int_0^{\pi/4} (\log 2 - \log (1 + \tan u))\, du$$

$$= \int_0^{\pi/4} \log 2.\, du - \int_0^{\pi/4} \log (1 + \tan u)\, du$$

$= (\pi/4) \log 2 - I$

So, $2I = (\pi/4) \log 2$

i.e. $I = (\pi/8) \log 2$

Example 3 :

Show that $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx = \frac{\pi^2}{4}$

Put $I = \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$ (1)

Substituting $\pi - x$ for x

$$I = \int_0^{\pi} \frac{(\pi-x) \sin (\pi-x)}{1+\cos^2 (\pi-x)} dx$$

i.e. $I = \int_0^{\pi} \frac{(\pi-x) \sin x}{1+\cos^2 x} dx$ (2)

Adding (1) and (2) we get

$I + I = \int_0^{\pi} \frac{\pi \sin x}{1+\cos^2 x} dx$ i.e., $2I = \pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx$

i.e. $I = \frac{\pi}{2} \int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx$

Set $\cos x = z$. Then $dx = \frac{dz}{-\sin x}$

Also $x = 0 \Rightarrow z = 1$ and $x = \pi \Rightarrow z = -1$

So, $I = \frac{\pi}{2} \int_1^{-1} \frac{\sin x}{1+z^2} \cdot \frac{dz}{-\sin x} = -\frac{\pi}{2} \int_1^{-1} \frac{dz}{1+z^2}$

$= \frac{\pi}{2} \int_{-1}^{1} \frac{dz}{1+z^2}$ by property (1) of the textbook

$$= \frac{\pi}{2} \left[\tan^{-1} z \right]_{-1}^{1}$$

$$= \frac{\pi}{2} (\tan^{-1} 1 - \tan^{-1}(-1))$$

$$= \frac{\pi}{2} \left(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right)$$

$$= \frac{\pi}{2}\left(\frac{\pi}{4}+\frac{\pi}{4}\right) = \frac{\pi}{2}\left(\frac{\pi}{2}\right)$$

$$= \frac{\pi}{4}$$

Example 4 :

Show that $\int_0^{\pi} x \sin x \, dx = \pi \int_0^{\pi/2} \cos x \, . \, dx$

Solution:

$I = \int_0^{\pi} x \sin x \, . \, dx = \int_0^{\pi} (\pi - x) \sin(\pi - x) \, . \, dx$ by result No.4 of the text.

$$= \int_0^{\pi} (\pi - x) \sin x \, . \, dx$$

$$= \pi \int_0^{\pi} \sin x . \, dx - \int_0^{\pi} x \sin x \, . \, dx$$

$$= \pi \int_0^{\pi} \sin x . \, dx - I$$

$= 2\pi \int_0^{\pi/2} \sin x \, . \, dx - I$ by result No.4(i) of this booklet.

$$= 2\pi \int_0^{\pi/2} \sin\left(\frac{\pi}{2} - x\right) dx - I$$

$$= 2\pi \int_0^{\pi/2} \cos x \, . \, dx - I$$

i.e. $2I = 2\pi \int_0^{\pi/2} \cos x . \, dx$

i.e. $I = \pi \int_0^{\pi/2} \cos x . \, dx$

Example 5 :

Show that $\int_0^{\pi/2} \frac{x \sin x \cdot \cos x}{\cos^4 x + \sin^4 x} dx = \frac{\pi^2}{16}$

Solution :

$$I = \int_0^{\pi/2} \frac{x \cdot \sin x . \cos x}{\cos^4 x + \sin^4 x} dx$$

$$= \int_0^{\pi/2} \frac{(\pi/2 - x) \cos x \cdot \sin x}{\sin^4 x + \cos^4 x}$$ by result No.4 of the text

$$= \frac{\pi}{2} \int_0^{\pi/2} \frac{\cos x \cdot \sin x}{\sin^4 x + \cos^4 x} dx - I$$

i.e. $2I = \frac{\pi}{2} \int_0^{\pi/2} \frac{\cos x - \sin x}{\sin^4 x + \cos^4 x} dx$

Now, $\sin^4 x + \cos^4 x = (\sin^2 x + \cos^2 x)^2 - 2 \sin^2 x \cdot \cos^2 x$

$$= 1 - \frac{\sin^2 2x}{2} = +1 - \frac{(1 - \cos^2 2x)}{2}$$

So, $2I = \frac{\pi}{4} \int_0^{\pi/2} \frac{\sin 2x}{1 - \left(\frac{1 - \cos^2 2x}{2}\right)} . dx$

set $\cos 2x = z$. Then $-2 \sin 2x.dx = dz$,

$x = 0 \Longrightarrow z = 1$ and $x = \pi/2 \Rightarrow z = -1$

So, $2I = \frac{\pi}{4} \int_{+1}^{-1} \frac{-dz}{1 - \left(\frac{1-z^2}{2}\right)} = \frac{\pi}{4} \int_{+1}^{-1} \frac{-dz}{1 + z^2}$

$$= \frac{\pi}{4} \int_{-1}^{+1} \frac{dz}{1 + z^2}$$

by result No.1 of the text.

$= \frac{\pi}{4} \left[\tan^{-1} z \right]_{-1}^{+1} = \frac{\pi}{4} \left[\tan^{-1} 1 - \tan^{-1}(-1) \right]$

$= \frac{\pi}{4} \left[\frac{\pi}{4} - \left(-\frac{\pi}{4} \right) \right]$

$= \frac{\pi}{4} \left[\frac{\pi}{4} + \frac{\pi}{4} \right]$

$= \frac{\pi}{4} \times \frac{\pi}{2} = \pi^2/8$

i.e. $I = \frac{\pi^2}{16}$

Example 6 :

Show that $\int_{-a}^{+a} \frac{t.e^{t^2}}{1+t^2} . dt = 0$

The given integral

$I = I_1 + I_2$, where $I_1 = \int_{-a}^{0} \frac{t.e^{t^2}}{1+t^2} . dt$ and $I_2 = \int_{0}^{a} \frac{t.e^{t^2}}{1+t^2} dt$

Now $I_1 = \int_{-a}^{0} \frac{t.e^{t^2}}{1+t^2} . dt$

$= -\int_{c}^{a} \frac{z . e^{z^2} dz}{1 + z^2}$ where $z = -t$ ($t = -a \Rightarrow z = a$)

$= - \int_{0}^{a} \frac{z. e^{z^2} dz}{1 + z^2}$ by result No.1 of the text

$= - \int_{0}^{a} \frac{t.e^{t^2} . dt}{1 + t^2}$ by result No.1 of the booklet

$= -I_2$ i.e. $I_1 + I_2 = 0$ i.e., $I = 0$.

Assignments :

1. Show that $\int_a^b f(a+b-x) = \int_a^b f(x).dx$

2. Show that $\int_{a-c}^{b-c} f(x+c) = \int_a^b f(x)$

3. Show that $\int_a^b f(nx)\ dx = 1/n \int_{na}^{nb} f(x).dx$

4. Show that

$$\int_0^{\pi/2} (a\cos^2 x + b\ \sin^2 x)\ dx = \frac{\pi}{4}(a+b)$$

5. Show that $\int_0^{\pi} xf(\sin\ x)\ dx = \frac{\pi}{2} \int_0^{\pi} f(\sin\ x)\ dx$

6. $\int_0^{\pi} t\ .\ \sin^2 t\ dt = \frac{\pi^2}{4}$

7. Show that

$$\int_0^{\pi} \frac{\sin 4\theta}{\sin\theta}\ .\ d\theta = 0$$

8. Show that $\int_0^1 \log \sin\left(\frac{\pi\theta}{2}\right)\ .\ d\theta = -\log 2$

9. Show that $\int_{-a}^{a} t\sqrt{a^2-t^2}\ .\ dt = 0$

10. Show that

$$\int_0^{\pi/2} \frac{\sin^{3/2}\theta}{\sin^{3/2}\theta + \cos^{3/2}\theta}\ d\theta = \frac{\pi}{4}$$

11. Show that

$$\int_0^{\pi/2} f(\sin\ x)\ .\ dx = \int_0^{\pi/2} f(\cos\ x)\ dx$$

12. Show that $\int_0^a f(x^2)dx = 1/2 \int_{-a}^{a} f(x^2).dx$

13. Show that $\int_{0}^{1} x^{m}(1-x)^{n}dx = \int_{0}^{n} x^{n}(1-x)^{m}dx, m > 0 , n > 0$

14. Show that $\int_{-\frac{\pi}{2}}^{\pi/2} x^{3}.\sin^{-2}x \, . \, dx = 0$

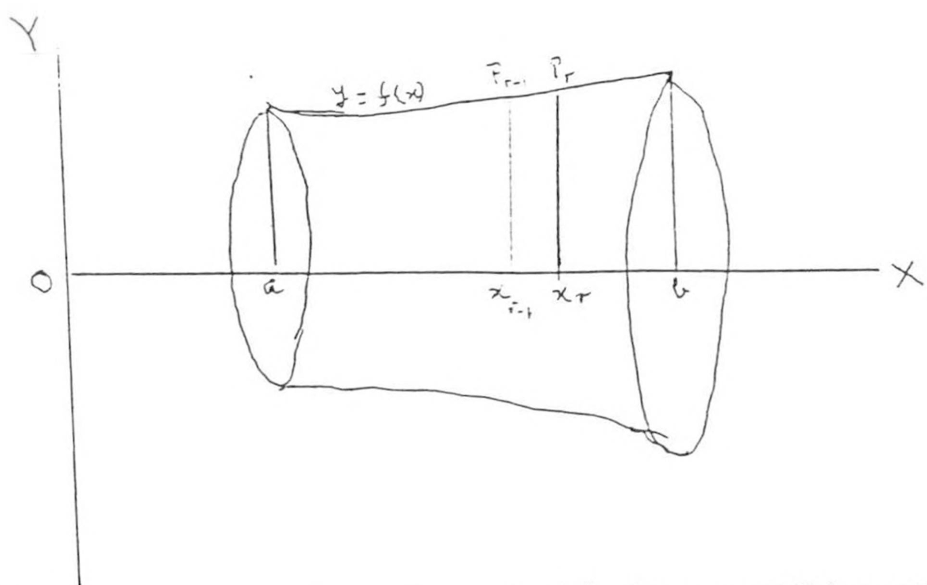
15. Show that $\int_{0}^{\pi} \frac{x}{a^{2}\cos^{2}x + b^{2}\sin^{2}x} \, dx = \frac{\pi^{2}}{2ab}$

EVALUATION OF VOLUMES OF SOLIDS OF REVOLUTION BY DEFINITE INTEGRALS

Key Concepts

1. Volume of a solid by revolution

Let an area be bound by the continuous curve $y = f(x)$, x-axis, the lines $x = a$ and $x = b$. Suppose that this area is revolved about the x-axis. Then a solid of revolution is generated. Here we are to find an expression for the volume of this solid of revolution.



Let $\{a = x_1, x_1, x_2, \ldots x_{n-1}, x_{n-1}, x_n = b\}$ be a partition of the intervals $[a,b]$ into n sub-intervals. Let $\delta x_r = x_r - x_{r-1}$.

Let P_{r-1}, P_r be the points on the curve $y = f(x)$ corresponding to the points x_{r-1}, x_r respectively on the x-axis. Thus the area under the curve $y = f(x)$ between the points x_{r-1} and x_r generates a disc of thickness δx_r. Clearly, the volume of this disc can be taken as

$\pi [f(x_{r-1})]^2 \delta x_r$ or $\pi [f(x_r)]^2 \delta x_r$

Since δx_r is very small, and f(x) is continuous, the volume of this disc of infinitesimal thickness is given by

$$\delta V = \pi [f(t_r)]^2 \, \delta x_r \,, \text{ when } x_{r-1} \leq t_r \leq x_r .$$

Taking the sum of volumes of all such discs, we have

$$V = \sum_1^n \pi [f(t_r)]^2 \delta x_r , \text{ where } x_{r-1} \leq t_r \leq x_r$$

Let n $\longrightarrow \infty$ so that max. $\delta x_r \longrightarrow 0$. Then we have

$$V = \lim_{n \to \infty} \sum_1^n \pi [f(t_r)]^2 \delta x_r$$

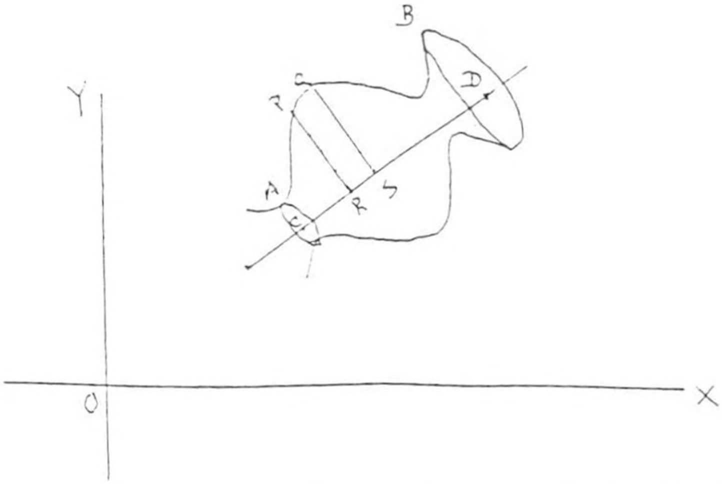
$$= \int_a^b \pi (f(x))^2 \, dx$$

$$= \int_a^b \pi y^2 \, dx$$

2. Suppose that an area is bound by the curve x = g(y), y = c, y = d, and y - axis. Let this area be revolved about y-axis. Then we get a solid of revolution generated by this area. By proceeding as in (1), we can show that the total volume of this solid of revolution is given by

$$V = \int_c^d \pi x^2 . dy$$

3.



Let AB be a curve which is being revolved about a line CD in the plane of the curve. Then a solid of revolution is generated and CD is the axis of this solid of revolution. Now it is required to find an expression for the volume V of this solid of revolution.

Let P and Q be points on the generating curve so that the distance PQ is an infinitesimal. Draw PR and QS perpendiculars on CD such that R and S are feet of the perpendiculars. Then the total volume of the solid of revolution is clearly given by

$$V = Lt \sum \pi . PR^2 . RS = \pi \int_C^{CD} PR^2 . d(CR)$$

Solved Examples :

1. Find the volume of the solid of revolution generated by revolving about the x-axis, the area bound by $y = 5x-x^2$ and x-axis.

Solution:

The equation to the curve can be written $y = 5x-x^2$.

i.e., $y = -(x^2 - 5x)$ i.e. $y = -\left[(x - \frac{5}{2})^2 + \frac{25}{4}\right]$

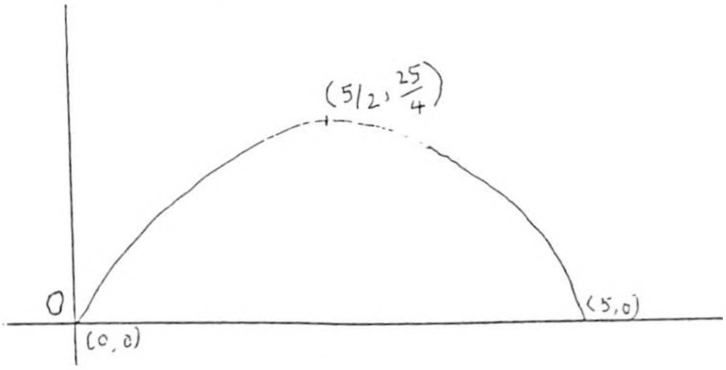
i.e., $y - \frac{25}{4} = -(x - \frac{5}{2})^2$ (1)

The x-coordinates of the points of intersection of this curve with x-axis, i.e. $y = 0$ is given by

$5x - x^2 = 0$ i.e., $x(5-x) = 0$

i.e. $x = 0$ or 5 (2)

Considering the information given by (1) and (2), we can draw the graph of the generating curve as follows :



The generating curve is a parabola with vertex at (5/2, 25/4) and intersecting x-axis at (0,0) and (5,0). So the total volume of the solid of revolution is given by

$$V = \pi \int_0^5 (5x - x^2)^2 \, dx$$

$$= \pi \int_0^5 (25x^2 - 10x^3 + x^4) \, dx$$

$$= \pi \left[25 \frac{x^3}{3} - 10 \frac{x^4}{4} + \frac{x^5}{5}\right]_0^5$$

$$= \pi = \left[25 \cdot \frac{5^3}{3} - 10 \cdot \frac{5^4}{4} + \frac{5^5}{5}\right] - 0$$

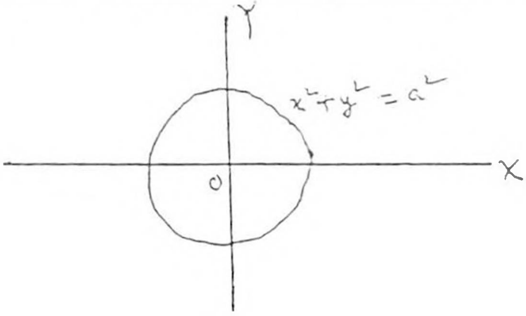
$$= \pi \cdot 5^4 \left[5/3 - \frac{5}{2} + 1\right]$$

$$= 625 \, \pi \left[\frac{10 - 15 + 6}{6}\right]$$

$$= 625 . \pi . \frac{1}{6}$$

$$= \frac{625 \pi}{6}$$

2. Show that the volume of a sphere of radius a is $\frac{4}{3}$ a^3.



A sphere is generated by revolving the region bounded by the circle

$$x^2+y^2 = a^2 \qquad (1)$$

about the y-axis.

So, the volume of the sphere

$$= \int_0^a x^2 dy \qquad =$$

$$= \pi \int_{-a}^{a} (a^2-y^2) dy = \pi \left[a^2y - \frac{y^3}{3} \right]_{-a}^{a}$$

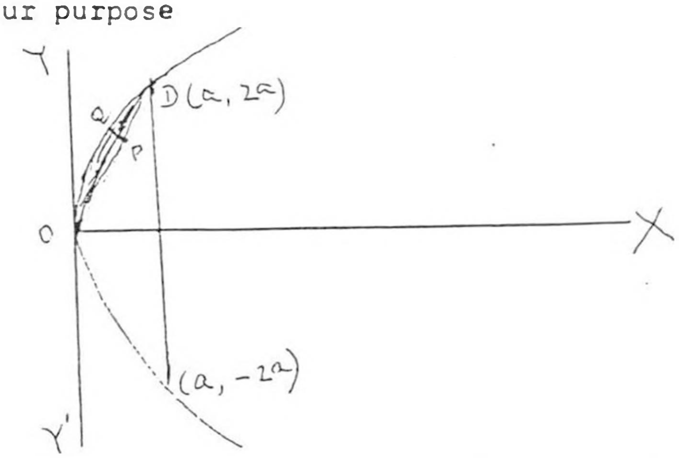
$$= \pi \left[a^3 - \frac{a^3}{3} + a^3 - \frac{a^3}{3} \right]$$

$$= \pi \left(2a^3 - \frac{2a^3}{3} \right)$$

$$= \frac{4}{3} \pi a^3.$$

3. The area cut off from the parabola $y^2 = 4ax$, by the chord joining the vertex to an end of the latus rectum rotates about the chord. Find the volume of the solid so formed.

<u>Solution:</u> The equation to the latus rectum of the parabola $y^2 = 4ax$ is $y = 2a$. So the latus rectum intersects the parabola $y^2 = 4ax$ at points whose x-coordinates are given by $(2a)^2 = 4ax$ i.e. $4a^2 = 4ax$ i.e., $x = a$. Correspondingly, y-coordinates of the points of intersection are given by $y^2 = 4a^2$ i.e., $y = \pm 2a$. So the points of intersection are $(a, 2a)$ and $(a, -2a)$. Let us consider the point D $(a, 2a)$ for our purpose



Now, OD is the line joining O (0,0) the origin and D(a, 2a). The equation to OD is given by

$$\frac{y}{x} = \frac{2a}{a} \text{ i.e. } y = 2x, \text{ i.e. } y - 2x = 0.$$

Let P (x', y') be a point on the parabola $y^2 = 4ax$ and PQ be perpendicular to OD with Q on OD. Clearly, the length PQ is given by

$$PQ = \frac{y' - 2x'}{\sqrt{5}}$$

Now the area shaded in the figure is rotated about OD and the volume of the solid so formed is to be evaluated.

The elem-entary length along OD is $\sqrt{5}\,.\,dx$.

So the volume V of the solid of revolution is given by

$$V = \pi \int_0^a PQ^2 . \sqrt{5} . dx$$

$$= \pi \int_0^a \left(\frac{y - 2x}{5}\right)^2 \sqrt{5}\; dx$$, suppressing the dashes in x^1, y^1

$$= \pi \int_0^a \frac{y^2 - 4xy + 4x^2}{5} \sqrt{5}\; dx$$

$$= \pi \int_0^a \frac{(4ax - 8\sqrt{a}x^{3/2} + 4x^2)}{\sqrt{5}}\, dx$$

$$= \frac{\pi}{\sqrt{5}} \left[\frac{4ax^2}{2} - \frac{2}{5} . 8\sqrt{a}\, x^{5/2} + \frac{4x^3}{3}\right]_0^a$$

$$= \frac{\pi}{\sqrt{5}} \left[\frac{4a^3}{2} - \frac{16}{5} a^3 + \frac{4a^3}{3}\right]$$

$$= \frac{\pi a^3}{\sqrt{5}} \left(2 - \frac{16}{5} + \frac{4}{3}\right)$$

$$= \frac{\pi}{\sqrt{5}} a^3 \left(\frac{30 - 48 + 20}{15}\right) = \frac{2\pi a^3}{15\sqrt{5}}$$

Assignments :

Find the volumes of solids generated by revolving about the x-axis, the areas bounded by the following curves and lines.

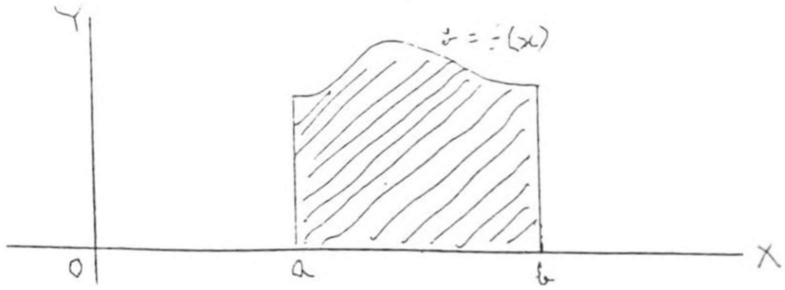
1. $y = \sin x;\ x = 0,\ x = \pi$
2. $y = 5x - x^2,\ \ x = 0,\ x = 4$
3. $y^2 = 9x,\ x = 4$
4. $x^2+y^2 = 4,\ \ x = 1,\ y = 0$
5. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
6. Prove that the volume of a right circular cone of height h and base of radius r is $\frac{1}{3}\pi r^2h$.
7. An arc of a parabola is bounded at both ends by the latus rectum of length 4a. Find the volume of the solid generated by rotating the arc about the latus rectum.
8. The area cut off by the line $x+y = 1$ from the parabola $\sqrt{x} + \sqrt{y} = 1$ is revolved about the same line. Find the volume of the solid so generated.
9. Show that the volume of the solid of revolution generated by revolving the cycloid $x = a(\theta + \sin\theta),\ \ y = a(1 + \cos\theta)$ about its base is equal to $5\pi^2a^3$.
10. Show that the volume of the solid generated by revolving the cardioide $r = a(1 - \cos\theta)$ about the initial line is equal to $\frac{8}{3}\pi a^3$.

9. Evaluation of Plane Areas by Definite Integrals

Key Concepts

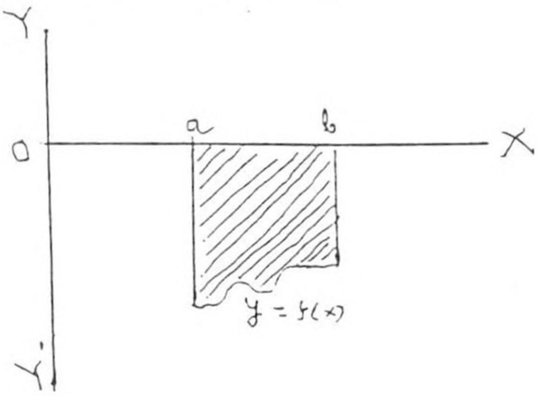
1. Let a region by bounded by the graph of $y = f(x)$, x-axis, the lines $x=a$ and $x=b$, $(a < b)$. Then area A of this region is given by

$$A = \int_a^b f(x).dx \text{ if } f(x) \geq 0 \text{ for } a \leq x \leq b$$



2. If $f(x) \leq 0$ for all $x \in [a,b]$, then $-f(x) \geq 0$ for all x in $[a,b]$ and the area A bounded by the graph of this function, $x = a$, $x = b$ and x - axis $(a < b)$ is given by

$$A = -\int_a^b f(x).\ dx$$

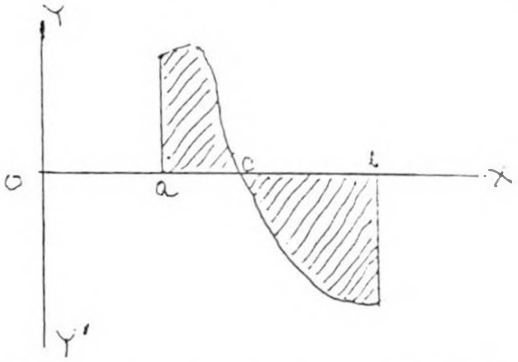


The proofs of the above two assertions are very much similar to the extended definition of $\int_a^b f(x).dx$ given in lesson 1 and the reader can frame the proofs themselves based on the definition of $\int_a^b f(x).\ dx$.

The above two assertions immediately lead to the following :

3. If $f(x) \geqslant 0$ for $x \in [a,c]$ and $f(x) \leqslant 0$ for $x \in [c, b]$ then the total area A bounded by $y = f(x)$, $x = a$, $x = b$ and y-axis is given by

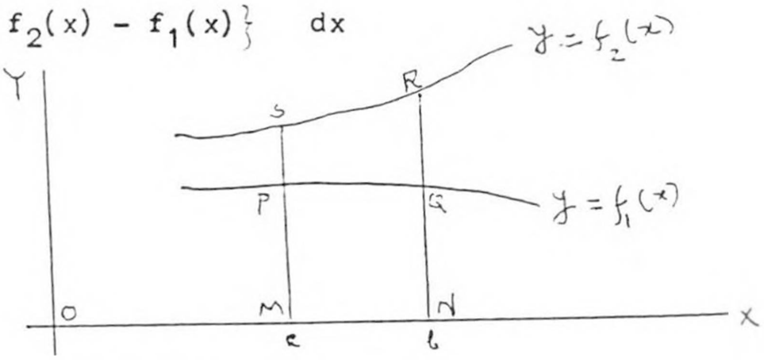
$$A = \int_a^c f(x)\,dx - \int_c^b f(x)\,dx$$



Similar results can be stated for the function $x = g(y)$.

4. The area A bounded by the graphs of the functions $y = f_1(x)$ and $y = f_2(x)$, and the ordinates $x = a$ and $y = b$, $(a < b)$ where $f_1(x) \leq f_2(x)$ for all $x \in [a,b]$ is given by

$$A = \int_a^b \{ f_2(x) - f_1(x) \}\, dx$$



The figure is self-explanatory.

Clearly, area PQRSP

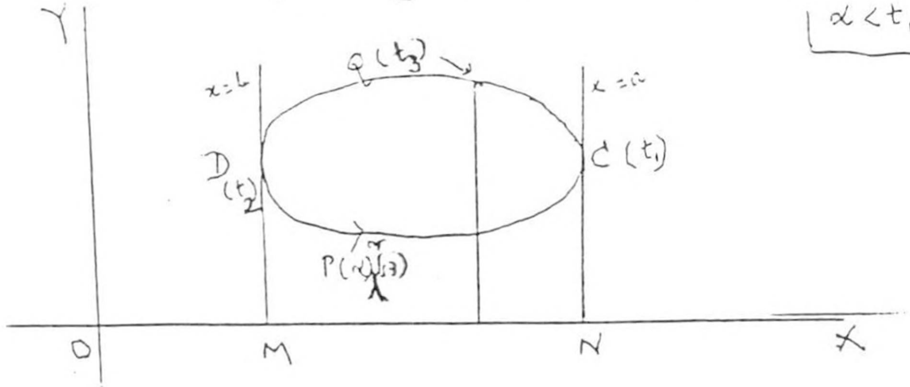
= area MNRSM - area MNQPM

$$= \int_a^b f_2(x)\,dx - \int_a^b f_1(x)\,dx$$

$$= \int_a^b \{ f_2(x) - f_1(x) \}\, dx$$

5. Area enclosed by a plane curve (equations given in parametric form):

Let a closed curve be given by $x = f(t)$, $y = g(t)$, $\alpha \leq t \leq \beta$ so that $f(\alpha) = f(\beta)$ and $g(\alpha) = g(\beta)$. Let us suppose that the closed curve starts (corresponding to α) and ends (corresponding to β) at the point P. Let any line parallel to y-axis (intersecting the curve) intersect the curve in exactly two points. Let the lines $x = a$ and $x = b$ touch the curve in points D and C, where these points correspond to t_1 and t_2 (values of t) respectively so that $\alpha < t_1 < t_2 < \beta$.



Let Q be a point on the curve corresponding to t_3 such that $t_1 < t_3 < t_2$.

Now the area of the region

= area of region MNC Q DM - area of region MNCPDM

$= S_2 - S_1$

where S_2 = area of region MNCQDM

where S_1 = area of region MNCPDM

Also $S_2 = \int_a^b y\,dx$, covering the region MNCQDM

$$= \int_{t_2}^{t_1} y(t)\,\frac{dx}{dt}\,dt + \int_{t_3}^{t_1} y(t)\cdot\frac{dx}{dt}\cdot dt$$

Similarly,

$$S_1 = \int_{t_2}^{\beta} y(t) \cdot \frac{dx}{dt}\, dt + \int_{t_1} y(t) \cdot \frac{dx}{dt} \cdot dt ,$$

considering the areas under the arcs DP and PC respectively.

So, $S = S_2 - S_1$

$$= \left(\int_{t_2}^{t_3} y \cdot \frac{dx}{dt}\, dt + \int_{t_3}^{t_1} y \cdot \frac{dx}{dt}\, dt \right) - \left(\int_{t_2}^{\beta} y \cdot \frac{dx}{dt} \cdot dt - \int_{\alpha}^{t_1} y \cdot \frac{dx}{dt} \cdot dt \right)$$

$$= -\int_{\alpha}^{t_1} y \cdot \frac{dx}{dt} \cdot dt - \int_{t_1}^{\tau} y \cdot \frac{dx}{dt}\, dt - \int_{t_3}^{t_2} y \cdot \frac{dx}{dt} \cdot dt - \int_{t_2}^{\beta} y \cdot \frac{dx}{dt}\, dt$$

$$= -\int_{\alpha}^{\beta} y \cdot \frac{dx}{dt}\, dt \quad \text{- - - - - - - - - - - (1)}$$

Similarly, considering tangents to the closed curve parallel to x-axis, we can show that

$$S = \int_{\alpha}^{\beta} x \cdot \frac{dy}{dt} \cdot dt \qquad (2)$$

Adding (1) and (2), we get

$$2\,S = \int_{\alpha}^{\beta} x \frac{dy}{dt} \cdot dt - \int_{\alpha}^{\beta} y \cdot \frac{dx}{dt} \cdot dt$$

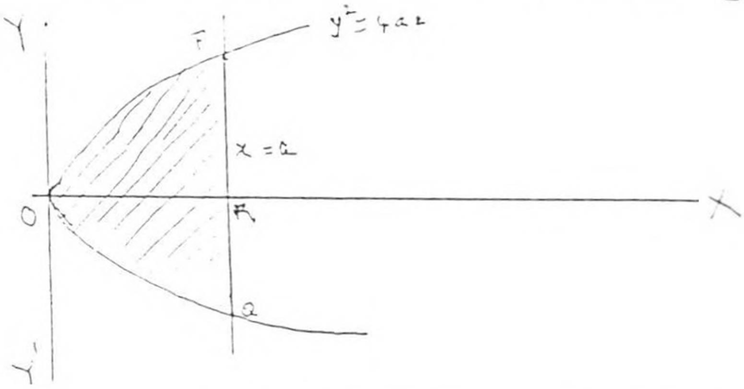
$$= \int_{\alpha}^{\beta} \left(x \frac{dy}{dt} - y \frac{dx}{dt}\right) dt$$

Hence the area enclosed in the closed curve

$$= \frac{1}{2} \int_{\alpha}^{\beta} x \frac{dy}{dt} - y \frac{dx}{dt} \quad dt$$

Solved Examples :

1. Determine the area bounded by the parabola $y^2 = 4ax$ and $x = b$.



The required area is the shaded portion in the figure which is self-explanatory. The parabola $y^2=4ax$ is symmetrical about x-axis. So, the required area

$= 2 \times \text{area } QPR$

$= 2 \int_0^b y \, . \, dx$

$= 2 \int_0^b \sqrt{4ax} \, . \, dx$ (y is taken as the positive side of the area is considered here)

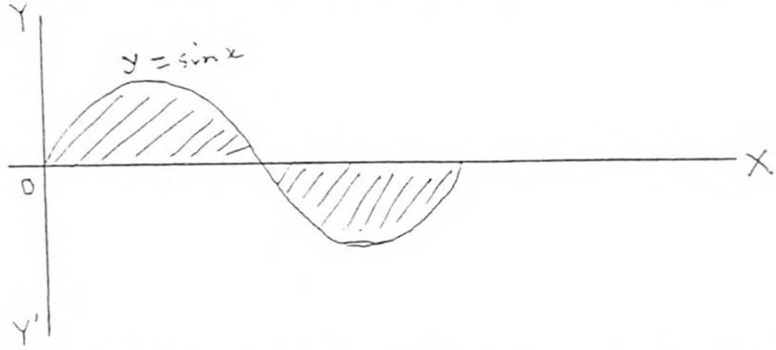
$= 2.2 \sqrt{a} \int_0^b x^{1/2} \, . \, dx$

$= 4\sqrt{a} \, \frac{2}{3} \left[x^{3/2} \right]_0^b \qquad = 4 \, \frac{2}{3} \sqrt{a} . \, b^{3/2}$

$= \frac{8}{3} \sqrt{a} \, . \, b^{3/2}$

$= \frac{8}{3} \sqrt{a \, b^{3/2}}$

2. Find the area under the curve $y = \sin x$ between 0 and 2 .



Here, we note that between 0 and π, $\sin x \geq 0$; and between o and 2π, $\sin x \leq 0$.

So the required area

$$= \int_0^{\pi} \sin x \, . \, dx - \int_{\pi}^{2\pi} \sin x . \, dx$$

$$= \Big[\cos x\Big]_0^{\pi} - \Big[\cos x\Big]_{\pi}^{2\pi}$$

$$= (1-0) - (0-1)$$

$$= 1 + 1 = 2$$

3. Find the area enclosed by a loop of the curve
$a^2y^2 = x^2 (a^2-x^2)$

Solution: Here the equation of the curve is

$$a^2y^2 = x^2 (a^2-x^2) \qquad (1)$$

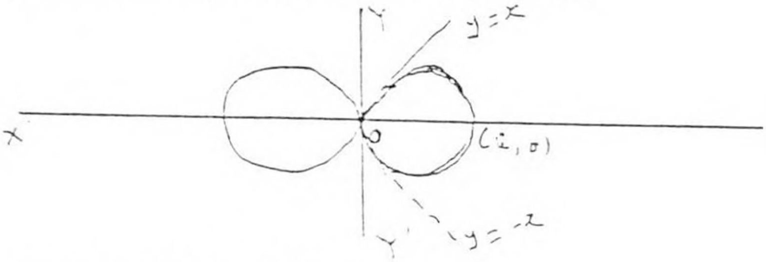
The curve (1) intersects $y = 0$ in the points given by
$0 = x^2(a^2-x^2)$ i.e. $x = 0$, $x = \pm a$.

The tangents at the origin is given by
$x^2-y^2 = 0$

which shows that the origin is a node.

So, a loop of the curve is
$a^2y^2 = x^2 (a^2-x^2)$, $\quad 0 \leq x \leq a$

Also the loop is symmetric about x- axis



Thus the area of the loop is

$$= 2 \int_0^a y \; dx = \frac{2}{a} \int_0^a x \sqrt{a^2 - x^2} \, . \, dx$$

$$= 2/a \int_0^{\pi/2} a \sin\theta, \; a \sin\theta, \; a\cos\theta \; d\theta$$ by putting

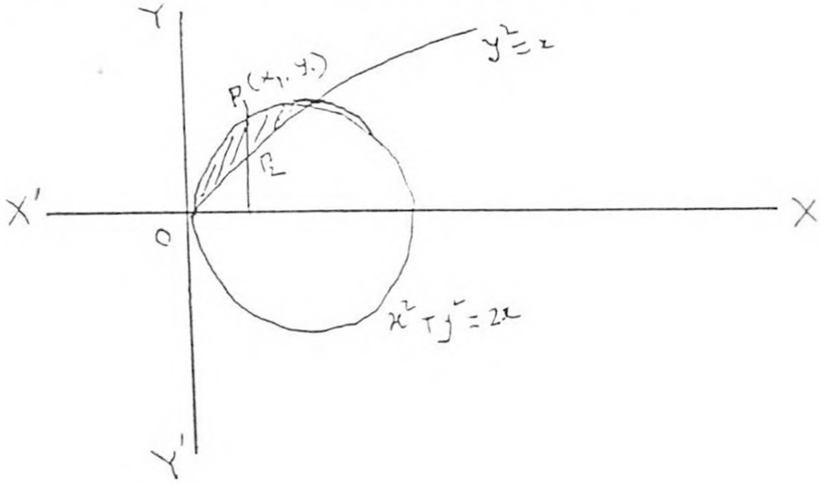
$x = a \sin\theta$

$$= 2 \, . \, a^3 \int_0^{\pi/2} \cos^2\theta, \; \sin\theta \, . \, d\theta$$

$$= 2a^2 \Big[- \cos^3\theta \Big]_0^{\pi/2}$$

$$= \frac{2a^2}{3}$$

4. Find the area above the x-axis, of the region bounded by the parabola $y^2=x$ and the circle $x^2+y^2 = 2x$.



The x-coordinates of points of intersection of the parabola $y^2 = x$ and the circle $x^2+y^2 = 2x$ are given by

$x^2+x = 2x$ i.e., $x^2-x = 0$ i.e. $x(x-1) = 0$ i.e. $x = 0$ and $x = 1$.

So we have to find the area bounded by the given curve above the x-axis so that for the points of the region

$0 \leq x \leq 1$.

Thus the required area

$$= \int_0^1 (y_1-y_2)\, dx, \text{ where } y_1 = 2x-x^2 \text{ and } y_2^2 = x$$

$$= \int_0^1 \left(\sqrt{2x - x^2} - \sqrt{x}\right) dx$$

$$= \int_0^1 \sqrt{2x - x^2} \cdot dx - \int_0^1 \sqrt{x} \cdot dx$$

For integrating $\int_0^1 \sqrt{2x - x^2} \cdot dx$, set $x = 2 \sin^2 \theta$. Then $dx = 4 \sin \theta \cos\theta \cdot d\theta$

and $x = 0 \implies \theta = 0$,

$x = 1 \implies \theta = \frac{\pi}{4}$

Then $\int_0^1 \sqrt{2x-x^2} \cdot dx$

$$= \int_0^{\pi/4} \sqrt{(2.2 \sin^2\theta - 4 \sin^4\theta)}\; 4 \sin \theta \cdot \cos \theta \cdot d\theta$$

$$= \int_0^{\pi/4} 2 \sqrt{\sin^2\theta \; (1-\sin^2\theta)}\; 4 \sin \theta \cos \theta \cdot d\theta$$

$$= \int_0^{\pi/4} 8 \sqrt{\sin^2\theta - \cos^2\theta} \cdot \sin \theta \cdot \cos \theta \cdot d\theta$$

$$= \int_0^{\pi/4} 8 \sin^2\theta \ \cos^2\theta \ . \ d\theta = \int_0^{\pi/4} 2 \sin^2 2\theta \ . \ d\theta$$

$$= \int_0^{\pi/4} (1-\cos 4\theta) \ d\theta = \left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\pi/4} = \frac{\pi}{4}$$

Also, $\int_0^1 \sqrt{x} \ . \ dx = \left[\frac{2}{3} x^{3/2}\right]_0^1 = \frac{2}{3}$

Therefore, the required area $= \frac{\pi}{4} - \frac{2}{3}$

5. Find the area enclosed by the curve given by

$x (1 + t^2) = 1 - t^2$, $y (1 + t^2) = 2t$

Solution :

Here it is a variable parameter taking its values from to . So we can set $t = \tan \theta$ where

Then $x = \dfrac{1 - t^2}{1 + t^2} = \dfrac{1 - \tan^2\theta}{1 + \tan^2\theta} = \cos 2\theta$

and $y = \dfrac{2t}{1 + t^2} = \dfrac{2 \tan \theta}{1+\tan^2\theta} = \sin 2\theta$, where $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

Note that the parametric equation represents a closed curve. Hence the required area

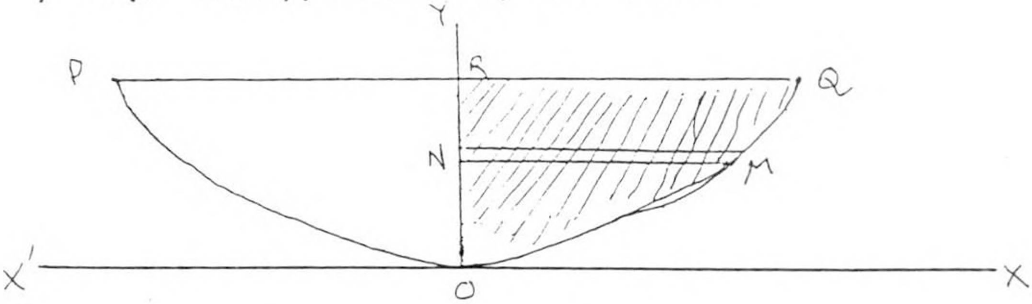
$$= \frac{1}{2} \int_{-\pi/2}^{\pi/2} \left(x \frac{dy}{d\theta} - y \frac{dx}{d\theta}\right) d\theta$$

$$= \frac{1}{2} \int_{-\pi/2}^{\pi/2} (\cos 2\theta \ . \ 2 \cos 2\theta - \sin \theta \ . \ (-2 \sin \theta)) \ d\theta$$

$$= \frac{1}{2} \cdot 2 \int_{-\pi/2}^{\pi/2} (\cos^2 2\theta + \sin^2 2\theta) \ d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\pi/2} d\theta \quad = \left[\theta\right]_{-\frac{\pi}{2}}^{\pi/2} \quad = \pi/4$$

6. Find the whole area of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$, bounded by its base.



Here the area of half the cycloid i.e., the shaded portion in the figure is the region bounded by the cycloid, $y = 0$ and $y = 2a$. Hence the total area of the cycloid

= 2 (area of the shaded portion in the figure).

$$= \quad 2 \ x \int_{0}^{a} dy$$

$$= \ 2 \int_{0}^{\pi} a\,(\theta + \sin\theta),\ a \sin\theta \ .\ d\theta$$

$$\left[\begin{array}{l} \text{for } x = a\,(\theta + \sin\theta), \\ \quad dy = a .\ \cos\theta \ .\ dx , \\ \quad y = 0 \Rightarrow \theta = 0 , \\ \quad y = 2a \Rightarrow \theta = \pi \end{array} \right]$$

$$= 2\,a^2 \int_{0}^{\pi} (\theta \ .\ \sin\theta + \sin^2\theta) \ d\theta$$

$$\int_0^{\pi} \theta \sin\theta \, d\theta = -\theta\cos\theta - \int_0^{\pi} (-\cos\theta)\, d\theta$$

$$= \Big[-\theta\cos\theta + \sin\theta\Big]_0^{\pi}$$

$$\int_0^{\pi} \theta \cdot \sin^2\theta \cdot d\theta = \int_0^{\pi} \frac{(1-\cos 2\theta)}{2} \, d\theta$$

$$= \left[\frac{\theta - \frac{\sin 2\theta}{2}}{2} \right]_0^{\pi}$$

Hence the required area

$$= 2a^2 \left[-\theta\cos\theta + \sin\theta + \tfrac{1}{2}\left(\theta - \tfrac{\sin 2\theta}{2}\right)\right]_0^{\pi}$$

$$= 2a^2 \Big[-\pi\cos\pi - \sin\pi + \tfrac{1}{2}\left(\pi - \tfrac{\sin 2\pi}{2}\right) + 0\cdot\cos 0 - \sin 0 - \tfrac{1}{2}\left(0\cdot \tfrac{\sin 0}{2}\right)\Big]$$

$$= 2a^2 \left[\pi + 0 + \frac{\pi}{2} - 0 + 0 - 0 - 0 \right]$$

$$= 2a^2 \cdot \frac{3\pi}{2} = 3a^2\pi$$

Note: Here the parametric equations of the cycloid do not represent a closed curve.

Assignments :

1. Find the area of the segment cut off from y^2
   line $y = 2x$.

2. Find the area of the portion of the circle $x^2+y^2 = 1$ which lies inside the parabola $y^2 = 1-x$.

3. Find the area bounded by the curves $y^2 - 4x - 4 = 0$ and $y^2 + 4x - 4 = 0$.

4. Find the area included between the ellipses $x^2+2y^2=1$ and $2x^2+y^2 = 1$.

5. Find the areas enclosed by the following curves :

a) $x = a \cos t + b \sin t$, $y = a^1 \cos t + b^1 \sin t$

b) $x = a \sin 2t$, $y = a \sin t$

c) $x = a(1-t^2)$, $y = at(1-t^2)$ $(-1 \leq t \leq 1)$

d) $x = \dfrac{1 - t^2}{1 + t^2}$, $y = t\dfrac{(1 - t^2)}{(1 + t^2)}$, $(-1 \leq t \leq 1)$

6. Find the area bounded by the axis x, part of the curve $y = (1 + \frac{8}{x^2})$ and the ordinates at $x = 2$ and $x = 4$. If the ordinates at $x = a$ divides the area into two equal parts, find a.

7. Find the area bounded by the curves $x^2+y^2 = 25$, $4y = |4-x^2|$ and $x=0$, above the x-axis.

Answers :

1. 8/3

2. $\left((1/2)\pi + \frac{4}{3}\right)$

3. 16/3

4. $2\sqrt{2}$

5. a) $\pi(ab^1 - a^1b)$

   b) $\frac{8}{3}a^2$

   c) $\frac{8a^2}{15}$

   d) $2 - \frac{\pi}{2}$

6. Area $\wedge$ = 4 sq. units, $a = 2\sqrt{2}$

7. $4 + 25\sin^{-1}\frac{4}{5}$ sq. units

DIFFERENTIAL EQUATIONS

1. Differential Equations, their Classification and Terminology

2. Methods of Solving First Order Differential Equations

3. Applications of First Order Differential Equations

by

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DIFFERENTIAL EQUATIONS

An Introduction :

1. A body is falling freely under gravity.
2. A body is falling under air resistance.
3. The bob of a simple pendulum is pulled aside and let go.
4. A hot body cools according to certain law.
5. A chain of given length hangs over the smooth edge of a table and begins to slide off the table.

Here are a few situations where we need to discuss the problem. The problem may be the motion of the body or the bob of the simple pendulum or the temperature of the cooling body at a given moment or the motion of the chain sliding off the table on which it is lying. A Differential Equation set up to describe each of these problems is the mathematical formulation of the problem itself. Consequently, solving the differential equation is equivalent to solving the problem itself.

Differential equations occur in the context of numerous problems which one comes across in different branches of science and engineering.

Some of them are the problem of determining

a) the motion of a projectile, rocket, satellite or planet.

b) the current in an electric circuit.

c) the conduction of heat in a rod or a slab.

d) the vibrations of a wire or a membrane

e) the flow of a liquid

f) the rate of decomposition of a radioactive substance or the rate of growth of a population.

g) the reaction of chemicals

h) the curves which have certain geometrical properties.

The mathematical formulation of such problems gives rise to differential equations. In each of the situations cited above, the objects involved obey certain laws of nature or scientific laws. These laws involve various rates of change of one or more quantities with respect to other quantities. Such rates are expressed as various derivatives and the scientific laws themselves become mathematical equations involving the derivatives, that is, differential equations.

"The vital ideas of mathematics.... were created by the solitary labour and individual genius of a few remarkable men.... A few of the greatest mathematicians of the past three centuries are Fermat, Newton, the Bernoullis, Euler, Lagrange, Laplace, Gauss, Abel, Hamilton, Liouville, Chebyshev, Hermite, Riemann and Poincare".

An elementary course on differential equations as this, aims at familiarising to its students, basic terminology and methods and techniques of solving first order equations of the type

$\frac{dy}{dx} = f(x,y)$ in easy cases.

Further, a student at the end of this course should be able to apply the concepts and techniques of solving differential equations of first order to problems arising in real life situations, some of which have been mentioned already.

The prerequisites for the course are

i) working knowledge of differentiation and integration

ii) familiarity with plane curves.

Differential Equations and Their Classification - Terminology

An equation involving an unknown function of one or more (independent) variables and the derivatives of the unknown function w.r.t. the independent variable(s) is called a differential equation.

Some examples :

1. $\frac{d^2y}{dx^2} + x + \left(\frac{dy}{dx}\right)^2 = 0$

2. $\frac{d^2y}{dt^2} + 5 \cdot \frac{dx}{dt} + 6x = e^t$

3. $\frac{\partial v}{\partial s} + \frac{\partial v}{\partial t} = 0$

4. $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$

5. $\frac{dx}{dt} = y; \quad \frac{dy}{dt} = -x.$

A differential equation involving ordinary derivatives of one independent variable w.r.t. the independent variable is called an ordinary differential equation (or equation).

Examples: In the earlier set of examples, equations (1), (2) and (5) are ordinary equations.

In (1) y is the dependent variable or the unknown function of x while x is the lone independent variable.

In (2) x is the dependent variable and t is the independent variable.

In (5) x and y are both dependent variables and t is the independent variable.

A differential equation involving partial derivatives of one dependent variable w.r.t. more than one independent variables is called a partial differential equation.

Examples: In the set of examples already given, equations (3) and (4) are partial differential equations.
In (3) v is the dependent variable and s and t are independent variables. In (4) z is the dependent variable and x,y are independent variables.

More examples of Differential Equations :

1. $\frac{dy}{dx} = -ky$

2. $m \cdot \frac{d^2x}{dt^2} = mg - k \cdot \frac{dx}{dt}$

3. $\frac{dy}{dx} + 2xy = e^{-x^2}$

4. $\frac{d^2y}{dx^2} - 5 \cdot \frac{dy}{dx} + 6x = 0$

5. $(1-x^2) \cdot \frac{d^2y}{dx^2} - 2x \cdot \frac{dy}{dx} + p(p+1)y = 0.$

6. $x^2 \cdot \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2-p^2)y = 0.$

7. $a^2\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = \frac{\partial w}{\partial t}.$

8. $$a^2\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)=\frac{\partial^2 w}{\partial t^2}$$

9. $$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}=0.$$

10. $$L\cdot\frac{d^2 Q}{dt^2}+R\frac{dQ}{dt}+\frac{1}{C}Q=E$$

Some of these equations are classical. (5) and (6) are called Legendre's equation and Bessel's equation respectively.

The equations (7), (8) and (9) are the classical heat equation, wave equation and Laplace's equation respectively.

Readily it is seen that (1) to (6) and (10) are ordinary equations while (7) to (9) are partial equations.

Order and Degree of a Differential Equation :

The order of the highest ordered derivative found in a differential equation is called the order of the equation.

The degree of the highest order derivative in a differential equation which is free from radicals and fractions in its derivatives is called the degree of the equation.

In the examples (1) to (10) we had earlier easily we can recognise the order and degree of each equation.

The equations (1) and (3) are of order 1 and degree 1. The other equations are of order 2 and degree 1.

More examples :

$$\left(\frac{d^2y}{dx^2}\right)^2 + K\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = 0$$

has its order and degree 2 each.

$\left(\frac{dy}{dx}\right)^3 + y = e^x$ has order 1 and degree 3.

The equation $\frac{dy}{dx} + \frac{1}{dy/dx} = 2x$

has to be rewritten as $\left(\frac{dy}{dx}\right)^2 + 1 = 2x\left(\frac{dy}{dx}\right)$.

Then the order and degree are respectively 1 and 2.

The equation

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

has to be rewritten as

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = \rho^2\left(\frac{d^2y}{dx^2}\right)^2$$

Then the order and degree of the equation are both 2. A Linear Equation of nth order. An ordinary Linear differential equation of nth order is given by

$$a_o(x)\, y^{(n)} + a_1(x)\, y^{(n-1)} + \dots + a_n(x)y = b(x) .$$

$y^{(k)} = \frac{d^ky}{dx^k}$ = the kth derivative of y w.r.t. x.

The equation is (1) said to be homogeneous if $b(x) \equiv 0$.

(2) said to be a linear equation with constant coefficients if all the coefficients $a_o(x), a_1(x), \dots, a_n(x)$ are

constants. An equation which is not homogeneous is called a non-homogeneous or inhcmogeneous equation.

Examples :

1. $y''' + 3x^2y'' + 3xy' + 2y = 0$

   is a linear homogeneous equation where $1 = \frac{d}{dx}$, $11 = \frac{d^2}{dx^2}$, etc.

2. $y'' + y' + xy = 0$

   is a homogeneous linear equation with variable coefficients.

3. $y^{(4)} + y'' + y = e^x$

   is a non homogeneous linear equation with constant coefficients.

4. $x^3y''' + 2x^2 y'' + 3xy' + 4y = \text{Sin } x$

   is a non homogeneous equation with variable coefficients.

5. $y^{11} + xy^2 = 0$ is not a linear equation.

6. $(y^1)^2 + y = e^x$ is also not linear.

Note:

1. y and its derivatives in the linear equation occur in first degree only.
2. Consequently a linear equation is necessarily of first degree.
3. No products of y and/or any of its derivatives are present.
4. No transcendental functions of y and/or its derivatives occur.

More examples :

1. $\frac{d^2y}{dx^2} + 5 \cdot \frac{dy}{dx} + 6y = 0$

2. $\frac{d^2y}{dx^2} + x^2 \cdot \frac{d^2y}{dx^2} + x^3 \cdot \frac{dy}{dx} = xe^x$

are ordinary linear equations.

An ordinary differential equation which is not linear is called a non linear ordinary differential equation.

$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y^2 = 0$ is a non linear ordinary equation.

A general ordinary differential equation of nth order is a relation of the type: $F\ (x,\ y,\ y',\ y'',\ldots,\ y^{(n)}) = 0$.

Formation of Differential Equations

Problems

1. Suppose that a body of mass m falls freely under gravity. In this case the only force acting on the body is its weight mg. If x is the distance through which the body falls in time t, then its acceleration is $\frac{d^2x}{dt^2}$.

Then the equation of motion of the falling body is

$$m\frac{d^2x}{dt^2} = mg \text{ or } \frac{d^2x}{dt^2} = g \qquad \ldots \qquad (1)$$

2. If there is a resisting force by air (say) proportional to the velocity, then the total force acting on the body is mg − K $\frac{dx}{dt}$ (− because the air resistance opposes the motion). In this case, the equation of motion becomes,

$$m \cdot \frac{d^2x}{dt^2} = mg - k\frac{dx}{dt}$$

or $$m \cdot \frac{d^2x}{dt^2} + k \cdot \frac{dx}{dt} = mg$$

or $$\frac{d^2x}{dt^2} + \left(\frac{k}{m}\right)\frac{dx}{dt} - g = 0 \quad ---(2)$$

3. Consider a pendulum consisting of a bob of mass m at the end of an inelastoc string or rod of negligible mass and of length a. If the bob is pulled aside through an angle and released, then by the principle of conservation of energy

$$1/2\, mv^2 = mg\,(a \cos\theta - a \cos\alpha)$$

$$s = a\theta, \quad \frac{ds}{dt} = v = a \cdot \frac{d\theta}{dt}$$

The equation of motion becomes

$$\frac{1}{2}a^2\left(\frac{d\theta}{dt}\right)^2 = ag\,(\cos\theta - \cos\alpha)\,; \quad \alpha > \theta$$

or $$\left(\frac{d\theta}{dt}\right)^2 = \frac{2g}{a}\,(\cos\theta - \cos\alpha)$$

Or $$\frac{d\theta}{dt} = \sqrt{\frac{2g}{a}(\cos\theta - \cos\alpha)} \quad ---(3)$$

4. Assume that a hot body cools at a rate proportional to the difference between the temperatures of the body and the surroundings. This law is known as Newton's law of cooling.

Let θ denote the temperature of the body at any moment t and θ_0 the temperature of the surroundings of the body. Then the rate of cooling is $\frac{d\theta}{dt}$ and this is proportional to $(\theta - \theta_0)$. Then the cooling of the body is governed by the equation

$$\frac{d\theta}{dt} = -k(\theta - \theta_0), \quad k > 0.$$

$$\text{or} \quad \frac{d\theta}{dt} + k\theta = k\theta_0 \quad --- \quad (4)$$

5. A tank contains 50 gal of pure water initially. A brine containing 2 lb of dissolved salt per gallon flows into the tank at the rate of 3 gals./min. The mixture is kept uniform by constant stirring and the well-stirred mixture simultaneously flows out of the tank at the same rate.

Let x denote the amount of salt in the tank at time t. Then the equation for the rate of change of x is

$$\frac{dx}{dt} = \text{Inflow-outflow} \quad \ldots \quad \text{(i)}$$

The brine flows at the rate of 3 gals/min and each gallon contains 2 lbs salt.

Then, Inflow = (2lb/gals) x (3 gal/min) = 6 lb/min... (ii)

Since the rate of outflow = the rate of inflow, the tank contains 50 gal of mixture in time t. This 50 gal. contains x lbs of salt in time t. Therefore, the concentration of salt at time t = $1/50$ x lb/gal.

Then, the outflow = (x/50 lb/gal) (3 gal/min) =

$\frac{3x}{50}$ lb/min. ..(iii)

Hence, (i), (ii) and (iii)

$$\Rightarrow \frac{dx}{dt} = 6 - \frac{3x}{50} \quad --- (5)$$

which is the equation governing the rate of change of salt content.

The above discussed problems illustrate how a differential equation describes the problem. In other words, in these illustrations, the mathematical formulation of the problem is the differential equation.

In each problem above, we can recognise the following important steps leading to the mathematical formulation of the problem, that is, the differential equation.

1. Identification of the law/laws, operating in the problem.
2. Analysis of the problem.
3. Representing the attributes by symbols.
4. Formation of the equation using the relationships or laws in the problem.

Differential Equations for Families of Curves :

1. Consider the family of concentric circles with their centre at the origin.

The circles are all given by

$$x^2 + y^2 = a^2 \qquad \ldots \qquad (1)$$

As a takes various values, we get different members of the family of circles. We describe a as the parameter of the family of circles.

Differentiating (1) w.r.t. x, we get

$$2x + 2y \frac{dy}{dx} = 0 \quad \text{or} \quad x + y \cdot \frac{dy}{dx} = 0 \quad --(2)$$

The differential equation (2) represents the family of circles.

We note: 1. that (2) is free from the parameter. In other words, the parameter a is eliminated in getting the differential equation.

2. The number of parameters in (1) is equal to the order of the differential equation (2), each being one.

2. Consider the family of circles through the origin with their centres on the x-axis.

Each circle of the family is given by $x^2 + y^2 = 2cx$. - - (1)

As c takes different values, we get different circles, c is the parameter of the family of circles.

Differentiating (1) w.r.t. x

$$2x + 2y \cdot \frac{dy}{dx} = 2c \quad \text{or} \quad x + y \cdot \frac{dy}{dx} = c \quad --(2)$$

Eliminating c between (1) and (2), we get

$$x^2+y^2 = 2\left(x + y\frac{dy}{dx}\right)x$$

$$= 2x^2 + 2xy\frac{dy}{dx}$$

or $$y^2 - x^2 = 2xy\cdot\frac{dy}{dx}$$

or $$\frac{dy}{dx} + \frac{1}{2}\left(\frac{x}{y} - \frac{y}{x}\right) = 0 \quad \text{-- (3)}$$

This differential equation represents the family of circles. Again we notice that (3) is a first order equation got by eliminating the single parameter c of the family of circles.

3. Consider the family of parabolas : $y = (x+c)^2$... (1)

c being the parameter of the family.

Differentiating (1), $\frac{dy}{dx} = 2\ (x+c)$

Eliminating c from (1)and (2), we get

$$\frac{dy}{dx} = 4\ (x+c)^2 = 4y$$

or $$\frac{dy}{dx} - 4y = 0 \qquad (3)$$

is the differential equation representing the family of parabolas.

Solution of a differential equation :

An Illustration : Consider the function $y = ae^{2x} + be^{-2x}$ (1)

where a,b are arbitrary constants.

Differentiating w.r.t. x we get $y = 2ae^{2x} - 2be^{-2x}$

Differentiating w.r.t. x again, $y = 4ae^{2x} + 4be^{-2x}$

$$= 4(ae^{2x} + be^{-2x})$$

or $y = 4y$ - - (2)

The function (1) satisfies the differential equation (2) for all constants a and b. (1) is a solution of the differential equation (2) for all values of a and b.

Consider an nth order ordinary differential equation

$$F(x, y, y', y'', ---, y^{(n)}) = 0 \text{ -- (1)}$$

where F is a real function of x, y, y', y",....$y^{(n)}$

$$y^{(n)} = \text{The } n^{th} \text{ derivative of } y \text{ w.r.t. } x = \frac{d^n y}{dx^n}.$$

A real function y = f(x) (2) is called a solution of the differential equation over some interval I if y is differentiable n times and 'satisfies the differential equation'

i.e. $F(x, f(x), f'(x), \ldots, f^{(n)}(x)) = 0$ for all $x \in I$.

The phrase 'satisfies the differential equation' means that when $y, \frac{dy}{dx}, \ldots,$ are replaced by $f(x), f'(x), \ldots f^{(n)}x$ respectively in (1), the equation (1) becomes an identity.

A differential equation is said to be solved if a solution of the equation is found.

Another Illustration :

The differential equation : $\frac{d^2y}{d\,x^2} + m^2x = 0$ has its solution $y = a \cos m\,x + b \sin m\,x$ where a and b are arbitrary constants.

Verification :

$$y = a\cos mx + b\sin mx$$

$$\frac{dy}{dx} = -ma\sin mx + mb\cos mx$$

and $$\frac{d^2y}{dx^2} = -m^2a\cos mx - m^2b\sin mx$$

$$= -m^2(a\cos mx + b\sin mx)$$

$$\frac{d^2y}{dx^2} = -m^2y \quad \text{or} \quad \frac{d^2y}{dx^2} + m^2y = 0$$

In the illustrations, the constants a and b of the solutions can take any values. Such a solution of a differential equation containing arbitrary constants (as a and b) is called the general solution of the differential equation.

A solution got from the general solution for particular values of the arbitrary constants is called a particular solution of the differential equation.

Initial Value Problem :

$y = x^2 + c$, c being an arbitrary constant, is the general solution of $\frac{dy}{dx} = 2x$. The particular solution satisfying the condition y = 4 when x = 1 is got from the general solution $y = x^2+c$. Putting x = 1, y = 4, 4 = 1 + c or c = 3. Hence the particular solution required is $y = x^2 + 3$.

A given differential equation together with an additional condition as in the above is called an Initial Value Problem (I.V.P.)
Thus, $\frac{dy}{dx} = 2x$
together with y = 4 when x = 1 is an initial value problem.
The above initial value problem is written as

$\frac{dy}{dx} = 2x$ The differential equation
$y(1) = 4$ The initial condition   I.V.P.

The condition in the initial value problem is called an initial condition of the problem. For the initial value problem :
$\frac{dy}{dx} = 2x$, $y(1) = 4$, $y = x^2 + 3$ is the solution.

Thus a solution of an initial value problem is a solution of the differential equation of the problem. In addition to this, the solution must satisfy the initial condition also.

Another Example : $\frac{d^2y}{dx^2} + y = 0$ has the general solution
y = a cos x + b sin x. Suppose y(0) = 2, y'(0) = 3, then a=2, b=3.
Thus, y = 2cos x + 3 sin x is a particular solution of the differential equation. This particular solution satisfies the conditions y(0) = 2, and y'(0) = 3.
Therefore, $\frac{d^2y}{dx^2} + y = 0$
with y(0) = 2 and y'(0) = 3.

is an initial value problem having the solution
$y = 2 \cos x + 3 \sin x$.

A general nth order initial value problem is of the type

$F(x, y, y, y, \ldots, y^{(n)}) = 0$ over I. (1)

$y(x_0) = y_0, \; y'(x_0) = y_0', \ldots, \; y^{(n-1)}(x_0) = y_0^{(n-1)}$ ... (2)

for some value $x = x_0 \in I$.

The set of conditions in (2) is the set of initial conditions of the initial value problem. Here,
$y_0, y_0, y_0, \ldots, y_0^{(n-1)}$ are given values.

Geometrical Meaning :

A differential equation represents a family of curves. Given a family of curves

$f(x,y,a,b) = 0$ ... (1)

by eliminating a and b, by differentiating (1), we get the differential equation.

$F(x, y, y', y'') = 0$ ... (2)

(1) is the general solution of (2) and represents the family of curves. Each curve of the family is a particular solution of the differential equation (2).

A solution of an initial value problem is a particular curve of the family of curves given by the differential equation (2).

Points to stress while teaching :

1. The difference between

a) the ordinary and partial equations

b) order and degree equations

c) Linear and non-linear equations.

d) Linear homogeneous and non homogeneous equations

e) General solution and particular solutions

f) Formation of an equation and solving an equation

g) Solving an equation and an Initial Value Problem

2. The geometrical meanings of

a) a differential equation : $\frac{dy}{dx} = f(x,y)$

b) the general solution of an equation

c) a particular solution of an equation

3. Information of a differential equation for a physical problem

a) identification of the law/laws operating

b) analysis of the problem

c) symbols and notations

4. Solution of an equation

a) Verification of a function as a solution of a given equation.

b) Formation of the equation from a given solution .

Assignments and Self Test :

I. 1. Classify the differential equations as ordinary or partial differential equations.

2. State the order and the degree.

3. Determine whether the equation is linear or non linear.

4. If the equation is linear, whether it is homogeneous or non-homogeneous.

i) $y' + x^2 y = x e^{x}$

ii) $y''' + 4y'' + 5y' + 3y = \sin x$

iii) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

iv) $x^2 dy + y^2 dx = 0$

v) $\frac{\partial^2 u}{\partial x^2} + c\,\frac{\partial u}{\partial t} = 0$

vi) $y^{(4)} + 3y'' + 5y^2 = 0$

vii) $y'' + y \sin x = 0$

viii) $y'' + x \sin y = 0$

ix) $\left(\frac{dr}{ds}\right)^2 = \sqrt{\frac{d^2 r}{ds^2} + 1}$

x) $\frac{dy}{dx} + \frac{dx}{dy} = 1$

xi) $xy' = y' \sqrt{1 - x^2 y^2}$

xii) $\frac{dy}{dx} = -\frac{xy}{x^2 + y^2}$

xiii) $y' = xe^{x^2}$

xiv) $\frac{dy}{dx} + \sqrt{\frac{1 - y^2}{1 - x^2}} = 0$

xv) $y''' + 4y'' - 5y' + 3y = \sin x$

II. Form the differential equation for the following problems.

a) The population (P) of a bacteria is increasing at a rate proportional to the population at the moment.

b) A moth ball evaporates at a rate proportional to its surface.

c) The air resistance on a falling body exerts a retardation proportional to the square of the velocity.

d) A chain 4 feet long starts sliding off the smooth table when 1 foot of the chain hangs over the edge which is supposed to be smooth (no friction).

e) A tank has 100 gallons of pure water. Brine containing 1 lb/gal. runs into the tank at the rate of 1 gal/min. The mixture is constantly stirred and flows out at the same rate as inflow.

f) An amount of invested money draws interest compounded continuously (i.e. the amount of money increases at a rate proportional to the amount present).

g) A chemical reaction converts a certain chemical into another chemical at a rate proportional to the amount of the unconverted chemical amount present at any time.

h) The rate at which radioactive nuclei decay is proportional to the number of such nuclei that are present in a given sample.

III. Show that the family of curves given by the first equation is represented by the corresponding differential equation.

1. $y = 2 + ce^{-2x^2}$, $\frac{dy}{dx} + 4xy = 8x$.
2. $y = (c+x^3)\, e^{-3x}$, $\frac{dy}{dx} + 3y = 3x^2 e^{-3x}$
3. $y = a\, e^{4x} + be^{-2x}$, $y'' - 2y' - 8y = 0$.
4. $y^2 = 4ax$, $2xy' = y$
5. $y = c_1 \sin 2x + c_2 \cos 2x$, $y'' + 4y = 0$
6. $xy = c$, $xy' + y = 0$
7. $y^2 = 4c\,(x+c)$, $(2x + yy')\, y' = y$
8. $y = c_1 e^{x} + c_2 e^{-x}$, $y'' = y$

IV. Verify that each function is a solution of the corresponding differential equation.

1. $y = x \operatorname{Tan} x$, $xy' = x^2 + y^2 + y$
2. $y = \log_e x$, $xy' = 1$
3. $y = 1 + 1/x$, $x^2 y' + 1 = 0$
4. $y = ce^{y/x}$, $x\,(y-n)\, y = y^2$
5. $x + y = \operatorname{Tan}^{-1} y$, $1 + y^2 + y^2 y' = 0$
6. $y = cx^n$, $x \frac{dy}{dx} = ny$
7. $y = cx + a/c$, $y = x \frac{dy}{dx} + a \frac{dx}{dy}$
8. $y = x^3 + ax^2 + bx + c$, $y''' = 6$
9. $y = x^2 - cx$, $2xyy' = x^2 + y^2$
10. $y = x + 3e^{-x}$, $y' + y = x + 1$
11. $y = 2e^{3x} - 5e^{4x}$, $y'' - 7y' + 12\, y = 0$
12. $y = e^{x} + 2x^2 + 6x + 7$, $y'' - 3y' + 2y = 4x^2$
13. $y = (1+x^2)$, $(1+x^2)y'' + 4xy' + 2y = 0$

V. Verify that the function given is a solution of the corresponding initial value problem.

a) $x^2 + y^2 = 25; \frac{dy}{dx} + \frac{x}{y} = 0, y(3) = 4$

b) $y = y x, xy' + y = 0 , y(1) = 1$

c) $y = (2+x^2)e^{-x}, \frac{dy}{dx} + y = 2xe^{-x}, y(0) = 2$

d) $y^2 = 4 \sec 2x, \frac{dy}{dx} = y \text{ Tan } 2x, y(0) = 2.$

e) $y^2 = 16x^3; 2xy' = 3y, y(1) = 4.$

f) $\text{Sin } y = x; y' = \text{Sec } y, y(0) = 0$

g) $y = e^{-x}; y' + y = 0, y(0) = 1.$

h) $y = \text{Tan}^{-1}x; y' = y (1+x^2), y(0) = 0$

VI. Assuming the given general solution of the differential equation, find the particular solution satisfying the additional (initial) condition.

a) $y' + y = 2xe^{-x}, y (c+x^2)e^{-x}, y(-1) = 3+e$

b) $xy' = 2y, y = cx^2, y(1) = 1$

c) $yy' = e^{2x}, y^2 = e^{2x} + c, y(0) = 1$

d) $y + xy' = x^4 (y')^2, y = c^2 + c/x, y(1) = 0$

<u>Key</u> : Ord - ordinary equation, part - partial equation,

1,1 - 1st order, 1st degree

L - Linear, H = homogeneous, NH = non homogeneous, NL = Non linear

i) Ord, 1,1, L, NH

ii) Ord, 3,1, L, NH

iii) Part, 2,1, L, H

iv) Ord, 1,1, N, L

v) Part, 2,1, L H

vi) Ord, 4,1 L, H

vii) Ord, 2,1, L, H

viii) Ord, 2,1, NL

ix) Ord, 2,1, NL

x) Ord, 2,1,NL

xi) Ord, 1,1, NL

xii) Ord, 1,1, NL

xiii) Ord, 1,1, L, N, H

xiv) Ord, 1,1, NL

xv) Ord, 3,1, L, NH

II.

a) $dp/dt = kP, \quad k > 0$

b) $m \cdot dv/dt = mg - kv^2$

c) $\frac{dv}{dt} = ks, \quad k < 0$

d) $\frac{d^2x}{dt^2} = gx$

x = the length of the hanging chain at any moment t.

e) If x lb is the amount of salt present in the tank at time t,

$$\frac{dx}{dt} = 1 - \frac{x}{100}$$

f) $\frac{dA}{dt} = KA, \; K > 0$ A = The amount at any moment t.

g) $\frac{d}{dt} = K(x_o - x), \; x_o$ = The amount of the chemical present initially.

h) $\frac{dx}{dt} = Kx$, x = the no. of radioactive nuclei disintegrating.

METHODS OF SOLVING FIRST ORDER DIFFERENTIAL EQUATIONS.

In this lesson, we discuss some first order differential equations and methods of solving them. A first order equation is of the type

$$\frac{dy}{dx} = f(x,y)$$

or the type $Mdx + Ndy = 0$ (1)

where $M = M(x,y)$, $N = N(x,y)$ (i.e. functions of x,y)

Equations with variables separable are of the form

$Mdx + Ndy = 0$

Methods of solution

where $M = M(x)$ = a function of x only

where $N = N(y)$ = a function of y only

The solution of the equation of this type is got by direct integration of the equation

The solution of (1) is $\int Mdx + \int Ndy = C$ - - - (1)

C being an arbitrary constant.

Note: (2) is the general solution of the equation (1). The solutions got from (2) by substituting particular values for C are particular solutions of the equation.

Illustrations: Solve the following problems.

1. $(1 + x^2)\, dx + (1 + y^2)\, dy = 0$

The equation is of the type (1) where $M = 1 + x^2$, $N = 1+y^2$

The solution is $\int (1+x)\, dx + \int (1+y^2) dy = C$

or $x + \frac{1}{3} x^3 + y + \frac{1}{3} y^3 = C$

or $x^3+y^3 + 3(x+y) = 3C = K$ (say)

2. $$\frac{d^2y}{dx^2} + \frac{1+y^2}{\sqrt{1-x^2}} = 0$$

The equation can be reduced to an equation in which the variables are separated, by manipulation.

Accordingly we get, $$\frac{dx}{\sqrt{1-x^2}} + \frac{dy}{1+y^2} = 0$$

Integrating $$\int \frac{dx}{\sqrt{1-x^2}} + \int \frac{dy}{1+y^2} = C$$

or $\sin^{-1}x + \tan^{-1}y = c$ is the solution.

3. $y \log x \, dx + x \log y \, dy = 0$

Rewriting the equation, $\left(\frac{\log x}{x}\right) dx + \left(\frac{\log y}{y}\right) dy = 0$.

Integrating $\int \frac{\log x}{x} dx + \int \log y/_y \, dy = c$; put $\log x = t$

Now $\int \frac{\log x}{x} dx = \int t \, dt = \frac{1}{2} t^2 = \frac{1}{2}(\log x)^2$

Similarly we get $\int \frac{\log y}{y} dy = \frac{1}{2}(\log y)^2$

Hence, $\frac{1}{2}(\log x)^2 + \frac{1}{2}(\log y)^2 = c$

or $(\log x)^2 + (\log y)^2 = K$ is the solution

4. $\frac{dy}{dx} + Ky = 0$ or $dy + Kydx = 0$

$\Longrightarrow \int \frac{dy}{y} + k \int dx = c$

$\Longrightarrow \log_e y + kx = c \Longrightarrow \log y = c - kx$

or $y = e^{c-kx} = e^c . e^{-kx} = ae^{-kx}$.

The solution is $y = ae^{-kx}$

a being the constant of integration.

Homogeneous differential equation of the type

$M(x,y) \, dx + N(x,y) \, dy = 0$ - - - (1B)

Homogeneous expressions/functions: Homogeneous equations

Consider (1) $f(x,y) = x^2 + xy + y^2$

We can write $f(x,y) = x^2(1 + y/x + (y/x)^2)$

or $f(x,y) = x \, f(1, y/x)$

Since $f(1, y/x) = 1 + 1 . y/x + (y/x)^2 = 1 + y/x + y^2/x^2$

f is a homogeneous function of degree 2 in x and y.

2. $f(x,y) = x^3 + 3x^2y + y^3$

$= x^3(1 + 3 \, y/x + (y/x)^3) = x^3 \, f(1,y/x)$

and f(x,y) is a homogeneous function of degree 3 in x and y.

3. $f(x,y) = x + \sqrt{xy} + y$

$= x\left[1 + \sqrt{y/x} + y/x\right] = x\, f(1,\ y/x)$

so that f(x,y) is a homogeneous function of degree 1 in x and y.

4. $f(x,y) = x \sin(y/x) + y \cos(y/x)$

$= x\left[\sin(y/x) + (y/x)\cos(y/x)\right]$

$= x\, f(1,\ y/x)$

f(x,y) is a homogeneous function of degree 2 in x and y.

In general, a homogeneous function of degree n in x and y, f(x,y) has the property, $f(x,y) = x\ f(1,\ y/x)$

Putting $y = v\,x$ or $y/x = v$

$f(x,y) = x\ f(1,v)$

Note: In a homogeneous function, each term is of the same degree.

$f(x,y) = x^2 + x + y + y^2$

is <u>not</u> a homogeneous function.

Since $f(x,y) = x^2 + x + y + y^2$

$= x^2(1 + 1/x + y/x^2 + y^2/x^2)$

This part is not a function of (y/x). Thus we cannot write $f(x,y) = x^n\ f(1,\ y/x)$ for any x.

<u>Definition</u> : $M(x,y)\,dx + N(x,y)\,dy = 0$
is called a <u>homogeneous equation</u> of 1st order if M(x,y) and N(x,y) are <u>homogeneous functions</u> of same degree.

If the differential equation is a homogeneous equation, then we can write the equation as

$$\frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)} = -\frac{x^n M(1,\ y/x)}{x^n N(1,\ y/x)} = -\frac{M(1,\ y/x)}{N(1,\ y/x)}$$

or $$\frac{dy}{dx} = f(y/x).$$

<u>Method of solving a homogeneous differential equation</u> :

Given the homogeneous equation

$M(x,y)\,dx + N(x,y)\,dy = 0$ -- (1)

Put $y = vx$ - - - (2)

$\frac{dy}{dx} = v.1 + x\frac{dv}{dx}$ or $dy = v\,dx + x\,dv$

This substitution converts the equation (1) into an equation in v and x with separated variables. Then the equation can be solved.

Illustrations: Solve the following equations.

1. $x \dfrac{dy}{dx} = x + y$ - - - *

By checking the coefficient function, it is easily seen that the equation is a homogeneous equation.

Put $y = vx \quad \therefore \dfrac{dy}{dx} = v + x \cdot \dfrac{dv}{dx}$ in *.

$$x\left(v + x \cdot \frac{dv}{dx}\right) = x + vx$$

$$\Rightarrow v + x \cdot \frac{dv}{dx} = 1 + v \Rightarrow x \cdot \frac{dv}{dx} = 1$$

$$\text{or} \quad dv = \frac{dx}{x}$$

On integration of the equation, we get

$v = \log x + c$

or $y/x = \log_e x + c$.

Hence $y = x(\log_e x + c)$ is the solution of the given differential equation.

2. $\dfrac{dy}{dx} = (x^2 + xy)/(y^2 + xy) \Rightarrow (xy + y^2)\dfrac{dy}{dx} = (xy + x^2)$.

x^2 = a homogeneous function of degree 2 and

x + xy + y = a homogeneous function of degree 2.

Hence the equation is a homogeneous equation.

Put $y = vx \quad \dfrac{dy}{dx} = v + x \cdot \dfrac{dv}{dx}$

$$\therefore (v^2x^2 + v^2x^2)\left(v + x \cdot \frac{dv}{dx}\right) = vx^2 + x^2$$

$$\Rightarrow v(1+v)\left(v + x \cdot \frac{dv}{dx}\right) = (v+1)$$

$$\Rightarrow v^2 + vx \cdot \frac{dv}{dx} = 1 \Rightarrow vx \cdot \frac{dv}{dx} = 1 - v^2$$

Separating the variables, we get $\frac{v}{1-v^2}dv = \frac{dx}{x}$.

Or. $\frac{dx}{x} + \frac{v.dv}{v^2-1} = 0$.

$\therefore \int \frac{dx}{x} + \int \frac{v\,dv}{v^2-1} = C$

or $\log x + \frac{1}{2}\log(v^2-1) = C$

or $2\log x + \log(v^2-1) = 2C \Rightarrow \log_e x^2(v^2-1) = 2C$

or $y^2 - x^2 = k = e^{2C}$

is the solution of the equation.

3. $x^2 \frac{dy}{dx} = x^2 + xy + y^2$

The equation is obviously a homogeneous equation.

Put $y = vx \quad \therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$

$x^2(v + x.\frac{dv}{dx}) = x^2 + x^2v + x^2v^2$

$\Rightarrow \not{v} + x\frac{dv}{dx} = 1 + v^2 + \not{v}$

$\therefore x.\frac{dv}{dx} = 1 + v^2$.

Separating the vari ables, we get

$\frac{dx}{x} = \frac{dv}{1+v^2} \quad \therefore \int \frac{dx}{x} = \int \frac{dv}{1+v^2} + C$

On integration

Hence the solution is $\log_e x = \text{Tan}^{-1}(y/x) + C$

or $y = x\,\text{Tan}\,(k + \log_e x)$, k being the constant of integration.

If the given problem is an initial value problem, then we need to find the particular solution of the differential equation which satisfies the initial condition also.

4. $x(1-y') + y(1+y') = 0$

with $y(1) = 0$

$y' = \dfrac{x+y}{x-y}$ which is a homogeneous equation.

Putting $y = vx$, $y = v + xv$

$v + xv' = \dfrac{x(1+v)}{x(1-v)}$

$xv' = \dfrac{1+v}{1-v} - v = \dfrac{1+v^2}{1-v}$

$dv = \dfrac{dx}{x}$

On integrating we get

$\tan^{-1} v - 1/2 \log(1+v^2) = C + \log x$

or $\tan^{-1}(y/x) = C + \log(x\sqrt{1+v^2})$

or $\tan^{-1}(y/x) = C + \log\sqrt{x^2+y^2}$ putting $x=1$, $y=0$ so that $C = 0$.

Hence the solution of the equation is $\tan^{-1}(y/x) = \log\sqrt{x^2+y^2}$

5. $x \sin(y/x) \dfrac{dy}{dx} = y \sin(y/x) + x$, $y(1) = \pi/2$

Putting $y = vx$, $\dfrac{dy}{dx} = v + x \cdot \dfrac{dv}{dx}$

$\sin v \left(v + x\dfrac{dv}{dx}\right) = v \sin v + x$

$v \sin v + x \sin v = v \sin v + 1$

$\sin v\, dv = dx/x$

On integrating we get $\log x + c = -\cos v$

Therefore, the general solution $\cos(y/x) + \log x + c = 0$

Putting $x=1$, $y = \pi/2$, $\cos(\pi/2) + \log + C = 0$ $C = 0$

Hence the solution is $\cos(y/x) + \log_e x = 0$

6. $xy' = y + 2x e^{-y/x}$ with $y(1) = 0$, putting $y = vx$, $y' = v + xv'$

$x(v + xv') = vx + 2x e^{-v}$

$v + xv' = v + 2e^{-v}$ $dv = 2dx/x$

On integrating we get $e^{v} = 2 \log x + c$

The solution is $e^{y/x} = 2 \log x + c$

Put $x = 1$, $y = 0$, $= 1 = 2 \log + C$ $c = 1$

$\exp(y/x) = 2 \log x$ is the solution.

7. $(y + \sqrt{x^2+y^2}) - xy' = 0$, $y(1) = 0$

The equation being homogeneous, put $y = vx$, $y' = v + xv'$

$$vx + \sqrt{x^2 + v^2 x^2}) - x(v + xv') = 0$$

$$v + \sqrt{1 + v^2} - v - x v' = 0$$

$$\Rightarrow \frac{dv}{\sqrt{1+v^2}} = \frac{dx}{x}$$

On integrating, $\sinh^{-1} v = C + \log x$

Or $\sinh (y/x) = C + \log x$

Putting $x = 1$, $y = 0$, $\sinh^{-1} (0) = C + \log 1$ $\therefore$ $C = 0$

The solution of the initial value problem is

$\sinh^{-1} (y/x) = \log_e x$

or $y = x \sinh (\log_e x)$

8. $(x \tan (y/x) + y) = x \frac{dy}{dx}$ $y(1) = \pi/2$

Putting $y = vx$, $y' = v + xv'$

$x \tan v + v x = x (v + xv')$

$\tan v + v = v + x v'$

Separating the variables,

$dx/x = (\cos v/\sin v)dv$

On integrating $\log x = C + \log \sin v$

or $\log x = C + \log \sin (y/x)$

Putting $x = 1$, $y = \pi/2$, we get $C = 0$

The solution of the initial problem is

$\log x = \log \sin (y/x)$

or $y = x \sin^{-1} x$

<u>Equations reducible to homogeneous equations</u> :

1. $$\frac{dy}{dx} = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}, \qquad a_1 b_2 \neq a_2 b_1$$

In this case, put $x = X + h$, $h = Y + K$ and choose (h,k) such that $a h + b k + c = 0$

$a h + b k + c = 0$

With the substitutions, the equation becomes $\frac{dy}{dx} = \frac{a_1 X + b_1 Y}{a_2 X + b_2 Y}$

This equation is homogeneous and can be solved by putting $Y = vX$.

An illustration :

$$\frac{dy}{dx} = \frac{x + y - 5}{x - y + 1}$$

Put $x = X - h$, $y = Y + k$

such that $h + k - 5 = 0$

$h - k + 1 = 0$

$$\frac{dy}{dx} = \frac{X + Y}{X - Y}$$

Solving $h = 2$, $l = 3$,

Hence, $x = X + 2$

$y = Y + 3$

Put $Y = vx$, $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$\frac{dy}{dx} = v + x.\frac{dv}{dx}$ so that $v + x.\frac{dv}{dx} = \frac{X + vX}{X - vX} = \frac{1+v}{1-v}$

$\therefore \quad x.\frac{dv}{dx} = \frac{1+v}{1-v} - v = \frac{1+v^2}{1-v} \Rightarrow \frac{dx}{x} = \left(\frac{1-v}{1+v^2}\right)dv$

On separating the variables, we get

$$\frac{dx}{x} = \left(\frac{1-v}{1+v^2}\right)dv \Rightarrow \int\frac{dx}{x} = \int\frac{1-v}{1+v^2}dv = \int\frac{dv}{1+v^2} - \int\frac{v\,dv}{1+v^2}$$

Integrating $\log X = \text{Tan}^{-1} v - \frac{1}{2}\log(1 + v^2) + C$

$\log X + \log\sqrt{1 + v^2} = \text{Tan}^{-1} v + c$

i.e. $\log\sqrt{X^2 + Y^2} = \text{Tan}^{-1}(y/x) + C$

The solution is $\log\sqrt{(x-2)^2 + (y-3)^2} = \text{Tan}^{-1}\left(\frac{y-3}{x-2}\right) + C$

2. $\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$, $a_1b_2 - a_2b_1 = 0$

$a_1b_2 = a_2b_1 \Rightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{1}{k}$ (say)

$\therefore \quad a_2 = ka_1, \quad b_2 = kb_1$

$$\therefore \quad \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2} = \frac{a_1x + b_1y + c_1}{k(a_1x + b_1y) + c_2}$$

$$= \frac{1}{k}\left(\frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2/k}\right) = \frac{1}{k}\left(\frac{a_1x + b_1y + c_1}{a_1x + b_1y + c_3}\right)$$

where $C_3 = C_2/K$.

Note: $(a_2x + b_2y)$ is $(a_1x + b_1y)$ for some constant. Substituting $Z = a_1x + b_1y$ the equation can be solvec.

Illustration : $\frac{dy}{dx} = \frac{x+y+4}{2x+2y-5}$, put $x + y = Z$

$\therefore 1 + \frac{dy}{dx} = \frac{dz}{dx}$

$$\frac{dz}{dx} - 1 = \frac{Z+4}{2Z-5} \Rightarrow dx = \left(\frac{2Z-5}{3Z-1}\right)dz$$

$$\text{i.e,}\quad \frac{dz}{dx} = \left(A + \frac{B}{3Z-1}\right)dz$$

WHERE $\frac{2z-5}{3z-1} = A + \frac{B}{3z-1}$

so that $2z-5 = A(3z-1) + B$

Put $z = 0$, $-A + B = -5$, $A = 2/3$

Put $z = 1$, $2A + B = -3$, $B = -13/3$

$$\therefore dx = \int\left(A + \frac{B}{3z-1}\right)dz \Rightarrow x + c = Az + \frac{B}{3}\log(3z-1)$$

Integrating, $x + c = Az + \frac{B}{3}\log(3z - 1)$

$x + C = A(x + y) + \frac{B}{3}\log(3x + 3y - 1)$

$x + C = 2/3\,(x+y) - 13/9\,\log(3x + 3y - 1)$

The solution is $9x + k = 6(x+y) - 13\log(3x + 3y - 1)$

or $(3x - 6y) + 13\log(3x + 3y - 1) + K = 0$.

First order Linear Equations :

Type : $dy/dx + py = Q$ - - - - . (1)

where $p = P(x)$, $Q = Q(x)$

(i.e. P, Q are functions of x only).

Let $\mu = \mu(x)$ be a function such that (1) becomes an exact* differential equation on multiplication by .

Multiplying (1) by μ

(1) becomes $\mu \frac{dy}{dx} + \mu P y = \mu Q$

or $\mu\, dy + \mu P y\, dx = \mu Q\, dx$ --(2)

By definition of exact equation,
the L.H.S. of the equation (2) can be written as

$d(\phi) = (\mu py)\, dx + \mu\, dy$ --- (3)

By the chain rule,

Hence, $\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = (\mu P y) dx + \mu\, dy$

$\frac{\partial \phi}{\partial x} = \mu P y, \ \frac{\partial \phi}{\partial y} = \mu$ --- (4)

$\therefore \frac{\partial^2 \phi}{\partial y \partial x} = \frac{\partial}{\partial y}(\mu P y), \ \frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial \mu}{\partial x}$

Since $\frac{\partial^2 \phi}{\partial y \partial x} = \frac{\partial^2 \phi}{\partial x \partial y}, \ \frac{\partial}{\partial y}(\mu P y) = \frac{\partial \mu}{\partial x}$ --(5)

Since $\mu = \mu(x), \ \frac{\partial \mu}{\partial x} = \frac{d\mu}{dx}$

(5) becomes $\mu P \cdot 1 + y \frac{\partial(\mu P)}{\partial y} = \frac{d\mu}{dx}$

Since p is a function of x only $\frac{\partial}{\partial y}(\mu P) = 0$.

$\therefore \frac{d\mu}{dx} = \mu P$

On integration we get $\mu = e^{\int P dx}$ --- (6)

From (2) and (3) $d\phi = \mu Q\, dx$

---

* By exact equation, we mean that the L.H.S. is the total derivative of some function of n.

Integrating the second equation : $\frac{\partial \phi}{\partial y} = \mu$ w.r.t y

$\phi = \mu y$

$d\phi = d(\mu y) = \mu Q dx$

integrating $\mu y = c + \int \mu Q dx$

but $\mu = \exp(\int P dx)$

Hence, $y = e^{-\int P dx}\left[C + \int Q e^{\int P dx} dx\right]$ - - (7)

in the solution of (1).

Working Rule: Given (1)

i) identify P and Q.

ii) compute $\int P dx$

iii) compute $\exp \int p dx = e^{\int P dx}$

iv) compute $\int Q e^{\int P dx} dx$

v) Fit in (7) to get the solution

A particular case of (1) is got when P is a constant. The equation (1) is then called a <u>first order linear equation with constant coefficients</u>.

Then, $\int p dx = px$

$e^{\int p dx} = e^{px}$

The formula (7) becomes $y = e^{-Px}\left[C + \int Q e^{Py} dx\right]$ - - - (8)

<u>Illustrations</u>: Solve the following equations ;

1. dy/dx + 2y = 4x

Here, P = 2, Q = 4x

$e^{Px} = e^{2x}$

$\int Q e^{Px} dx = \int 4x e^{2x} dx = 4\left[\frac{1}{2} x e^{2x} - \frac{1}{2}\int e^{2x} dx\right]$

$= 2\left[x e^{2x} - \frac{1}{2} e^{2x}\right] = (2x-1) e^{2x}$

The solution $y e^{2x} = C + (2x-1) e^{2x}$

or $y = ce^{-2x} + (2x - 1)$

2. $y' + y = \cos x$

Here $P = 1$, $Q = \cos x$ $\therefore e^{Px} = e^x$

$$\int Q e^{Px} dx = \int e^x \cos x \, dx$$

Let $I = \int e^x \cos x \, dx$

Then $I = e^x \sin x - \int e^x \sin x \, dx$ (Integrating by parts).

$= e^x \sin x - [e^x(-\cos x) + \int e^x \cos x \, dx]$

$= e^x \sin x + e^x \cos x - I$

or $I = e^x (\sin x + \cos x) - I$

$2I = e^x (\cos x + \sin x)$

$I = \int e^x \cos x \, dx = \frac{1}{2} e^x (\cos x + \sin x)$

The solution is $y e^x = C + \frac{1}{2} e^x (\cos x + \sin x)$ or

$y = C e^{-x} + \frac{1}{2} (\sin x + \cos x)$

3. $y' - y = 1$

$P = -1$, $Q = 1$

$e^{Px} = e^{-x}$, $\int Q e^{Px} dx = \int 1 . e^{-x} dx = -e^{-x}$

The solution is $y e^{-x} = C - e^{-x}$

or $y + 1 = C e^x$

4. $y' + 2y = 6e^x$, $P = 2$, $Q = 6e^x$ $e^{Px} = e^{2x}$

$\therefore \int Q e^{Px} dx = \int 6e^x e^{2x} dx = 6 \int e^{3x} dx$

$\therefore \int Q e^{Px} dx = 2e^{3x}$

The solution is $y e^{2x} = C + 2e^{3x}$ or $y = C e^{-2x} + 2e^x$

5. $(1 + \cos x) \, dy/dx = (1 - \cos x)$

$$dy/dx = \frac{1 - \cos x}{1 + \cos x} = \tan^2 (x/2)$$

or $dy = \tan^2 (x/2) \, dx = (\sec^2 (x/2) - 1) \, dx$

Integrating $y = C + \int (\sec^2 (x/2) - 1) \, dx$

$y = C + 2 \tan (x/2) - x$

is the solution of the equation.

6. $\left(y - x\cdot\frac{dy}{dx}\right) = a\left(y^2 + \frac{dy}{dx}\right)$

$\Rightarrow\ y - x\cdot\frac{dy}{dx} = ay^2 + a\cdot\frac{dy}{dx}$

$\Rightarrow\ (y - ay^2) = (x+a)\frac{dy}{dx}$

$\Rightarrow\ \frac{dx}{a+x} = \frac{dy}{y-ay^2}$

$\therefore\ \int\frac{dx}{a+x} + C = \int\left(\frac{1}{y} + \frac{a}{1-ay}\right)dy$

$\frac{1}{y-ay^2} = \frac{1}{y(1-ay)}$

$= \frac{1}{y} + \frac{a}{1-ay}$

or $\log(a+x) + c = \log y - \log(1-ay)$

or $\log\left(\frac{y}{1-ay}\right) = \log(a+x) + c = \log(a+x) + \log k$

or $\log\left(\frac{y}{1-ay}\right) = \log\left[k(a+x)\right]$

or $y = K(a+x)(1-ay)$

is the solution.

7. $2xy' = 3y,\ y(1) = 4.$

$2x\frac{dy}{dx} = 3y \Rightarrow 2\frac{dy}{y} = 3\frac{dx}{x}$

$2\log y = 3\log x + \log C$

$y^2 = Cx^3$ put $x = 1,\ y = 4,\ c = 16$

The particular solution satisfying $y(1) = 4$, is $y^2 = 16x^3$

8. $y' = 2e^x y^3\ \ y(0) = 1/2$

$\Rightarrow\ \frac{dy}{y^3} = 2e^x dx \Rightarrow \int\frac{dy}{y^3} = 2\int e^x dx + C$

$\Rightarrow\ -\frac{1}{2}y^{-2} = C + 2e^x$

Put $x = 0$ $\quad y = 1/2,\ -1/2$

$-1/2\,(1/2)^{-2} = C + 2e^0 = C + 2$

i.e. $C + 2 = -1/2 \times 2^2 = -2$ or $c = -4$

The particular solution required is $-\frac{1}{2y^2} = 2e^x - 4$

or $\therefore\ y^2 = \frac{1}{8-4e^x}$

9. $\frac{dy}{dx} = x e^{x} \quad y(1) = 3$

$$dy = x e^{x} dx$$

$$\int dy = C + \int x e^{x} dx$$

$$C + (x e^{x} - \int e^{x} 1.dx = C + (x-1) e^{x}$$

or $y = C + (x-1) e^{x}$

Putting $x = 1$, $y = 3$, $3 = C + 0 \implies C = 3$

$y = 3 + (x-1) e^{x}$ is the solution of the initial value problem.

10. $\frac{dy}{dx} + x e^{x^2-y} = 0, \quad y(0) = 0$

$$e^{y} dy + xe^{x^2} dx = 0$$

$$\int e^{y} dy + \int x e^{x^2} dx = C$$

$$e^{y} + \tfrac{1}{2} e^{x^2} = C$$

Put $x = 0$, $y = 0$, $1 + \tfrac{1}{2}.1 = C$ or $C = 3/2$

The solution of the initial problem is $2e^{y} + e^{x^2} = 3$

Equations reducible to the form : $y' + py = Q$, where $P = P(x)$, $Q = Q(x)$.

Bernoulli's equation: $\frac{dy}{dx} + py = Qy^{n}$ - - - (1)

dividing the equation by y^n

$$\frac{1}{y^n}\frac{dy}{dx} + P\frac{y}{y^n} = Q$$

$$\Rightarrow \frac{1}{y^n}\frac{dy}{dx} + P\left(\frac{1}{y^{n-1}}\right) = Q$$

Put $Y = \frac{1}{y^{n-1}}$ - - - (2) $\therefore -(n-1)\frac{dy}{dx} = \frac{dY}{dx}$

Then, on substitution, the given equation becomes

$$\frac{dY}{dx} - (n-1) Y.P = -Q (n-1) \text{ - - - (3)}$$

Put $P_1 = -(n-1) P$, $Q_1 = -(n-1) Q$

Clearly, $P_1 = P_1 (x)$, $Q_1 = Q_1 (x)$ and (3) becomes

$$\frac{dY}{dx} + P_1 Y = Q_1 \text{ - - (4)}$$

This equation can be solved since it is an equation of the form

$$\frac{dY}{dx} + PY = Q.$$

1. Solve : $\frac{dy}{dx} + yx = x/y$

$\Rightarrow y\frac{dy}{dx} + y^2x = x$

Put $y^2 = Y$ $\quad 2y\frac{dy}{dx} = \frac{dY}{dx}$

The equation becomes

$\frac{1}{2}\, dY/dx + xY = x$

or $dY/dx + 2xY = 2x$

Comparing it with $dy/dx + py = Q$,

$P = 2x \int Pdx = x^2,\ Q = 2x$

$\int Q e^{\int Pdx} = \int e^{x^2} \cdot 2xdx = e^{x^2}$

The solution is $Y e^{x^2} = c + e^{x^2}$

or $Y = c\, e^{-x^2} + 1$

Since $Y = y^2$, the solution is $y^2 = c\, e^{-x^2} + 1$.

2. Solve : $xy' + y = x^4y^3$

Dividing by xy^3, $\frac{1}{y^3}y' + \frac{1}{xy^2} = x^3$

Put $Y = \frac{1}{y^2}$ $\quad \frac{dY}{dx} = -\frac{2}{y^3}y'$ or $\frac{1}{y^3}y' = -\frac{1}{2}\frac{dY}{dx}$

The equation becomes, $\quad -\frac{1}{2}\frac{dY}{dx} + \frac{Y}{x} = x^3$

or $\quad \frac{dY}{dx} - \left(\frac{2}{x}\right)Y = -2x^3$

Comparing this equation with $\quad \frac{dy}{dx} + Py = Q$

$P = -2/x,\ Q = -2x^3$

$\int Pdx = -2\log_e x = \log_e (1/x^2)$ $\therefore$ $e^{\int Pdx} = 1/x^2$

The solution is $Y e^{\int Pdx} = C + \int Q\, e^{\int pdx}\, dx$

$\int Q\, e^{\int Pdx}\, dx = \int -2x^3 \cdot 1/x^2\, dx = -x^2$

Therefore, $Y \cdot 1/x^2 = C - x^2$

or $Y = x^2(c - x^2)$

or $1/y^2 = x^2(c - x^2)$ is the solution or $\therefore y^2 = \frac{1}{x^2(c-x^2)}$

3. Solve $(e^y - 2xy)\, y^1 = y^2$

Here it is necessary to treat x as the dependent variable and y as th… dependent variable.

Noting $y' = dy/dx = \frac{1}{dx/dy}$

The equation becomes $(e^y - 2xy) \Big/ \left(\frac{dx}{dy}\right) = y^2$

$$e^y - 2xy = y^2 \frac{dx}{dy}$$

Here $P = P(y) = 2/y$, $Q = Q(y) = e^y / y$

$\int P dy = \int 2/y\, dy = 2 \log y$

$e^{\int P dx} = y^2$

$\therefore \int Q e^{\int P dy} dy = \int e^y / y^2 \cdot y^2 dy = e^y$

The solution is $x e^{\int P dy} = C * \int Q e^{Pdy} dy$

The solution is $xy^2 = C + e^y$

4. Solve the initial value problem :

$$\frac{dy}{dx} + y = xy^3, \quad y(0) = \sqrt{2}$$

$$\Rightarrow \frac{1}{y^3} \frac{dy}{dx} + \frac{1}{y^2} = x$$

Put $1/y^2 = Y$

$dY/dx - 2Y = -2x$

$P = -2$, $Q = -2x$

$\int P dx = -2x \qquad e^{\int P dx} = e^{-2x}$

$$\int Q e^{Pdx} dx = \int -2x e^{-2x} dx$$

$$= -2 \left[x \left(\frac{e^{-2x}}{-2} \right) - \int \frac{e^{-2x}}{x} \cdot 1\, dx \right]$$

$$= x e^{-2x} + \frac{1}{2} e^{-2x} = \left(x + \frac{1}{2}\right) e^{-2x}$$

The solution is $Y e^{-2x} = C + (x + 1/2)\, e^{-2x}$

or $1/y^2 = x + 1/2 + ce^{+2x}$.

Using the initial condition

$y(0) = \sqrt{2}$

$\frac{1}{2} = \frac{1}{2} + c \qquad c = 0$

The solution is $\frac{1}{2}y^2 = x + \frac{1}{2}$

5. $\cos y \dfrac{dy}{dx} + \dfrac{\sin y}{x} = 1$

Put $\sin y = Y \qquad \cos y \dfrac{dy}{dx} = \dfrac{dY}{dx}$

$$\frac{dY}{dx} + \frac{Y}{x} = 1$$

$P(x) = \frac{1}{x}, \; Q(x) = 1$

$\int P(x)\,dx = \log_e x \Rightarrow e^{\int P(x)dx} = x$

$\int Q\, e^{\int P(x)\,dx} = \int 1.\, x \; dx = \frac{1}{2} x^2$

Hence the solution is $Y = e^{-\int P(x)\,dx}\left[C + \int Q\, e^{\int P(x)}\, dx\right]$.

or $\sin y = \frac{1}{x}\,(c + \frac{1}{2} x^2)$

or $\sin y = c/x + x/2$

6. $(y+1)\, dy/dx + (y^2+2y)\, x = x$

Put $y^2 + 2y = Y$

$(2y+2)\, dy/dx = dY/dx$ } using these in the given d.e.

$\frac{1}{2}\, dY/dx + Yx = x$

or $\dfrac{dY}{dx} + (2x)\, Y = 2x$

$P = 2x, \; Q = 2x$

$e^{\int P(x)\,dx} = e^{x^2}$

$\int Q\, e^{\int P(x)\,dx} = \int 2x \; e^{x^2}\, dx = e^{x^2}$

Hence the solution is $Y = e^{-\int P(x)\,dx}\left[C + \int Q\, e^{\int P(x)\,dx}\right]$

i.e. $(y^2 + 2y) = e^{-x^2}\,(c + e^{x^2})$

or $y^2 + 2y = c\, e^{-x^2} + 1$

Theoretical Problems on linear first order equations :

1. If f and g are two solutions of $dy/dx + py = 0$ —(1) then $c_1f + c_2g$ is also a solution of the equation for any arbitrary constants c_1 and c_2.

Proof : f is a solution. $\Longrightarrow$ $df/dx + p.f. = 0$ $\times c_1$

g is a solution $\Rightarrow$ $dg/dx + p.g. = 0$ $\times c_2$

$$c_1 \frac{df}{dx} + c_2 \cdot \frac{dg}{dx} + p(c_1 f + c_2 g) = 0$$

or $d/dx\,(c_1 f + c_2 g) + P\,(c_1 f + c_2 g) = 0$

Hence $c_1 f + c_2 g$ is also a solution of $dy/dx + py = 0$.

Note : The result can be extended. Accordingly, for any solution, f, g, h,... of the equation, $c_1f + c_2g + c_3h + \ldots$ is also a solution of the equation for any arbitrary constants c_1, c_2, c_3,...

2. Consider the differential equation $dy/dx + py = 0$ where $P = P(x)$. Show that

a) $f(x) \equiv 0$ for all x is a solution of the equation.

b) if f(x) is a solution of the equation such that $f(x_0) = 0$ for some value $x = x_0$, then $f(x) \equiv 0$ for all x.

c) if f and g are solutions such that $f(x_0) = g\,(x_0)$ for some $x = x_0$, then $f(x) \equiv g(x)$ for all x.

Note: The solution $f(x) \equiv 0$ of the equation (1) is called the zero solution or trivial solution. Any other solution than this is called a non zero or non trivial solution of (1).

a) Putting $y = 0$ in the equation, the equation is satisfied.
Hence $f(x) \equiv 0$ is a solution of the equation.

b) consider $dy/dx + py = 0$
Separating the variables, $dy/y + P(x)\,dx = 0$
Integrating the equation, we get $y = c\,e^{\int P(x)\,dx}$

is the general solution (i.e. all solutions are of this form).
Let f(x) be a solution. Then for some c, $f(x) = c\,e^{\int P(x)\,dx}$.

Let $f(x_0) = 0$ for some $x = x_0$, then putting $x = x_0$
$f(x_0) = c\,e^{\int P(x)\,dx} = 0 \Rightarrow c = 0$

Then $f(x) \equiv 0$ for all x.

c) Let $f(x)$, $g(x)$ be two solutions such that $f(x_o) = g(x_o)$.
Then $f(x) = c_1 e^{\int P(x)\, dx}$

$g(x) = c_2 e^{\int P(x) dx}$

$f(x_o) = g(x_o)$

$\Rightarrow f(x) - g(x) = 0$. Also $f(x) - g(x)$ is also a solution of the equation.
Hence from (b), $f(x) - g(x) \equiv 0$ for all x

$f(x) \equiv g(x)$ for all x.

3. Let $f_1(x)$ be a solution of $\frac{dy}{dx} + P(x)\, y = Q_1\,(x)$ (1)

and $f_2(x)$ be a solution of $\frac{dy}{dx} + P(x)\, y = Q_2(x)$ (2)

Then prove that $f_1(x) + f_2(x)$ is a solution of

$$\frac{dy}{dx} + P(x)\, y = Q_1\,(-x) + Q_2(x\,) \quad (3)$$

Since $f_1(x)$ and $f_2(x)$ are solutions of the differential equations (1) and (2) respectively.

$$\frac{df_1}{dx} + P(x)\, f_1 = Q_1(x)$$

$$\frac{df_2}{dx} + P(x)\, f_2 = Q_2(x)$$

Adding: $d/dx\,(f_1 + f_2) + P(x)\,(f_1 + f_2) = Q_1(x) + Q_2(x)$ which shows that $f_1(x) + f_2(x)$ is a solution of the differential equation (3).

A Uniqueness Theorem :

4. If $P(x)$ and $Q(x)$ are continuous functions of x, then show that
$dy/dx + P(x)\, y = Q(x)$.
has a unique solution $y(x)$ satisfying the initial condition.
$y\,(x_o\,) = y_o$

Let $y_1(x)$ and $y_2(x)$ be two solutions of the initial value problem

$dy/dx + P(x)\, y = Q(x)$

$y(x_o) = y_o$

Then $y_1(x) - y_2(x)$ is a solution of $dy/dx + P(x) = 0$

Also, $y_1(x_o) - y_2(x_o) = y_o - y_o = 0$

Hence, $y_1(x) - y_2(x)$ is a solution of the homogeneous equation $dy/dx + P(x)\, y = 0$ satisfying the condition $y_1(x_o) - y_2(x_o) = 0$.

Hence, $y_1(x) - y_2(x) \equiv 0$ (i.e. for all x)

$$y_1(x) \equiv y_2(x) \text{ for all } x.$$

Hence the solution is unique.

Assignment and Self Test :

1. Solve the differential equations.

a) $(x-4)\, y^4 dx - (y^2-3)\, x^3 dy = 0$

b) $x \sin y\, dx + (x^2+1) \cos y\, dy = 0$

c) $4xy + (x^2+1)\, y^1 = 0$

d) $(e^y +1) \cos x \quad + e^y (\sin x+1) \frac{dy}{dx} = 0$

e) $\text{Tan}\, \theta \; dr + 2\, r\, d\theta = 0$

f) $(x+y)\, dx - x dy = 0$

g) $(2xy + 3y^2) - (2xy + x^2 y^1 = 0$

h) $(x^2-2y^2) + xyy^1 = 0$

i) $x^2 \frac{dy}{dx} = 3\,(x^2+y^2)\, \text{Tan}^{-1} (y/x) + xy$

j) $(xy^1 - y) \sin (y/x) = x$

k) $xy^1 = y + 2x\, e^{-y/x}$

l) $(x\, \text{Tan}\, (y/x) + y)\, dx - x\, y = 0$

2. Solve the Initial Value Problem

a) $(y+2\ )\, dx + y\, (x+u)\, dy = 0,\ y(-3) = -1$

b) $(x^2+3y^2)\, dx - 2xy\, dy = 0,\quad y(2) = 6$

c) $(2x-5y)dx + (4x-y)\, dy = 0,\ y(1) = 4$

d) $(3x+8)\, (y^2+4)\, dx - 4y\, (x^2+5x+6)\, dy = 0,\ y(1) = 2$

e) $(3x^2+9xy +5y^2) - (6x^2 + 4xy) \frac{dy}{dx} = 0,\ y(2) = -6.$

3. Solve :

a) $(x+2y-3)\frac{dy}{dx} + (2x-y-1) = 0$

b) $(x+y-1)\,dx + (2x-y-8)\,dy = 0$

c) $(x+y)\frac{dy}{dx} + (2+y-x) = 0$

d) $(x+y+1)\,dy + (2-x-y)\,dx = 0$

e) $(x+2y+3)\frac{dy}{dx} + (2x+4y+3) = 0$

4. Solve :

a) $\frac{dy}{dx} + \frac{3y}{x} = 6x^2$

b) $x\frac{4dy}{dx} + 2x^3y = 1$

c) $\frac{dy}{dx} + 3y = 3x^2e^{-3x}$

d) $\frac{dy}{dx} + 4xy = 8x$

e) $\frac{dx}{dt} \quad \frac{x}{t^2} = \frac{1}{t^2}$

f) $(u^2+1)\frac{du}{du} + 4u\ v = 34$

g) $x\frac{dy}{dx} + \frac{2x+1}{x+1}y = x-1$

h) $(x^2+x-2)\frac{dy}{dx} + 3(x+1)y = (x-1)$

i) $\frac{dr}{d\theta} + r\,\text{Tan}\,\theta = \text{Cos}\,\theta$

j) $\frac{dy}{dx} - \frac{y}{x} + \frac{y^2}{x} = 0$

k) $x\frac{dy}{dx} + y + 2x^6y^4 = 0$

l) $\frac{dy}{dx} + x\,(4y - 8/y^3) = 0$

m) $\frac{dx}{dt} + \left(\frac{t+1}{2t}\right) x = \frac{t+1}{xt}$

n) $x\,\frac{dy}{dx} - 2y = 2x^4$, $y(2) = 8$

o) $\frac{dy}{dx} + 3x^2 y = x^2$, $y(0) = 2$

p) $\frac{dy}{dx} + \frac{y}{2x} = \frac{x}{y^3}$, $y(1) = 2$

q) $x\,\frac{dy}{dx} + y = (xy)^{3/2}$, $y(1) = 4$

r) $\frac{dy}{dx} + y = f(x)$, when $f(x) = \begin{cases} 2, & 0 \leq x < 1 \\ 0 & x \geq 1 \end{cases}$
$y(0) = 0$

s) $(x+2)\,\frac{dy}{dx} + y = f(x)$ when $f(x) = \begin{cases} 2x, & 0 \leq x < 2 \\ 4 & x \geq 2 \end{cases}$
$y(0) =$

APPLICATIONS OF FIRST ORDER EQUATIONS

Geometrical Applications - orthogonal Trajectories

Given a first order equation

$$\frac{dy}{dx} = f(x,y) \qquad (1)$$

the general solution of (1) is given by

$$F(x,y,c) = 0 \qquad - - - (2)$$

c being an arbitrary constant.

(2) represents a family of curves (a one-Parameter family) in the x y plane.

(1) gives the slope of a curve of the family at (x,y).

Definition : Given a C_1 - family of curves, a C_2 - family of curves is called an orthogonal family of curves to C_2 if each curve of C_2 cuts every curve of C_1 orthogonally (i.e. at right angles).

Note : Since orthogonality (i.e. Perpendicularity) is a symmetric relation, if C_2 - family is orthogonal to C_1 - family, then C_1 - family is orthogonal to C_2 - family.

Given the family of curves C_1 by the differential equation (1), the orthogonal trajectories to C_1 are got by

$$\frac{dy}{dx} = -\frac{1}{f(x,y)} \qquad - - (3)$$

(Recall that for two curves to be orthogonal, the Product of slopes= -1).

Procedure for finding the orthogonal trajectories of a given family of curves :

Step 1 : From the equation (given) $F(x,y,c) = 0$ (i) of the given family of curves, find the differential equation of the family : $\frac{dy}{dx} = f(x,y)$ (ii)

Step 2 : Replace in (ii), $f(x,y)$ by $-1/f(x,y)$ to get $\frac{dy}{dx} = -1 \quad f(x,y)$ --- (iii)

This is the differential equation of the orthogonal trajectories of (1).

Step 3 : Solve the equation (iii) to get the equation of the family of orthogonal trajectories - a one - parameter family of curves

$$G(x,y,c) = 0 \qquad (iv)$$

Caution : In step 1, in finding the equation (ii) be sure of eliminating C.

Illustration :

1. Obtain the orthogonal trajectories of the family of circles :

$x^2 + y^2 = C^2$ - - - (1)

(1) represents the family of concentric circles centred at the origin.

Differentiating (1) we get

$$2x + 2y \frac{dy}{dx} = 0$$

$$\text{or} \quad \frac{dy}{dx} = -\frac{x}{y} \quad \text{- - (2)}$$

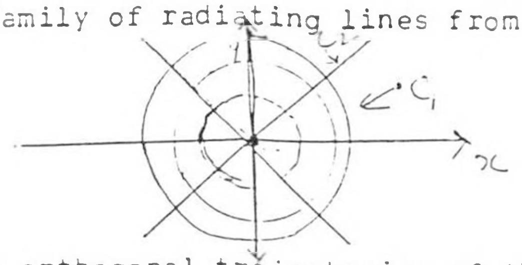
changing $-x/y$ by $-\left(-\frac{1}{x/y}\right) = y/x$

The orthogonal trajectories of (1) are given by

$$\frac{dy}{dx} = \frac{y}{x} \quad \text{- - - (3)}$$

Separating the variables in (3) and integrating (3) we get $y = c\,x$. (4)

This is the family of radiating lines from the origin.



(2). Find the orthogonal trajectories of the family of Parabola $Y = cx^2$ - - (1)

Differentiating $\frac{dy}{dx} = 2\,cx$

Eliminating C, $\frac{dy}{dx} = 2\,y/x$ - - - (2)

The orthogonal trajectories are given by

$$dy/dx = -\frac{x}{2y} \quad \text{- - - (3)} \implies x\,dx + 2y\,dy = 0$$

Integrating this equation

$$2y^2 + x^2 = \text{constant or} \quad x^2 + 2y^2 = c^2 \qquad (4)$$

which are ellipses.

(3) Find the orthogonal trajectories of the curves given $y^2 = 2\,Cx + C^2$

Consider $Y^2 = 2\,Cx + c^2$ - - (1)

Substituting for C in (1)

$$Y^2 = 2(yy^1)x + y^2y^{1^2}$$

$$y^2 = 2xyy^1 + y^2y^{1^2} \quad \text{- - -} \quad (2)$$

Replacing y^1 by $-\frac{1}{y'}$

We get $y^2 = -\frac{2xy}{y'} + \frac{y^2}{y'^2}$

$y^2 y^{1^2} + 2xyy^1 = y^2$ (3)

(2) and (3) are identical. Hence the orthogonal trajectories of the given curves are themselves i.e. given by (1) itself.

Definition : A given family of curves is said to be self -orthogonal if its family of orthogonal trajectories is the same as the given family. In the above example, the given family of Parabolas

$y^2 = 2cx + c^2$ is self-orthogonal.

Miscellaneous Examples :

(4) Find the curves such that the portion of the tangent intercepted by the axes is bisected at the point of contact.

Let Y = f(x) in a curve with the property.

The equation of the tangent at $P(x_1, y_1)$ is

$$(y-y_1) = \left(\frac{dy}{dx}\right)_P \cdot (x-x_1) \qquad (1)$$

Putting Y = 0 in (1) $-y_1 = y^1_P \; (x-x_1)$ (1)

or $x = -x_1 - \frac{y_1}{y'_P}$

$A = \left(x_1 - \frac{y_1}{y'_P}, 0\right)$

Putting x = 0 in (1), $y-y_1 = y_P' \cdot (-x_1)$

or $y = y_1 - x_1 y'_P$

$B = (0, y_1, -x_1 y'_P)$

Since P is the mid point of AB,

$$(x_1, y_1) = \left[\frac{1}{2}\left(x_1 - \frac{y_1}{y'_P} + 0\right), \frac{1}{2}\left(y_1 - x_1 y'_P + 0\right)\right]$$

$$\Rightarrow x_1 = \frac{1}{2}\left(x_1 - \frac{y_1}{y'_p}\right), \quad y_1 = \frac{1}{2}\left(y_1 - x_1 y'_p\right)$$

$$\Rightarrow 2x_1 = x_1 - \frac{y_1}{y'_p}, \quad 2y_1 = y_1 - x_1 y'_p$$

$$x_1 y'_p + y_1 = 0$$

Then the curves with the property are given by the differential equation $x\,y' + y = 0$

Solving it by separating the variables, we get the curves to be $x\,y = C^2$.

(5) Find the curves for which the subnormal at any point of the curve is of length 1.

The sub normal at any point $P(x,y)$ is given by $y\,y^1$.

Therefore, the differential equation of the curves whose subnormal is 1

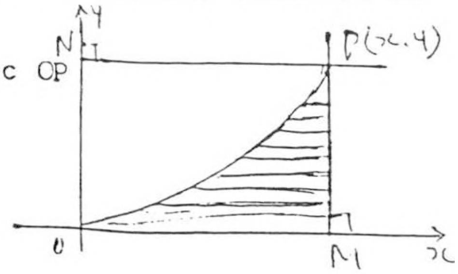
is $y\,y^1 = 1 \Rightarrow y\,\frac{dy}{dx} = 1$

Integrating $y^2 = 2x + c$

are the required curves.

6. A curve rises from the origin in the xy plane in the first quadrant. The area under the curve from (0,0) to (x,y) is one-third the area of the rectangle with the points on opposite points. Find the equation of the curve.

Given: $1/3$ (Area of OMPN) = Area under the arc OP



The curve is such that the area under the arc OP, $P(x,y)$

$= 1/3$ (The area of the rectangle OMPN)

$= 1/3\ xy$

The area under the curve $= \int_0^1 y\,dx$

Hence, $\int_0^1 y\,dx = 1/3\ xy$

Differentiating w.r.t. x

$$y = \frac{1}{3}\ \frac{d}{dx}(x.y)$$

or $3y = x\,y^1 + y$

or $2\,y = xy^1 \Rightarrow \frac{2dx}{x} = \frac{dy}{y}$

This is the differential equation of the curve.
Integrating the equation we get its equation to be $y = x^2$ or $2x$.

Falling Body Problems/Pendulum

(a) Free ball: If m is the mass of a falling body and a is the acceleration of the body, then the force acting on the body is given by $F = ma$ by the second law of motion. Accordingly, if v is the velocity of a freely falling body which has fallen through a distance x, then the equation of motion is $m \frac{dv}{dt} = mg$ or $\frac{dv}{dt} = g$ (1)

Integrating the equation $v = v_0 + gt$ (1)
v_0 being the initial velocity (at t=0).

Since $v = \frac{dx}{dt}$ (2) becomes $\frac{dx}{dt} = v_0 + gt$

Integrating again, $x = v_0 t + \frac{1}{2} gt^2$ (3)
Since $x = 0$, when $t = 0$
the motion of the freely falling body is governed by the equations (1), (2) and (3).

(b) Retarded fall : If we assume that air exerts a force opposing the motion of the falling body and that this opposing force varies directly as the velocity of the body, then the quation of falling body becomes

$$\frac{dv}{dt} = g - cv \quad (1) \qquad (c > 0)$$

or $$\frac{dv}{g - cv} = dt$$

Integrating (1) $-\frac{1}{c}\log(g-cv) = t + c_1$ or $g - cv = c_2 e^{-ct}$

Taking the initial velocity as zero i.e. $V(0) = 0$

$$C_2 = g$$

$$V = \frac{g}{c}(1 - e^{-ct}) \quad \ldots \quad (2)$$

C is +ve. Hence $v \to g/c$ on $t \to \infty$

This limiting value of v is called the terminal velocity .

Since $v = \frac{dx}{dt}$

(2) becomes $\frac{dx}{dt} = \frac{g}{c}(1 - e^{-ct})$

Integrating again, $x = c_3 + \frac{g}{c}(t + \frac{1}{c} e^{-ct})$

since $x = 0$ when $t = 0$ $C_3 + g/c^2 \cdot 1 = 0$ or $C_3 = -g/c^2$

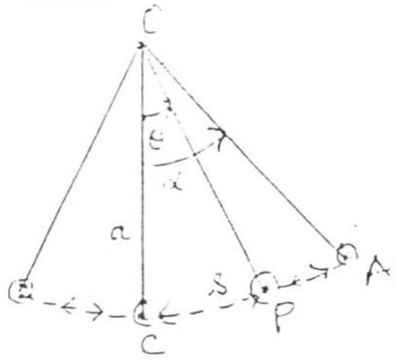
$$x = \frac{g}{c}\left[t + \frac{1}{c}\left(e^{-ct} - 1\right)\right] \quad \ldots (3)$$

(c) <u>The motion of a simple pendulum</u> : Let m be the mass of the bob and a the length of the pendulum. The bob is pulled aside through an angle α (measured from the plumb line). If V is the velocity of the bob when the string makes   with the plumb line, then by the principle of Conservation of energy

$$\frac{1}{2} mv^2 = mg \quad (a \cos\theta - a \cos\alpha)$$

But $s = a\theta \qquad v = \frac{ds}{dt} \quad = a \frac{d\theta}{dt}$

The equation becomes $\frac{1}{2} a^2 \left(\frac{d\theta}{dt}\right)^2 = ag(\cos\theta - \cos\alpha)$ - - - (1).



That is the equation of motion of the pendulum

Differentiating (1) w.r.t. t

$$a \cdot \frac{d\theta}{dt} \frac{d^2\theta}{dt^2} = -ag \sin\theta \cdot \frac{d\theta}{dt}$$

or $$\frac{d^2\theta}{dt^2} = -\frac{g}{a} \sin\theta \quad - - - (2)$$

(i) <u>Case of small oscillations</u>

Assuming that the oscillations are small, (i.e. θ is small) We replace : sin θ by θ (since θ is small, sin θ is almost θ itself) This equation becomes

$$\frac{d^2\theta}{dt^2} = -\frac{g}{a}\theta \quad - - (3)$$

This is the equation of motion of a simple pendulum for <u>small oscillations</u>.

Assuming that $\theta = \alpha$ and $\frac{d\theta}{dt} = 0$ when $t = 0$

$$\theta = \alpha \cos\left(\sqrt{g/a}\, t\right) \quad - (4)$$

<u>Simple electric circuits</u> :

Consider a simple electric circuit consisting of

(i) a source of electromotive force (emf) E

(ii) a resistor of resistance R which opposes the current producing a drop in emf of magnitude E_R. If I = the current, then $E_R = RI$ (This equation is called Ohm's law).

(iii) An inductor of inductance L, which opposes any change in the current by producing a drop in emf of magnitude

$$E_L = L \frac{dI}{dt}$$

(iv) a capacitor (or condenser) of capacitance C which stores the charge Q. The charge accumulated by the capacitor resists the inflow of additional charge, and the drop in emf arising in this way is

$$E_C = \frac{1}{C} Q.$$

Furthermore, since the current is the rate of flow of charge at which charge builds up on the capacitor, we have $I = \frac{dQ}{dt}$

These elements act in accordance with Kirchoff's Law, which states that the algebraic sum of the emfs around a closed circuit is zero.
This principle yields

$$E - E_L - E_R - E_C = 0$$

or $E - RI - L\frac{dI}{dt} - \frac{1}{C} Q = 0$

or $L\frac{dI}{dt} + RI + \frac{1}{C} Q = E$ (1)

Replacing I by $\frac{dQ}{dt}$

(1) becomes $L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E$ .. (2)

When no capacitor is present the equation (1) becomes the first order differential equation : $L\frac{dI}{dt} + RI = E$ (3)

We solve (3) assuming that initial current I_o and a constant emf E_o is impressed on the circuit at time t = 0.
The equation governing the flow of current is

$$L\frac{dI}{dt} + RI = E_o$$

separating the variables

$$\frac{dI}{E_o - RI} = \frac{1}{L} dt$$

on integrating using the initial condition $I = I_o$, when t = 0.

We get $\log (E_o - RI) = -\frac{R}{L} t + \log (E_o - RI_o)$

so $I = \frac{E_o}{R} + \left(I_o - \frac{E_o}{R}\right)\exp\left(-\frac{Rt}{L}\right)$

Note that the current I consists of a steady state Part E_o/R and a transient Part $\left(I_o - \frac{E_o}{R}\right) \exp\left(\frac{-Rt}{L}\right)$ that approaches zero as $t \to \infty$. Consequently, Ohm's law $E_o = RI$ is nearly true for large values of t. If $I_o = 0$, then $I = \frac{E_o}{R}(1 - e^{-Rt/L})$

and if $E_o = 0$, then $I = I_o e^{-Rt/L}$

OTHER RATE PROBLEMS

a) Mixture Problem: A tank initially contains 50 gallons of pure water. Starting at time t = 0 brine containing 2 lb of dissolved salt per gallon flows into the tank at the rate of 3 gallon/minute. The mixture is kept uniform by constant stirring and the mixture simultaneously flows out of the tank at the same rate.

1. How much salt remains in the tank at any time $t > 0$?
2. How much salt remains at the end of 25 minutes ?
3. How much salt remains eventually (after a long time) ?

Let x denote the amount of salt in the tank at five t. The basic equation governing the flow is

$$\frac{dx}{dt} = \text{Inflow} - \text{outflow} \quad \text{(i)}$$

Since the inflow is at the rate 3 gallon/minute and each gallon contains 2 lb of sal t

Thus inflow = 2 lb/gal X 3 gal/min = 6 lb/min (ii)

Since the rate of outflow equals the rate of inflow the tank contains 50 gal. of the mixture at any time.

Then 50 gallons contains x lb of salt.

The concentration of salt at time t = $\frac{x}{50}$ lb/gal

The outflow = $\left(\frac{x}{50} \text{ lb/gal}\right) \times (3 \text{ gal/min}) = \frac{3x}{50}$ lb/min. (iii)

Thus the differential equation governing the flow is

$$\frac{dx}{dt} = 6 - \frac{3x}{50} \quad \text{(iv)}$$

Initially there was no salt in the tank. Hence x = 0 when t = 0 (or X(0) = 0).

To solve (iv) separating the variables, $\frac{dx}{100-x} = \frac{3}{50}\,dt$

Integrating $x = 100 + C\,e^{-3t/50}$

S ince x (0) = 0, C = -100

$$x = 100\,(1 - e^{-3t/50}) \quad \text{(v)}$$

This answers question (1).
For the question (2), put $t = 25$.

$\therefore x(25) = 100\,(1 - e^{-1.5}) \approx 78$ lb

The question (3) is solved by letting $t \to \infty$ in (v). Then $x = 100$.

b) A certain chemical is converted into another chemical by a chemical reaction. The rate at which the first chemical is converted (into the second) is proportional to the amount of this chemical present at any instant. Ten percept of the original amount of the first chemical has been converted in 5 minutes.

i) What percent of first chemical will have been converted in 20 min?

ii) In how many min. will 60% of the 1st chemical has been converted ?

Let x_0 in the amount of first chemical present initially. Let x be the amount of chemical undergoing reaction at the end of time t.
Then $(x_0 - x)$ is the amount of the chemical left out at the end of time t.

By the hypothesis, the rate of charge of x is prop. to $(x_0 - x)$.

Therefore, the differential equation of the reaction is

$$\frac{dx}{dt} = K\,(x_0 - x) \qquad (1)$$

Separating the variables and integrating $x = x_0 - C\,e^{-kt}$

Since $x = 0$ when $t = 0$, $c = x_0$

$$\therefore X = x_0\,(1 - e^{-kt}) \qquad (2)$$

Now $x = \frac{x_0}{10}$ when $t = 5$ min.

$$\frac{x_0}{10} = x_0\,(1 - e^{-5k})$$

$$e^{-5k} = 0.9 \qquad e^{k} = (0,9)^{1/5}$$

Hence (2) becomes $x = x_0\,(1 - (0.9)^{t/5})$ ... (3)

(i) At the end of 20 min. $x = x_0\,(1 - (0.9)^{20/5}) = x_0\,(1, (0,9)^{4}$

Thus at the end of 20 min. $\frac{100x}{x_0} = 100\,(1 - (0.9)^{4})$

Percent of the chemical is converted into the second chemical.

(ii) If $x = \frac{6x_0}{10}$ (60% of the first chemical)

Then from (3) $\frac{6x_0}{10} = x_0\left(1 - (0.9)^{t/5}\right)$

$$0.6 = (1 - (0.9)^{t/5}$$

$$\therefore (0.9)^{t/5} = 0.4$$

$$t/5 = \frac{\log 0.4}{\log 0.9}$$

$t = 5\left(\frac{\log 0.4}{\log 0.9}\right)$ min.

c) Assume that the rate at which a hot body cools is proportional to the difference between its temperature and that of the surrounding (this law is called Newton's law of cooling). A body is heated to 110°C and placed in air at 10°C. After one hour its temperature is 60°C. How much additional time is required for it to cool to 30°C ?

Let θ°C be the temperature of the body at time t. from start. Since the temperature of the surrounding air is 10°C, by hypothesis

$$\frac{d\theta}{dt} = k(\theta - 10) \quad (1), \quad k > 0$$

Integrating, $\log_e(\theta - 10) = -kt + \text{constant}$

or $= 10 + c\, e^{-kt}$

when $t = 0$, $\theta = 110$, $110 = 10 + c\, e = 107$ C or $C = 100$

$$\theta = 10 + 100\, e^{-kt} \quad (2)$$

when t = 1 hour, $\theta = 60°C$

$60 = 10 + 100\, e^{-k \cdot 1}$

$$e^{-k} = 0.5$$

$$\theta = 10 + 100\,(0.5)^t \; --- \; (3)$$

If $\theta = 30°$ C, $30 = 10 + 100\,(0.5)^t$

$$(0.5)^t = \frac{20}{100} = (0.2)$$

$$t = \frac{\log 0.2}{\log 0.5} \text{ hr}$$

The additional time required $\left(\frac{\log 0.2}{\log 0.5} - 1\right)$ hrs $= \left(\frac{\log 5}{\log 2} - 1\right)$ hrs.

d) A sum of money is deposited in a bank that pays interest at an annual rate r compounded continuously.

1. Find the time required for the original sum to double.
2. Find the interest rate that must be paid if the initial amount doubles in 10 years.

If A = a(t), be the amount at any time.

then, $\frac{dA}{dt} = \frac{r}{100} A$ - - - (1)

Integrating $A = A_0 e^{\frac{rt}{100}}$ - - - (2)

$A_0 = A(0)$ = the initial deposits.

1. If t = T, when A = $2A_0$ Then $2A = A_0 e^{rt/100}$

$rT/100 = \log_e 2$

$T = \frac{100}{r} \log_e 2$ - - - (3)

2. If T = 10, then $10 = \frac{100}{r} \log_e 2$

$r = 10 \log_e 2$ (4) is the rate of compound interest.

(e) In a certain chemical reaction a substance A is converted into another substance X. Let a be the initial concentration of A and x = x(t) in the concentration of X at time t. Then a - x(t) in the concentration of A at t. If the reaction is jointly proportional to x and a-x (i.e., the reaction is simulated by the substance being produced, when the reaction is described on auto catalytic) and $x(0) = x_0$, find x(t).

The rate of reaction is governed by

$$\frac{dx}{dt} = kx(a-x) \quad \text{-- (1)} \qquad k>0$$

$$\frac{dx}{x(a-x)} = k.dt \Rightarrow \left(\frac{1}{x} + \frac{1}{a-x}\right) dx = akdt$$

$$\text{Log}\,x - \text{Log}(a-x) = akt + \text{Const}$$

$$\left(\frac{a-x}{x}\right) = c\exp(-akt)$$

or $\frac{a}{x} = 1 + Ce^{-akt}$

or $x = \frac{a}{[1 + Ce^{-akt}]}$

At $t = 0$, $x = x_0 = \frac{a}{1+C}$

$= C = \frac{a}{x_0} - 1$

$$x = \frac{a}{1 + (a/x_0 - 1)e^{-akt}} = \frac{a x_0}{x_0 + (a - x_0)e^{-akt}}$$

(f) A moth ball whose radius was originally 1/4 inch is found to have a radius 1/8 inch after one month. Assuming that it evaporates at a rate proportional to its surface, find the radius at any time. After how many months will the moth ball disappear altogether.

If x is the radius of the moth ball, V its volume and S its surface a at time t,

$$\frac{dv}{dt} = -KS \qquad K > 0 \quad \cdots (1).$$

$$V = \frac{4}{3}\pi x^3, \quad S = 4\pi x^2$$

$\frac{dv}{dx} = 4\pi x^2 \frac{dx}{dt} = S\frac{dx}{dt}$. Hence (1) becomes $S \cdot \frac{dx}{dt} = -KS$

$$\Rightarrow \frac{dx}{dt} = K \quad --(2)$$

Integrating $x = C - Kt$

When $t = 0$, $x = 1/4$, $1/4 = C$ $\quad x = 1/4 - Kt$

When $t = 1$, $x = 1/8$, $1/8 = 1/4 - K$ $\quad$ or $K = 1/8$

$$x = \frac{1}{4} - \frac{1}{8}\, t = \frac{2-t}{8}$$

$$x = x(t) = 1/8\,(2-t) \qquad (3)$$

when $t = 2$, $x = 0$ and the moth ball disappears. The moth ball disappears after 2 months.

g) The rate at which radioactive nuclei decay is proportional to the number of such nuclei present in a given sample. Half of the original radio active nuclei have undergone disintegration in 1500 years.

h) What percentage of the original radio active nuclei will remain after 4500 years ?

2. In how many years will only one-tenth of the original nuclei remain ?

If $x = x(t)$ is the amount of radio active nuclei remaining after t years and $x(0) = x_o$, the original amount of the nuclei, then the disintegration of the radioactive nuclei is governed by

$$\frac{dx}{dt} = -Kx \quad (1) \qquad K > 0,$$
$$x(0) = x_o$$

Integrating the equation (1) we get, $x = x(t) = C(\exp(-kt))$.

Putting $t = 0$, $x = x_o$, we get $x = x_o e^{-kt}$ - - - (2)

It is given that when $t = 1500$, $x = 1/2\, x_o$

$$\frac{1}{2} x_o = x_o . e^{-1500\,k}$$

$$\frac{1}{2} = e^{-1500k} \quad \text{or} \quad (e^{-k})^{1500} = \frac{1}{2}$$

$$e^{-k} = -(\tfrac{1}{2})^{\frac{t}{1500}}.$$

(2) becomes $x = x_o\left(\frac{1}{2}\right)^{\frac{t}{1500}}$ - (3)

(1) when $t = 4500$, $x = x_o\left(\frac{1}{2}\right)^3 = \frac{x_o}{8}$

12.5% of the original amount remains after 4500 years.

(2) when $x = \frac{1}{10} x_o$, $\quad \frac{1}{10} x_o = x_o \left(\frac{1}{2}\right)^{\frac{t}{1500}}$

Taking Logarithm

$$\log\left(\frac{1}{10}\right) = \frac{t}{1500}\left(\log\frac{1}{2}\right)$$

$$t = 1500\,\frac{\log\left(\frac{1}{10}\right)}{\log\left(\frac{1}{2}\right)} = 1500\,\frac{\log 10}{\log 2} \approx 4985 \text{ years.}$$

h) The rate at which a certain substance dissolves in water in proportional to the product of the amount undissolved and the difference $C_1 - C_2$ where C_1 is the concentration in the saturated solution and C_2 is the concentration in the actual solution. If saturated, 50 gm of water would dissolve 20 gm of the substance.

If 10 gm of the substance is placed in 50 gm of water and half of the substance is then dissolved in 90 min., how much will be dissolved in 3 hour ?

Since 20 gm of the substance dissolves in 50 gm saturated solution, the concentration in the saturated solution

$$= C_1 = \frac{20}{50} \qquad \text{(i)}$$

Let x gm be the substance dissolved in 50 gm of water at time t. Then (10 - x) gm of the substance is undissolved at time t.

.The concentration of the substance in 50 gm of water at time

$$t = C_2 = \frac{x}{50} \qquad \text{(ii)}$$

The substance dissolves in water according to the law

$$\frac{dx}{dt} = K\,(C_1 - C_2)\,(10-x)$$

$$= K\left(\frac{20}{50} - \frac{x}{50}\right)(10-x)$$

$$\text{or} \quad \frac{dx}{dt} = \frac{k}{50}\,(20-x)\,(10-x)$$

Separating the variables

$$\frac{dx}{(10-x)(20-x)} = \frac{k}{50}\,dt \quad \text{or}$$

$$\frac{1}{10}\left(\frac{1}{10-x} - \frac{1}{20-x}\right) = \frac{k}{50}\,dt$$

Integrating $\log_e\left(\frac{20-x}{10-x}\right) = \frac{k}{5}\,t + \text{const.}$

$$\text{or} \quad \frac{20-x}{10-x} = C\,e^{k/5\,t}$$

When t = 0, x = 0, C = 2

$$\frac{20-x}{10-x} = 2e^{\frac{k}{5}t} \quad \ldots \ (1)$$

Since half the substance (i.e. 5 gm) is dissolved in 90 m in
Putting x = 5, t = 90

$3 = 2e^{18k}$, $\frac{3}{2} = e^{18k}$ or $e^{k} = \left(\frac{3}{2}\right)^{\frac{1}{18}}$

$$\frac{20-x}{10-x} = 2e^{\frac{k}{5}t} = 2(e^{k})^{t/5} = 2\left(\frac{3}{2}\right)^{\frac{t}{90}}$$

Now we express x in terms of t.

$$20-x = 2\left(\frac{3}{2}\right)^{t/90}(10-x)$$

$$20-20\left(\frac{3}{2}\right)^{t/90} = x\left[1-2\left(\frac{3}{2}\right)^{\frac{t}{90}}\right]$$

$$x = 20\left[\frac{\left(\frac{3}{2}\right)^{t/90}-1}{2\left(\frac{3}{2}\right)^{t/90}-1}\right] \quad \text{---} (2)$$

When t = 3 hrs = 180 min.

$$x = 20\left[\frac{(3/2)^{2}-1}{2(3/2)^{2}-1}\right] \text{gm} = 20\left[\frac{9-4}{18-4}\right] = \frac{100}{14} \quad \text{or} \quad x = 7.14 \text{ gm.}$$

P R O B A B I L I T Y

1. Basic Terminology
2. Some theorems on Probability
3. Random Variables and Probability Distribution

by

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PROBABILITY

In our day-to-day life we perform certain activities to verify certain known facts or to observe certain phenomena. Such activities usually we call as experiments. In certain experiments, we can predict results exactly before conducting the experiment and in other it will not be possible. The experiments where the results can be predicted exactly are known as deterministic experiments and the experiments where the prediction is not exact are known as non-deterministic or random or probabilistic experiments. For example, a train is running at a uniform speed of sixty k.m. per hour, then we can predict with hundred percent surety that it will cover one hundred twenty kilometers after two hours, assuming that it never stopped during these hours. Similarly, for a perfect gas, PV = constant (P is pressure, V is volume).

In case of non-deterministic experiments, we cannot make predictions with complete reliability. The results are based on some 'chance element'. For example, if we toss a coin, will it show 'head up' or 'tail up'? Although we cannot predict anything with complete surety, yet if we throw the coin a large number of times, it is very likely that the head will turn up fifty percent of the times and also it is very unlikely that the head turns up in every case.

Consider another example of a trained parachuter who is ready to jump. When he jumps then either his parachute will open or it will not. But experience says that most of the time it opens, though there are occasions on which it does not i.e. the uncertainty associated with the head or tail coming up when we toss a coin.

How will you proceed in answering the following questions ?

1. How should a businessman order for replenishment (filling once again) of his stocks (inventory) so that he has not carried very large stocks, yet the risk of refusing customers is minimized ? (Inventory problem).
2. At what intervals should a car owner replace the car so that the total maintenance expenses are minimized ? (Replacement problem).
3. How many trainees should a large business organisation recruit and train them in certain intervals so that at any time it does not have a large number of trained persons whom it cannot employ and yet the risk of its being without sufficient persons when needed is minimized ?
4. How should the bus service in a city be scheduled so that the queues do not become too long and yet the gains by the bus company are maximized ? (Queing problem).
5. How many booking counters should be provided at a station to serve in the best way the interests of both the railways and the travelling public ? (Queing problem).

6. What should be the strength of a dam (or a bridge) so that its cost is reasonable and yet the risk of its being swept away by the floods is minimized ?
7. How many telephone exchanges should be established in a given city so as to give the best service at a given cost ?
8. Which variety is the best out of given varieties of wheat, on the basis of yields from experimental fields ?
9. What should be the minimum premia charged by an insurance company so that the chance of its running into loss is minimized ?
10. How to decide whether a given batch of items is defective when only a sample of the batch can be examined ?

Answers for all such questions are based upon certain facts and then try to measure the uncertainty associated with some events which may or may not materialise. The theory of probability deals with the problem of measuring the uncertainty associated with various events rather precisely, making it these by possible today, to a certain extent of course, to control phenomena depending upon chance.

The 'measure of uncertainty' is known as probability.

History of Probability Theory

Probability had its birth in the seventeenth century and over the last three hundred years, it has progressed rapidly from its classical heritage of simple mathematical and combinatorial methods to its present rigorous development based on modern functional analysis. The probability had its origin in the usual interest in gambling that pervaded France in the seventeenth century. Eminent mathematicians were led to the quantitative study of games of chance. The Chevalier de Mere, a French nobleman and a notorious gambler, posed a series of problems to B Pascal (1623-1662) like the following :

> Two persons play a game of chance. The person who first gains a certain number of points wins the stake. They stop playing before the game is completed. How is the stake to be divided on the basis of the number of points each has got ?

Though Galileo (1564-1642) had earlier solved a similar problems, this was the beginning of a systematic study of chance and regularity in nature. Pascal's interest was shared by Fermat (1601-1665), and in their correspondence the two mathematicians laid the foundation of the theory of probability. Their results aroused the interest on the Dutch physicist Huyghens (1629-1695) who started working on some difficult problems in games of chance, and published in 1654 the first book on the theory of probability. In this book, he introduced the concept of mathematical expectation which is basic to the modern theory of probability. Following this, Jacob Bernoulli (1654-1705) wrote his famous 'Art Conjectandi' the result of his work of over twenty years. Bernoulli approached this subject from a very general point of view and clearly foresaw the wide applications of the theory. Important contributions were made by Abraham de Moivre (1667-1754) whose book 'The Doctrine of Chance' was published in 1718. Other main contributors were T.Bayse (Inverse Probability), P.S. Laplace (1749-1827) who after extensive research published 'Theoric Analytique des probabilities' in 1812. In addition to these Levy, Mises and R.A.Fisher were the main contributors. It was, however, in the work of Russian mathematicians Tschebyshev (1821-1874), A Markov (1856-1922), Liapounov (Central Limit theorem), A Kintchine (Law of Large Numbers) and A Kolmogorov that the theory made great strides. Kolmogoroff was the person who axiomised the calculus of probability.

The probability theory itself has developed in many directions, but at present the dominant area is the stochastic processes, which has wide applications in physics, chemistry, biology, engineering, management and the social sciences.

Calculus of Probability

In our day-to-day vocabulary we use words such as 'probably', 'likely', 'fairly good chances', etc. to express the uncertainty as indicated in the following example. Suppose a father of a XII class student wants to know his son's progress in the studies and asks the concerned teacher about his son. Teacher may express to the father about the student's progress in any one of the following sentences.

It is certain that he will get a first class.
He is sure to get a first class.
I believe he will get a first class.
It is quite likely that he will get a first class.
Perhaps he may get a first class.
He may or he may not get a first class.
I believe he will not get a first class.
I am sure he will not get a first class.
I am certain he will not get a first class.

Instead of expressing uncertainty associated with any event with such phrases, it is better and exact if we express uncertainty mathematically. The measure of uncertainty or probability can be measured in three ways and these are known as the three definitions of probability. These methods are

Mathematical or Classical or Priori Probability
Statistical or Empirical Probability and
Axiomatic Probability

Before discussing those methods, we define some of the terms which are useful in the definition of probability.

Experiment : An act of doing something to verify some fact or to obtain some result. (Ex. Throwing a die to observe which number will come up (Die is a six-faced cube).

Trial : Conducting experiment once is known as the trial of that experiment. Ex. Throwing a die once.

Outcomes : The results of an experiment are known as outcomes. Ex. In throwing a die, getting '1' or '2' or '6' are the outcomes.

Events: Any single outcome or set of outcomes in an experiment is known as an event.

Ex: 1. Getting '1' in throwing of a die is an event.

Also getting an even number in throwing a die is also an event.

Ex: 2. Drawing two cards from a well shuffled pack of cards is a trial and getting of a king and a queen is an event.

Exhaustive Events : The total number of possible outcomes in any trial are known as exhaustive events.

Ex: 1. In tossing a coin there are two exhaustive events.

2. In throwing a die, there are six exhaustive cases viz (1,2,3,4,5,6).

Favourable Events (Cases): The number of outcomes which entail the happening of an event are known as the favourable cases (events) of that event.

Ex: In throwing two dice, the number of cases favourable for getting a sum of 5 are (1,4), (2,3), (3,2) and (4,1).

Mutually Exclusive Events : Events are said to be mutually exclusive or incompatible if the happening of any one of them precludes or excludes the happening of all others.

Ex: In tossing a coin, the events head and tail are mutually exclusive (because both cannot occur simultaneously).

Mathematical or Classical or 'a priori' probability

If a trial results in 'n' exhaustive, mutually exclusive and equally likely cases and 'm' of them are favourable to the happening of an event E, then the probability 'p' of happening of E is given by

p = Favourable number of outcomes / Total No of outcomes = m/n

We write p = P(E).

Ex:1. Probability of getting head in tossing of a coin once is ½ because the number of exhaustive cases are 2 and these are mutually exclusive and equally likely (assuming the coin is made evenly) and of these only 1 case is favourable to our event of getting head.

Ex: 2. The probability of getting a number divisible by 3 in throwing of a fair (evenly made) die is 2/6 because the favourable cases are 3 (viz. 3 and 6) and exhaustive cases are 6.

The probability 'q' that E will not happen is given by

$$q = \frac{n-m}{n} = 1 - \frac{m}{n} = 1 - p$$

Always $0 \leq p \leq 1$.

If p = P(E) = 1, E is called a certain event and if P(E) = 0, E is called an impossible event.

In this method, the mathematical ratio of two integers is giving the probability and therefore, this definition is known as mathematical definition. Here we are using the concept of probability in the form of 'equality likely cases' and therefore, this definition is a classical definition. Before using this definition, we should know about the nature of outcomes (viz. Mutually exclusive, exhaustive and equally likely) and therefore, it is also known as 'a priori' probability definition.

The definition of mathematical or classical probability definition breaks down in the following cases: 1. If the various outcomes of the trial are not equally likely. 2. If the exhaustive number of cases in a trial is infinite.

Ex:1. When we talk about the probability of a pass of a candidate, it is not ½ as the two customers 'pass' and 'fail' are not equally likely.
Ex: 2. When we talk about the probability of a selected real number is to be divided by 10, the number of exhaustive cases are infinite.

In such above mentioned circumstances it is not possible to use mathematical probability definition. Therefore, probability is defined in the other way as below :

Statistical or Empirical Probability :

If a trial is repeated a number of times under essentially homogeneous and identical conditions, then the limiting value of the ratio of the number of times an event happens to the number of trials, as the number of trials becomes indefinitely large, is called the probability of happening that event.

Mathematically, we write

$$p = P(E) = \lim_{n \to \infty} \left(\frac{m}{n}\right)$$

Here n is the number of trials and m is the number of times of the occurrence of event E. The above limit should be finite.

Ex: When you throw a die 10000 times and if you get 1600 times the number '1', then the probability of getting '1' is 1600/10000. This ratio is nothing but the relative frequency of '1'.

But this definition is also not applicable always because it is very difficult to maintain the identical conditions throughout the experiment. Therefore, the probability is defined in another way by using certain axioms. This definition is known as 'Axiomatic Probability' definition.

Here we define some of the terms which are useful in the 'Axiomatic Probability' definition.

Sample Space: The set of all possible outcomes of an experiment is known as the sample space of that experiment. Usually we denote it by S. Ex: In tossing a coin, S = { H, T }.

Sample Point : Any element of a sample space is known as a sample point.

Ex: In tossing a coin experiment, H or T is a sample point.

Event: Any subset of a sample space is an event.
Ex: In throwing a die, (1,3,5), (2,4,6) or (5,6) are the events where S = {1,2,3,4,5,6}.

If A and B are any two events then $\overline{A}$, $\overline{B}$, AUB, $A \cap B$ are also events because they are also subsets of S.

The event S (entire sample space) is known as certain event and the event Φ (empty set) is known as impossible event.

Mutually Exclusive Events : Events are said to be mutually exclusive if the corresponding sets are disjoint.
Ex: In throwing of a die experiment, if A = (1,3,5) and B = (2,4,6) then A and B are mutually exclusive because we cannot get both odd number and even number simultaneously. That is, if $A \cap B = \Phi$, then A and B are mutually exclusive events.

Axiomatic Probability :

Let S be a sample space and ξ be the class of events. Also let P be a real valued function defined on ξ. Then P is called a probability function and P(A) is called the probability of the event A if the following axioms hold :

i) For every event A, $0 \leq P(A) \leq 1$.
ii) P(S) = 1.
iii) If A and B are mutually exclusive events, then P(AUB) = P(A) + P(B).
iv) If $A_1, A_2, \ldots$ is a sequence of mutually exclusive events, then
$P(A_1 U A_2 \ldots) = P(A_1) + P(A_2) + \ldots\ldots$

In the above definition axiom (iv) may seem to be not necessary. But it is necessary to stress that axiom (iii) should be extended to more than two events.

Theorem 1 : If Φ is the empty set, then $P(\Phi) = 0$.

Proof : We know that $S = S \cup \Phi$ and $P(S) = P(S\cup\Phi) = P(S) + P(\Phi)$. (because S and Φ are disjoint and according to axiom (iii)). But $P(S) = 1$ and therefore, $1 = 1 + P(\Phi)$.

$\therefore P(\Phi) = 0$.

Theorem 2 : If $\bar{A}$ is the complement of an event A, then

$P(\bar{A}) = 1 - P(A)$.

Proof : $A \cup \bar{A} = S$.

$P(A \cup \bar{A}) = P(A) + P(\bar{A}) = P(S)$ (A and $\bar{A}$ are disjoint).

But $P(S) = 1$, therefore,
$P(A) + P(\bar{A}) = 1$
Or $P(\bar{A}) = 1 - p(A)$.

Theorem 3 : If $A \subseteq B$, then $P(A) \leq P(B)$.
Proof : We know that if $A \subsetneq B$, then
$B = A \cup (B - A)$ (here we may use the notation B/A)
So, $P(B) = P(A) + P(B-A)$
But from axiom i, $P(B-A) \geq 0$
$\therefore P(B) \geq P(A)$.

Theorem 4 : If A and B are any two events, then
$P(A-B) = P(A) - P(A \cap B)$

Proof: We can write, $A = (A \cap B) \cup (A - B)$

But $(A \cap B)$ and $(A - B)$ are disjoint and according to axiom (iii).
$P(A) = P(A \cap B) + P(A - B)$.
Or $P(A - B) = P(A) - P(A \cap B)$.

Theorem 5 : (Addition Theorem)

If A and B are any two events, then

$$P(A\cup B) = P(A) + P(B) - P(A \cap B)$$

Proof: We can write, AUB = BU(A-B). But B and (A-B) are disjoint and therefore, by axiom (iii),
P(AUB) =P(B) + P(A-B).

Also, from theorem 4, P(A-B) = P(A) – P(A $\cap$ B)

Hence, P(AUB) = P(B) + P(A-B)
= P(B) + P(A) – P(A $\cap$ B)

This theorem is known as addition theorem and it can be extended to any number of events as follows :

Theorem 6 : (Addition Theorem in case of n events)

If $A_1, A_2, \dots, A_n$ are any n events, then

$$P(A_1 \cup A_2 \cup \dots A_n) = \sum_{ij=1}^{n} P(A_i \cap A_j) - \sum_{i,j,k=1;\ i<j<k}^{n} P(A_i \cap A_j \cap A_k) + \dots + (-1)^{n-1} P(A_1 \cap A_2 \cap \dots \cap A_n)$$

Proof: This theorem can be proved by the method of induction. For the events A_1 and A_2 we have from theorem 5,

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$$

$$= \sum_{i=1}^{2} P(A_i) + (-1)^{2-1} P(A_1 \cap A_2)$$

Hence the theorem is true for n = 2.
Now, suppose the theorem is true for n = r, say

Then,

$$P(A_1 \cup A_2 \cup \dots \cup A_r) = \sum_{i=1}^{r} P(A_i) - \sum_{ij=1,\ i<j}^{r} P(A_i \cap A_j) + \dots + (-1)^{r-1} P(A_1 \cap A_2 \cap \dots \cap A_r)$$

Now,

$$P(A_1 \cup A_2, \dots, \cup A_r \cup A_{r+1}) = P(A_1 \cup A_2 \cup \dots \cup A_r) \cup A_{r+1})$$

$$= P(A_1 \cup A_2 \cup \dots \cup A_r) + P(A_{r+1}) - P((A_1 \cap A_{r+1}) \cup (A_2 \cap A_{r+1}) \dots \cup (A_r \cap A_{r+1}))$$

$$= \sum_{i=1}^{r} P(A_i) - \sum_{ij=1,\ i<j}^{r} P(A_i \cap A_j) + \dots (-1)^{r} P(A_1 \cap A_2 \cap \dots A_{r+1})$$

$$+ P(A_{r+1}) - \sum_{i=1}^{r+1} P(A_i \cap A_{r+1}) + \sum_{ij=1,\ i<j}^{r+1} P(A_i \cap A_j \cap A_{r+1}) + \dots + (-1)^{r} P(A_1 \cap A_2 \cap \dots \cap A_{r+1})$$

$$= \sum_{i=1}^{r+1} P(A_i) - \sum_{ij=1,\ i<j}^{r+1} A(A_i \cap A_j) + \dots + (-1)^{r} P(A_1 \cap A_2 \cap \dots A_{r+1})$$

Hence, if the theorem is true for n=r, it is also true for n=r+1. But we have proved that the theorem is true for n = 2. Hence by the method of induction, the theorem is true for all positive integer values of n.

Corollary 1 : If A and B are two mutually exclusive events, then,
P (AUB) = P(A) + P(B).

Corollary 2 : If A_1, A_2,....A_n are n mutually exclusive events,

Then $P(A_1 \cup A_2 \ldots \cup A_n) = P(A_1) + P(A_2) + \ldots + P(A_n)$

Conditional Probability :

So far, we have assumed that no information was available about the experiment other than the sample space while calculating the probabilities of events. Sometimes, however, it is known that an event A has happened. How do we use this information in making a statement concerning the outcome of another event B ?

Consider the following examples.

Ex.1: Draw a card from a well-shuffled pack of cards. Define the event A as getting a black card and the event B as getting a spade card. Here P(A) = ½ and P(B) = ¼ . Suppose the drawn card is a black card then what is the probability that card is a spade card? That is, if the event A has happened then what is the probability of B given that A has already happened? This probability symbolically we write as P(B/A). In the given example,

$$P(B/A) = \frac{1}{2} = \frac{P(A \cap B)}{P(A)} = \frac{(1/4)}{(1/2)}$$

Because probability of simultaneous occurrence of A and B is ¼ and probability of A is ½.

Ex.2: Let us toss two fair coins. Then the sample space of the experiment is S = { HH, HT, TH, TT}. Let event A = { both coins show same face } and B = { at least one coin shows H }. Then P(A) = 2/4. If B is known to have happened, this information assures that TT cannot happen, and P {A, conditional on the information that B has happened } =

$$P(A/B) = 1/3 = \frac{1/4}{3/4}$$

$$= \frac{P(A \cap B)}{P(B)}$$

In the above two examples, we were interested to find the probability of one event given the condition that the other event has already happened. Such events based on some conditions are known as conditional events. In the above examples B/A and A/B are the conditional events. The probability of a conditional event is known as conditional probability of that event. We write the conditional probabilities as P(A/B), P(E/F), etc.

Definition of conditional probability : The conditional probability of an event A, given B, is denoted by P(A/B) and is defined by

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

Where A, B and $A \cap B$ are events in a sample space S, and $P(B) \neq 0$.

From the definition of conditional probability we know that

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

Therefore, we can write from the above

$$P(A \cap B) = P(B)\, P(A/B)$$

Also, we know that $P(A \cap B) = P(B \cap A)$ and

$$P(B \cap A) = P(A)\; P(B/A)$$

Hence we can write

$$P(A \cap B) = P(A)\; P(B/A) \text{ or } P(B)\; P(A/B).$$

The above result is known as multiplication law of probabilities in case of two events.

Multiplication Theorem of Probabilities : If A and B are any two events of a sample space S, then

$$P(A \cap B) = P(A)\; P(B/A) \text{ or } P(B)\; P(A/B).$$

The above theorem can be extended to any n events as follows :

If $A_1, A_2, \ldots, A_n$ are any n events, then

$$P(A_1 \cap A_2 \ldots \cap A_n) = P(A_1)\; P(A_2/A_1)\; P(A_3/A_1 \cap A_2) \ldots\ldots P(A_n / A_1 \cap A_2 \ldots \cap A_{n-1}).$$

This theorem can be proved by method of induction or generalization.

Baye's Theorem : If $E_1, E_2, \ldots, E_n$ are mutually exclusive events with $P(E_i) \neq 0$, $(i = 1,2,\ldots n)$ then for any arbitrary event A which is a subset of $\bigcup_{i=1}^{n} E_i$ such that $P(A) > 0$, we have

$$P(E_i/A) = \frac{P(E_i)\, P(A/E_i)}{\sum_{i=1}^{n} P(E_i)\, P(A/E_i)} \quad \text{for all i.}$$

Proof: Since $A \subset \bigcup_{i=1}^{n} E_i$ we have

$$A = A \cap (\bigcup_{i=1}^{n} E_i) = \bigcup_{i=1}^{n} (A \cap E_i)$$ (by distributive law).

Since $(A \cap E_i) \subset E_i$ (for i = 1,2,...., n) are mutually exclusive events, we have by additional theorem of probability

$$P(A) = P[\bigcup_{i=1}^{n} (A \cap E_i)] = \sum_{i=1}^{n} P(A \cap E_i) = \sum_{i=1}^{n} P(E_i)\, P(A/E_i)$$

(By multiplication theorem in case of two events.)

Also, we have

$$P(A \cap E_i) = P(A) \quad P(E_i/A) \quad \text{and}$$

$$P(E_i/A) = \frac{P(A \cap E_i)}{P(A)} = \frac{P(E_i)\, P(A/E_i)}{P(A)}$$

Hence, $$P(E_i/A) = \frac{P(E_i)\, P(A/E_i)}{\sum_{i=1}^{n} P(E_i)\, P(A/E_i)}$$

This theorem is very useful in calculating the conditional probabilities in certain situations.

If $P(A \cap B) = P(A)\, P(B)$, then we see that $P(B/A) = P(B)$ and hence we say that the probability of B is not depending upon the happening of A. That is the conditional probability of B is same as the unconditional probability of B. Such events are called independent events.

Two events A and B are independent if and only if

$P(A \cap B) = P(A)\; P(B)$.

Ex: Let two fair coins be tossed and let

A = { head on first coin }, B = { head on the second coin }.

Then P(A) = P {HH, HT } = ½

P(B) = P { HH, TH } = ½ and

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{1/4}{1/2} = 1/2 = P(A)$$

Thus,

P (A ∩ B) = P(A) P(B).

and we know that the probability of getting head on the first coin does not depend upon the probability of getting head on the second coin. Hence A and B are independent. Also we see that the condition P(A ∩ B) = P(A) P(B) is both necessary and sufficient for those events A and B to be independent.

If there are three or more than three events, we will have the situation where every pair of these events are independent or the situation where the events in every set of events are independent. In the first case, we call the events as pairwise independent and in the second case we call as complete or mutual independent events.

Geometric Probability :

Sample space can be countably finite or countably infinite or uncountably finite or uncountably infinite depending upon the situation. If the sample space is countably finite, then it is easy to calculate the probability of any event by using either mathematical probability or axiomatic probability definition. Even if the sample space is countably infinite say S = (e_1, e_2,...) we obtain a probability space assigning to each $e_i \in S$ is a real number p_i, called its probability, such that

$$P_i \geq 0 \text{ and } p_1 + p_2 + \ldots = \sum_{i=1}^{\infty} p_i = 1$$

The probability P(A) of any event A is then the sum of the probabilities of its points.

Consider the sample space S ={ 1,2,... } of the experiment of tossing a coin till a head appears; here n denotes the number of times the coin is tossed. A probability space is obtained by

$$P(1) = 1/2, \quad P(2) = 1/4, \ldots, P(n) = \frac{1}{2^n}, \ldots$$

But the calculation of probability of events regarding an uncountably finite or infinite sample space is not so easy.

Consider a situation of selecting a point at random on a line segment of length 'l'. Here the sample space is uncountably finite and the procedure to find the probability of any event in case of countable sample space is not applicable.

Consider another example. Suppose that two friends have agreed to meet at a certain place between 9 a.m. to 10 a.m. They also agreed that each would wait for a quarter of an hour and, if the other did not arrive, would leave. What is the probability that they meet ?

In the above example also both the sample space and the given event are uncountable and the ordinary procedures of calculation of probability are not applicable. So we need different procedure in such cases.

If the sample space is uncountably finite, we present that sample space by some geometrical measurement, m(S) such as length, area of volume, and in which a point is selected at random. The probability of an event A, i.e. the selected point belongs to A, is then the ratio of m(A) to m(S) is

$P(A) = \frac{length\ of\ A}{length\ of\ S}$ or $P(A) = \frac{area\ of\ A}{area\ of\ S}$ Or $P(A) = \frac{volume\ of\ A}{volume\ of\ S}$

Such probability is known as 'geometrical probability'.

Solved Problems :

1. A bag contains 5 red, 4 white and 3 blue balls. What is the probability that two balls drawn are red and blue ?

Sol : Total number of balls = 5 + 4 + 3 = 12

The number of ways of drawing two balls out of 12 balls = $12^{c}2 = \frac{12\times 11}{2} = 66$ ways

The number of ways of drawing 1 red ball out of 5 red balls = 5 ways.

The number of ways of drawing 1 blue ball out of 3 blue balls = 3 ways.

The number of ways of drawing 1 red ball out of 5 red balls and 1 blue ball out of 3 blue balls = 5 x 3 = 15 ways.

The required probability = 15/66 = 5/22, by using Mathematical probability definition.

2. If the letters of the word 'STATISTICS' are arranged at random to form words, what is the probability that three S's come consecutively ?

Sol: Total no. of letters in the word 'STATISTICS' = 10. Total no. of arrangements of these 10 letters in which 3 are of one kind (viz. S) , 3 are of second kind (viz. T), 2 are of third kind (viz. I), 1 of fourth kind (viz. A) and 1 of fifth kind (viz. C).

$$= \frac{10!}{3!\ 3!\ 2!\ 1!\ 1!}$$

Following are the 8 possible combinations of 3 S's coming consecutively.

i) in the first three places

ii) in the second, third and fourth places

iii) in the eighth, ninth and tenth places

Since in each of the above cases, the total number of arrangements of the remaining 7 letters, viz. TTTIIAC of which 3 are of one kind, 2 of second kind, 1 of third kind and 1 of fourth kind

$$= \frac{7!}{3!\ 2!\ 1!\ 1!}$$

and the required number of favourable cases = $\frac{8 \times 7!}{3!\ 2!\ 1!\ 1!}$

Hence the required probability

$$= \frac{\textit{Favourable Cases}}{\textit{Total No of cases}} = \frac{8 \times 7!}{3!\ 2!\ 1!\ 1!} \Big/ \frac{10!}{3!\ 3!\ 2!\ 1!\ 1!}$$

$$= \frac{8 \times 7! \times 3!}{10!} = \frac{1}{15}$$

3. What is the probability that a leap year selected at random will contain 53 Sundays ?

Sol : In a leap year, there are 366 days of 52 complete weeks and 2 days more. In order that a leap year selected at random should contain 53 Sundays, one of these extra 2 days must be Sunday. But there are 7 different combinations with these two extra 2 days viz. Sunday and Monday, Monday and Tuesday, etc. Out of these 7 possible ways, only in 2 ways we are having an extra Sunday.

∴ Required probability =2/7.

4. Two dice are thrown simultaneously. What is the probability of obtaining a total score of seven?

Sol: Six numbers (1,2,3,4,5,6) are on the six faces of each die. Therefore, there are six possible ways of outcomes on the first die and to each of these ways, there corresponds 6 possible number of outcomes on the second die.

Hence the total number of ways, n = 6 x 6 = 36. Now we will find out of these, how many are favourable to the total score of 7. This may happen only in the following ways (1,6), (6,1), (2,5), (5,2), (3,4) and (4,3) that is, in six ways where first number of each ordered pair denotes the number on the first die and second number denotes the number on the second die.

m = 6.

Hence required probability = $\dfrac{Favourable\ No\ of\ Cases}{Total\ No\ of\ cases}$

$$= \frac{m}{n} = \frac{6}{36} = \frac{1}{6}$$

5. Two digits are selected at random from the digits 1 through 9. If the sum is even find the probability, p that both numbers are odd.

Sol: If both numbers are even or if both numbers are odd, then the sum is even. In this problem, there are 4 even numbers (2,4,6,8) and hence there are $4^{C}2$ ways to choose two even numbers. There are 5 odd numbers (1,3,5,7,9) and hence there are $5^{C}2$ ways to choose two odd numbers. Thus there are $4^{C}2 + 5^{C}2 = 16$ ways to choose two numbers such that their sum is even. Since 10 of these ways occur when both numbers are odd, the required probability,

$$p = \frac{10}{16} = \frac{5}{8}$$

6. Six boys and six girls sit in a row randomly. Find the probability that a) the six girls sit together, b) the boys and girls sit alternately.

Sol: a) Six girls and six boys can sit at random in a row in 12 ways. Consider six girls as one object and the six boys as six different objects. Now these seven objects can be arranged in 7! different ways. But the six girls in the first object can be arranged in 6! ways. Thus the favourable number of cases to the event of sitting all girls together is 7! 6! ways.

Therefore, the required probability = $\dfrac{Favourable\ No\ of\ Cases}{Total\ No\ of\ Cases} = \dfrac{7!\ \ 6!}{12!} = \dfrac{1}{132}$

b) Since the boys and girls can sit alternately in 6! 6! ways if we begin with a boy and similarly they can sit alternately in 6! 6! ways if we begin with a girl. Thus the total number of ways sitting the boys and girls alternately = 2 6! 6!.

$\therefore$ The required probability = $\dfrac{\textit{Favourable No of Cases}}{\textit{Total No of Cases}} = \dfrac{2\ \ 6!\ \ \ 6!}{12!} = \dfrac{1}{462}$

7. Out of (2n+1) tickets consecutively numbered, three are drawn at random. Find the chance that the numbers on them are in A.P.

Sol: Suppose that the smallest number among the three drawn is 1. Then the groups of three numbers in A.P. are (1,2,3), (1,3,5), (1,4,7)....., (1, n+1, 2n+1) and they are n in number.

Similarly, if the smallest number is 2, then the possible groups are (2,3,4), (2,4,6),...(2, n+1, 2n) and their number is n-1. If the lowest number is 3, the groups are (3,4,5), (3,5,7),....(3, n+2, 2n+1) and their number is n-1.

Similarly, it can be seen that if the lowest numbers selected are 4,5,6,..., 2n-2, 2n-1, the number of selections respectively are (n-2), (n-2), (n-3), (n-3),..., 2,2,1,1. Thus the favourable ways for the selected three numbers are in A.P.

$= 2\,(1 + 2 + 3 + \ldots + n\text{-}1) + n$

$= \dfrac{2\,(n-1)\,n}{2} + n = n^2$

Also the total number of ways of selecting three numbers out of (2n+1) numbers

$= \dbinom{2n+1}{3} = \dfrac{(2n+1)\,(2n)\,(2n-1)}{1\,.\,2\,.\,3} = \dfrac{n\,(4n^2-1)}{3}$

Hence the required probability = $\dfrac{\textit{Favourable No of cases}}{\textit{Total No of cases}} = \dfrac{n^2}{n\,(4n^2-1)/3} = \dfrac{3n}{4n^2-1}$

8. If a coin is tossed (m+n) times (m >n), then show that the probability of at least m consecutive heads is $\dfrac{n+2}{2^{m+1}}$.

Sol: Let us denote by H the appearance of head and by T the appearance of tail and let X denote the appearance of head or tail. Now P(H) = P(T) = 1/2 and P(X) = 1.

Suppose the appearance of m consecutive heads starts from the first throw, we have

(H H H....m times) (X X ...... n times)

The chance of this event = (½ . ½ .... m times) = $\dfrac{1}{2^m}$

If the sequence of m consecutive heads starts from the second throw, the first must be a tail and we have

T (H H .... m times) (X X ....(n-1) times)

The chance of this event = ½ (½ . ½ .. m times) = $\frac{1}{2^{m+1}}$

If the sequence of m consecutive heads starts from the (r+1)th throw, the first (r-1) throws may be head or tail but rth throw must be tail and we have

(X X,,,, r-1 times) T (H H .... m times) (X X... (m+n-r) times)

The probability of this event = $\frac{1}{2}\frac{1}{2^m} = \frac{1}{2^{m+1}}$

In the above case, r can take any value from 1,2,... n. Since all the above cases are mutually exclusive, the required probability when r takes 0,1,2,... n

$$= \frac{1}{2^m} + \left(\frac{1}{2^{m+1}} + \frac{1}{2^{m+1}} + \ldots n\ \text{times}\right)$$

$$= \frac{n+2}{2^{m+1}}$$

Hence the result.

9. What is the probability that in a group of N people, at least two of them will have the same birthday ?

Sol: We first find the probability that no two persons have the same birthday and then subtract from 1 to get the required probability. Suppose there are 365 different birthdays possible in a year (excluding leap year).

Any person might have any of these 365 days of the year as birthday. A second person may likewise have any of these 365 birthdays and so on. Hence the total number of ways of N people to have their birthdays = $(365)^N$.

But the number of possible ways for none of these N birthdays to coincide is =

$365 . 364 \ldots (365 - \overline{N-1})$

$$= \frac{(365)!}{(365-N)!}$$

The probability that no two birthdays coincide is

$$\frac{(365)!}{(365-N)!} \Big/ (365)^N$$

Hence the required probability (for at least two people to have the same birthday)

$$= 1 - \frac{(365)!}{(365-N)!\ (365)^N}$$

10. A and B are two independent witnesses (i.e. there is no collusion between them) in a case. The probability that A will speak the truth is x and the probability that B will speak the truth is y. A and B agree in a certain statement. Show that the probability that the statement is true is $xy / (1 - x - y + 2xy)$.

Sol: A and B agree in a certain statement means either both of them speak truth or make false statement. But the probability that they both speak truth is xy and both of them make false statement is $(1 - x)(1 - y)$.

Thus the probability of their agreement in a statement

$$= xy + (1 - x)(1 - y) = 1 - x - y - 2xy$$

Therefore, the conditional probability of their statement is true $= \dfrac{xy}{1 - x - y + 2xy}$

(by using the definition $P(A/B) = \dfrac{P(A \cap B)}{P(B)}$, where A is the event of correct statement and B is the event of common-statement).

11. Two friends have agreed to meet at a certain place between nine and ten O' clock. They also agreed that each would wait for a quarter of an hour and, if the other did not arrive, would leave. What is the probability that they meet ?

Sol: Suppose x is the moment one person arrives at the appointed place, and y is the moment the other arrives.

Let us consider a point with coordinates (x,y) on a place as an outcome of the rendezvous.

Every possible outcome is within the area of square having side corresponds to an hour as shown in the figure.

The outcome is favourable (the two meet) for all points (x,y) such that $|x - y| \leq 1/4$. These points are within the shaded part of the square in the above figure 1.

All the outcomes are exclusive and equally possible, and therefore, the probability of the rendezvous equals the ratio of the shaded area to the area of the square. That is, $m(A) = 7/16$ and $m(S) = 1$.

Hence by geometric probability, the required probability $= \dfrac{m(A)}{m(S)} = \dfrac{7/16}{1} = \dfrac{7}{16}$

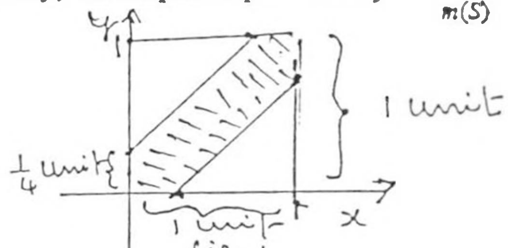


fig 1

Exercises :

1. A factor of 60 is chosen at random. What is the probability that it has factors of both 2 and 5 ?

2. The numbers 3,4 and 5 are placed on three cards and then two cards are chosen at random.

   a) The two cards are placed side-by-side with a decimal point in front. What is the probability that the decimal is more than 3/8 ?

   b) One card is placed over the other to form a fraction. What is the probability that the fraction is less than 1.5 ?

   c) If there are 4 cards with numbers 3,4,5 and 6, then what are the probabilities of the above two cases ?

3. A vertex of a paper isosceles triangle is chosen at random and folded to the midpoint of the opposite side. What is the probability that a trapezoid is formed ?

4. A vertex of a paper square is folded onto another vertex chosen at random. What is the probability that a triangle is formed ?

5. Three randomly chosen vertices of a regular hexagon cut from paper are folded to the centre of the hexagon. What is the probability that an equilateral triangle is formed?

6. A piece of string is cut at random into two pieces. What is the probability that the short piece is less than half the length of the long piece ?

7. A paper square is cut at random into rectangles. What is the probability that larger perimeter is more than 1 ½ times the smaller ?

8. The numbers 2, 3 and 4 are substituted at random for a,b,c in the equation $ax + b = c$.

9. Each coefficient in the equation $ax^2 + bx + c = 0$ is determined by throwing an ordinary die. Find the probability that the equation will have real roots.

10. The numbers 1, 2 and 3 are substituted at random for a,b and c in the quadratic equation $ax^2 + bx + c = 0$.

    a) What is the probability that $ax^2 + bx + c = 0$ can be factored?

    b) What is the probability that $ax^2 + bx + c = 0$ has real roots ?

11. Two faces of a cube are chosen at random. What is the probability that they are in parallel planes ?

12. Three edges of a cube are chosen at random. What is the probability that each edge is perpendicular to the other two ?

13. A point P is chosen at random in the interior of square ABCD. What is the probability that triangle ABP is acute ?

14. Find the probability of the event of the sine of a randomly chosen angle is greater than 0.5.

15. Suppose you ask individuals for their random choices of letters of the alphabet. How many people would you need to ask so that the probability of at least one duplication becomes better than 1 in 2 ?

16. Six boys and six girls sit in a row randomly. Find the probability that i) the six girls sit together, ii) the boys and girls sit alternately ?

17. If the letters of the word 'MATHEMATICS' are arranged at random, what is the probability that there will be exactly 3 letters between H and C ?

18. The sum of two non-negative quantities is equal to 2n. Find the probability that their product is not less than ¾ times their greatest product.

a) What is the probability that the solution is negative ?

b) If c is not 4, what is the probability that the solution is negative ?

19. If A and B are independent events then show that $\bar{A}$ and $\bar{B}$ are also independent events.

20. Cards are dealt one by one from well-shuffled pack of cards until an ace appears,. Find the probability of the event that exactly n cards are dealt before the first ace appears.

21. If four squares are chosen at random on a chess-board, find the chance that they should be in a diagonal line.

22. Prove that if P(A/B) < P(A) then P(B/A) < P(B) ?

23. If n people are seated at a round table, what is the chance that the two named individuals will be next to each other ?

24. A and B are two very weak students of Mathematics and their chances of solving a problem correctly are 1/8 and 1/12 respectively. If the probability of their making common mistake is 1/1001 and they obtain the same answer, find the chance that their answer is correct.

25. A bag contains an unknown number of blue and red balls. If two balls are drawn at random, the probability of drawing two red balls is five times the probability of drawing two blue balls. Furthermore, the probability of drawing one ball of each colour is six times the probability of drawing two blue balls. How many red and blue balls are there in the bag ?

26. A thief has a bunch of n keys, exactly one can open a lock. If the thief tries to open the lock by trying the keys at random, what is the probability that he requires exactly k attempts, if he rejects the keys already tried ? Find the probability of the same event when he does not reject the keys already tried.

27. A problem in Mathematics is given to three students and their chances of solving it are ½, 1/3 and ¼. What is the probability that the problem will be solved ?

28. A bag A contains 3 white balls and 2 black balls and other bag B contains 2 white and 4 black balls. A bag and a ball out of it are picked at random. What is the probability that the ball is white ?

29. Cards are drawn one-by-one at random from a well-shuffled pack of 52 cards until 2 aces are obtained for the first time. If N is the number of cards required to be drawn, then show that

$$p(N = n) = \frac{(n-1)\ (52-n)\ (51-n)}{50.59.\ 17.\ 13}$$

Where $2 \le n \le 50$.

30. A,B, C are events such that

$P(A) = 0.3$, $P(B) = 0.4$, $P(C) = 0.8$, $P(A \cap B) = 0.08$, $P(A \cap C) = 0.28$,

$P(A \cap B \cap C) = 0.09$

If $P(A \cup B \cup C) \ge 0.75$, then show that $P(B \cap C)$ lies in the interval (0.23, 0.48).

31. A man takes a step forward with probability 0.4 and backwards with probability 0.6. Find the probability that at the end of eleven steps, he is one step away from the starting point.

32. Huyghens Problem. A and B throw alternately a pair of dice in that order. A wins if he scores 6 points before B gets 7 points, in which case B wins. If A starts the game, what is his probability of winning ?

33. A Doctor goes to work following one of three routes A, B,C. His choice of route is independent of the weather. If it rains, the probabilities of arriving late, following A, B, C are 0.06, 0.15, 0.12 respectively. The corresponding probabilities, if it does not rain, are 0.05, 0.10, 0.15.

a) Given that on a sunny day he arrives late, what is the probability that he took route C ? Assume that, on average, one in every four days is rainy.

b) Given that on a day he arrives late, what is the probability that it is a rainy day.

34. Bonferroni's Inequality. Given n(>1) events $A_1, A_2, \ldots A_n$ show that

$$\sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j) \le P\left(\bigcup_{i=1}^{n} A_i\right) \le \sum_{i=1}^{n} P(A_i)$$

35. Show that for any n events $A_1, A_2, \ldots, A_n$

i) $P\left(\bigcap_{i=1}^{n} A_i\right) \ge 1 - \sum_{i=1}^{n} P(\overline{A_i})$

ii) $P\left(\bigcap_{i=1}^{n} A_i\right) \ge \sum_{i=1}^{n} P(A_i) - (n-1)$

36. If A and B are mutually exclusive and P(AUB) ≠ 0, then prove that

$$P(A/A \cup B) = \frac{P(A)}{P(A) + P(B)}$$

37. If 2n boys are divided into two equal groups, find the probability that the two tallest boys will be a) in different subgroups, and b) in the same subgroup.

38. A small boy is playing with a set of 10 coloured cubes and 3 empty boxes. If he puts the 10 cubes into the 3 boxes at random, what is the probability that he puts 3 cubes in one box, 3 in another box, and 4 in the third box ?

39. The sample space consists of the integers from 1 to 2n, which are assigned probabilities to their logarithms. A) Find the probabilities, b) Show that the conditional probability of the integer 2, given that an even integer occurs is

$$\frac{\log 2}{n \log 2 + \log n!}$$

40.a) Each of n boxes contains four white and six black balls, while another box contains five white and five black balls. A box is chosen at random from the (n+1) boxes, and two balls are drawn from it, both being black. The probability that five white and three black balls remain in the chosen box is 1/7. Find n.

40b). A point is selected at random inside a circle. Find the probability p that the point is closer to the centre of the circle than to its circumference.

41. What is the probability that two numbers chosen at random will be prime to each other?

42. In throwing n dice at a time, what is the probability of having each of the points 1,2,3,4,5,6 appears at least once ?

43. A bag contains 50 tickets numbered 1,2,3,..., 50 of which five are drawn at random and arranged in ascending order of magnitude ($x_1 < x_2 < x_3 < x_4 < x_5$), what is the probability that $x_3 = 30$?

44. Of the three independent events, the probability that the first only to happen is ¼, the probability that the second only to happen is 1/8 and the third only to happen is 1/12. Obtain the unconditional probabilities of the three events.

45. What is the least number of persons required if the probability exceeds ½ that two or more of them have the same birthday (year of birth need not match) ?

46. If m things are distributed among 'a' men and 'b' women, then show that the chance that the number of things received by men is

$$\frac{1}{2}\,\frac{(b+a)^m - (b-a)^m}{(b+a)^m}$$

47. A pair of dice is rolled until either 5 or a 7 appears. Find the probability that a 5 occurs first.

48. In a certain standard tests I and II, it has been found that 5% and 10% respectively of 10^{th} grade students earn grade A. Comment on the statement that the probability is $\frac{5}{100}\;\frac{10}{100} = \frac{1}{200}$ that a 10^{th} grade student chosen at random will earn grade A on both tests.

49. A bag contains three coins, one of which is coined with two heads while the other two coins are fair. A coin is chosen at random from the bag and tossed four times in succession. If head turns up each time, what is the probability that this is the two headed coin ?

50. A man stands in a certain position (which we may call the origin) and tosses a fair coin. If a head appears he moves one unit of length to the left. If a tail appears, he moves one unit to the right. After 10 tosses of the coin, what are his possib. positions and what are the probabilities ?

51. There are 12 compartments in a train going from Madras to Bangalore. Five friends travel by the train for some reasons could not meet each other at Madras statio. before getting aboard. What is the probability that the five friends will be _ different compartments ?

52. The numbers 1,2,3,4,5 are written on five cards. Three cards are drawn in succession and at random from the deck, the resulting digits are written from left to righ. What is the probability that the resulting three digits number will be even ?

53. Suppose n dice are thrown at a time. What is the probability of getting a sum 'S' of points on the dice ?

54. A certain mathematician always carries two match boxes, each time he wants a match stick he selects a box at random. Inevitably, a moment comes when he finds a box empty . Find the probability that the movement the first box is empty, the second contains exactly r match sticks (assume that each box contain N match-stick. initially).

55. There are 3 cards identical in size. The first card is red both sides, the second one is black both sides and the third one red one side and black other side. The cards are mixed up and placed flat on a table. One is picked at random and its uppe. (visible) side was red. What is the probability that the other side is black ?

56. N different objects 1,2,...., n are distributed at random in n places marked 1,2,... n. Find the probability that none of the objects occupies the place corresponding to its number.

Answers :

1. ½
2. A) 2/3 b) 5/6 c) 3/4, 3/4
3. 1/3
4. 1/3
5. 1/10
6. 2/3
7. 2/5
8. a) ½ b) 3/4
9. 43/216
10. a) 1/3 b) 1/3
11. 1/5
12. 2/55
13. $1 - \pi/8 = 0.6073$
14. 2/3
15. 7

16. i) $\frac{7! \quad 6!}{12\,!}$ ii) $\frac{2 \quad (6!)^2}{12!}$

17. 7/55

18. ½

20. $\frac{4\ (51-n)\ (50-n)\ (49-n)}{52.51.50.49}$

21. $\frac{91}{158844}$

23. $\frac{2}{n-1}$

24. 13/24

25. Red = 6, Blue = 3

26. 1/n, 1/n $\left(1 - \frac{1}{n}\right)^{k-1}$

27. 3/4

28. 7/15

31. $(0.4)^5\,(0.6)^5$

32. 30/61

33. a) 0.5 b) 41/131

37. a) $\frac{n}{2n - 1}$ b) $\frac{n - 1}{4n - 2}$

38. $\frac{3 \quad 10!}{3! \quad 3! \quad 4! \quad 3^{10}}$

39. a) K log 2i b) (log 2i) (n log 2 + log n!)

40. a) 4 b) 1/4

41. $\pi \left(1 - \frac{1}{r^2}\right) = \frac{6}{\pi^2}$

42. $1 - n\left(\frac{5}{6}\right)^n + \binom{n}{2}\left(\frac{4}{6}\right)^n - \binom{n}{3}\left(\frac{3}{6}\right)^n + \binom{n}{4}\left(n\frac{2}{6}\right)^n - \binom{n}{5}\left(\frac{1}{6}\right)^n$

43. $\dfrac{\binom{29}{2}\binom{20}{2}}{\binom{50}{2}}$

44. ½, 1/3, 1/4

45. 23

47. 2/5

49. 8/9

50.

Distance from origin	-10	-8	-6	-4	-2	0	2	4	6	8	10
Prob	$\left(\frac{1}{2}\right)^{10}$	$\binom{10}{1}\left(\frac{1}{2}\right)^{10}$	$\binom{10}{2}\left(\frac{1}{2}\right)^{10}$	$\binom{10}{3}\left(\frac{1}{2}\right)^{10}$	$\binom{10}{4}\left(\frac{1}{2}\right)^{10}$	$\binom{10}{5}\left(\frac{1}{2}\right)^{10}$	$\binom{10}{6}\left(\frac{1}{2}\right)^{10}$	$\binom{10}{7}\left(\frac{1}{2}\right)^{10}$	$\binom{10}{8}\left(\frac{1}{2}\right)^{10}$	$\binom{10}{9}\left(\frac{1}{2}\right)^{10}$	$\left(\frac{1}{2}\right)^{10}$

51. 55/144

52. 1/5

53. $(-1)^k \binom{n}{k} \binom{s-6k-1}{n-1} | 6^n$

54. $\dfrac{\binom{2n-1}{n}}{2^{2n-1}}$

55. ½

56. $\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} = \ldots + (-1)^n \frac{1}{n!}$.

RANDOM VARIABLES AND PROBABILITY DISTRIBUTIONS

In the earlier pages, the idea of a function, subject to certain postulates, which assigned weights called probabilities, to the points of the sample space, was introduced. We then had a probability function which allowed us to compute probabilities for events. Now we deal with the concept of Random Variable.

Random Variable :

Scientific theories on models are our way of depicting and explaining how observations come about. Such theories are simplified statements containing essential features and make for easier comprehension and communication. In statistics, we use a mathematical approach since we quantify our observations. Random variable is the result of such mathematical approach dealing with the probabilities assigning to different events of a random experiment. The set of possible outcomes for a random experiment can be described with the help of a real-valued variable by assigning a single value of this variable to each outcome. For a two coin tossing experiment, the outcomes are two tails, a tail and a head , a head and a tail, or two heads. The sample space can be represented as (TT, TH, HT, HH). Here we express the outcomes by using the number of heads and so assigning the values (0,1,1,2) respectively to those outcomes. Therefore, the outcomes of this experiment can be denoted by the different values of the real-valued variable viz. 0,1,2.

Any function or association that assigns a unique, real value to each sample point is called a chance or random variable. The assigned values are the values of the random variable.

Random variables are symbolised by capital letters, most often X, and their values by lower case letters. The outcome of a random experiment determines a point i.e., the sample space, called the domain of the random variable, and the function transform each sample point to one of a set of real numbers. This set of real numbers is called the range of the random variable. If the sample space is discrete, then the outcomes will be denoted by certain discrete values. The random variable associated with a discrete sample space is known as discrete random variable. Similarly, the random variable associated with continous sample space is known as continuous random variable.

Probability Function :

The association of probabilities with the various values of a discrete random variable is done by reference to the probabilities in the sample space and through a system of relationships or a function is called a probability set function or, simply, a probability function.

Let the discrete random variable X assume the values $x_1, x_2 \ldots x_n$ Then the system of relations can be written as

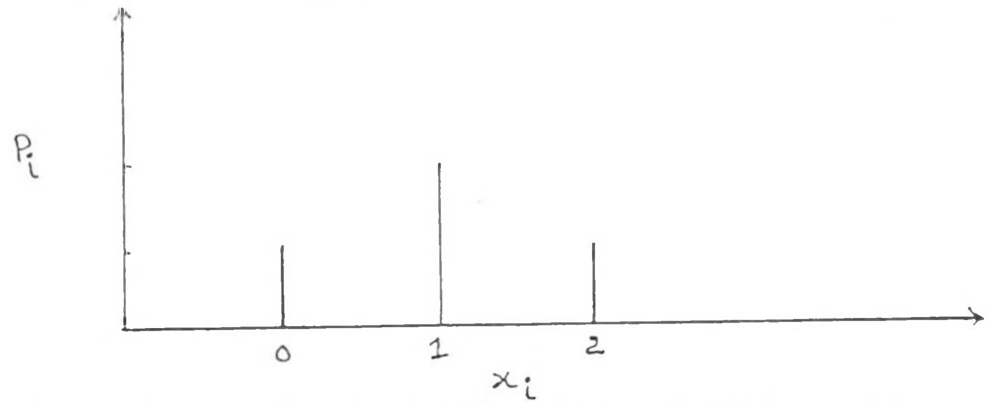
$$P(X = x_i) = p_i$$

This is read as 'the probability that the random variable X takes the value of x_i is p_i'. The set of ordered pairs (x_i, p_i) constitutes a probability function with numerical values to be provided for the x_i and p_i's such that $p_i \geq 0$ for all i and $\sum_i p_i = 1$.

A discrete probability function is a set of ordered pairs of values of a random variable and the corresponding probabilities.

For a two coin experiment, X takes the values 0,1,2 with the probabilities ¼, ½, ¼ respectively.

Sometimes probability function can be represented by a graph or a mathematical function. In case of above example, the X values and the corresponding probabilities can be represented with the help of the following graph.



Suppose X assume the values 1 and 0 with the probabilities p and 1-p respectively. This information can be given with the help of the following function p(x) defined by

$$P(x) = p^x(1-p)^{1-x},\ x = 0, 1.$$

This type of function which gives the probabilities of the different values assumed by a random variable is known as probability mass function or simply probability function. Therefore, a function p(x) is said to be a probability function of random variable or a distribution if

i) $p(x) \geq 0$ for all x.

$$\sum_x p(x) = 1$$

where p(x) denotes the probability of the events that the random variable X assumes the value x.

Distribution Function :

The law of probability distribution of a random variable is the rule used to find the probability of the event related to a random variables. For instance, the probability that the variable assumes a certain value or falls in a certain interval. The general form of the distribution law is distribution function, which is the probability that a random variable X assumes a value smaller than a given x i.e. $F(x) = P(X \leq x)$.

The distribution function F(x) for any random variable possesses the following properties :

i) $F(-\infty) = 0$

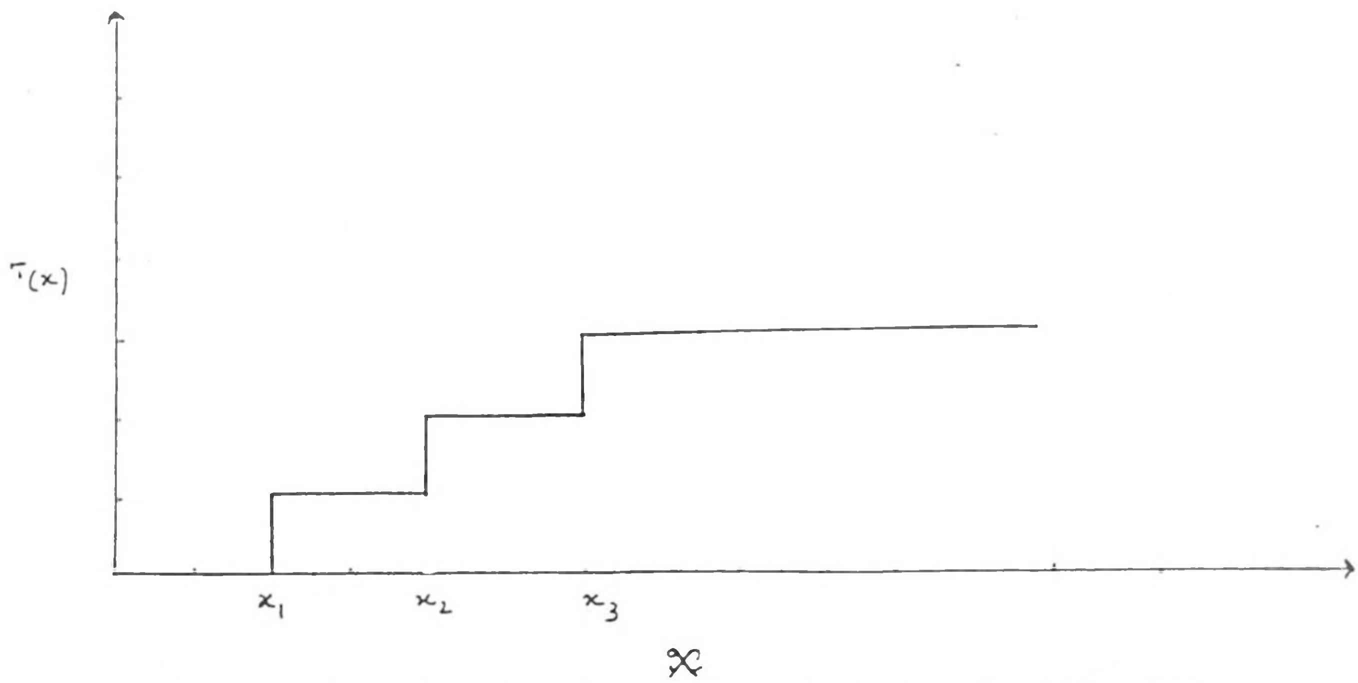
ii) $F(+\infty) = 1$

iii) F(x) does not decrease with an increase in x.

In the case of discrete random variable

$$F(x_k) = \sum_{i=1}^{k} p(x_i)$$

Where $x_1, x_2, \ldots, x_k \ldots$ are the values of the random variable. The graph of F(x) in discrete random variable case is generally as shown below :



It is seen from the above figure that the graph of F(x) is a 'step function' having jump $p(x_i)$ at $x = x_i$ and is constant between each pair of values of x. It can also be proved that

$$F(x_i) - F(x_{i-1}) = p(x_i)$$

Therefore, distribution function can also be used to indicate the distribution of the random variable instead of probability function.

Example :

A student is to match three historical events (Mahatma Gandhi's birth year, India's freedom, and first Word War) with three years (1947, 1914, 1869). If he guesses, with no knowledge of the correct answers, what is the probability distribution of the number of answers he gets correctly ?

Solution : Here the number of correct answers is the random variable, say X. Therefore, X assumes the values 0,1,2,3 because there are three events to match with only three years. Suppose the events are E_1,E_2, E_3 and the corresponding correct years are Y_1, Y_2, Y_3. Student gets the correct answers when he/she matches E_1 to Y_1, E_2 to Y_2 and E_3 to Y_3.

All matchings are wrong only when he/she matches E_1 to Y_2, E_2 to Y_3, E_3 to Y_1 or E_1 to Y_3, E_2 to Y_1, E_3 to Y_2. But the total possible matchings are 6. Therefore, the probability of all matchings to go wrong is 2/6 =1/3. That is, the probability that X to take the value '0' is 1/3.

Similarly X assumes the value '1' with probability 3/6 (= ½) the value '2' with 0 probability and the value '3' with 1/6 probability.

So the probability distribution of the correct answers in the given matching is

No of correct answers (x)	0	1	2	3
Probability	1/3	½	0	1/6

Example : Suppose a number is selected at random from the integers 10 through 30. Let X be the number of its divisors. Construct the probability function of X. What is the probability that there will be 4 or more divisors ?

Solution : X is the number of divisors of randomly selected number from the integers 10 through 30. Therefore, X is a random variable. The possible values tht X assumes are :

2, 3,4, 5,6 depending upon the selected number. For example, if the selected number is either 1,2,3,5,7,11,13,17,19 then X takes the value 2. Similarly when the selected number is 4,6,8,10,14,15 X takes 4. Therefore, the different values of X and the number of their appearances we get the following :

X values	1	2	3	4	5	6
No of appearances out of 20	1	8	3	4	1	3

Now the required probability distribution is

x	1	2	3	4	5	6
p(x)	1/20	8/20	3/20	4/20	1/20	3/20

The probability of X to take 4 or more

$= P(x = 4 \text{ or } 5 \text{ or } 6) = P(x = 4) + P(x = 5) + P(x = 6)$

$$= \frac{4}{20} + \frac{1}{20} + \frac{3}{20} = \frac{8}{20} = \frac{2}{5}$$

Mean, Variance, Standard Deviation of the Random Variable.

Let X be a random variable with probability function as follows :

x	x_1	x_2	..	..	..	x_n
p(x)	$p(x_1)$	$p(x_2)$	..	..	..	$p(x_n)$

The mean of X is defined as

$$x_1\ p(x_1) + x_2\ p(x_2) + \ldots + x_n\ (n_n)$$

or

$$\sum_{i=1}^{n} x_i\ p(x_i)$$

This is also known as mean of the distribution and generally denoted by μ.

The variance of X is defined as

$$\sum_{i=1}^{n} x_i^2\ p(x_i) - \left[\sum_{i=1}^{n} x_i\ p(x_i) \right]^2$$

$$\sum_{i=1}^{n} x_i^2 \; p(x_i) - \mu^2$$

where μ is the mean of X.

The variance is generally denoted by σ^2.

The standard deviation is the positive square root of variance and is denoted by σ.

Example : A single 6-sided die is tossed. Find the mean and variance of the number of points on the top face.

Solution : Let X represents the number of points on the top face. The probability function of X is

x	1	2	3	4	5	6
p(x)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

The mean, μ is given by

$$\sum_{i=1}^{n} x_i \; p(x_i) = x_1 \; p(x_1) + x_2 \; p(x_2) \ldots + x_n \; p(x_n)$$

$$\text{Here } \mu = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} \cdot 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6}$$

$$= \frac{1}{6}(1 + 2 + 3 + 4 + 5 + 6)$$

$$= \frac{1}{6} \; \frac{6 \times 7}{2} = \frac{7}{2}$$

Variance, σ^2 is given by

$\sum_{i=1}^{n} x_i^2 \; p(x_i) - \mu^2$ where μ is mean.

Here

$$\sum_{i=1}^{n} x_i^2 \; p(x_i) = 1^2 \cdot \frac{1}{6} + 2^2 \cdot \frac{1}{6} + 3^2 \cdot \frac{1}{6} + 4^2 \cdot \frac{1}{6} + 5^2 \cdot \frac{1}{6} + 6^2 \cdot \frac{1}{6}$$

$$= \frac{1}{6}\left[1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2\right]$$

$$= \frac{1}{6}\ \frac{6\times7\ \ (2\times6\times1)}{6} = \frac{91}{6}$$

Variance $\Rightarrow \sigma^2 = \frac{91}{6} - \left(\frac{7}{2}\right)^2 \qquad (\because\ \mu = \frac{7}{2})$

$$= \frac{91}{6} - \frac{49}{4} = \frac{35}{12}$$

Exercises :

1. One cube with faces numbers 1,2,3,4,5 and 6 is tossed twice, and the recorded outcome consists of the ordered pair of numbers on the hidden faces at the first and second tosses.

   a) Let the random variable X takes on the value 0 if the sum of the numbers in the ordered pair is even and 1 if odd. What is the probability function for this random variable ?

   b) Let the random variable X takes on the value 2 if both numbers in the ordered pair are even, 1 if exactly one is even, and 0 if neither is even. What is the probability distribution of this random variable ?

   c) Let the random variable X be the number of divisors in the sum of the two faces. What is the probability function of X ?

2. Of six balls in a bag, two are known to be black. The balls are drawn one at a time from the bag and observed until both black balls are drawn. If X is the number of trials (draws) required to get the two black balls. Obtain the probability distribution of X.

3. Suppose that the random variable X has possible values 1,2,3,... and $P(x = j) = \frac{1}{2^j}$, j = 1,2,...

   i) compute P(x is even), ii) compute P(x is divisible by 3).

4. The probability mass function of a random variable X is zero except at the points x= 0,1,2,. At these points has the values $p(0) = 3c^3$, $p(1) = 4c - 10c^2$ and $p(2) = 5c - 1$ for some $c > 0$.

i) Determine the value of c.

ii) Compute P (1 < X ≤ 2).

iii) Describe the distribution function and draw its graph.

iv) Find the largest x such that F(x) < ½.

5. Let X denote the profits that a man makes in business. He may earn Rs.3000 with probability 0.5, he may lose Rs.5000 with probability 0.3 and he may neither earn nor lose with probability 0.2. Calculate his average profits.

6. A man wins a rupee for head and loses a rupee for tail when a coin is tossed. Suppose that he tosses once and quits if he wins but tries once more if he loses on the first toss. What are his expected winnings ?

7. Three boxes contain respectively 3 red and 2 black balls, 5 red and 6 black balls and 2 red and 4 black balls. One ball is drawn from each box. Find the average number of black balls drawn.

8. If the random variable, X takes the values 1,2,....n respectively with probabilities $\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}$ find the mean and variance of X.

Answers :

1. a)

X	Prob
0	½
1	½

b)

X	Prob
0	¼
1	½
2	¼

c)

X	Prob
2	15/36
3	12/36
4	8/36
6	1/36

2.

X	Prob
2	1/15
3	2/15
4	3/15
5	4/15
6	5/15

3. i) 1/3 ii) 1/7

4. i) 1/3 ii) 2/3 iii) 1

5. 0

6. 0

7. $\frac{266}{165}$

8. Mean = $\frac{(n+1)}{2}$, Variance = $\frac{n^2-1}{12}$

DISCRETE DISTRIBUTIONS

In the previous pages, we discussed about 'random variable', 'probability function', etc. Here we discuss some theoretical discrete distributions in which variables are distributed according to some definite law which can be expressed mathematically.

Bernoulli Distribution : Suppose you want to study the probability of different events corresponding to tossing of a single coin experiment. The two possible events are getting a head or getting a tail. Define a random variable x assuming the values 1 and 0 corresponding to these two events viz. Head and tail respectively. If the probability of getting a head in tossing that coin is 'p' then the probability that the random variable to take '1' is p and the probability that the random variable to take '0' is 1-p. Therefore, the distribution of the random variable X becomes

X	Prob
1	p
0	1-p

Any experiment where there are only two possible outcomes viz. Success and failure is called as Bernoulli experiment. A single trial of a Bernoulli experiment is known as Bernoulli trial.

Corresponding to any Bernoulli experiment, it is possible to define a random variable X as given above.

A random variable X which takes two values 0 and 1, with probability q(=1-p) and p respectively is called Bernoulli variate and is said to have a Bernoulli distribution.

Binomial Distribution :

Let a Bernoulli experiment be performed repeatedly and let the occurrence of an event in any trial be called a success and its non-occurrence a failure. Consider a series of n independent Bernoulli trials (n being finite), in which the probability 'p' of success in any trial is constant for each trial. Then q = 1-p is the probability of failure inany trial. Let the random variable X be the number of successes in these trials.

The probability of x successes and consequently (n-x) failures in n independent trials, in a specified order (say) SS FF SSS .... FSFF (where S represents success and F failure) is given by compound probability as given below :

P (SSFF, ... FSFF) = P(S) P(S) P(F) P(F) ....P(F) P(S) P(F) P(F)
= p.p.qq....q p q q
= pp ....p qq ....q (x p's and (n-x) q's)
$= p^x q^{n-x}$

But x successes in n trials can occur in $\binom{n}{x}$ ways and the probability for each of these ways is $p^x q^{n-x}$. Hence the probability of x successes in n trials in any order whatsoever is given by the addition of individual probabilities and is given by $\binom{n}{x} p^x q^{n-x}$. The number of successes in n trials will be either 0 or 1 or 2 ... or n in any experiment.

$p(x) = P(X = x) = \binom{n}{x} p^x q^{n-x}$

Is true for all x = 0,1,2,...n.

This function $p(x) = \binom{n}{x} p^x q^{n-x}$, x = 0, 1,...., n is called the probability mass function of the Binomial distribution, for the obvious reason that the probabilities of 0,1,2,...n successes, viz. q^n, $\binom{n}{1} q^{n-1} p$, $\binom{n}{2} q^{n-2} p^2$,...., p^n Are the successive terms of the binomial expansion $(q + p)^n$.

A random variable X is said to follow binomial distribution if its probability mass function is given by

$P(X = x) = p(x) = \binom{n}{x} p^x q^{n-x}$, x = 0,1,2,.....n; q = 1 - p.

The values n and p of this distribution are known as the parameters of the distribution.

Mean and Variance of Binomial Distribution

We know, mean of any discrete distribution

$= \sum_r r\, p(r)$

where p(r) is the probability that the random variable X to take the value r. In case of binomial distribution x takes the values r = 0,1,2,...., n and $p_r = \binom{n}{r} p^r q^{n-r}$ where n and p are the parameters of the binomial distribution.

$$\therefore \text{Mean} = \sum_{r=0}^{n} r \binom{n}{r} p^r q^{n-r}$$

$$= \sum_{r=0}^{n} r \frac{n!}{r!\,(n-r)!} p^r q^{n-r}$$

$$= np \sum_{r=1}^{n} \frac{(n-1)!}{(r-1)!\,(n-r)!} p^{r-1} q^{n-r}$$

$= n p \left[q^{n-1} + (n-1) C_1 q^{n-2} + \ldots + p^{n-1} \right]$

$= n p (q+p)^{n-1}$

$= np \quad (\because p + q = 1)$

Also we know Variance $= \sum_r r^2 p(r) - \left[\sum_r r \, p(r) \right]^2$

$= \sum_r r^2 p(r) - (Mean)^2$

In case of binomial distribution

Variance $= \sum_{r=0}^{n} r^2 \binom{n}{r} p^r q^{n-r} - (n p)^2$

$= \sum_{r=0}^{n} r^2 \frac{n!}{r! \, (n-r)!} p^r q^{n-r} - (np)^2 \quad (\because Mean = np)$

$= \sum_{r=0}^{n} [r(r-1) + r] \frac{n!}{r! \, (n-r)!} p^r q^{n-r} - (n p)^2$

$= \sum_{r=0}^{n} r(r-1) \frac{n!}{r! \, (n-r)!} p^r q^{n-r} + \sum_{r=0}^{n} r \frac{n!}{r! \, (n-r)!} p^r q^{n-r} - (n p)^2$

$= \sum_{r=2}^{n} \frac{n!}{(r-2)! \, (n-r)!} p^r q^{n-r} + np - (n p)^2$

$(\because \sum r \frac{n!}{r! \, (n-r)!} p^r q^{n-r} = np \text{ provedabove})$

$= n(n-1) p^2 \left[\sum_{r=2}^{n} \frac{(n-2)!}{(r-2)! \, (n-r)!} p^{r-2} q^{n-r} \right] + np - (n p)^2$

$= n(n-1) p^2 \left[q^{n-2} + (n-2)C_1 + (n-2) C_2 + \ldots p^{n-2} \right] + np - (np)^2$

$= n(n-1) p^2 (q+p)^{n-2} + np - (np)^2$

$= n(n-1) p^2 + np - (np)^2$

$= np \, [(n-1) p + 1 - np]$

$= n p \, [np - p + 1 - np]$

$= np \, [1 - p] = n p q$

So, Mean = np

Variance = npq

Standard Deviation = $\sqrt{Variance} = \sqrt{npq}$

Example : The mean and variance of binomial distribution with parameters n and p are 16 and 8. Find i) P (x = 0), ii) P (x ≥ 2).

Solution : We know mean = np and variance = npq.

∴ np = 16 and npq = 8

Solving for n and p we get n = 32 and p = ½

Now $P(X = 0) = \binom{n}{0} p^0 q^{n-0} = q^n$

(Because $p(x = r) = \binom{n}{r} p^r q^{n-r}$

$\therefore P(x = 0) = (1-p)^n = \left(1 - \frac{1}{2}\right)^{32} = \left(\frac{1}{2}\right)^{32}$

$(\because n = 32, \; q = 1-p = 1 - \frac{1}{2})$

ii) $P(x \geq 2) = 1 - P(x < 2) = 1 - [P(x=0) + P(x=1)] = 1 - P(x=0) - P(x=1)$

But P (x = 0) = $\left(\frac{1}{2}\right)^{32}$ (As obtained above)

and $P(x=1) = \binom{n}{1} p^1 q^{n-1} = 32\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{32-1}$

$\therefore P(x \geq 2) = 1 - \left(\frac{1}{2}\right)^{32} - 32\left(\frac{1}{2}\right)^{32} = 1 - 33\left(\frac{1}{2}\right)^{32}$

Example : A perfect cube is thrown a large number of items in sets of 8. The occurrence of a 2 or 4 is called a success. In what proportion of the sets would you expect 3 successes.

Solution : In this problem we have to find the probability of getting 3 successes out of 8 trials. Tossing of a single cube is our trial. The probability of success, p is getting either 2 or 4. The number of cubes in the set is the number of trials. If we define x as the number of successes in 8 trials, then x is distributed as a binomial variate with parameters 8 and p where p is the probability of success.

The probability of getting either 2 or 4 in tossing of a perfect cube = 2/6 = 1/3.

∴ p = 1/3

Hence $P(x=r) = \binom{n}{r} p^r q^{n-r}$

and $P(x=3) = \binom{n}{3} p^3 q^{n-3}$ (∵ x is a binomial variate)

$= \binom{8}{3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^{8-3}$ $\quad$ ($\because$ n = 8, p = 1/3, q = 1-p)

$= 8\times 7\left(\frac{1}{3}\right)^8 2^5$

$= 56\times 32\left(\frac{1}{3}\right)^8$

= 0.2731

$\therefore$ The proportion of sets in which we expect 3 successes = 27.31 %.

Example :The probability of a man hitting a target is ¼.

i) If he fires 7 times, what is the probability of his hitting the target at least twice ?

ii) How many times must he fire so that the probability of his hitting the target at least once is greater than 2/3 ?

Solutions :

i) Consider 'firing once' as a Bernoulli trial. Firing 7 times is the Binomial experiment with 7 independent Bernoulli trials. If X is the number of hits in 7 trials, then the required probability of hitting the target at least twice = P (X ≥ 2).

We know,

P(X ≥ 2) = 1 – p (X < 2)

= 1- P(X = 0) – P (X = 1)

and $P(X = x) = \binom{n}{x} p^x q^{n-x}$ where n = 7, p =1/4, and q = 1 – p =3/4.

$P(X = 0) = (3/4)^7$

$P(X = 1) = \binom{7}{1}\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^6 = 7\,\frac{3^6}{4^7}.$

The required probability

$= 1 - \left(\frac{3}{4}\right)^7 - 7\,\frac{3^6}{4^7} = \frac{4547}{8192}.$

ii) p = ¼, q = ¾

We want to find n such that P(X ≥ 1) > 2/3

Or 1 – P (X < 1)> 2/3

Or 1 –P (X = 0) > 2/3

Or $1 - q^n > 2/3$ when $q = 3/4$

$\Rightarrow (3/4)^n < 1/3$

$\Rightarrow n = 4.$

POISSON DISTRIBUTION

There are many situations where we must count the number of individuals possessing a certain characteristic yet have difficulty in defining the basic experiment. In turn, it becomes difficult to say what is the probability of the occurrence of a single event. For example i) number of telephone calls received at a particular telephone exchange, ii) emission of radioactive particles, iii) number of printing mistakes in a book. In all these situations, it is easy to count the events, but what are the non events.

In situations like those mentioned above, we customarily resort to specifying a unit size or a time interval in which to observe the events etc. We find then that we are observing events that fluctuate around some mean value that might be defined in terms of some sort of underlying binomial parameters p and n as np, a product never separable into its component parts and simply give the mean value. Therefore, in such situations, we assume that for a short enough unit of time or space, the probability of an event occurring is proportional to the length of time or size of the space. We also assume that for non overlapping units, the results in one unit are of no value in predicting when or where another event will occur (independently). The above assumptions underlie the probability function given by

$$P(X = x) = \frac{e^{-\lambda} \lambda^x}{x!}, \quad x = 0,1,2,3,\ldots$$

where λ is the average number of times an event occurs in a unit interval and is called the parameter of a Poisson distribution. Poisson Distribution as a limiting case of Binomial Distribution.

The above mentioned Poisson distribution can be viewed as a limiting case of the binomial distribution under the following conditions.

i) n, the number of trials in the binomial experiment is infinitely large i.e. $n \to \infty$.
ii) p, the probability of success in each trial is indefinitely small, i.e. $p \to 0$.
iii) $np = \lambda$ is finite so that $p = \frac{\lambda}{n}$, $q = 1 - \frac{\lambda}{n}$.

We know , if X is a binomial variate with parameters n and p then

$$P(X=x) = p(x) = \binom{n}{x} p^x q^{n-x}, \; x = 0, 1, 2, \ldots n$$

where $n \to \infty$ and $p \to 0$.

Therefore, this probability

$$= \lim_{\substack{n \to \infty \\ p \to 0}} \binom{n}{x} p^{x} q^{n-x}$$

$$= \lim_{n \to \infty} \binom{n}{x} \left(\frac{\lambda}{n}\right)^{x} \left(1 - \frac{\lambda}{n}\right)^{n-x} \qquad \left(\because\ p = \frac{\lambda}{n},\ q = 1 - \frac{\lambda}{n}\right)$$

$$= \lim_{n \to \infty} \frac{n!}{x!\ (n-x)!} \left(\frac{\lambda}{n}\right)^{x} \left(1 - \frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{\lambda^{x}}{x!} \lim_{n \to \infty} \frac{n\ .\ (n-1)\ \ldots\ (n-\overline{x-1})}{n^{x}} \frac{\left(1 - \frac{\lambda}{n}\right)^{n}}{\left(1 - \frac{\lambda}{n}\right)^{x}}$$

$$= \frac{\lambda^{x}}{x!} \lim_{n \to \infty} \left[1 . \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \ldots \left(1 - \frac{x-1}{n}\right) \left(1 - \frac{\lambda}{n}\right)^{x}\right]$$

$$\lim_{n \to \infty} \frac{1}{\left(1 - \frac{\lambda}{n}\right)^{x}}$$

$$= \frac{\lambda^{x}}{x!} \lim_{n \to \infty} \left(1 - \frac{\lambda}{n}\right)^{n} \qquad \left(\because\ \lim_{n \to \infty} \frac{1}{\left(1 - \frac{\lambda}{n}\right)^{x}} = 1\right)$$

$$= \frac{\lambda^{x}}{x!} \lim_{m \to \infty} \left(1 + \frac{1}{m}\right)^{-m\lambda}$$

$$= \frac{\lambda^{x}}{x!} \lim_{m \to \infty} \left[\left(1 + \frac{1}{m}\right)^{m}\right]^{-\lambda}$$

$$= \frac{\lambda^{x}}{x!} e^{-\lambda}$$

$$\therefore\ P(X = x) = \frac{e^{-\lambda} \lambda^{x}}{x!}, \quad x = 0, 1, 2, \ldots$$

This function is known as the Probability function of the Poisson distribution and λ is the parameter of the distribution.

Mean and Variance of the Poisson Distribution :

$$\text{Mean} = \sum_{x=1}^{\infty} x\ P(x_i)$$

$$= \sum_{x=1}^{\infty} x \frac{e^{-\lambda} \lambda^{x}}{x_i!} \qquad \left(\because P(x) = \frac{e^{-\lambda} \lambda^{x}}{x!} \text{ In case of Poisson distribution}\right)$$

$$= \sum_{x=0}^{\infty} x \frac{e^{-\lambda} \lambda^x}{x!}$$ (∵ the values of Poisson variate are 0,1,2,...)

$$= \lambda e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^{x-1}}{x!}$$

$$= \lambda e^{-\lambda} \left(1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \ldots \right)$$

$$= \lambda e^{-\lambda} e^{\lambda}$$

$$= \lambda$$

$$\text{Variance} = \sum_{i=1}^{\infty} x_i^2 \, p(x_i) - \left[\sum_{i=1}^{\infty} x_i \, p(x_i) \right]^2$$

Continued in last page.

Exercises :

1. A random variable X has a binomial distribution with parameters n = 4 and p = 1/3

i) Describe the probability mass function and sketch its graph.

ii) Compute the probabilities $P(1 < X \leq 2)$ and $P(1 \leq X \leq 2)$.

2. In a binomial distribution consisting of 5 independent trials, probabilities of 1 and 2 successes are 0.4096 and 0.2048 respectively. Find the parameter p of the distribution.

3. The probability of a man hitting a target is 1/3.

i) If he fires 5 times what is the probability of hitting the target at least twice ?

ii) How many times must he fire so that the probability of hitting the target at least once is more than 90% ?

4. The random variable X has a binomial distribution with n = 4, p = 0.5. Find $\{ |X-2| \geq 1 \}$

Answers :

1. 8/27, 56/81
2. 0.2
3. I) 131/243
   ii) 6
4. 5/16

L I N E A R   P R O G R A M M I N G

1. Linear Inequations and Convex Sets
2. Formulation of L.P. Problems
3. Applications of Linear Programming

by

Dr.G.RAVINDRA

LINEAR PROGRAMMING

Introduction :

Mathematical Programming constitutes one of the most important problem areas of Operational Research (OR). It encompasses a wide variety of optimization problems. The basic problem of Mathematical Programming is to find the optimum (maximum or minimum) of a non-linear/linear function (called the objective function variously known as cost function, gain, measure of efficiency, return function, performance index, utility measure, etc. depending on the context) in a domain determined by a given system of non-linear and linear inequalities and equalities (called constraints).

Linear Programming (LP) is a Mathematical Programming problem where the objective function and the constraints are all (at least approximated) Linear functions of the unknown variables.

In practical terms, mathematical programming is concerned with the allocation of scarce resources - men, materials, machines and money (commonly known as the 4 M's in OR) - for the manufacture of one or more products so that the products meet certain specifications and some objective function (cost/profit) is minimized or maximized. Whenever the objective function is a linear function of the decision variables and the restrictions on the utilization or availability of resources are expressible as a system of linear equations or inequations, we have a Linear Programming Problem (LPP). For example, in the case of manufacturing a variety of products on a group of machines, the production problem is to determine the most efficient utilization of available machine capacities to meet the required demand. The programming problem is to allocate the available machine resources to the various products so that the total production cost is minimum. To solve this problem, we need to know the unit production cost (cost for producing one item), unit production time, machine capacity and production requirements. This is an LPP (for more clarification see Section 3 on formulation of Linear Programming Problems for a similar example).

The standard technique of solving an LPP is by Simplex Method (due to George, B.Dantzig, 1947) which is quite complicated and is beyond the scope of this unit. However, LPP's involving two variables can be solved graphically. Moreover, there are certain special types of LPP's such as transportation and assignment problems which admit easier methods of solution. Recently, there have been some spectacular developments in the area of LP due to an Indian, Narendra Karmakar of Bell Telephone Labs, U.S.A, where he is able to reach the solution of an LPP considerably faster than simplex method.

In this unit, we confine our attention to formulation of LPP's and their solution by graphical method.

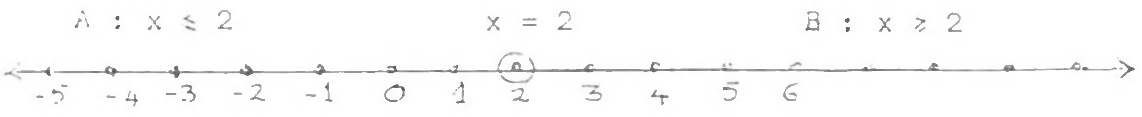
LINEAR INEQUATIONS AND CONVEX SETS :

The restrictions on the utilization (demand) or availability of resources in a linear programming problem (LPP) are expressed as a system of linear equations or linear inequations, and the set of feasible solutions of an LPP is convex set. Though any LPP (in any number of variables) could be solved by the famous Simplex Algorithm, the LPP in two variables can be solved in an easier way by graphical method essentially identifying the intersection of graphs of various linear inequations and testing the objective function for maximum or minimum at the vertices of such a graph. The graph of a linear inequation is essentially a convex set. Thus the concept of Linear Inequations (and their graphs) and convex sets play an important role in the study and the solution of Linear Programming Problem (especially in the two variables case).

Linear Inequation :

Consider the relation $2x=4$ in exactly one variable x on real number line. In this equation, the highest power of x is 1 and so it is a linear equation in one variable. The graph of the equation is the set of all those points on x axis (Real line, R) satisfying the condition $2x=4$. Since there is exactly one point satisfying the condition namely $x=2$, the graph of the equation consists of just one point namely $x=2$ and it divides the x-axis into exactly two parts A and B, where A is the set of points on the axis satisfying $2x \leq 4$ and B is the set of points on the axis satisfying $2x \geq 4$, $2x \leq 4$ and $2x \geq 4$ are linear inequations in one variable and their graphs are respectively A and B, which are two opposite rays with end point $x = 2$.

The following illustrates the graphs of equation $2x=4$ and inequations $2x \leq 4$ and $2x \geq 4$.



In general, $ax= b$, where a and b are real numbers, is a linear equation in one variable and its graph is just the point $x = b/a$ on x-axis (real line). Also, the point $x = b/a$ is common to the rays $ax \leq b$ and $ax \geq b$.

Consider another relation $2x+3y = 6$ in two variables. This is a linear equation in two variables. The graph of the equation is the set of all the points (x,y) in the cartesian plane (i.e. R^2 or xy-plane) which satisfy the equation $2x+3y = 6$. (3,0) and (0,2) are respectively the points of x-axis and y-axis satisfying $2x+3y = 6$. Thus the graph of $2x+3y = 6$ intersects the x-axis and y-axis respectively at (3,0) and (0,2). We know that the equation of the line passing through (3,0) and (0,2) is $2x+3y = 6$. Thus, the graph of $2x+3y = 6$ is essentially the straight line which intersects x-axis and y-axis respectively at (3,0) and (0,2). Further, the graph of $2x+3y = 6$ is the common edge of the two regions C and D where C is the set of points satisfying the inequation $2x+3y \leq 6$ and D is the set of points satisfying the inequation $2x+3y \geq 6$. C and D are called the graphs of $2x+3y \leq 6$ and

$2x + 3y \geq 6$ respectively. More precisely, we observe that the xy-plane has the following partitions.

1. The set of points satisfying $2x+3y < 6$.
2. The set of points satisfying $2x+3y = 6$.
3. The set of points satisfying $2x+3y > 6$.

Thus, if (x,y) is a point in the xy-plane, then it belongs to

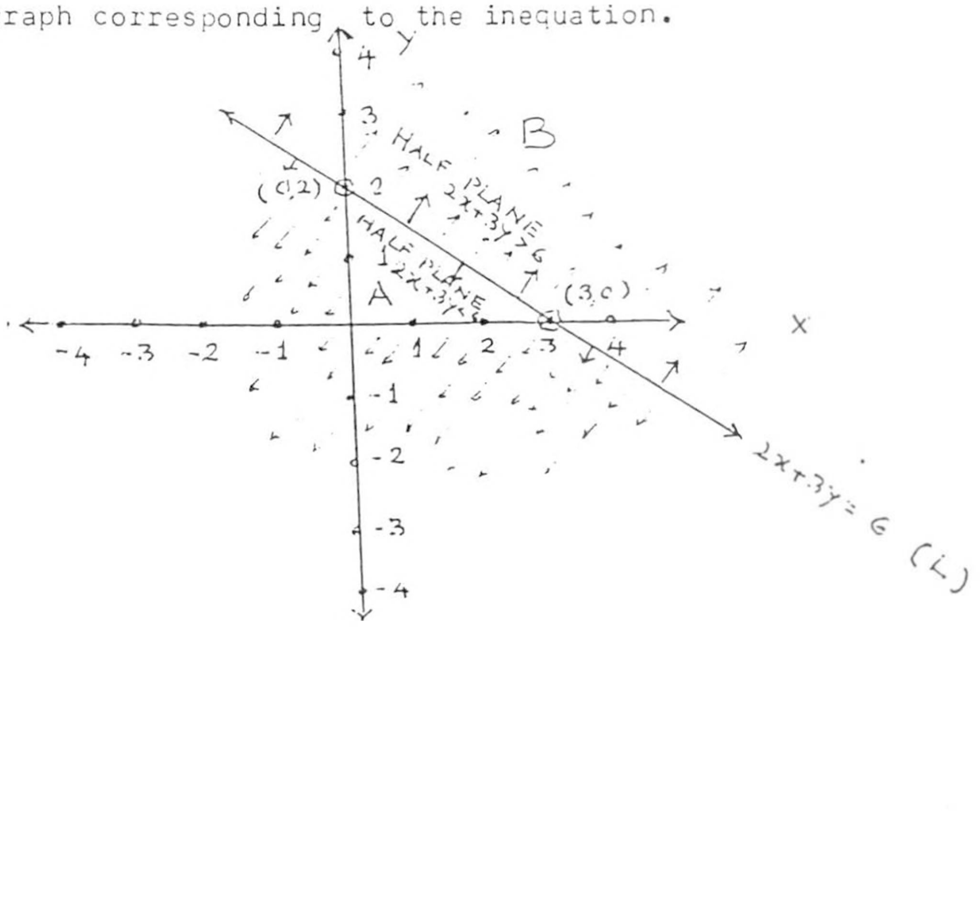
<u>either</u> i) the graph of $2x+3y < 6$

<u>or</u> ii) the graph of $2x+3y = 6$

<u>or</u> iii) the graph of $2x + 3y > 6$

This is the basic philosophy in identifying the graph of an inequation. We illustrate the same as follows :

Suppose we wish to identify the graph of the inequation $2x+3y < 6$. In the following figure, L represents the graph of the line $2x+3y = 6$. The graph of $2x+3y < 6$ could be either A or B (but not a portion of both). We have to mark which one of them is the exact graph corresponding to the inequation.



Here A and B are mutually disjoint. Choose a point which does not belong to L. (0,0) is one such point. The point (0,0) satisfies the inequation $2x+3y < 6$. Hence A is the graph of the inequation.

$A \cup L$ is the graph of the inequation $2x+3y \leq 6$. Suppose we wish to identify the graph $2x+3y > 6$. Since (0,0) which is in A does not satisfy the inequation, A cannot be the graph of the inequation. Therefore, B is the graph of the inequation. Also $B \cup L$ is the graph of the inequation $2x+3y \geq 6$.

In general, the graph of the linear equation $ax+by = c$ (in two variables) is the set of points on the line intersecting x-axis at $(c/a, 0)$ and y-axis at $(0,c/b)$. Further, the graph divides the xy-plane into two parts E and F, one of which is the graph of $ax+by \leq c$ and the other is the graph of $ax+by \geq c$. If a point in E (which is not on L) satisfies $ax+by \leq c$, then E is the graph of the inequation $ax+by \leq c$ and F is the graph of the inequation $ax+by \geq c$. Otherwise, E is the graph of the inequation $ax+by \geq c$ and F is the graph of the inequation $ax+by \leq c$.

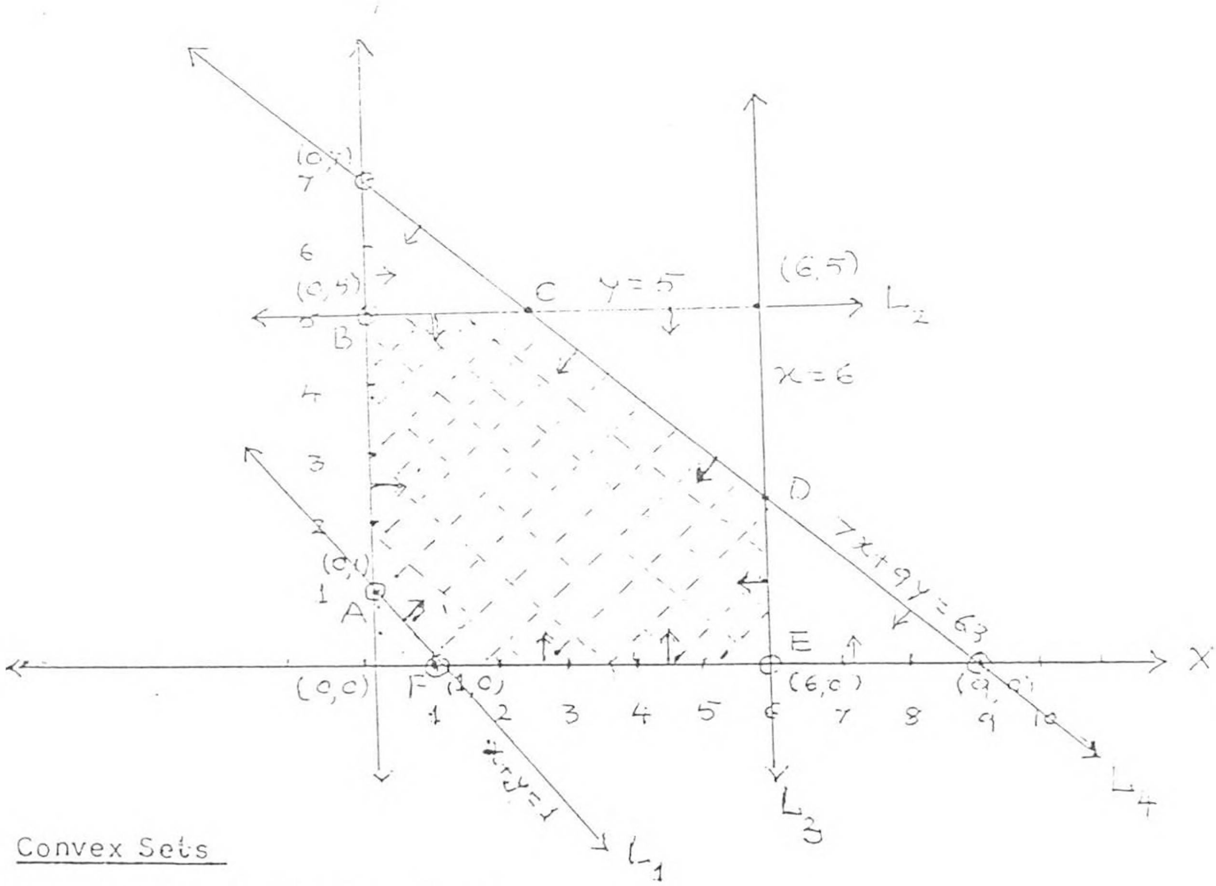
Consider the linear equation $ax+by+cz = d$ in three variables. The graph of this is a plane in the space R^3 and is common to the two parts A and B where A is a set of points (x,y,z) in R^3 satisfying $ax+by+cz \leq d$ and B is the set of the points (x,y,z) satisfying $ax+by+cz \geq d$. A and B are called <u>half planes</u>.

In general, the graph of $a_1x_1 + a_2x_2 + \dots + a_nx_n = b$ is called Hyper plane in the space R^n (i.e. n-dimensional Euclidean space) giving rise to two parts A and B where A is the set of points $(x_1, x_2, \dots, x_n)$ in R^n such that $a_1x_1 + a_2x_2 + \dots + a_nx_n \leq b$ and B is the set of points such that $a_1x_1 + a_2x_2 + \dots + a_nx_n \geq b$. A and B are called Half spaces.

In what follows, we shall mainly confine our discussion to equations and inequations in two variables only.

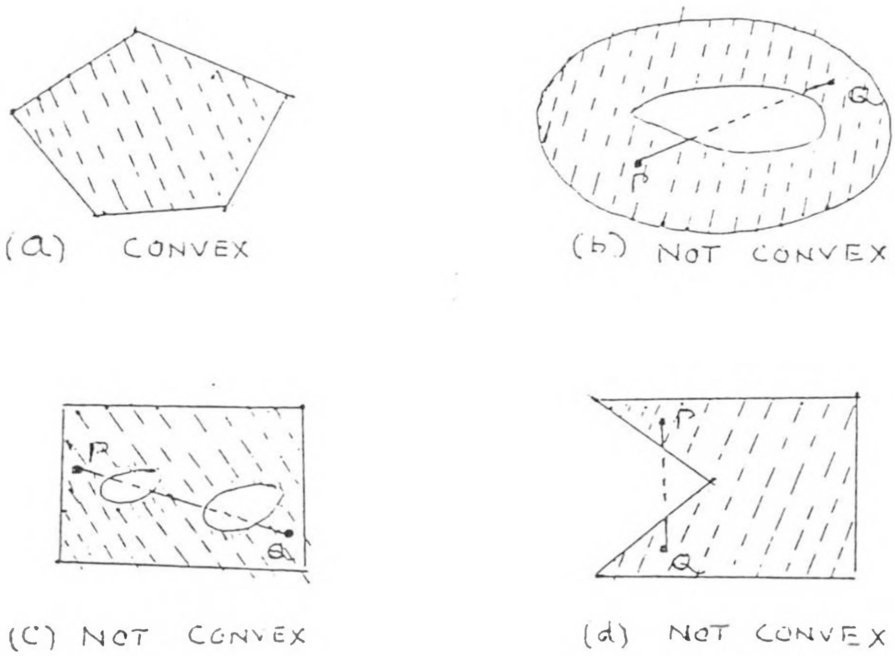
Example: Identify the intersection of graphs of the following linear inequations : $x + y \geq 1$, $y \leq 5$, $x \leq 6$, $7x + 9y \leq 63$, $x, y \geq 0$.

In the following figure, we have drawn arrow marks along the line L_1 representing $x+y = 1$ in such a way that the pointers of the arrows lie in the graph (region) of $x+y \geq 1$. The same is repeated for the rest of the inequations. The intersection of the graphs of these inequations is identified as that region which includes pointers corresponding to all the lines L_1, L_2, L_3, L_4 , X and Y. The region enclosed by the polygon ABCDEF is such a region and hence it is the required graph satisfying all the six inequations simultaneously. Note that the region S enclosed by CDF is not the required region as no pointer corresponding to L_4 lies in it. Note that the arrows corresponding to all the lines L_1, L_2, L_3, L_4, X and Y converge in the graph satisfying all the six inequations.



Convex Sets

Examine the following figures.



(a) CONVEX (b) NOT CONVEX

(C) NOT CONVEX (d) NOT CONVEX

The figure (a) is distinctly different from the other three. In the figure, the linear segment joining any two points is entirely within it, while the regions (b), (c) and (d) do not have the same property. For example, in (b) the line segment joining X and Y is not entirely in it, in (c) the line segment PQ is not in it and in (d) the line segment joining R and S is not entirely in it. Note that the dotted portion of the lines in (b), (c), (d) are not inside the regions. The figures like that of (a) are of special significance in the solution of LPP's and they are said to be <u>convex</u>. Speaking more precisely, a set of points C in the xy-plane (or R^n in general) is called a convex set if the line segment joining any two of its points is entirely contained in C.

<u>Examples of Convex Sets</u> :

i) xy-plane is a convex set.

ii) Circular region in xy-plane is convex but a circle is not convex. (by a circle, here we mean the set of points in xy-plane each of which is equidistant from a given point in the plane).

iii) Sphere, cube, cone, ellipsoid, paraboloid, etc. are convex sets in R^3.

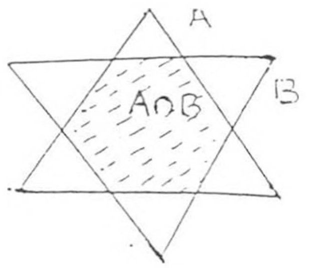
iv) Torus is not a convex set in R^3.

v) Hyperboloid is not a convex set in R^3.

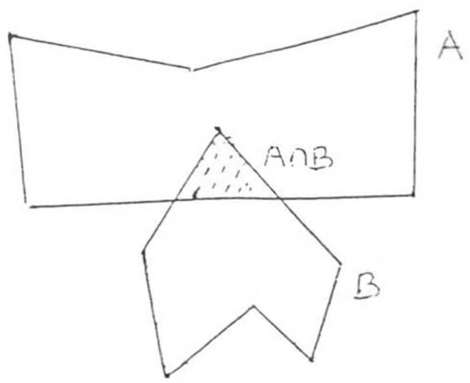
vi) The graphs of the inequations $ax+by \leq c$ and $ax+by \geq c$ are convex, i.e. half planes are convex.

vii) Half spaces in R^n are convex.

Now suppose A and B are any two sets with a given property P. The intersection of A and B may or may not have the property P, though it is part of the both. For example, if A and B are triangular regions in xy-plane their intersection is not necessarily a triangular region in the xy-plane. Similarly, if A and B are two sets in xy plane which are '<u>not convex</u>', their intersection need not have the same property, that is, it could be convex. The following figures illustrate this.



The Intersection A B is not a triangular region.



The Intersection A B is a convex set.

If A and B are convex, will the intersection of A and B also be convex ? We will verify whether this is true or false.

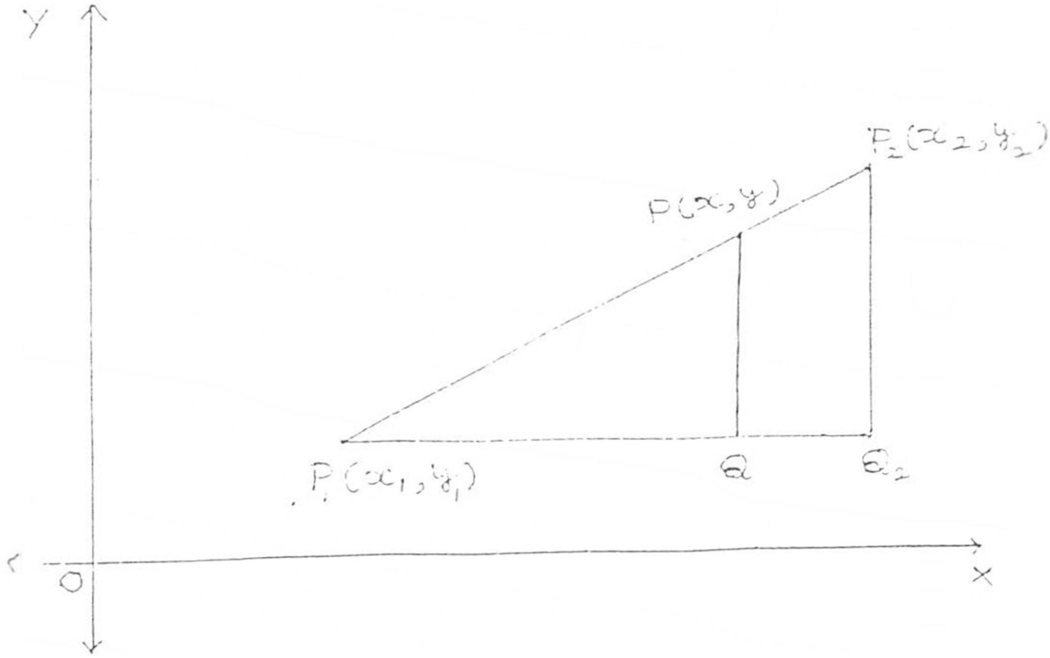
Let C be the intersection of A and B. Let P and Q be any two points in C. Let L be the line segment joining P and Q. Since A is convex, the points of L are contained in A. Since B is convex, the points of L are also contained in it. Thus the points of L are in both A and B. That is, the line segment joining any two points P and Q is entirely in C. This implies that C is a convex set. That is, the intersection of A and B is a convex set. We list the interesting result as

FACT : The intersection of any number of convex sets is also convex. Justification for this essentially follows from the above arguments, replacing sets A and B by any number of sets.

We now look for another way of defining convex sets which often helps in proving results concerning convex sets.

We know from coordinate geometry that (x,y) is a point on a line segment joining the points (x_1,y_1) and (x_2,y_2) if and only if $x = (1-t)\ x_1 + tx_2$ and $y = (1-t)\ y_1 + ty_2$, where $0 \leq t \leq 1$. Justification for the statement follows by considering the similar triangles P_1CP and $P_1P_2O_2$ and their implication viz.

$$\frac{P_1Q}{P_1Q_2} = \frac{CP}{Q_2P_2}$$



Let $X = (x,y)$, $X_1 = (x_1, y_1)$, $X_2 = (x_2,y_2)$, $t_1 = 1-t$, $t_2=t$.
Using these symbols, the above statement can be restated as follows:
X is a point on the line segment joining X_1 and X_2 if and only if
$X = t_1X_1 + t_2X_2$ such that $t_1 + t_2 = 1$, $t_1, t_2 \geq 0$.
(Since $X = (x,y) = ((1-t)\ x_1 + tx_2,\ (1-t)\ y_1 + ty_2)$
$= ((1-t)\ x_1,\ (1-t)y_1) + (tx_2,\ ty_2) = (1-t)\ (x_1,y_1) + t\ (x_2,\ y_2) =$
$(1-t)\ X_1 + tX_2$). The point X so expressed is said to be a <u>convex combination</u> of the points X_1 and X_2 in xy-plane.

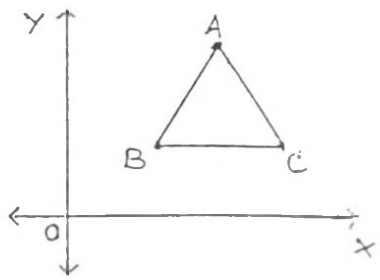
A <u>convex combination</u> of points X_1, $X_2, \ldots, X_n$ in xy-plane (or R^n in general) is a point $X = t_1X_1 + t_2X_2 + \ldots\ldots + t_nX_n$ where t_i's are non-negative real numbers and, $t_1 + t_2 + \ldots + t_n = 1$. As seen already, a point $X = (x_1,y_1)$ belongs to the line segment joining $X_1 = (x_1,y_1)$ and $X_2 = (x_2,y_2)$ if and only if X is a convex combination of X_1 and X_2. Thus a convex set can also be defined as follows:

A set C in xy-plane (or R^n) is a convex set if convex combination of any two points in C is also in it.

In fact, for a given convex set C any convex combination of any number of points in C is also in C.

Not every point in C is a convex combination of some points in C. For example, consider the triangle ABC in xy-plane (The following figure). There are no two distinct points in the triangle such that the line segment joining them contains A. That is, A is not an 'intermediate' point of any line segment in the triangle. Though A is a point on the line segment AB, it is not an intermediate point but one of the extreme points. Thus, A is not a convex combination of any other two distinct points in the triangle. Similarly, the points B and C have the same property. But any other point in the triangle is an intermediate point of same line segment in C. That is any point in the triangle other than A, B and C is a convex combination of some other two distinct points. The points A,B,C are extreme points in comparison with other points in the triangle.

A point X in a convex set is called an extreme point if X cannot be expressed as a convex combination of any other two distinct points in C.



Note that in the above example, the vertices A, B and C are the only extreme points of the triangle.

Examples :

1. The end points of a line segment are extreme points.
2. Vertices or corners of a cube in R^3 are extreme points.
3. Every point of the boundary of a circular region is an extreme point.
4. All the interior points of a circular region are not extreme points.
5. No point of a xy-plane is extreme in the plane.
6. The extreme points of a polygonal region are its vertices.
7. Any point in xy-plane is an extreme point of the singleton set containing the point.
8. The point of intersection of two line segments is not an extreme point of the line segments.

The extreme points play a very significant role in the solution of a LPP. In fact, the objective function of a LPP attains its optimum at at least one of the extreme points of its feasible region which is always convex.

Exercises :

1. Which of the given points belong to the graph of the given inequations ?

i) $x + y < 5$ (0,0); (3,2)

ii) $x - y > 6$ (4,3); (11,4)

iii) $3x+y \leq 2$ (0,0); (0,4)

2. State whether the solution set of the following system of linear inequations is a null set or not.

i) $x \leq 0$ and $x \leq 2$

ii) $x < 2$ and $x > 2$

iii) $y > 1$ and $y > -1$

3. State true or false.

i) The line $y = 10x + 50$ separates the xy-plane in two half planes.

ii) A half plane is the graph of the inequation.

iii) The graph of a linear inequation is a convex set.

iv) The union of two convex sets in xy-plane is also a convex set in xy-plane.

v) The intersection of two convex sets in xy-plane is a convex set in xy-plane.

vi) If A and B are two sets in R^2 which are not convex, their intersection is also not convex in R^2.

vii) Vertices of a cube are extreme points.

viii) If m is the number of linear inequations in two variables and if the intersection of their graphs is a polygonal region with n sides then $m = n$.

ix) If a point (x,y) in xy-plane is a convex combination of two points (r,s) and (p,q) in the plane, then it lies on the line joining the two points (p,q) and (r,s).

x) The converse of the above statement is generally not valid.

xi) The intersection of two convex sets could possible be disjoint union of two convex sets.

xii) Union of two convex sets is convex.

xiii) Every point in a convex set is a convex combination of two other points in it.

4. Find two points in xy-plane that satisfy each of the following.

i) $y = 5x$, ii) $y < 5x$ iii) $y > 5x$

5. Mark the region which represents the graph of following inequations.

a) $x < 3$ b) $y > 3$ c) $2x+ 4y \leq 8$

d) $x + y \leq 4$

6. State whether the region representing the following is bounded or unbounded.

$x \geq 0$, $y \geq 0$ and $x+y \leq 8$.

7. Let ABCD is a square in the first quadrant of xy-plane.

i) If $x + y = 1$ is the equation of the side AB, find the equations of the sides, BC, CD and DA.

ii) Write the inequations whose intersection is the interior of the square.

8. Let ABCDEF be a regular hexagon with length of each of its sides equal to 1 unit. Write the inequations whose intersection is the given hexagon.

9. Prove or disprove :

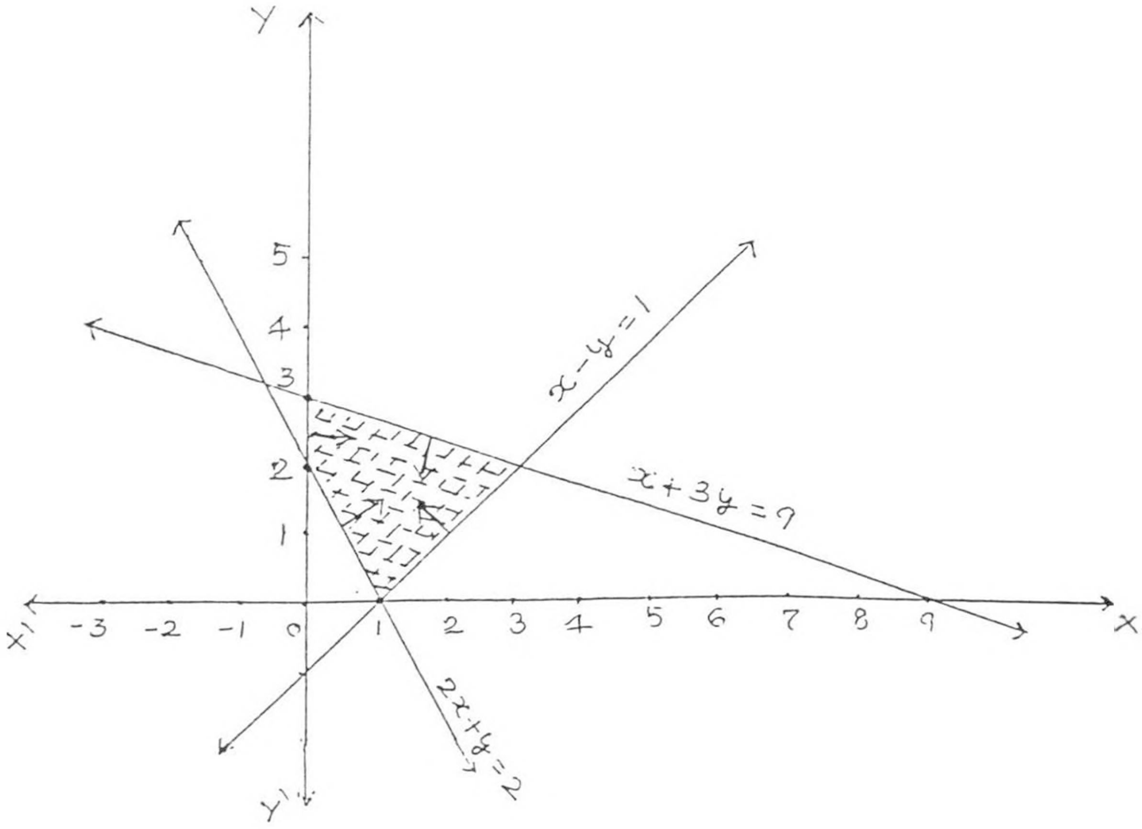
i) The circle $x^2+y^2 = a^2$ (a is a given real number) is a convex set.

ii) Every point on the boundary of a circular region is an extreme point.

iii) If G is the graph satisfying m linear inequations simultaneously, then G is a polygonal region having m sides.

iv) A set consisting of single element of R^2 is a convex set in R^2.

10. Find the linear constraints for which the shaded region in the following figure is the solution set.



FORMULATION OF LINEAR PROGRAMMING PROBLEMS

A large class of problems can be formulated as LP models. While formulating an LP model it is worth-while to remember the following 3-way rule suggested by Dantzig..

i) Identify the unknown activities to be determined and represent them by suitable algebraic symbols. Identify the inputs and outputs associated with each activity.

ii) Identify the restrictions (constraints) in the problem and express (at least approximate) them as linear algebraic equations/inequations.

iii) Identify the objective function and express it as a linear function of the unknown variables.

Proper definitions of the variables (step (i)) is a key step and will largely facilitate the rest of the work.

Let us illustrate the formulation by a few examples.

Example : Suppose we are concerned with a problem encountered by a man who sells oranges and apples in a running train. He has only Rs.120 with him and he decided to buy atleast 5 kgs of each item. One kg of apple costs Rs.10 and 1 kg of orange costs Rs.5. He can carry to the train only a maximum load of 15 kgs which his bag would hold. He expects a profit of Rs.2 per kg from applies and Rs.1 per kg from oranges. How much each of these two items should he buy (if he is wise enough) so as to get a maximum profit ?

Here, the ultimate goal or objective of the fruit seller is to get the maximum profit in his business, i.e. he wants to maximise his profit. To achieve this, he cannot purchase the items at random. The problem is to find out in what combinations should he buy apples and oranges so that the profit is maximum. Let us try to find out the possible combinations. The man can buy a total of 15 kgs of apples and oranges. Can he buy 15 kgs of oranges? Of course, not, because he has to buy at least 5 kgs of apples, i.e., he can buy a maximum of 10 kgs of oranges. Can he buy 15 kgs of apples ? He cannot because he should buy at least 5 kgs of oranges i.e. he can buy a maximum of 10 kgs of apples. He can purchase oranges from 5 kgs to 10 kgs and so also apples. We can list all the possible combinations of his purchase of apples and oranges and calculate the profit in each case. See the table below.

PURCHASE (in kgs)		COST			PROFIT		
Orange	Apple	Orange Rs.5	Apple Rs.10	Total	Orange Re.1	Apple Rs.2	Total
5	10	25.00	100.00	125.00	Not possible		
6	9	30.00	90.00	120.00	6.00	18.00	24.00
7	8	35.00	80.00	115.00	7.00	16.00	23.00
8	7	40.00	70.00	110.00	8.00	14.00	22.00
9	6	45.00	60.00	105.00	9.00	12.00	21.00
10	5	50.00	50.00	100.00	10.00	10.00	20.00

Look at the last column. The maximum profit is Rs.24. He gets this profit when he purchases 6 kgs of oranges and 9 kgs of apples.

This is the solution of the problem which maximises or optimises the profit. So we call it an optimal solution of the problem.

Optimal solution = 9 kgs of apples and 6 kgs of oranges.

Optimum profit = Rs.24.

After investigating the next example, where we maximise the profit as in this example, we will be able to see if we can arrive at the optimal solution by trial and error method. Before that let us formulate the above example in Mathematical terms (see Dantzig's 3-way rule).

i) <u>Definition of variables</u>

Let x be the number of kgs of oranges and y be the number of kgs of apples bought.

ii) <u>Constraints</u> : Since one cannot buy negative number of oranges or apples it is clear that $x \geqslant 0$ and $y \geqslant 0$.

Since one kg of orange costs Rs.5, x kgs of orange will cost Rs.5x. Similarly, y kgs of apple costs Rs.10y. Therefore, the total cost will be 5x + 10y. Since he has only Rs.120 with him we have,

$$5x + 10y \leq 120.$$

Since he has decided to buy atleast 5 kgs of each item,

$$x \geq 5, \quad y \geq 5.$$

As he cannot carry more than 15 kgs

$$x + y \leq 15.$$

iii) The objective function :

Since he expects a profit of Rs.2 per kg from apples and Re.1 per kg from oranges, his total profit would be x+2y which has to be maximised. The L.P. model is : Maximise $Z = x+2y$ subject to $x \geq 5$, $y \geq 5$, $5x+10y \leq 120$, $x-y \leq 15$; and $x,y \geq 0$. In this problem, the non-negativity restrictions are not necessary in view of the constraints $x,y \geq 5$.

Example : A company sells two different types of radios - 3 band types and 2 band types. Company has a profit of Rs.50 for each of the former type and Rs.30 for each of the second type. The production process has a capacity of 80,000 man hours in total. It takes 10 man hours labour to assemble 3-band type and 8 man hours for 2-band type. It is expected that a maximum of 6000 numbers of the former type and a maximum of 8000 of latter type can be sold out. How many of each type should be produced so as to maximise the profit ?

In this problem, the company aims at getting the maximum profit. i.e. profit is to be maximised. The problem is to find out in what combination should he produce 2-band radios and 3-band radios in order to achieve this objective. We know that the company gets more profit from the 3-band radios. Naturally, we can think of a possibility where all the radios produced are 3-band type. This could not be done since the maximum number of 3-band type radios should be six thousand. The other possibility is to think of another way. The man hours needed to produce a 2- band radio is smaller compared to 3-band radios. In that case, he should increase the number of 2-band radios, which should not exceed 8000. Naturally, a third question arises - can the company produce 6000, 3-band radios and 8000, 2-band radios. In that case, we have to take into consideration the man hours available. The man hours required for producing 8000, 2-band radios is 8x8000 = 64000. The total man hours required to produce 6000 3-band type and 8000 2-band type is 124000 which is greater than the man hours available. From the above discussion, we found that the number of 3-band radios can extend from 0 to 6000 and that of 2-band radios from 0 to 8000. To get a solution for this problem, we have to enumerate all the cases from 0 to 6000 and 0 to 8000, which evidently is laborious. Therefore, we have to find out an easier method to solve such problems.

We will now think of evolving an easy method to solve such problems. Before entering into the details of this method, let us explain the problem mathematically. In other words, let us try to write the LP formulation of the problem.

In the above problem, what we are expected to find is the number of 3-band radios and 2-band radios to be produced so as to get the maximum profit. Let us assume that the number of 3-band radios produced is 'x' and the number of 2-band radios produced is 'y'.

Number of 3-band radios = x
Number of 2-band radios = y

Once we know the number of each type of radios, we can calculate the total profit of the company. Profit from a 3-band radio is Rs.50 and the profit from a 2-band radio is Rs.30.

Total profit = $50x + 30y$.

The objective of the company is to get the maximum profit i.e. $50x + 30y$ should be maximum. We call this the <u>objective function</u> of the problem. Now the problem reduces to finding the maximum values of $50x + 30y$. In other words, we have to maximise $50x + 30y$.

What are the conditions to be satisfied ?
We know that 'x' and 'y' are the numbers of radios produced. So we can say that x and y cannot be negative. Mathematically, we put it as

$x \geq 0$ and $y \geq 0$

x is the number of 3-band radios. The maximum number of 3-band radios produced is 6000.

i.e. $x \leq 6000$

Similarly $y \leq 8000$

The total man hours available is only 80000. Man hours required to produce one 3-band radio is 10.
Man hours required for x radios = $10 \times X$
= $10X$

In a similar way, man hours needed for y, 2-band radios = $8y$.
The total man hours should not exceed 80000

i.e. $10x + 8y \leq 80000$

Thus the restrictions or conditions to be satisfied are

1. $x \geq 0$
2. $y \geq 0$
3. $x \leq 6000$
4. $y \leq 8000$
5. $10x + 8y \leq 80000$

These conditions are generally called <u>constraints</u> of the problem. The first two viz. $x \geq 0$ and $y \geq 0$ are called <u>non-negativity restrictions</u>. Each of these constraints is an inequation of degree 1. Hence, they are called <u>linear constraints</u>.

The mathematical formulation of the problem is as follows :

Maximise $50x + 30y$

subject to $x \geq 0$

$y \geq 0$

$x \leq 6000$

$y \leq 8000$

and $10x + 8y \leq 80000$

Here the objective function as well as the constraints are all linear (first degree).

<u>A typical LP Model</u> :

Suppose a company with two resources (labour and material) wishes to produce two kinds of items A and B.

Let t_1, t_2 units of time (hours or minutes) be respectively time required to produce one unit of A and B, m_1 and m_2 be the amount of unit material (in Kg or pounds or any unit of weight) respectively required for one unit of A and B, and Rs.p_1 and Rs.p_2 profit per unit of A and B. Suppose the daily availability of manpower (labour) is T hours and the supply of raw material is restricted to M Kgs per day. The problem of the company is :

How many items of kind A and how many items of kind B be produced everyday, so that the total profit is maximum ?

This kind of problem is generally known as <u>Product-Mix Problem</u>.

The entire information of the problem can be stored in matrix (tabular) form as follows :

Resources	Kinds of Items		Supply/availability
	A	B	
Labour (hours/unit)	t_1	t_2	T
Material (Kgs/unit)	m_1	m_2	M
Profit (Rs./unit)	p_1	p_2	

In view of the 3-way rule suggested earlier we have

Step 1 : Let x = Daily production of kind A

y = Daily production of kind B

Step 2 : Constraint corresponding to the first row :

$$t_1x + t_2y \leq T$$

Constraint corresponding to second row :

$$m_1x + m_2y \leq M$$

Non negativity conditions :

$$x, y \geq 0$$

Step 3 :

The third row corresponds to the objective function and is given by

$$Z = p_1x + p_2y$$

Thus the mathematical formulation of the problem is :

(I) - Find numbers x, y which will <u>maximize</u>

$$Z = p_1x + p_2y$$

subject to the constraints

$$t_1x + t_2y \leq T$$

$$m_1x + m_2y \leq M$$

and $x, y \geq 0$

Note that in the mathematical formulation (I) above we deal only with numbers, equations, inequations and the given situation (that is company's problem) is no longer under consideration.

The above typical problem can be adopted in many real life situations and thus a teacher can find a problem of linear programming according to the nature of the students (urban, rural, etc). For example, if the 'company' is an industry like "CHKAY" A could be taken as Idly mix and B could be taken as Dosa mix. The relevant information concerning resources and profit (possibly in terms of cost price and selling price) can be obtained in the form of a matrix. Such matrix will help in identification of the problem as well as in its mathematical formulation. If we consider a comfy in kitchen appliance,

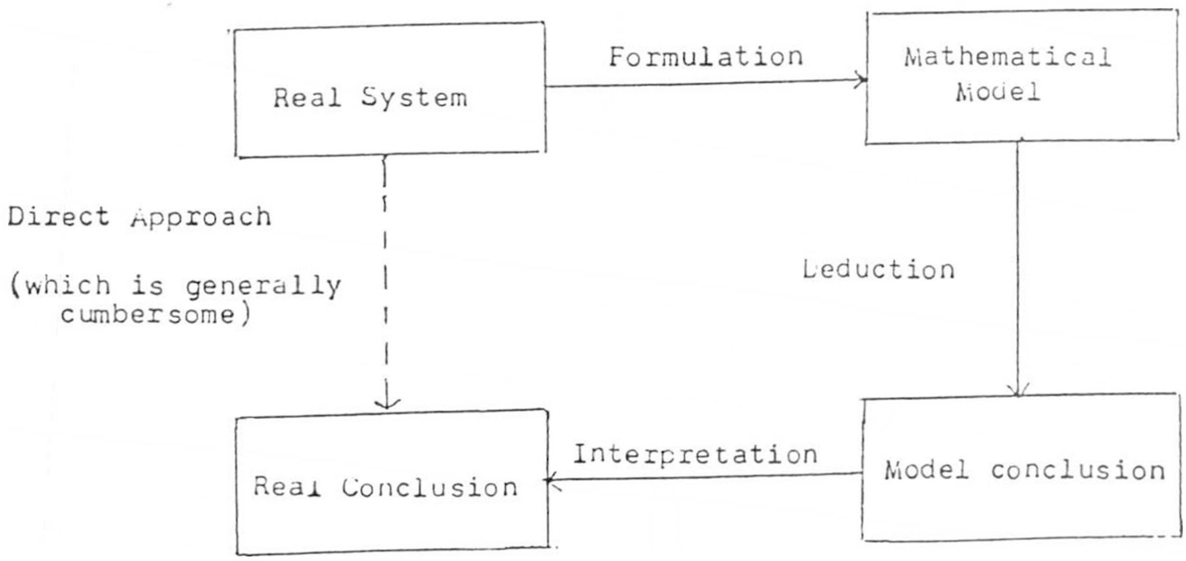
A could be considered as a pressure cooker

B could be considered as pressure pan.

If we want to have a farmer's problem, we can take A and B respectively to be areas of a given field for production of wheat and gram. The resource corresponding to material could be fertilizer. Here we will have an extra constraint viz. $x+y \leq a$ where 'a' is the area of the given field. Note that there could be any number of resources (and hence constraints) depending upon the situations.

Linear Programme a Mathematical Model :

A mathematical model is a symbolic representation of a real situation. The process of mathematical modelling is depicted in the following figure.



In example 1, the real situation is 'selling of oranges and apples'. In example 2, the real problem (situation) is 'to evolve a selling policy of two kinds of radios' and in the product mix problem the real situation is 'productive scheduling'. In all these problems, mathematical formulation is mathematical model. The mathematical models in the above examples consist of objective function and constraints which are expressed quantitatively or mathematically as functions of decision variables 'Mathematical conclusion' and 'Real conclusion' constitute the solution of a linear programming problem, which we would be dealing within the next section.

EXERCISES :

1. A company makes two kinds of leather belts A, B. Belt A is of higher quality and belt B is of lower quality. The respective profits are Rs.4 and Rs.3 per belt. Each belt of type A requires twice as much time as a belt of type B, and, if all belts were of type B, the company, could make 1000 per day. The supply of leather is sufficient for only 800 belts per day (both A and B combined). Belt A requires a fancy buckle and only 400 per day are available. There are only 700 buckles a day are available for belt B. Formulate this as a linear programming model.

2. Give an example of a real situation (other than those mentioned in this lesson) whose mathematical model is a linear programming model.

3. Give an example of a mathematical model which is not a linear programming model.

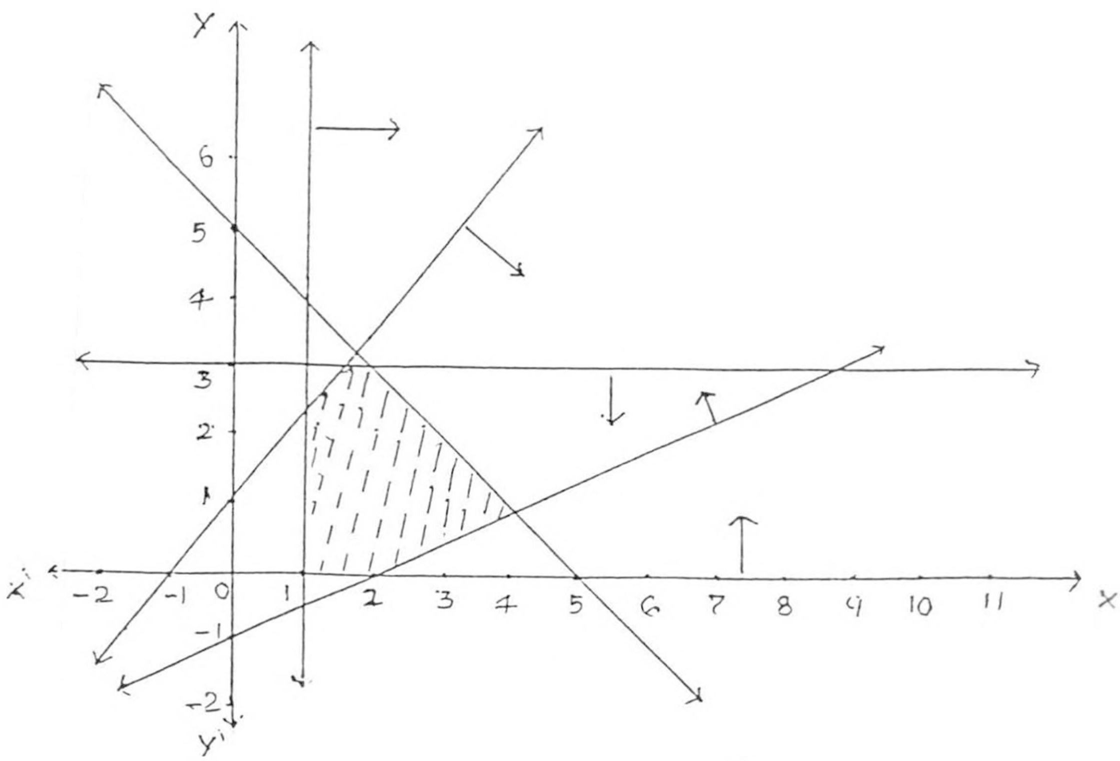
4. An Advertising company wishes to plan an advertising campaign in three different media - television, radio and magazines. The purpose of the advertising company is to reach as many potential customers as possible. Results of the market study are given below :

	Television		Radio	Magazine
	Day Time	Prime Time		
	Rs.	Rs.	Rs.	Rs.
Cost of an advertising unit	40,000	75,000	30,000	15,000
Number of potential customers reached per unit.	400,000	900,000	500,000	200,000
Number of women customers reached per unit	300,000	400,000	200,000	100,000

The company does not want to spend more than Rs.800,000 on advertising. It further requires that (i) atleast 2 million exposures take place among women, (ii) advertising on television be limited to Rs.500,000, (iii) atleast 3 advertising units be bought on day time television and two units during prime time; and (iv) the number of advertising units on radio and magazine should each be between 5 and 10.

Find different types of advertising units which minimize the total number of potential customers reached is maximum.
(Note: The problem involves four decision variables).

5. Write the constraints associated with the solution space shown in the following figure and identify all redundant constraints.



SOLUTION OF LINEAR PROGRAMMING PROBLEM BY GRAPHICAL METHOD :

Let us consider another example of an optimisation problem. We can examine whether this is a linear programming problem by formulating a mathematical model of the problem. We can also try to find the solution of the problem by graphical method.

Example 1 : A contractor has 30 men and 40 women working under him. He has contracted to move at least 700 bags of cement to a work site. Due to the peculiar nature of the work site he could employ at the maximum of 50 workers at a time. A man will carry 25 bags in a day and a woman will carry 20 bags in a day. A man demands Rs.45 a day and a woman demands Rs.35 a day as their wages. In what ratio should the contractor employ men and women so that the cost of moving the cement to the work site is minimum ?

Now the mathematical model of the problem is :

Minimize $Z = 45x + 35y$

subject to the conditions

$x \geq 0$

$y \geq 0$

$x \leq 30$

$y \leq 40$

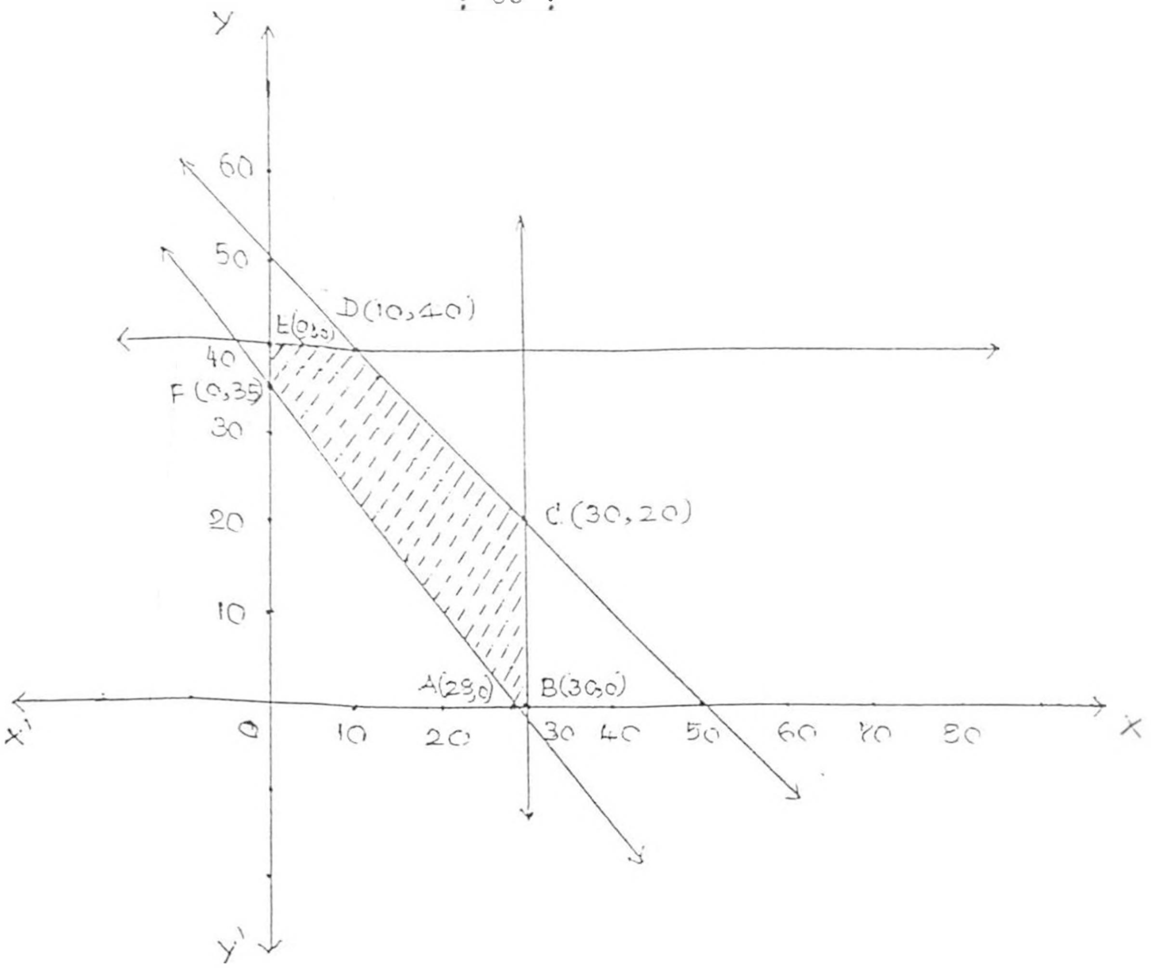
and $5x + 4y \geq 140$

(x and y are respectively the number of men and women employed and z is the total wage for them).

The above problem is an optimisation problem. The objective function as well as the constraints are linear. Hence, it is a LPP.

The next step is to find a value for x and a value for y such that $45x + 35y$ is minimum subject to the conditions laid down in the problem.

We first draw the graph of the inequations and see how the graph will give the solution of the problem.



The intersection of the graphs of the inequations is the region of the polygon ABCDEF, called the <u>feasible region</u>. Any point p (x,y) in the feasible region is a <u>feasible solution</u> of the LPP. The coordinates of such a point will satisfy all the inequations. Let us consider a point p(20, 20) in this region. We can easily verify that it satisfies all inequations. So we can consider the x-coordinate of P as a value of x and y - coordinate of P as a value of y. i.e. x = 20 (x- coordinate of P) and y = 20 (y - coordinate of P) is a feasible solution of the LPP. If we select another point say D (10,40) in the region, x = 10 and y = 40 is another feasible solution of the problem. We know that there are infinite number of points in the region ABCDEF. The coordinates of each point will give a feasible solution of the problem i.e.

the number of feasible solutions are infinite. The problem is to decide which one of these is optimal. For this, we make use of the following key result.

THEOREM . If there exists an optimal solution to an LPP, the objective function of the LPP always attains its optimum (minimum or maximum) at at least one of the corners (extreme points) of the feasible region.

Proof: We prove the validity of the theorem for two variables (coordinates) and in fact the same arguments can be extended to prove the theorem for any number of variables.

Let K be the set of feasible solutions of a linear programming problem. Let $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ be the extreme points (corners), of the feasible region corresponding to K. Let $Z(x, y) = C_1 x + C_2 y$ be the objective function of the linear programming problem.

Suppose for $x = x_0$ and $y = y_0$ the objective function attains its minimum.

That is, $Z(x_0, y_0) = C x_0 + C y_0$ is the minimum value of the objective function. Let $m = Z(x_0, y_0)$.

If (x_0, y_0) is one of the extreme points (corners) of the region representing K, the theorem is true. Therefore, we assume that (x_0, y_0) is not an extreme point. Hence, by the definition of extreme point, (x_0, y_0) can be expressed as a convex combination of extreme points of K.

That is,

$$(x_0, y_0) = t_1 (x_1, y_1) + t_2 (x_2, y_2) + \ldots + t_n (x_n, y_n) \qquad (1)$$

where $t_1 + t_2 + \ldots t_n = 1$ and $t_i \geqslant 0$.

This implies that

$$m = Z(x_0, y_0) = t_1 z(x_1, y_1) + t_2 z(x_2, y_2) + \ldots + t_n z(x_n, y_n)$$

Suppose $Z(x_r, y_r)$ be minimum along $Z(x_1, y_1), \ldots, z(x_n, y_n)$ so that

$$Z(x_i, y_i) \geq Z(x_r, y_r), \quad 1 \leq i \leq n \qquad (2)$$

Now (1) and (2) together imply that

$m \geq t_1 z(x_r, y_r) + t_2 z(x_r, y_r) + \ldots + t_n z(x_r, y_r)$

(Since t_i' s are non negative).

That is,

$m \geq (t_1 + t_2 + \ldots + t_n) \; Z(x_r, y_r)$

or $m \geq Z(x_r, y_r)$ (Since $t_1 + t_2 + \ldots + t_n = 1$) (3)

By definition of minimum

$m \leq Z(x,y)$ for every (x,y) in K and in particular

$m \leq Z(x_r, y_r)$ (4)

(3) and (4) together imply that

$m = Z(x_r, y_r)$ where (x_r, y_r) is an extreme point. Thus Z (the objective function) attains its minimum at an extreme point of the feasibility region.

Remark :

Let for $x = x^1$ and $y = y^1$, $z(x,y)$ (the objective function) attain its maximum. Then by definition of maximum

$z^1 \geq z(x,y)$ for every x,y in K (where $z^1 = z(x^1, y^1)$)

$\Rightarrow -z^1 \leq -z(x,y)$

$\Rightarrow -z^1$ is the minimum value of $-z(x,y)$

That is, $-z^1 = \min(-z(x,y))$

or $-(\max z(x,y)) = \min(-z(x,y))$

or $\max z(x,y) = -\min(-z(x,y))$

Thus minimisation problem can be converted to maximization problems by considering negative of the objective function $z(x,y)$. And accordingly, the above theorem is true in the case of maximisation problems also.

In view of the above theorem, it is sufficient to concentrate our attention only on the corner points of the polygon ABCDEF. Evaluating the objective function at each of the vertices of ABCDEF and selecting the minimum of these values, we get the minimum value of the objective function. The coordinates of the corresponding vertices will constitute an optimal solution. The details are shown in the table given below :

Corner Point	Value of the objective function Z = 45x + 35y	
A (28, 0)	45x28 + 35x0	= 1260
B (30,0)	45x30 + 35x0	= 1350
C (30,20)	45x30 + 35x20	= 2050
D (10,40)	45x10 + 35x40	= 1850
E (0,40)	45x0 + 35x40	= 1400
F (0,35)	45x0 + 35x35	= 1225

Thus, it is clear that when the contractor employs 35 women and no men the cost of moving cement to work-spot is minimum and the minimum cost is Rs.1225. Now let us solve a maximisation problem by graphical method.

Example2: If a young man rides his motor cycle at 25 km per hour, he has to spend Rs.2 per km on petrol; if he rides at faster speed of 40 km per hour, the cost increases to Rs.5 per km. He has Rs.100 to spend on petrol. What is the maximum distance he can travel within one hour ?

Let x = distance travelled by the young man in one day at the speed of 2 km/hour,

and y = distance travelled by the young man in one day at the speed of 40 km/hour.

Let Z = X+Y

Objective Function : Z = x+y (with the objective to maximize Z)

Constraints:

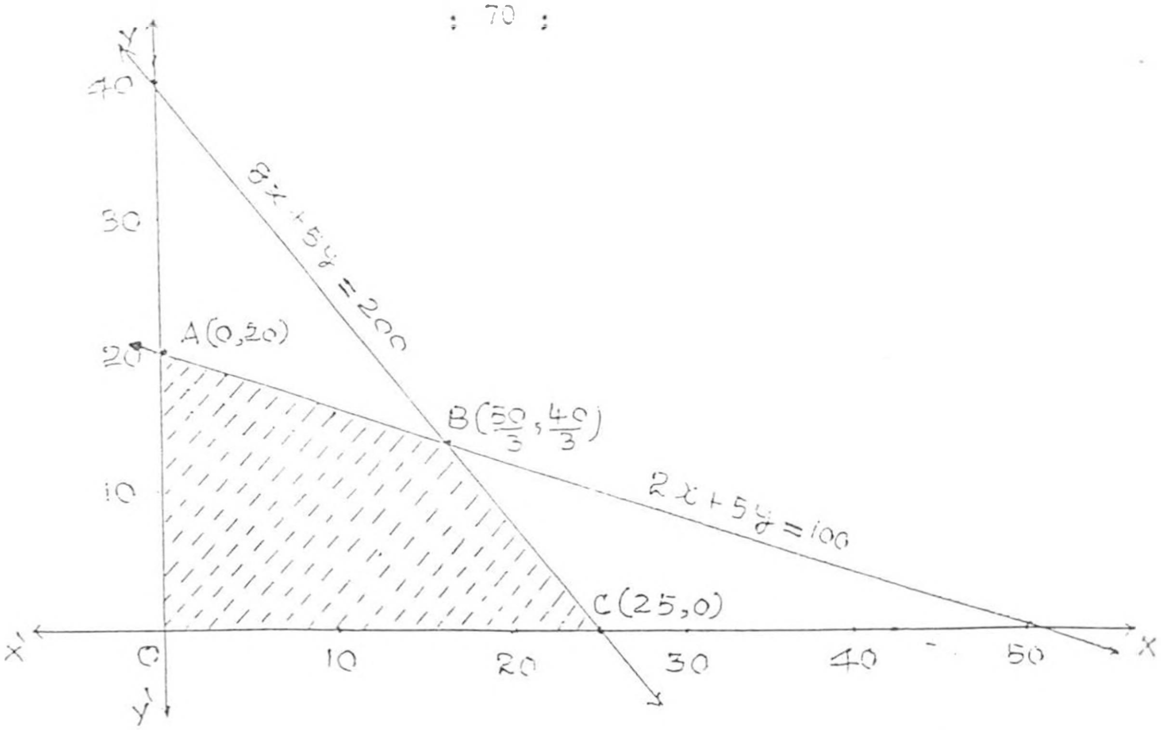
i) money spent on petrol = $2x+5y \leq 100$ (constraint due to money)

ii) total time of travel = $\frac{x}{25} + \frac{y}{40} \leq 1$ (constraint due to time)

or $8x + 5y \leq 200$

iii) non negativity conditions : $x \geq 0$, $y \geq 0$

We now draw the graph corresponding to the constraints.



The feasible region is the shaded region of the polygon OABC.

Corner point	Value of z = x+y
O (0,0)	0
A(0,20)	20
B ($\frac{50}{3}$, $\frac{40}{3}$)	30
C (25,0)	25

Therefore, 30 = Max z = the maximum distance the young man can travel in one day.

The procedures that we follow in solving a LPP (in two variables) by graphical method is summarised below :

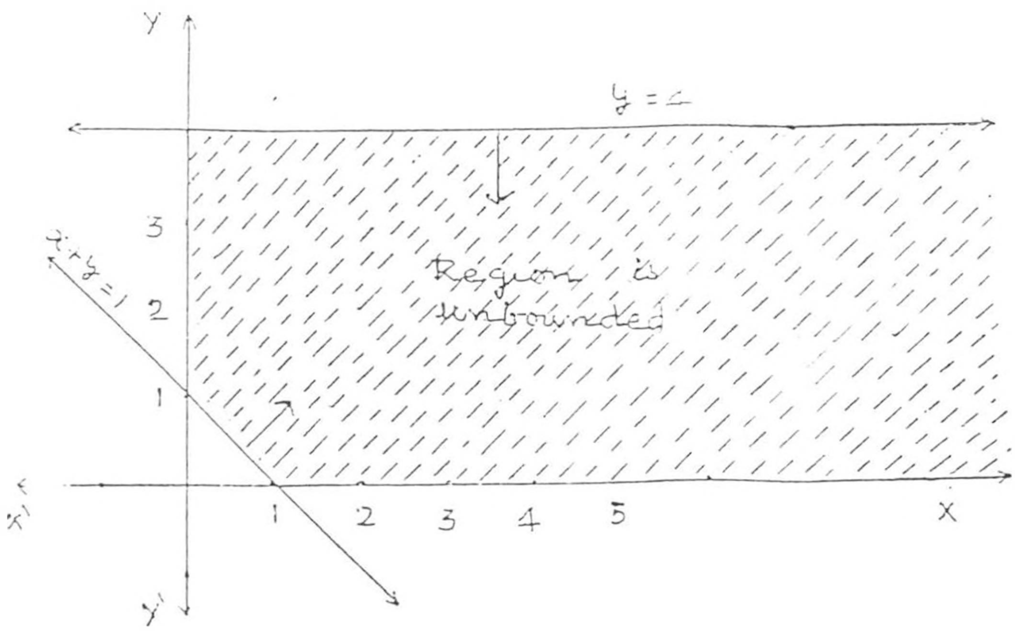
1. Mark the feasible region. (This is the intersection of the graphs of constraints).

2. Evaluate the objective function at each of the corner points of the feasible region and pick out the point which gives the minimum (maximum) value for the objective function as the case may be.

Theorem holds true if there exists an optimal solution to a LPP. There may be cases where the objective function has no finite optimal value. For example,

Maximise $Z = x + 2y$
subject to $x + y \geq 1$
$x \geq 0$, $y \geq 0$
$y \leq 4$

The shaded region in the following figure is the feasible region of the problem. Note that the feasible region is not a polygonal region, but is unbounded.



In this case, moving farther away from the origin increases the value of the objective function $Z = x + 2y$ and the maximum value of Z would tend to $+\infty$ i.e., Z has no finite maximum. Whenever a LPP has no finite optimal value (maximum or minimum), we say that it has an unbounded solution. Further, there could be a linear programming problem such that it has no feasible solution.

For example,

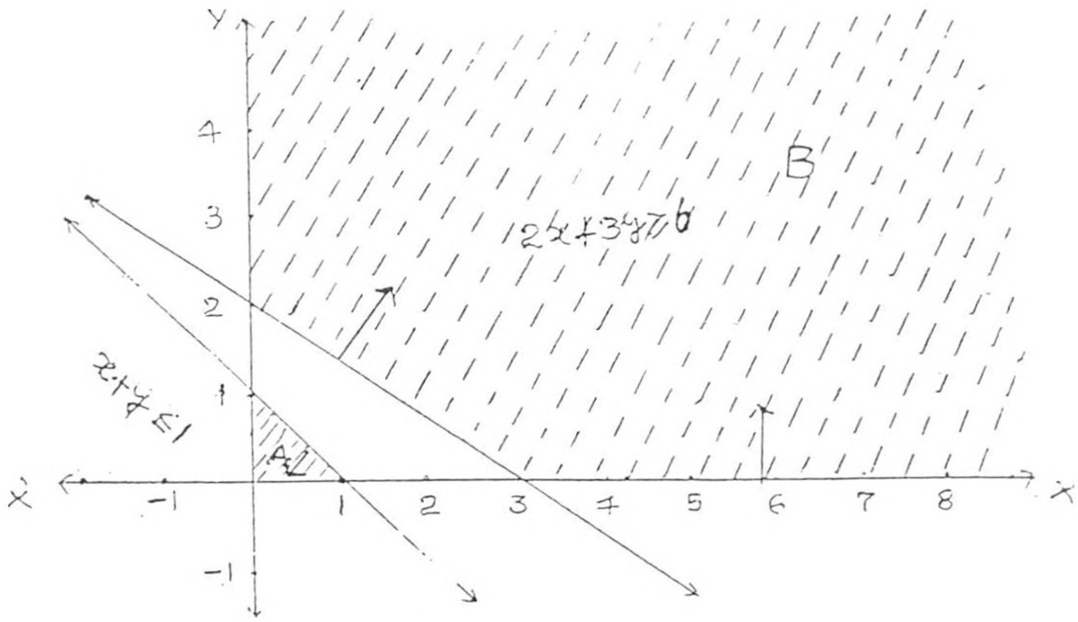
Maximise $Z = 4x + 3y$

subject to $x + y \leq 1$

$2x + 3y \geq 6$

$x \geq 0, y \geq 0$

The shaded regions A and B in the following figure indicate the graphs of the inequation $x+y \leq 1$ and the graph of the inequation $2x + 3y \geq 6$ respectively.



Obviously, the intersection of A and B is empty. Hence the LPP has no feasible solution.

The following LPP has or does not have a feasible solution depending upon the value of L.

Maximise $Z = x$

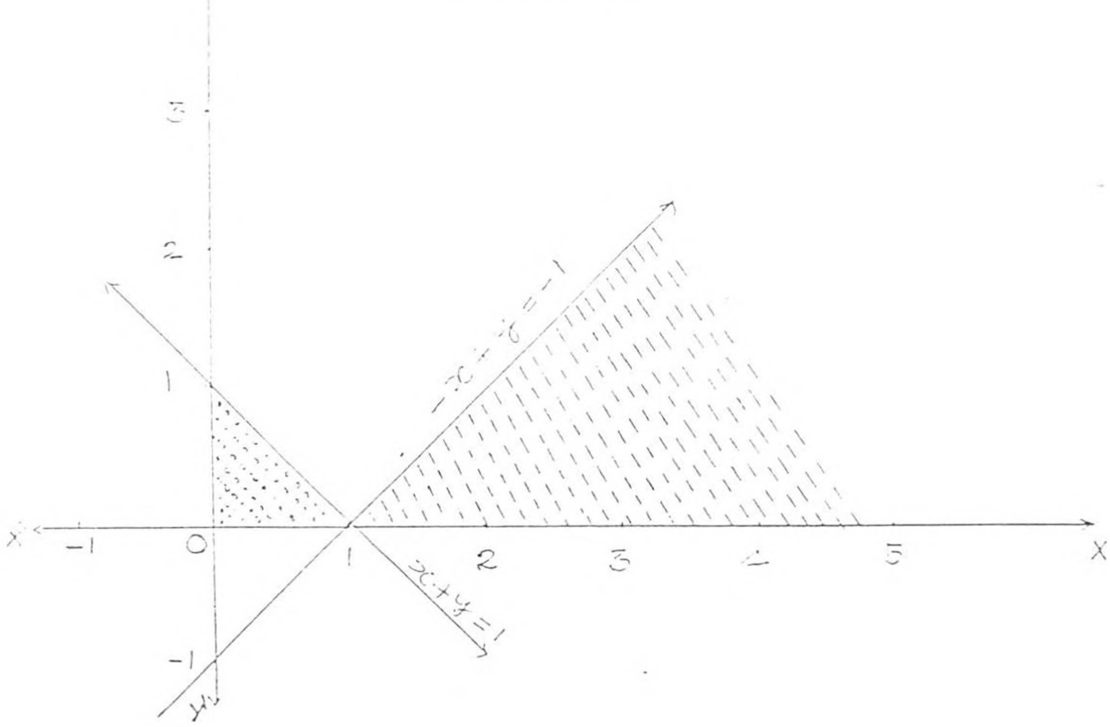
subject to $x+y \leq L$

$-x + y \leq -1$

$x \geq 0 , \quad y \geq 0$

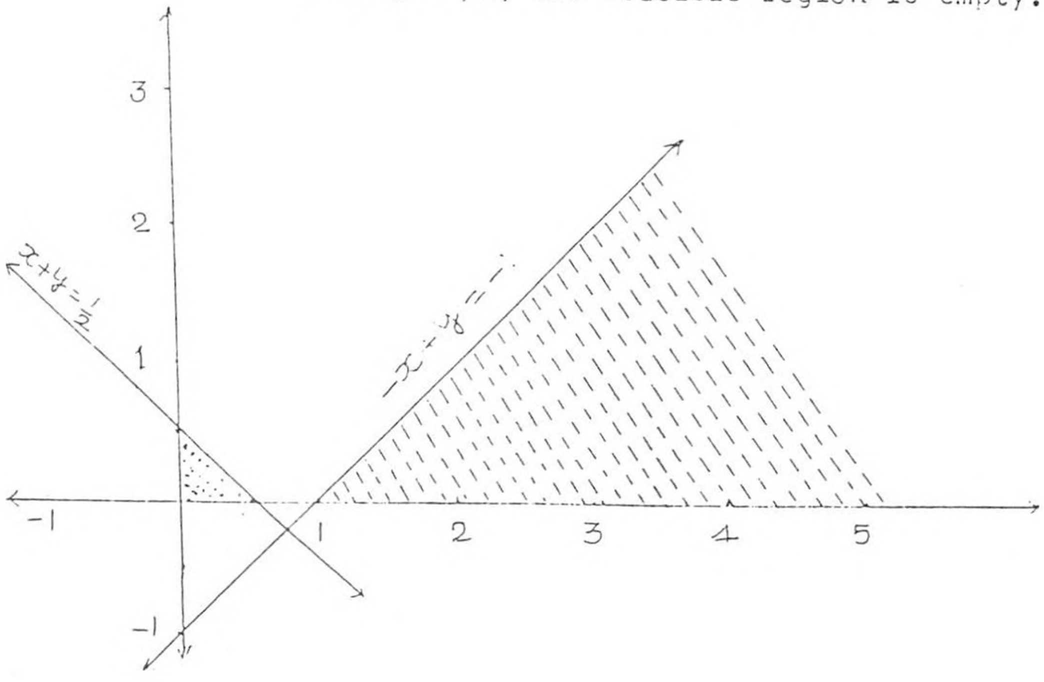
If $L = 1$, the feasible region of the problem consists of just one point (1,0) (See figure shown below).

when L=1, feasible region consists of one point only.



If $L = 1/2$, the feasible region is empty since there are no points satisfying the non-negativity restrictions.

when $L = 1/2$, the feasible region is empty.



In fact, for all values of $L < 1$ the feasible region corresponding to the given constraints is empty.

The above fact can also be verified analytically. For $L < 1$, suppose there exists a point (x_1, y_1) satisfying the constraints of the problem.

That is $x_1 + y_1 < 1$, since L is strictly less than 1

$-x_1 + y_1 \leq -1$

and $x_1 \geq 0$, $y_1 \geq 0$

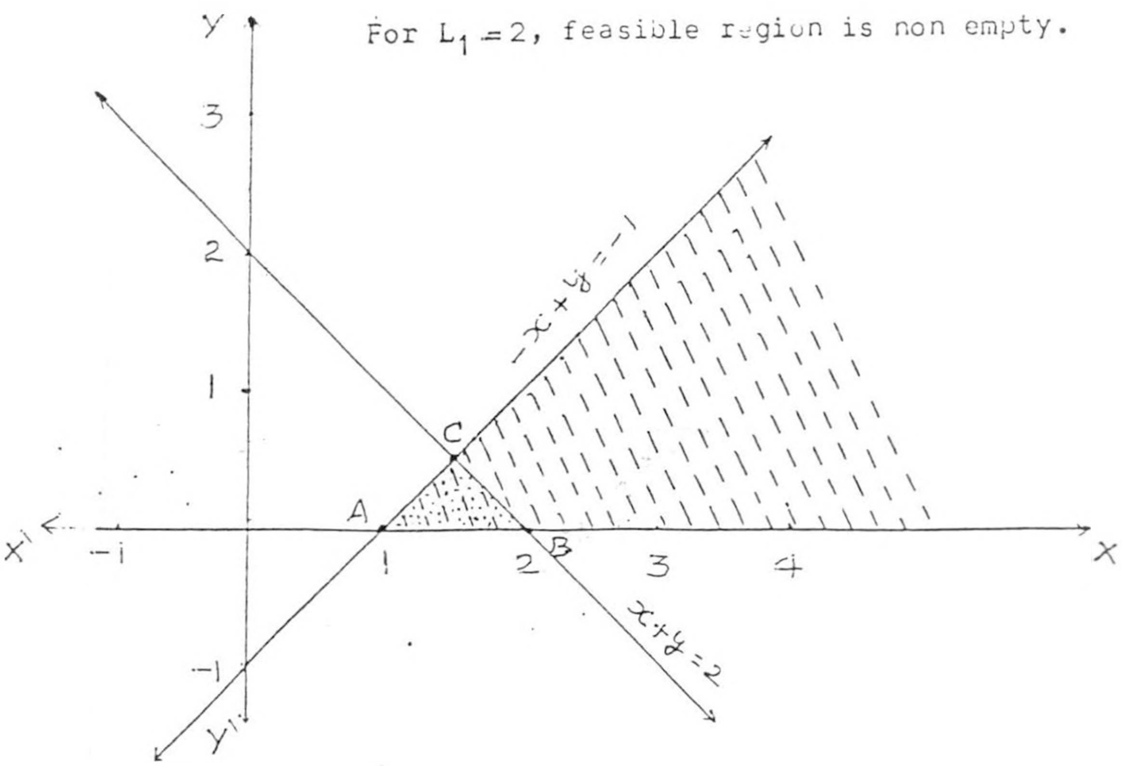
The first two inequalities imply (by adding them), that
$2y_1 < 0$.

In other words, $y_1 < 0$ which contradicts the fact that
$y_1 \geq 0$.

Thus we conclude that there is no point (x,y) which satisfies the given constraints whenever $L < 1$.

If $L = 2$, the feasible region is the shaded region ABC of the figure which is non empty.

For $L_1 = 2$, feasible region is non empty.



From the foregoing discussion, it is clear that the feasible region is non-empty for all values of $L \geq 1$.

If $L = 1$, it consists of just one point. If $L > 1$ it consists of infinitely many points.

We can verify this analytically also. Given constraints are

$x + y \leq L$

$-x + y \leq -1$

$x \geq 0, \; y \geq 0$

First two inequalities (by adding them) imply that

$2y \leq L-1$ or $L-1 \geq 2y$

This implies that

$L-1 \geq 0$, (since $y \geq 0$)

In other words, $L \geq 1$

If $L \geq 1$, choose non negative numbers x_1 and y_1 such that

$2x_1 = L + 1$

and $2y_1 = L - 1$

(This is possible since $L-1 \geq 0$)

These equations imply that

$2x_1 + 2y_1 = 2L$ and $-2x_1 + 2y_1 = -2$

That is, $x_1 + y_1 = L$ and $-x_1 + y_1 = -1$ obviously, such x_1 and y_1 satisfy the given constraints.

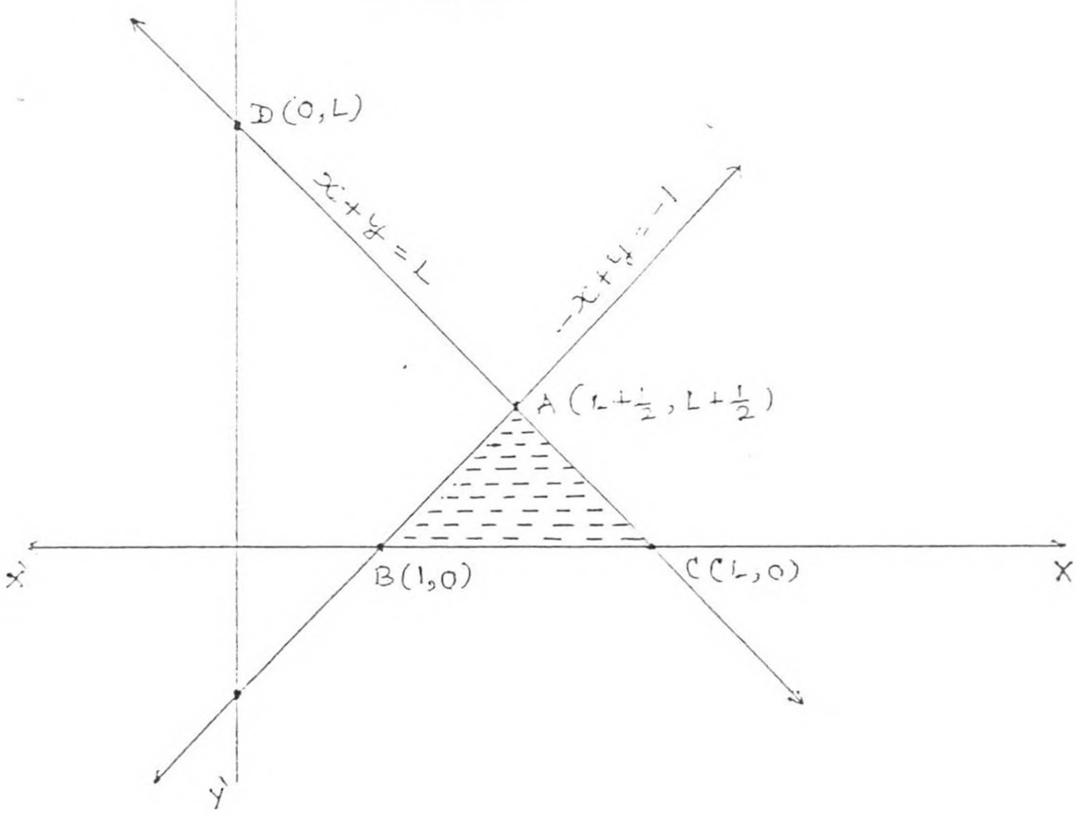
Thus we conclude that there exist numbers $x=x_1$ and $y=y_1$ satisfying the given constraints <u>if and only if</u> $L \geq 1$. That is the given L.P.P. has a feasible solution if and only if $L \geq 1$.

We now solve the given L.P.P.

If $L < 1$, then the given problem has no feasible solution. Therefore, let $L \geq 1$.

If $L = 1$, the feasible solution has just one point (1,0) and so the maximum value of Z is 1. The feasible region for any value of $L > 1$ will look like the shaded region ABC of the following figure.

For any value L 1, feasible region is non empty.



The coordinates of A are obtained by solving x+y = L and -x+y = -1.

i.e., $x = \frac{L+1}{2}$, $y = \frac{L-1}{2}$

Now,

the value of Z at A = $\frac{L+1}{2}$

The value of Z at B = 1

The value of Z at C = L

Since $L \geq 1$, $L+1 \geq 2$ and so $\frac{L+1}{2} \geq 1$

Also, since $L \geqslant 1$, $2L \geqslant L+1$ and so $L \geqslant \frac{L+1}{2}$

Thus, we have

$$1 \leq \frac{L+1}{2} \leq L$$

Therefore, $\max \left\{ 1, \frac{L+1}{2}, L \right\} = L$

That is maximum value of Z is L, and Z attains the maximum at C.

If the problem is to minimize Z=x with the same constraints, minimum value of Z is 1 and it is attained at B.

Exercises :

1. Choose the most appropriate answer.

i) The set of feasible solutions of a linear programming problem is

a) convex b) not a convex set c) convex or concave

b) bounded and convex

ii) The minimum number of inequations needed to find a feasible region in a linear programming problem is

a) 1, b) 2, c) 3, d) 4

iii) The maximum value of the objective function of a linear programming problem always occurs

a) exactly at one vertex of the feasibility region.

b) everywhere in the feasibility region.

c) at all the vertices of the feasibility region.

d) at some vertices of the feasibility region.

iv) The feasible region of a linear programming problem intersects

a) first quadrant b) second quadrant

c) third quadrant d) fourth quadrant

v) A factory has an auto lathe which when used to produce screws of larger size produces 400 items per week and when used to produce screws of smaller size produces 300 items per week. Supply of rods used in making these screws limits the total production of both types/week to 380 items in all. The factory makes a profit of 25 paise per large screw and 10 paise per small screw. How much of each type should be produced to get a maximum profit ? (Ans. 80,300)

vi) Using graphical method
maximise $Z = 3x + 4y$

subject to $4x + 2y \leqslant 80$

$2x + 5y \leqslant 180$

$x \geqslant 0, \quad y \geqslant 0,$

(Ans: $x = 2.5$; $y = 35$; maximum value = 147.5)

vii) Using graphical method

minimise $Z = 4x + 2y$

subject to $x + 2y \geq 2$

$3x + y \geq 3$

$4x + 3y \geq 6$

$x \geq 0, \quad y \geq 0$

(Ans: $x = .6$, $y = 1.2$, minimum value $= 4.8$)

viii) Consider the following problem :

Maximize $Z = 6x_1 - 2x_2$

subject to $x_1 - x_2 \leq 1$; $3x_1 - x_2 \leq 6$; $x_1, x_2 \geq 0$.

Show graphically that at the optimal solution the variables x_1, x_2 can be increased indefinitely, while the value of the objective function remains constant.

ix) Consider the following LPP :

Maximize $Z = 4x + 4y$

subject to $2x + 7y \leq 21$; $7x + 2y \leq 49$; $x, y \geq 0$.

Find the optimal solution (x,y) graphically. What are the ranges of variation of the coefficients of the objective function that will keep (x , y) optimal ?

x) Consider the following problem.

Maximize $Z = 3x + 2y$ subject to $2x + y \leq 2$, $3x + 4y \geq 12$, $x, y \geq 0$.

Show graphically that the problem has no feasible extreme points. What can one conclude concerning the solution of the problem ?

xi) Prove or disprove :

a) For some LPP, the set of feasible solutions is a disjoint union of convex sets.

b) The set of feasible solutions of every LPP is non empty.

c) Every L.P.P. is a mathematical model.

Applications of L.P.

LP is a powerful and widely applied technique to solve problems related to decision making. It was employed formally in three major categories - military applications, inter industry economics and zero sum two-person games. But, now the emphasis has been shifted to the industrial area. The following are a few of the applications of L.P.

1. Agricultural applications :
   Farm economics and Farm management - the first is related to the economy of a region whereas the second is related to individual farm.

2. Industrial applications :
   a) Chemical Industry - Production and inventory control - chemical equilibrium problem.
   b) Coal industry
   c) Airline operations
   d) Communication industry - optical design and utilisation of communication network
   e) Iron and steel industry
   f) Paper industry - for optimum newsprint production
   g) Petroleum industry
   h) Rail road industry
3. Economic analysis - Capital budgeting
4. Military - Weapon Selection and Target analysis
5. Personal assignment
6. Production scheduling - inventory control and planning cost controlled production
7. Structural designs
8. Traffic analysis
9. Transportation problem and network theory
10. Travelling salesman problem
11. Logical design of electrical network
12. Efficiency in the operations of a system of Dams

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N U M E R I C A L   M E T H O D S

1. The solution of Algebraic Equations

2. Numerical Integration

by
Dr.B.S.P.RAJU

NUMERICAL METHODS :

Introduction :

Numerical Methods are the methods or procedures that explains how to find approximate solution of problems from a given numerical data. These procedures use only basic arithmetic operations like addition , subtraction, multiplication, division and exponentiation and certain other logical operations such as algebraic comparisons unlike other methods where we use complicated techniques like differentiation and integration. This is because the procedures are so designed for the execution on a modern high speed digital computer.

The numerical data used in solving the problems of every day life are usually not exact, and the numbers expressing such data are therefore not exact. Not only the data, sometimes the methods and processes by which the desired result is to be found are also approximate. The numerical data is not exact because of one or many of the following reasons.

1. The imperfections in the instrument such as faulty graduation marks, warping of a wooden yardstick due to moisture, precision of the instrument.

2. Personal errors arising due to personal bias or judgement; lack of knowledge of the use of instruments.

3. Theoretical errors arising due to the use of the instruments other than those for which the instrument is designed or calibrated.

4. Accidental errors over which the observer has no control.

So all numerical calculations are approximate and so we study what are approximate numbers.

1.2 Approximate Numbers :

Approximate numbers will arise from measurement, from estimates, from rounding exact numbers or from computations with exact numbers. The rational numbers 5/6 and $0.83\frac{1}{3}$ are both exact but the natural numbers .83 is an approximate number for $\frac{5}{6}$. The irrational number $\sqrt{10}$ is an exact number, but the rational number 3·16 is an approximation of the irrational number $\sqrt{10}$.

Since approximate number implies the existence of an exact number, we naturally wish to know how "good" or how "close". The approximation is. This we read in the next two sections.

Significant figures :

A significant figure is any one of the digits 1,2,3,...,9 and 0 is a significant figure except when it is used to fix the decimal point or to fill the places of unknown or discarded digits.

It can be split it into the following ways :

1. All non zero figures are always significant wherever used.
2. Zeros occuring between non-zero figures are always significant.
3. Terminal zeros following the decimal point are always significant.
4. In a number less than 1, zeros immediately following the decimal point are not significant.

Ex: 48.3 all are significant figures.
46.05 -all are significant figures.
.002-two is the only significant figure.
0.20-two and zero are significant figures.

Rounding off numbers :

To round off a number to n significant figures discard all digits to the right of the nth place.

If the digit in the (n+1) th place is less than 5 leave the nth digit unchanged. If the digit in (n+1) th place is greater than 5 add 1 to the nth digit. If the digit in the n+1 th place is equal to 5 leave the nth digit unaltered if it is an even number, but increase it by 1 if it is an odd number.

We follow the above rule, to reduce the errors to a minimum.

Nos.	Rounded to 4 significant figures
65.634	65.63
65.63618	65.64
65.63501	65.64
65.68501	65.68

The solution of Numerical Algebraic Equations

Introduction : Finding the roots of an algebraic equation is one of the challenging, interesting and is fascinating to many since a long time. We have a well defined formula to find the roots of any quadratic equation and beyond this we do not have definite formula that helps to solve any equation. Well; though there is a method to solve a cubic equation, it is not that simple. There are methods to solve when the equations are of particular type say reciprocal equations etc. but no general method occurs beyond the degree 3.

Here we will discuss some methods in the following sections.

Equation: If n is a positive integer, and $a_0, a_1, a_2, \ldots, a_n$ are constants and $a_0 \neq 0$ an expression of the form

$$a_0 x^n + a_1 x^{n-1} + \cdots + a_n$$

is called a polynomial in x of nth degree.

The equation obtained by putting the polynomial equal to zero is called an algebraic equation.

$$a_0 x^n + a_1 x^{n-1} + \cdots + a_n = 0$$

of nth degree.

Root : Any value of x for which the polynomial f(x) vanishes is called a root of the equation

$$f(x) = 0$$

Relation between the roots and coefficients of equation :

Let the equation be

$$a_0 x^n + a_1 x^{n-1} + \cdots + a_n = 0$$

If $\alpha_1, \alpha_2, \cdots \alpha_n$ are the roots of this equation, then we have

$$\sum_{i=1}^{n} \alpha_i = -\frac{a_1}{a_0}$$

$$\sum_{i,j=1}^{n} \alpha_i \alpha_j = \frac{a_2}{a_0}$$

$$\alpha_1 , \alpha_2 \alpha_3 \cdots \alpha_n = (-1)^n \frac{a_n}{a_0}$$

Here we give some results which are useful in the coming sections :

1. f(x) defined above is a continuous function in x for all values of x.
2. Every equation f(x) = 0 of the nth degree has n and only n roots.
3. In an equation with real coefficients, imaginary roots occur in pairs.
4. In an equation with rational coefficients irrational roots occur in pairs.

Finding an approximate value of the roots (Initial approximation) :

All the methods in this chapter require one or two approximate values of a root (initial values) to begin with, which can be found in one of the following ways :

Given f(x) = 0;
We substitute at random successive values for x till for two successive values of x, f(x) changes sign.
This is based on the following theorem :

If f(x) is continuous from x = a to x = b and if f(a) and f(b) have opposite signs, then there is atleast one real root between a and b.

Let $f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n = 0$

The largest root may frequently be approximated by the root of the linear equation.

$$a_0x + a_1 = 0$$

or by the root larger in absolute value of the quadrative equation

$$a_0x^2 + a_1x + a_2 = 0$$

The smallest root may similarly be approximated by the root of the equation

$a_{n-1}x + a_n = 0$

or by the smaller root in absolute value of the quadratic equation

$a_{n-2}x^2 + a_{n-1}x + a_n = 0$

BISECTION METHOD OR HALF INTERVAL METHOD

The method has been explained in the text book. But here we give an algorithm.

Step 1 : Find out two values x_1 and x_u for the roots of the given equation so that f(x) have opposite signs

i.e. $f(x_1)\ f(x_u) < 0$

Step 2 : Compute $x_r = \frac{x_1 + x_u}{2}$

If this x_r is accurate enough to meet your requirement, go to Step 4.

Step 3 : If $f(x_1)\ f(x_r) < 0$, then the root lies in $[x_1, x_r]$ and set $x_u = x_r$ and to go to step 2.

If $f(x_1)\ f(x_r) > 0$; then the root lies in $[x_r, x_u]$ and set $x_1 = x_r$ and go to step 2.

If $f(x_1)\ f(x_r) = 0$; then the root is equal to x_r and go to step 4.

Step 4: x_r is one of the root of the equation.

Since each application of the iterative scheme reduces the length of the interval in x, by half, known to contain α.
Where α is the root of the equation, this procedure is called the half interval method.

2.7.1 To find the number of iterations or operations to carry out to get the root within a prescribed tolerance :

Let $[a_0, b_0]$ be the initial interval in which the root lies, and be the prescribed tolerance and so

$$\frac{b_0 - a_0}{2^n} \leq \varepsilon$$ where n is the number of iterations

Taking logarithms on both sides

$$\log \frac{b_0 - a_0}{2^n} \leq \log \varepsilon$$

i.e., $\log (b_0 - a_0) - \log 2^n \leq \log \varepsilon$

i.e., $\log (b_0 - a_0) - \log \leq n \log n$

i.e. $n \log 2 \geq \log\left(\frac{b_0 - a_0}{\varepsilon}\right)$

$$n \geq \frac{\log\left(\frac{b_0 - a_0}{\varepsilon}\right)}{\log 2}$$

Newton - Raphson Method :

We give here a method which is different from that of text book.

Let $y = f(x)$ be the equation whose roots are to be found. Let α be one of theroot, x_0 be the approximate value of the root and h denote the correction which must be applied to give the exact value of the root, so that

$$f(x_0 + h) = f(\alpha) = 0 \qquad (1)$$

By Taylor's Theorem,

$$f(x_0 + h) = f(x_0) + h\frac{f'(x_0)}{1} + \frac{h^2 f''(x_0)}{2} + \cdots$$

By neglecting the higher order terms from h^2 and by (1)

$$0 = f(x_0) + \frac{h\, f^1(x_0)}{1}$$

$$h = \frac{-f(x_0)}{f^2(x_0)}$$

If $x_1 = x_0 + h$, Then $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

Similarly, $x_2 = x_1 + h = x_1 - \frac{f(x_1)}{f'(x_1)}$

$\vdots$

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

To find the square root of a number by Newton Raphson Method :

Let K be the number whose square root is to be found.

i.e. Let $\sqrt{k} = x$

Squaring on both sides .

$$K = x^2$$

i.e. $x^2 - k = 0$

If we solve the equation we will get the square root of the given number k.

Let $f(x) = x^2 - k$

$f'(x) = 2x$

By applying the Newton Raphson method

$$x_{n+1} = x_n - \frac{x_n^2 - k}{2x_n}$$

$$= \frac{x_n^2 + K}{2x_n}$$

$$= \frac{1}{2}\left(x_n + \frac{k}{x_n}\right) \text{ for } n = 0,1,2,\ldots.$$

To find the reciprocals of numbers without division by Newton Raphson Method.

For a given number $k > 0$, we want to find the value of $1/k$ (we are not considering $k = 0$, because in that case $\frac{1}{k} = \frac{1}{0}$ is ∞ - undefined and $k < 0$; i.e. $k = -ve$ i.e. $\frac{1}{-k} = -\frac{1}{k}$ so we can add - sign to $+\frac{1}{k}$ and hence it is unnecessary).

Let $x = \frac{1}{k}$

i.e. $\frac{1}{x} = \frac{1}{1/k}$

(we are not considering $x - \frac{1}{k} = 0$ in which case

$f(x) = x - \frac{1}{k}$

and $f^1(x)$ becomes 1

$x_{n+1} = x_n - \frac{(x_n - 1/k)}{1}$ i.e, $x_{n+1} = \frac{1}{k}$)

i.e. $\frac{1}{x} - k = 0$

Let $f(x) = \frac{1}{x} - k$

$f^1(x) = -\frac{1}{x^2}$

Substituting $f(x)$ and $f^1(x)$ in Newton Raphson's formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f^1(x_n)}$$

$$x_{n+1} = x_n - \frac{\frac{1}{x_n} - k}{-\frac{1}{x_n^2}}$$

$$= x_n + \left(\frac{1}{x_n} - k\right) \times x_n^2$$

$$= x_n + x_n - K x_n^2$$

$$= 2x_n - k x_n^2 \qquad = x_n (2 - kx_n) \text{ for } n = 0,1,2,\ldots$$

By Newton Raphson method, we get the root in a few iterations that the methods we discussed earlier. But it is not always guaranteed that as the iterations increases it gives convergent root. It depends on the initial value chosen. So here we give the bounds between which the initial value should lie so that it gives a convergent root in the case of finding the reciprocal of a number by Newton Raphson method.

We have $x_{n+1} = x_n\ (2-kx_n)$

Let $F(x) = x\ (2-kx)$

$= 2x-kx^2$ so $F^1(x) = 2-2kx$

$-1 < F^1(x) < 1$

i.e. $-1 < 2\ (1-kx) < 1$

i.e. $-1 < 2\ (1-kx)$ | $2\ (1-kx) < 1$

$-1/2 < 1-kx$ | $1-kx < 1/2$

$-3/2 < -kx$ | $-kx < 1/2$

$3/2 > kx$ | $kx > 1/2$

$kx < 3/2$ | $x > 1/2K$

$x < 3/2K$

i.e. $\frac{1}{2k} < x < \frac{3}{2k}$

k is a positive integer

$0 < x < \frac{3}{2k} < \frac{2}{k}$

i.e. $0 < x < 2k^{-1}$

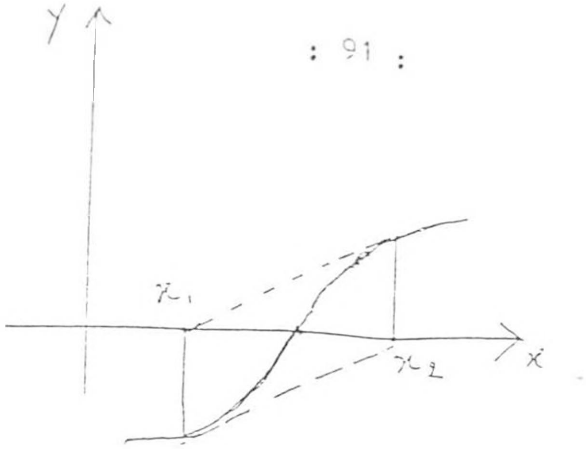
In order to get a convergent root it is better if we choose the initial value of x between 0 and 2/k.

As an example for the notes given above

If $f(x) = \begin{cases} \sqrt{x-r} & \text{for } x \geq r \\ -\sqrt{r-x} & x < r \end{cases}$

and $x_1 = r-h$; then $x_2 = r+h$

and successive approximations will move back and forth between the two values.



Remark: The method will fail at any time when $f^1(x_n) = 0$ for some n. Then in that case, choose a new starting value.

Assignment and Self Test

1. Find a root of the equation given below by Bisection method, false position method and Newton Raphson method.

1. $x = \dfrac{1}{(x+1)^2}$
2. $x = (5-x)^{1/3}$
3. $x^3-x-4 = 0$
4. $x^4-x - 10 = 0$
5. $x^3-100 = 0$
6. $-0.874x^2 + 1.75x + 2.627 = 0$
7. $t^3-4x^2 - 6t + 4 = 0$
8. $x^3-2x-5 = 0$
9. $x^3 - 5x + 3 = 0$

Find the square root of the following by Newton-Raphson method to four significant figures.

1. 3 2. 5 3. 7 4. 11

Find the reciprocals of the following numbers by Newton Raphson method without actual division.

1. 3 2. 6 3. e

NUMERICAL INTEGRATION

Introduction:

Given a polynomial we can be able to write the values of y for a given value of x. But in scientific applications generally we get the tabulated data. From the tabulated data we fit a polynomial using finite differences. Such polynomials are called inter-polating polynomials. There are several interpolation formulae like Newton's gregory forward and backward formulae, Bessel's Sterling's formula etc. which are very much useful in numerical differentiation and integration. Here in the following section we derive Newton-Cotes Integration formula, so that we can deduce the Simpson's 1/3 rule, Simpson's 3/8th rule, trapezoidal rule, Weddle's rule etc.

Newton-Cotes Integration Formula :

Let $x_0, x_1, x_2, \ldots, x_n$ are the n+1 evenly spaced base points from which an interpolating polynomial of degree n has been obtained with the help of the functional values at the base points. (Here we assume the polynomial. For an understanding of interpolating polynomials, readers can refer the book Numerical Analysis by SCARBOROUGH). Let a be the lower limits of integration coincides with the base point x_0. Let b, the upper limit of integration, be arbitrary for the moment.

Then,

$$\int_a^b f(x)\, dx = \int_{x_0}^{b} P_n(x)\, dx$$

$$\alpha = \frac{x - x_0}{h} \qquad \therefore \text{ Lower limit } = \frac{x_0 - x_0}{h} = 0$$

$$x = x_0 + h \qquad \therefore \text{ Upper limit } = \frac{b - x_0}{h}$$

$$dx = h . d\alpha \qquad \text{Let this be equal to } \bar{\alpha}$$

$$= h \int_0^{\bar{\alpha}} P_n (x_0 + h)\, d\alpha$$

$$\simeq h \int_0^{\bar{\alpha}} \left[f(x_0) + \alpha \Delta f(x_0) + \frac{\alpha(\alpha-1)}{\underline{2}} \Delta^2 f(x_0) + \cdots \right] d\alpha$$

$$\simeq h \left[\alpha f(x_0) + \frac{\alpha^2}{2} \Delta f(x_0) + \left(\frac{\alpha^3}{6} - \frac{\alpha^2}{4} \right) \Delta^2 f(x_0) + \left(\frac{\alpha^4}{24} - \frac{\alpha^3}{6} + \frac{\alpha^2}{4} \right) \Delta^3 f(x_0) + \cdots \right]_0^{\bar{\alpha}}$$

All the terms vanish at the lower limit and so

$$\int_0^h f(x)\,dx \simeq h \left[\bar{\alpha} f(x_0) + \frac{\bar{\alpha}^2}{2} \Delta f(x_0) + \left(\frac{\bar{\alpha}^3}{6} - \frac{\bar{\alpha}^2}{4} \right) \Delta^2 f(x_0) + \left(\frac{\bar{\alpha}^4}{24} - \frac{\bar{\alpha}^3}{6} + \frac{\bar{\alpha}^2}{4} \right) \Delta^3 f(x_0) + \cdots \right] \rightarrow (I)$$

If the upper limit b is chosen to coincide with one of the base points so that $b = x_m$ (say) then

$$\bar{\alpha} = \frac{x_m - x_o}{h} = \frac{x_o + mh - x_o}{h}$$

$$= \frac{mh}{h} = m$$

i.e. assumes the integral value m.

By giving various values to $\bar{\alpha}$ we get various formulas i.e.
by putting $\bar{\alpha} = 1$, we get trapezoidal rule
by putting $\bar{\alpha} = 2$, we get Simpson's 1/3rd rule
by putting $\bar{\alpha} = 3$, we get Simpson's 3/8th rule
by putting $\bar{\alpha} = 6$, we get Weddle's rule and so on.

<u>Trapezoidal Rule</u> :

Put $\bar{\alpha} = m = I$, in (I) $a = x_o$

$$\int_{x_0}^{x_0+h} f(x)\,dx = h \left[f(x_o) + \frac{1}{2} \Delta f(x_o) \right]$$

$$= h \left[f(x_o) + \left[\frac{f(x_o+h) - f(x_o)}{2} \right] \right]$$

$$= \frac{h}{2} \left[2 + (x_o) + f(x_o+h) - f(x_o) \right]$$

$$= \frac{h}{2} \left[f(x_o) + f(x_o+h) \right]$$

$$= \frac{h}{2}\left[f(x_0) + f(x_1)\right] \quad \text{where } x_1 = x_0 + h$$

$$\int_{x_0+h}^{x_0+2h} f(x)\,dx = \frac{h}{2}\left[f(x_1) + f(x_2)\right]$$

$$\vdots$$

$$\int_{x_0+(n-1)h}^{x_0+nh} f(x)\,dx = \frac{h}{2}\left[f(x_{n-1}) + f(x_n)\right]$$

Adding all these integrals we get

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n)\right]$$

$$= \frac{h}{2}\left[f(x_0) + 2\left\{f(x_1) + f(x_2) + \ldots + f(x_{n-1})\right\} + f(x_n)\right]$$

This is called the trapezoidal rule.

Note: If we increase the number of intervals then the trapezoidal rule gives a better value.

SIMPSON'S 1/3rd RULE :

Let $\bar{\alpha} = m = 2$ in (I) (derived earlier, then $b = x_2$; $a = x_0$)

$$\int_{x_0}^{x_2} f(x)\,dx = h\left[2f(x_0) + \frac{4}{2}\Delta f(x_0) + \left(\frac{2^3}{6} - \frac{2^2}{4}\right)\Delta^2 f(x_0)\right]$$

$$= h\left[2f(x_0) + 2\,\Delta f(x_0) + \frac{1}{2}\Delta^2 f(x_0)\right]$$

$$= h\left[2f(x_0) + 2\left\{f(x_1) - f(x_0)\right\} + 1/3\left\{f(x_2) - 2f(x_1) + f(x_0)\right\}\right]$$

$$= h\left[2f(x_0) + 2f(x_1) - 2f(x_0) + 1/3\, f(x_2 - 2/3\, f(x_1) + 1/3 f(x_0)\right]$$

$$= \frac{h}{3}\left[6f(x_1) + f(x_2) - 2f(x_1) + f(x_0)\right]$$

$$= \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]$$

$$\int_{x_2}^{x_4} f(x)\,dx = \int_{x_0+2h}^{x_0+4h} f(x)\,dx$$

$$= \frac{h}{3}\left[f(x_2) + 4f(x_3) + f(x_4)\right]$$

$$\int_{x_0+(n-2)h}^{x_0+nh} f(x)\,dx = \frac{h}{3}\left[f(x_{n-2}) + 4(x_{n-1}) + f(x_n)\right]$$

Adding all the above we get

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{3}\Big[f(x_0) + 4\left\{f(x_1)+f(x_3)+\ldots+f(x_{n-1})\right\}$$
$$+ 2\left\{f(x_2) + f(x_4) + \ldots + f(x_{n-2})\right\} + f(x_n)\Big]$$

This formula is known as Simpson's 1/3rd rule.

Note: The interval of integration should contain an even number of steps of width h.

By means of an example we explain how to choose the interval h so that the value obtained by Trapezoidal rule is correct to any number of given places.

Ex: Predict how tightly values of f(x) must be packed (what interval h) for the trapezoidal rule itself to achieve a correct result to four places,

for $\int_1^2 \frac{dx}{x}$

Ans: Given $f(x) = \frac{1}{x}$

$$f^1(x) = -\frac{1}{x^2}$$

$$f^{11}(x) = \frac{2}{x^3}$$

Max. $f^{11}(x) = 2$ for $x = (1,2)$

Truncation error $-\frac{(b-a)}{12} h^2 \underset{\text{Max}}{f^{11}(x)}$ (Reader can get it from any standard book on Numerical Analysis)

$$= -\frac{(2-1)}{12} h^2 \quad (2)$$

$$= -\frac{h^2}{6} < .00005 \quad \text{(given in problem)}$$

i.e. $h^2 < .0003$

i.e. $h < \frac{\sqrt{3}}{100}$

Evaluate the following integrals by using trapezoidal rule and Simpson's 1/3rd rule.

1. $\int_{1}^{3} (x^3 - 2x^2 + 7x - 5)\, dx$

2. $\int_{1}^{1.3} \sqrt{x}\, dx$

3. $\int_{1}^{2} \frac{dx}{x}$

4. $\int_{1}^{1.4} \frac{x}{\sin h\, x}\, dx$

5. $\int_{-3}^{3} x^4\, dx$

6. $\int_{0}^{1} \frac{dx}{1+x}$

7. Predict how many values of f(x), or how small an interval h will be needed for Trapezoidal rule to produce $\log_e 2$ connect to four places.

A L G O R I T H M S & F L O W C H A R T S

by
Dr.D.BASAVAYYA

ALGORITHMS AND FLOWCHARTS

Computers cannot think for themselves. They are automatically controlled and do the work of many human beings incredibly at high speeds. The really important thinking is done by the human who feed them with information and the desired instructions. A set of instructions given to a computer to solve a problem is known as a computer program. Before writing such computer programs, it is useful and necessary to list out the steps involved in solving the problem sequentially. A step-by-step list of instructions for solving a particular problem is known as the algorithm of that problem.

The concept of algorithm is basic to any computational scheme, numerical or non-numerical. The word 'algorithm' originated from the word 'algorism' - means the art of computing Arabic numerals. Some scholars relate the origin with the name of a famous Arabic mathematician, Abu Jafar Muhammed ibu Musa al-Khwarizmi (825 A.D.) who first suggested the method of adding decimal numbers by taking one digit from each of the operands and a previous carry digit.

Algorithm is a logical process of analysing a mathematical problem and data step by step so as to make it vulnerable for easy computing or for conversion of data into information. In our day-to-day work, the brain automatically performs the algorithm by experience but not in a systematic and error free way. Algorithm represented in many forms such as

i) step-by-step method
ii) flowcharting and
iii) by programming language.

Step by Step Method :

Example 1 : Suppose we want to find the largest of any three given numbers.

Assume that A, B and C are the given numbers. First compare A and B and identify the larger. Then compare this bigger number with C and identify the bigger among these two as the largest of all the three given numbers.

This method of solving this problem may be expressed as the following series of instruction.

Step 1 : Note the three given values as A, B, C.

Step 2 : Compare A and B.

Step 3 : If A is greater than B, then do the step 4 otherwise do the step 5.

Step 4 : Compare A and C and write the greater number and stop.

Step 5 : Compare B and C and write the greater number and stop.

Example 2 : Suppose you want to solve the following pair of equations for x and y given the values a,b,c,d,e,f.

$ax + by = c$ .... (1)

$dx + ey = f$ .... (2)

Assume that a,b,c,d,e,f are non zero. The procedure to solve these equations is as follows : From equation (1) we have

$x = (c-by)/a$ ... (3)

Substituting x in (2) we get

$$\frac{d\ (c - by)}{a} + ey = f$$

or $(e - db/a)\ y = f - \frac{dc}{a}$

Thus using this value of y in (3) we get the value of x. This method of solving the equations may be expressed as follows :

Step 1 : Note the values of a,b,c,d,e,f.

Step 2 : Calculate the value of $(e - \frac{db}{a})$ and call it as s.

Step 3 : Calculate the value of $(f - \frac{dc}{a})$ and call it as t.

Step 4 : If s = 0 then write 'No solution' and stop.

Step 5 : Set y = t/s.

Step 6 : Set $x = \frac{(c - by)}{a}$

Step 7 : Write the values of x and y as answers.

Step 8 : Stop.

The above set of 8 instructions is known as an algorithm of that problem.

A good algorithm should possess the following characteristics.

Algorithm 1. should be simple (simplicity).

2. should be clear without ambiguity (definiteness).

3. should lead to unique solution of the problem (uniqueness).

4. should involve a finite number of steps to arrive a solution (finiteness).

5. should have capability to handle some unexpected situations which may arise during the solution problem like division by zero (completeness).

6. should be effective in saving time (effectiveness) - using that particular algorithm, processing should be quicker.

The description of an algorithm (Step-by-step method) requires a suitable language. Although we can describe an algorithm in a combination of natural language (say, English) and mathematical notations, there are some drawbacks. They are

1. The algorithm is not usually concise.
2. A natural language is ambiguous.
3. The manner of expression does not reveal the basic structure of the algorithm.
4. Certain operations cannot be expressed by existing mathematical notations.
   Ex : Replace B by A.

Because of these drawbacks, sometimes, we have to prefer other methods. One of the most convenient languages which is effective for communication and description of an algorithm is the language of flow-charting.

Flowcharts :

Flowcharting is a technique for representing a succession of events in symbolic form. Flowchart is the diagrammatic representation of a sequence of events, usually drawn with conventional symbols (geometrical figures) representing different types of events and their interconnection. Flowchart is also known as flowdiagram.

In flowchart the essential steps of the algorithm are pictured by boxes of various geometrical shapes and the flow of data between steps is indicated by arrows, or flowlines. The flowchart is usually drawn so that the flow direction is downward or from left to right. The symbols themselves are of standardized shapes that indicate the type of action taking place at that step of the algorithm. In fact, each symbol is labelled by its algorithm step, written within the symbol. Flowcharts may be divided into systems flowcharts and program flowcharts. A chart that depicts the flow of data in the over -all data processing system, or phase of the system, is called a system flowchart or process chart. A chart that depicts the operations and logical decisions in a problem solving is called a program flowchart or logical flowchart. In our discussion we are concerned with program flowchart. The preparation of either type of chart is called flowcharting. The different symbols used in any flowchart (standard) are as follows :

Terminal Symbol : The oval symbol is used to indicate the beginning or end of an algorithm by (Start) or (Stop) respectively. Clearly, a flowchart can contain only one start symbol; however it can contain more than one stop symbol, since the algorithm may contain alternative branches. Sometimes, we omit the start/stop symbols if it is clear where the chart begins/ends.

Input/Output Symbol : The parallelogram symbol is used to indicate an input or an output operation. Specifically, we write

/ Read A,B,C,D /

to indicate that data are to be inputted into the memory locations A, B, C and D, in that order. Similarly,

/ Write X,Y,Z /

indicates that the data in the memory locations X,Y,Z are to be outputted. We can also output messages by including the message in quotation. e.g.

/ Write 'No solution' /

indicates that the message 'No solution' is to be outputted.

Process Symbol : The rectangle symbol is used to indicate a processing operation. This can be an assignment statement, defined below or it can be macroinstruction, whose programming language translation would otherwise require an entire list of computer statements; e.g.

Alphabetize Names	Find mean and standard deviation of marks

are macroinstructions.

The assignment statement has the form

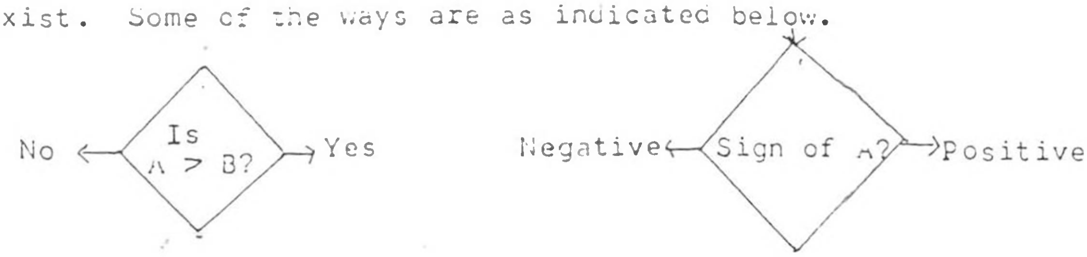
Variate = arithmetic expression

Another acceptable form for an assignment statement is

Variable ← arithmetic expression

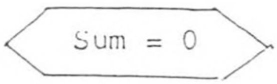
in which the backward arrow ← is used to instead of the equals sign.

Decision Symbol : The diamond-shaped (Rhombus) symbol is used to indicate a decision. This represents a point at which different process paths may be selected. The question is placed inside the symbol, and each alternative answer to the question is used to label the exist arrow which leads to the appropriate next step of the chart. We note that the decision symbol is the only symbol that may have more than one exist. Some of the ways are as indicated below.



Connector Symbol : A flow chart that is long and complex may require more than one sheet of paper, which means that certain flow lines cannot be drawn or the flow chart may contain crossing flow lines that could cause confusion. The connector symbol, a small circle  , is used to remedy such situations. Specifically, one assumes that a flow line exists between any part of identically labelled connector symbols such that the flow is out of the flow chart at one of the connectors (exist connector) and into the flow chart at the other connector (entry connector).

Preparation Symbol : This symbol indicates the preparation for some procedure by initializing certain variables, e.g.

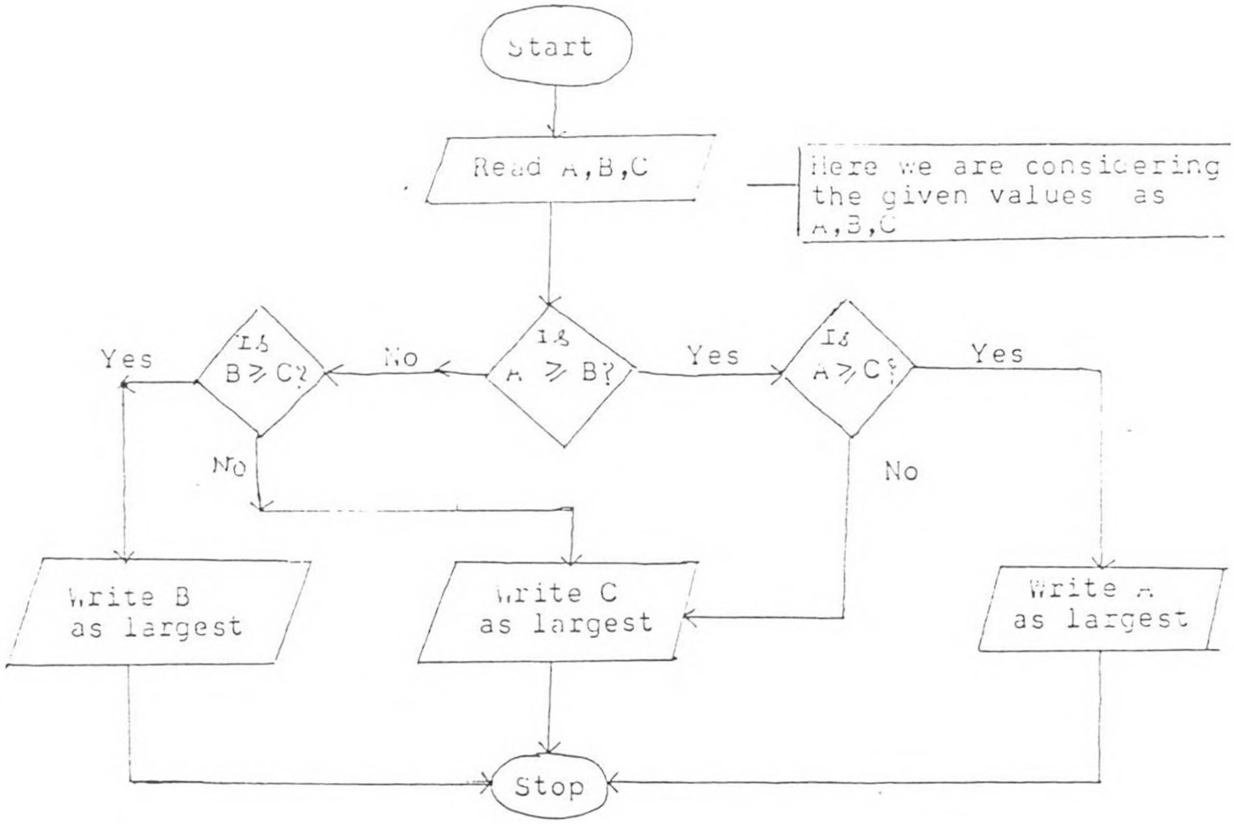


Many programmers indicate a preparation by means of the process symbol only.

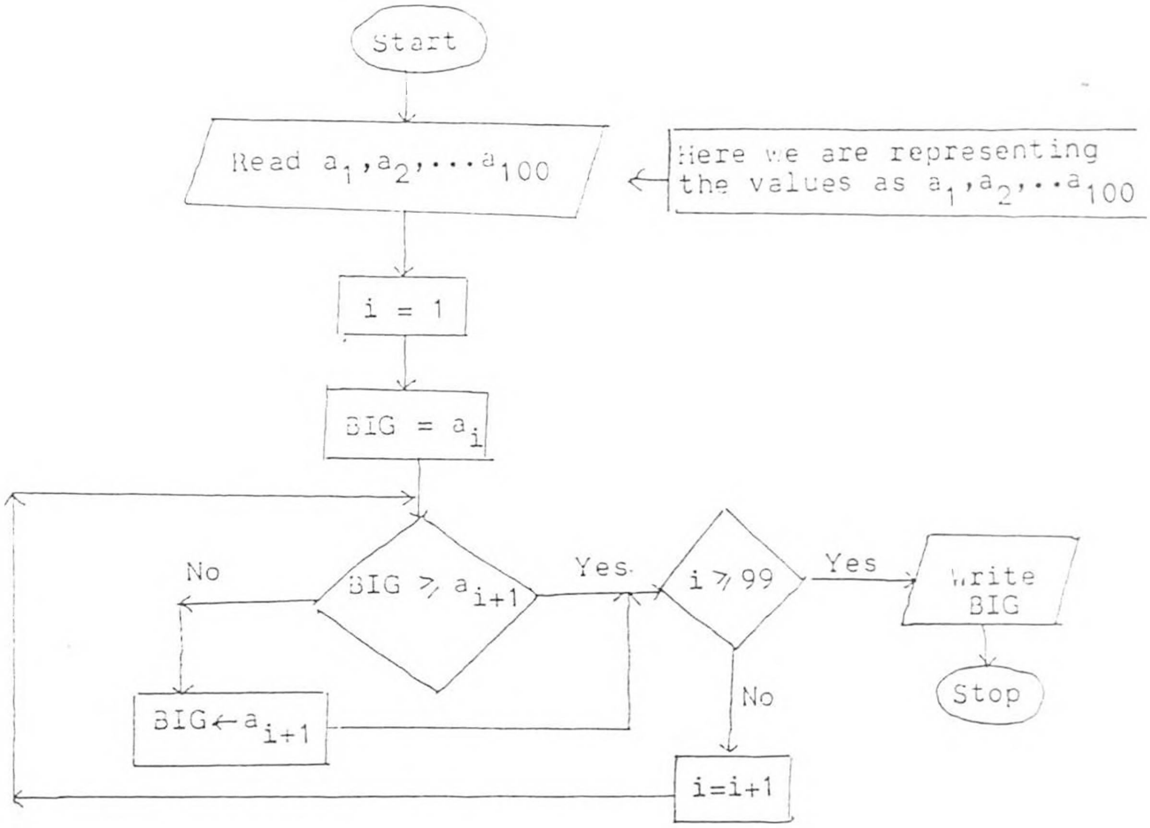
Comment Symbol : The following symbol used to indicate comments on the contents of a symbol.

Consider the following examples to explain the use of different symbols in writing a flowchart.

Example 1 : Pick up the largest of any three given numbers. The flowchart for this problem is as follows :



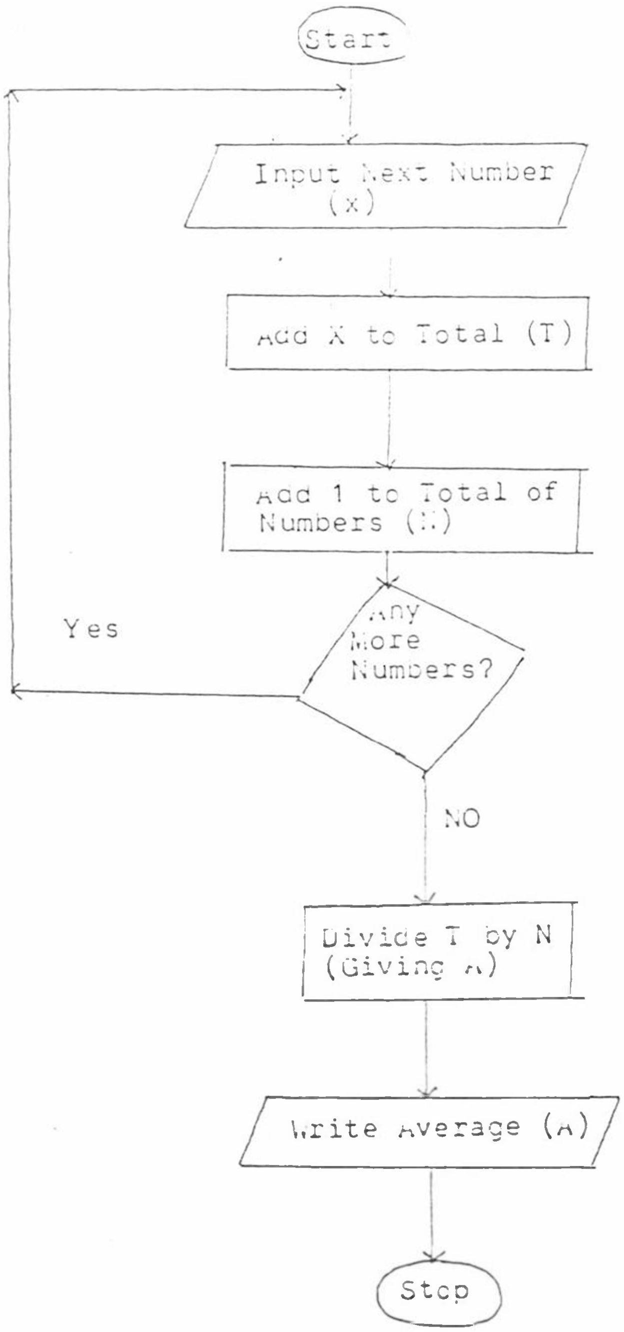
Example 2 : Pick up the largest of any given hundred numbers. The required flowchart is



Example 3 : Draw a flowchart to find the average of any number of numbers.

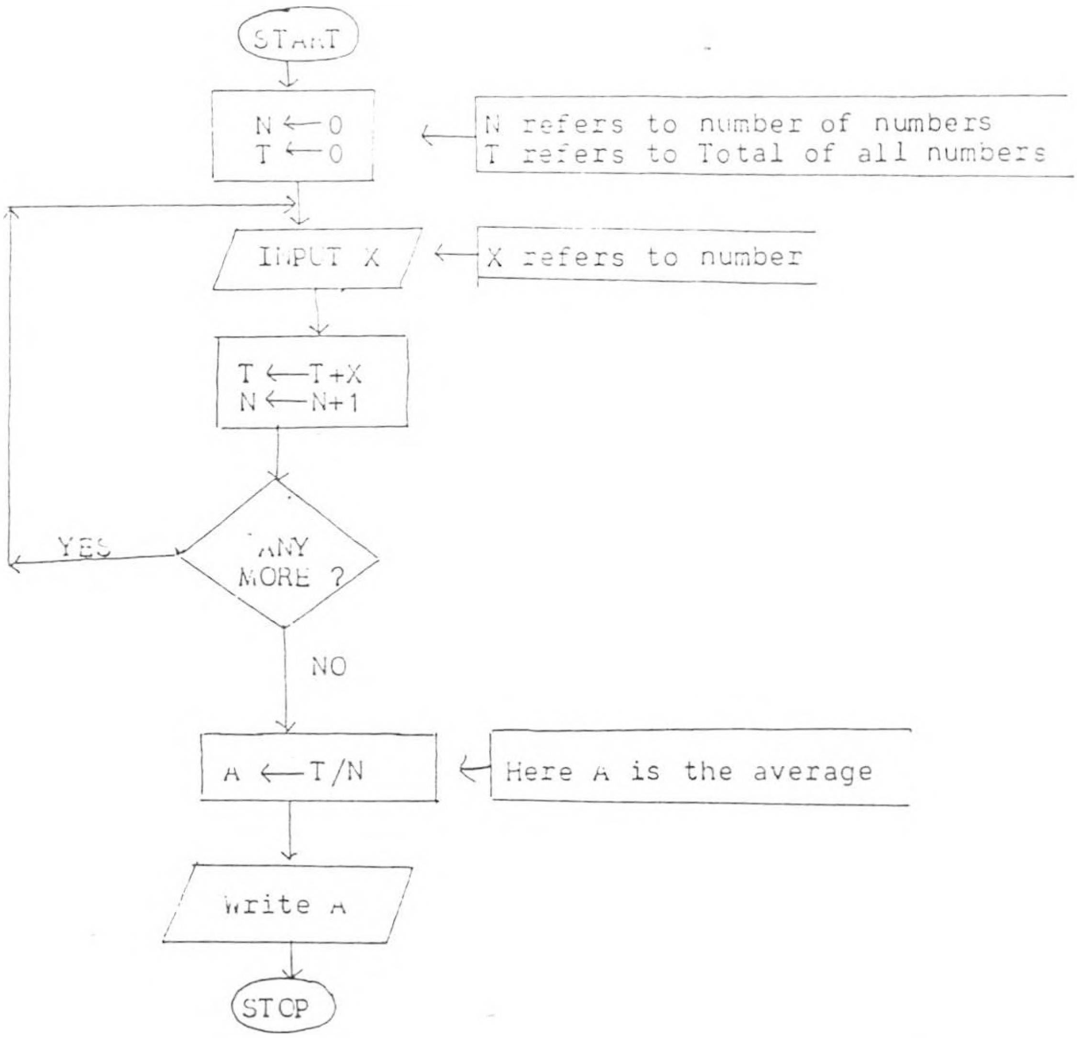
Here, we write the flowchart in two ways : i) by using English words and ii) by defining variables.

First type :



Start
Input Next Number (x)
Add X to Total (T)
Add 1 to Total of Numbers (N)
Any More Numbers?
Yes
NO
Divide T by N (Giving A)
Write Average (A)
Stop

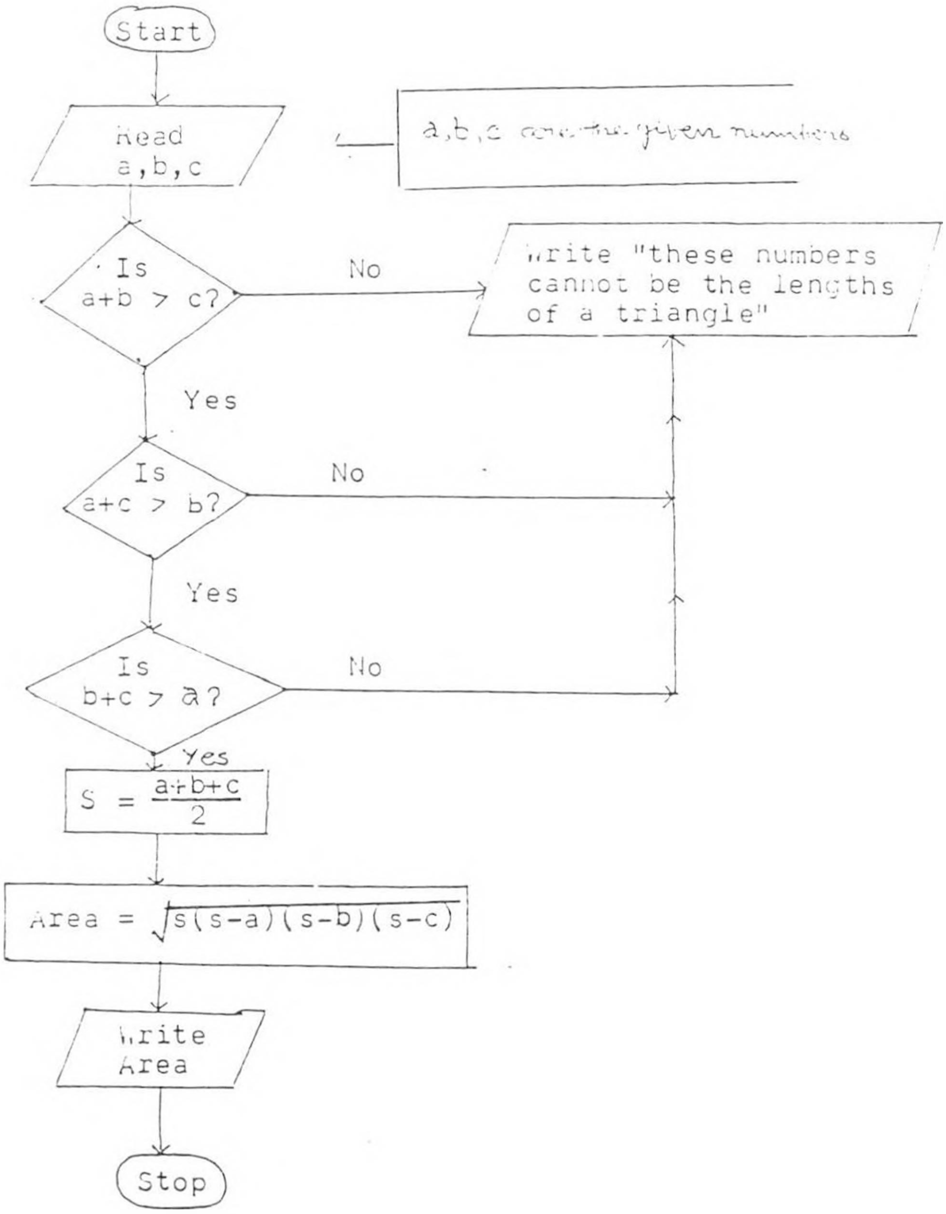
Second Type :



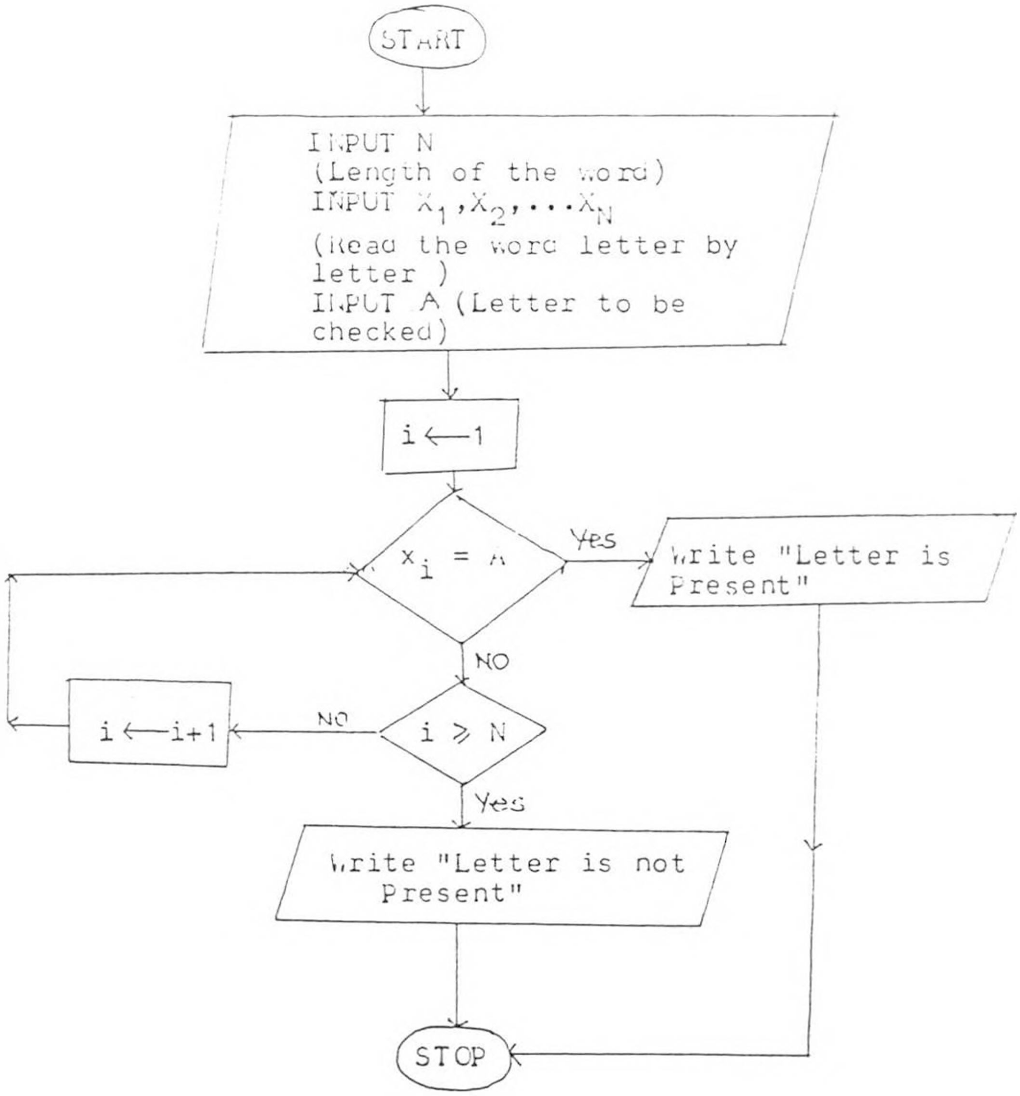
Example 4 : Prepare a flowchart to check the three given numbers a,b,c can be the lengths of the three sides of a triangle and if so to find the area of that triangle using the formula

$$\text{Area} = \sqrt{s(s-a)\ (s-b)\ (s-c)} \quad \text{where } s = \frac{a+b+c}{2}$$

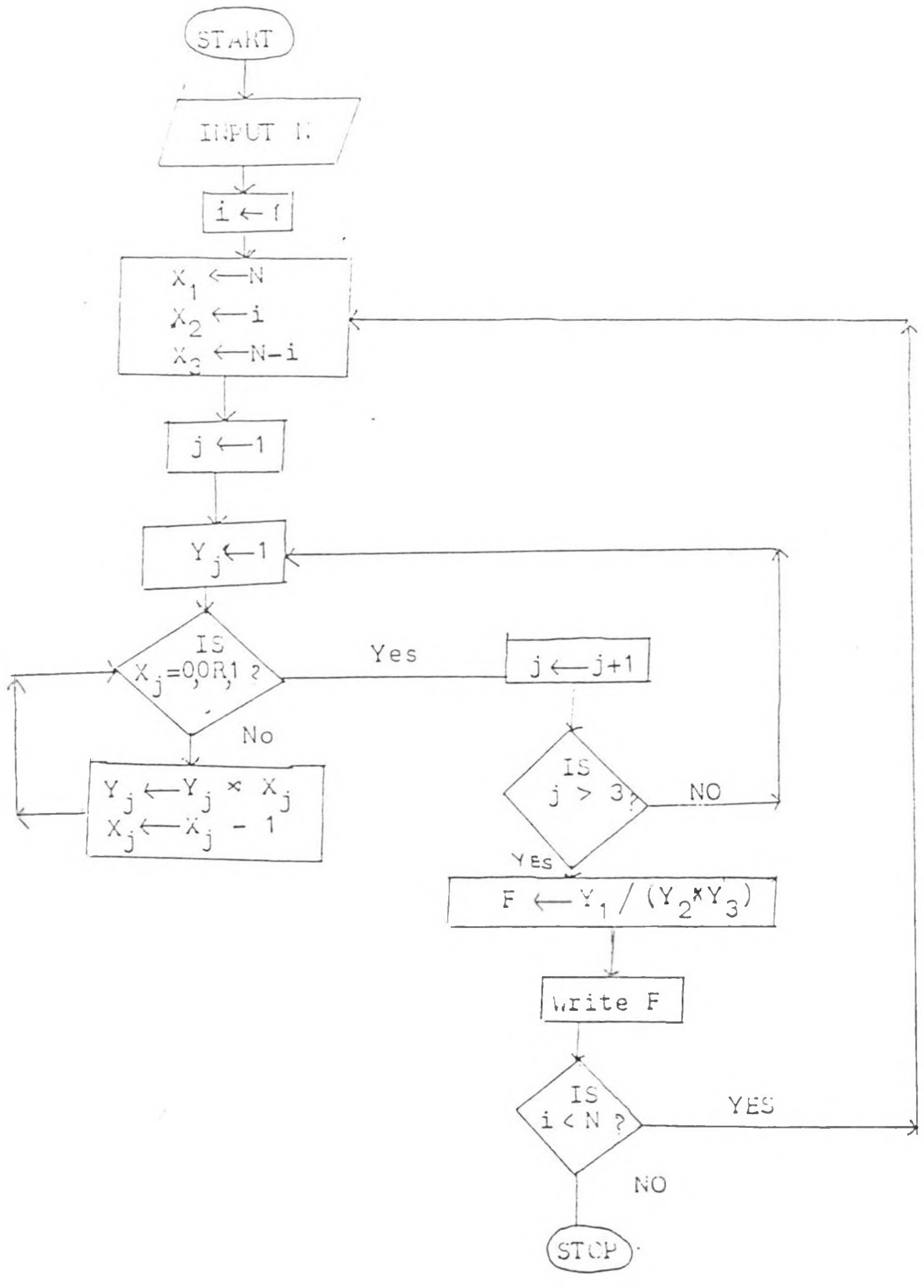
Solution :


Start
Read
a,b,c
a,b,c are the given numbers
Is
a+b > c?
No
Write "these numbers
cannot be the lengths
of a triangle"
Yes
Is
a+c > b?
No
Yes
Is
b+c > a?
No
Yes
S = (a+b+c)/2
Area = √(s(s-a)(s-b)(s-c))
Write
Area
Stop

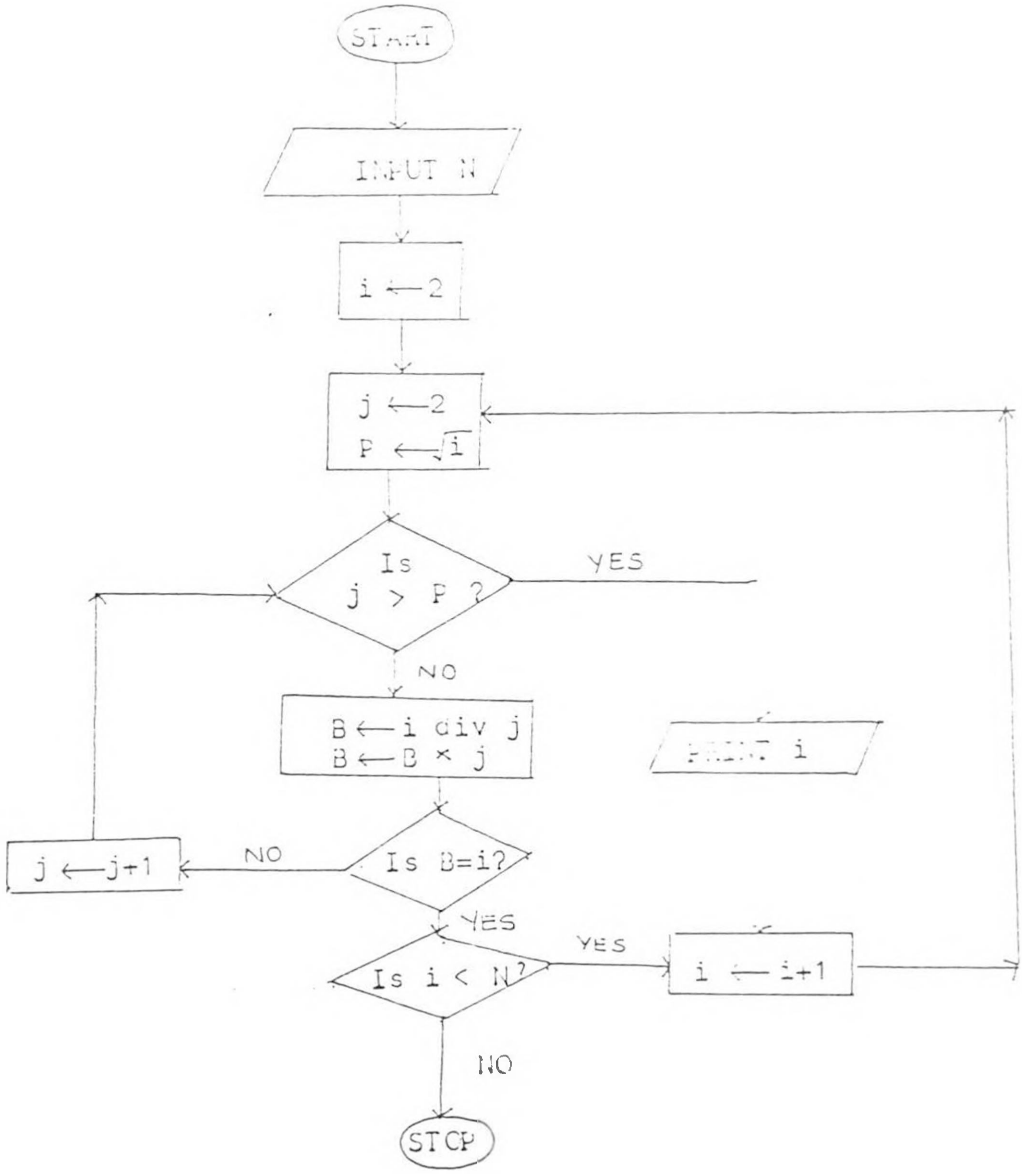
Flowchart to verify whether a given letter is existing in the given word or not.



Flowchart for the list of Binomial Coefficients.

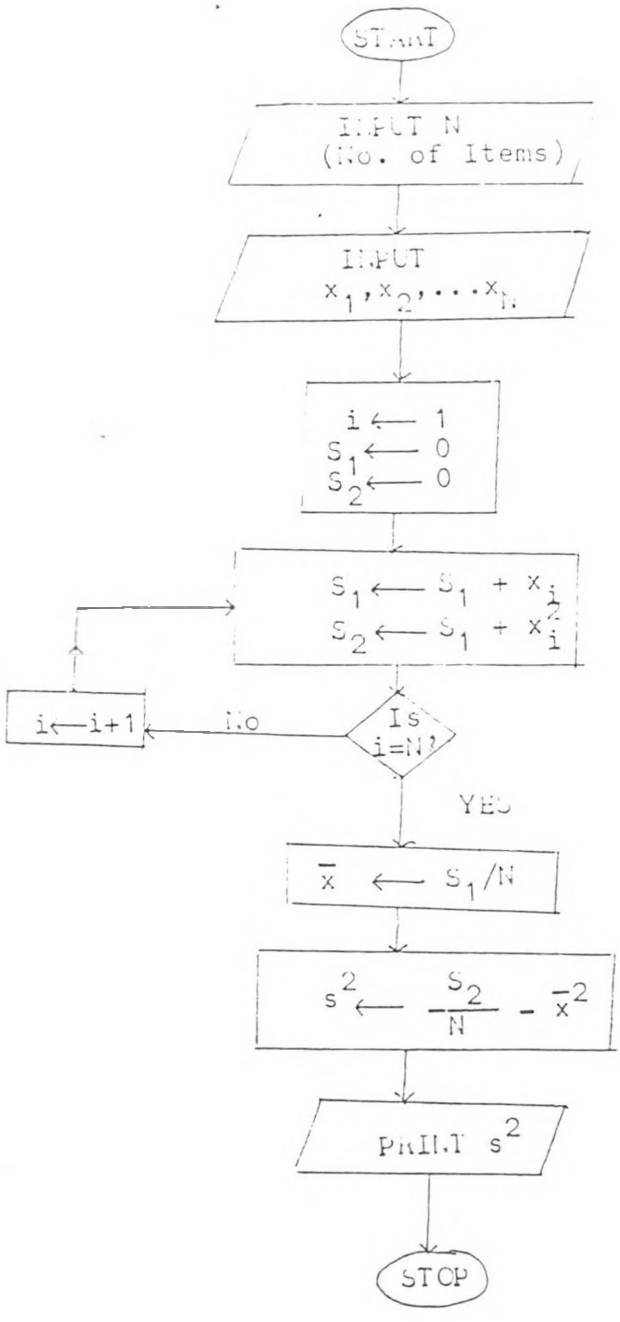


Flowchart for listing of Prime number below any given number N.

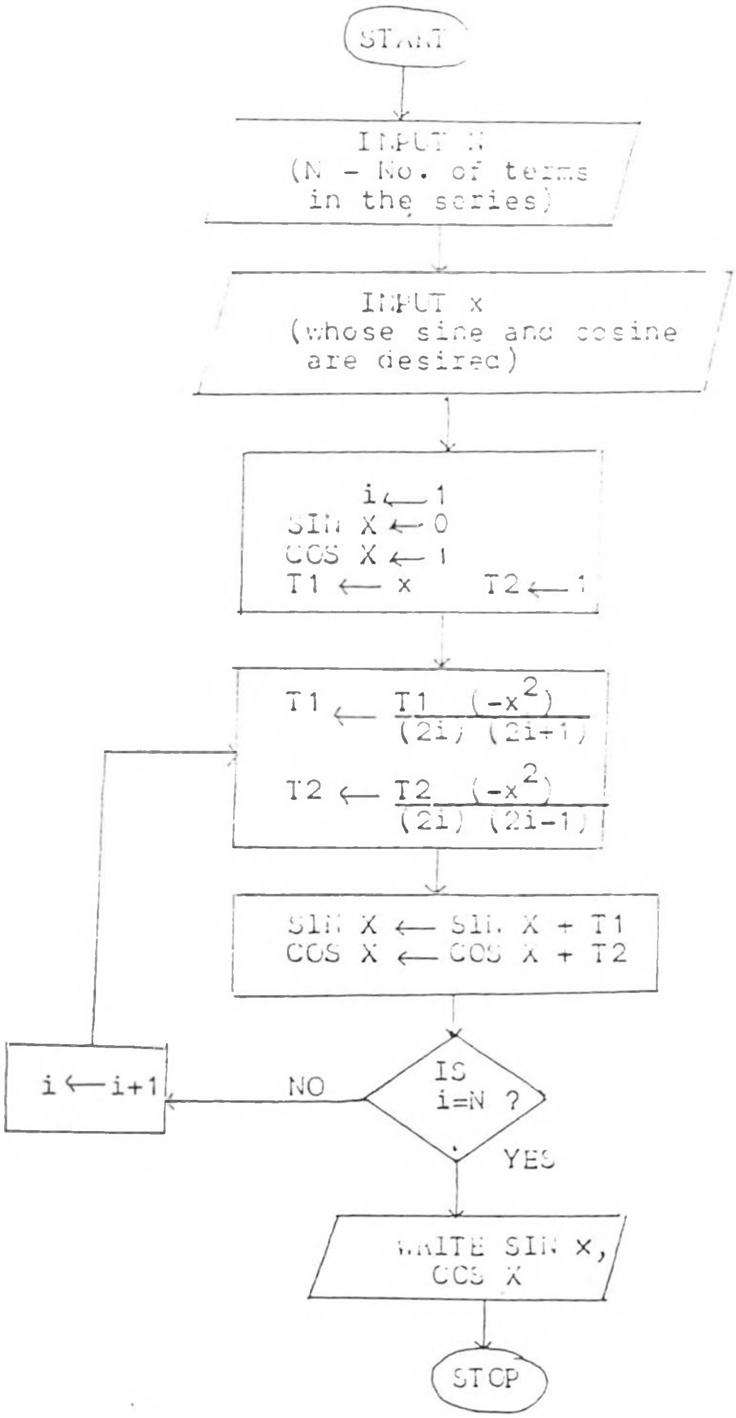


Flowchart for calculation of variance (s^2) using the formula

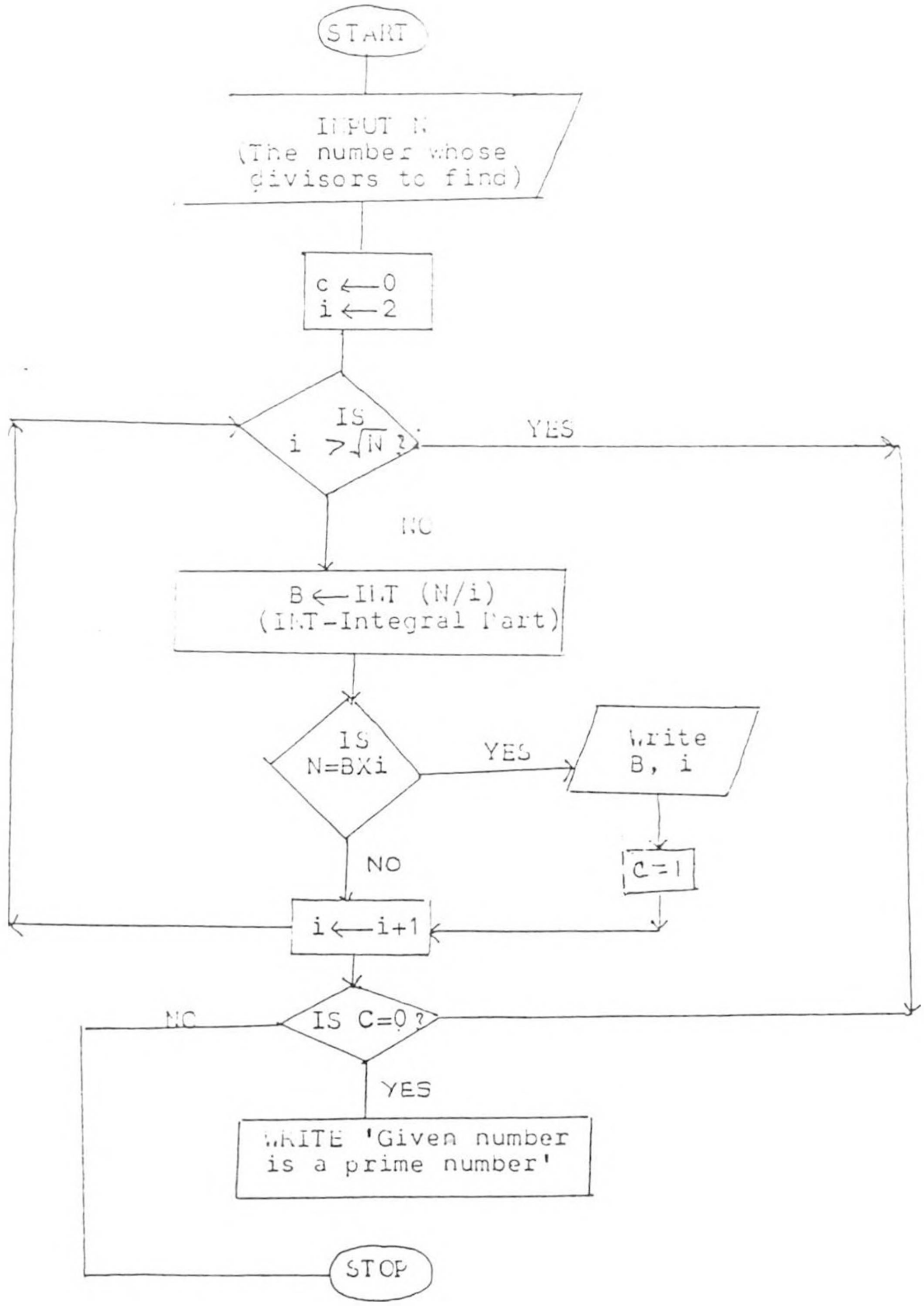
$$s^2 = \frac{\sum x_i^2}{n} - (\bar{x})^2.$$



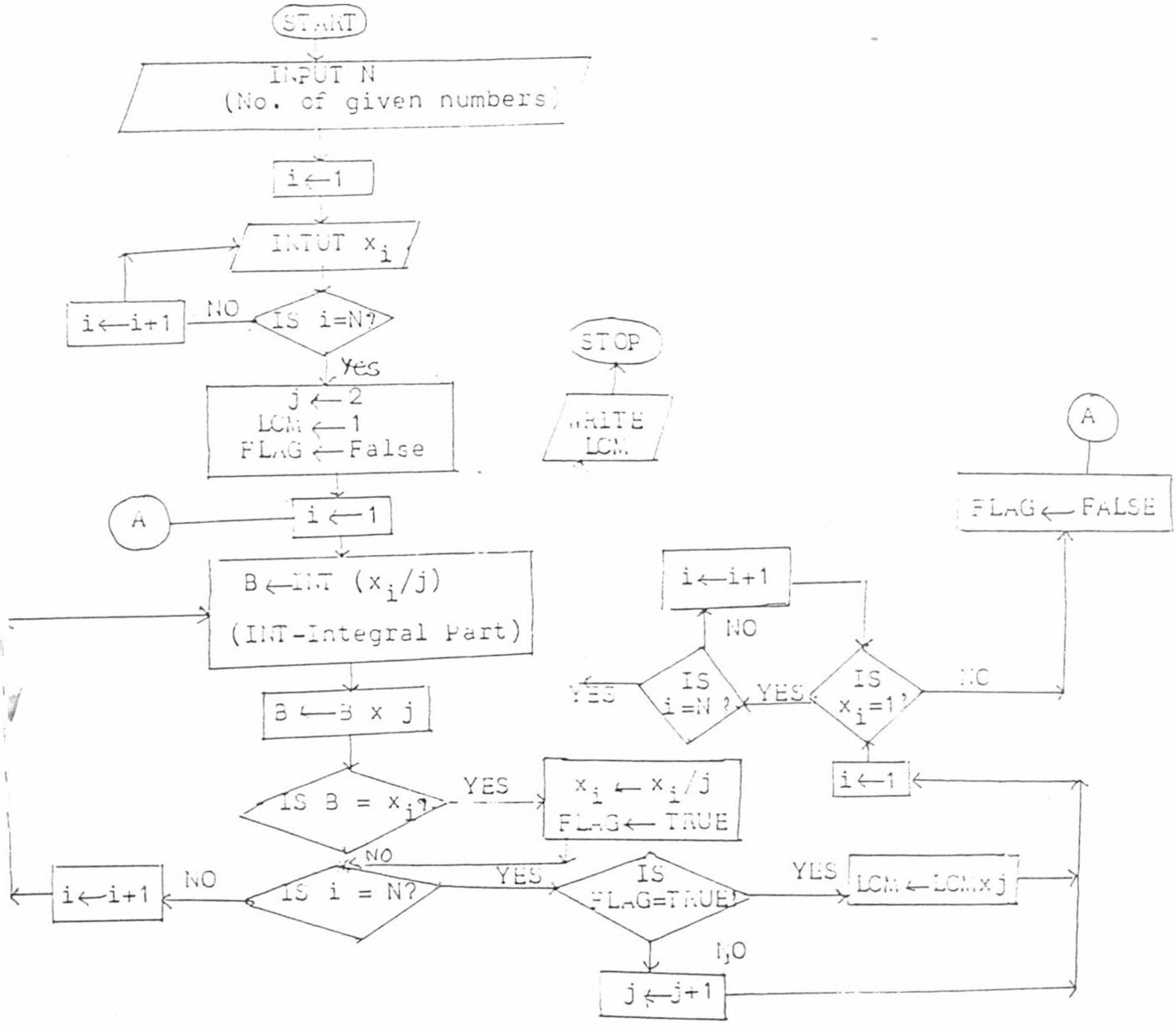
Flowchart to find the values of Sin(x) and Cos(x) for any given value of x.



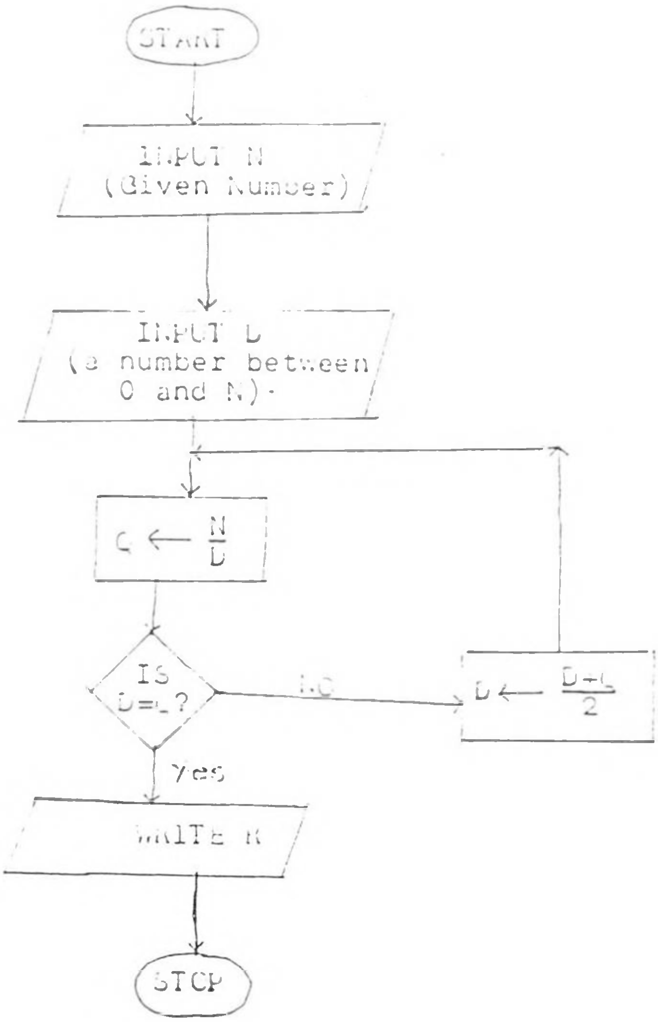
Flowchart to find all divisors of a given number.



Flowchart to find LCM of given numbers.



Flowchart to find square root of a given number.



Exercises :

- What is a flowchart ? Why is it called a flowchart ?
- What are the various symbols used in a flow chart ? What does each symbol represent ?
- Prepare a flow chart to add digits from 1 to 100.
- Prepare a flow chart to pick the largest of three given numbers.

5. What is a program flow chart ?
6. Prepare a flow chart/algorithm in each of the following situations.

a) Preparation of a multiplication table.

b) Preparation of a bill in any consumer shop.

c) To verify whether there exists any given letter in any given word or not.

d) Calculation of compound interest for any given principle, rate and time.

e) To verify whether the given triangle is an equilateral or not.

f) To calculate the possible number of linear arrangements of 8 students so that two particular students sit together.

g) To obtain the transpose of any given matrix.

h) To show that the opposite angles in a cyclic quadrilateral are supplementary to one another.

i) Calculation of sin(x) and cos(x) for different values of x.

j) To verify whether the given two straight lines are parallel or not.

k) To calculate the value of the following polynomial for a given value of x.

$$a_o x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_n$$

l) To find the product of any two given polynomials of degree 'n'.

m) To convert the given angle in radians to degrees.

n) To find divisors of any given number.

o) To write a given number in reverse order.

p) To find the d ay of a week for a given date.

q) To categorize a given triangle as an obtuse, acute or right angled triangle with given sides.

r) Calculation of square root of any given number.

s) Calculation of LCM of any given numbers.

t) To list the binomial coefficients for any given 'n'.

u) To calculate the variance (s^2) of any given set of values $x_1, x_2, \ldots, x_n$ using the formula

$$s^2 = (x_i - \bar{x})^2 \quad n \text{ where } \bar{x} = x_i/n$$

v) To calculate the area of trapezium with given sides.

w) Printing of all prime numbers below a given number.

x) To find the roots of any given quadratic equation.

y) To convert a decimal number into a binary number.

z) To solve the following differential equation

$$\frac{dy}{dx} = 2x+3$$

given $y = 4$ when $x = 1$.