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VOL III

PROBABILITY

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NUMERICAL METHODS

ALGORITHMS AND FLOW CHARTS

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P R O B A B I L I T Y

1. Basic Terminology
2. Some theorems on Probability
3. Random Variables and Probability Distribution

by
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PROBABILITY

In our day-to-day life we perform certain activities to verify certain known facts or to observe certain phenomena. Such activities usually we call as experiments. In certain experiments, we can predict results exactly before conducting the experiment and in other it will not be possible. The experiments where the results can be predicted exactly are known as deterministic experiments and the experiments where the prediction is not exact are known as non-deterministic or random or probabilistic experiments. For example, a train is running at a uniform speed of sixty K.M. per hour, then we can predict with hundred percent surety that it will cover one hundred twenty kilometers after two hours, assuming that it never stopped during these hours. Similarly for a perfect gas, PV = constant (P is pressure, V is volume).

In case of non-deterministic experiments we cannot make predictions with complete reliability. The results are based on some 'chance element'. For example, if we toss a coin, will it show 'head up' or 'tail up' ? Although we cannot predict anything with complete surety, yet if we throw the coin a large number of times, it is very likely that the head will turn up fifty percent of the times and also it is very unlikely that the head turns up in every case.

Consider another example of a trained parachuter who is ready to jump. When he jumps then either his parachute will open or it will not. But experience says that most of the time it opens, though there are occasions on which it does not i.e., the uncertainty associated with opening or not opening of a parachute when a parachuter jumps is lesser as compared with the uncertainty associated with the head or tail coming up when we toss a coin.

How will you proceed in answering the following questions ?

1. How should a businessman order for replenishment (filling once again) of his stocks (inventory) so that he has not carried very large stocks, yet the risk of refusing customers is minimized ? (Inventory problem).

2. At what intervals should a car owner replace the car so that the total maintenance expenses are minimized ? (Replacement problem).

3. How many trainees should a large business organisation recruit and train them in certain intervals so that at any time it does not have a large number of trained persons whom it cannot employ and yet the risk of its being without sufficient persons when needed is minimized ?

4. How should the bus service in a city be scheduled so that the queues do not become too long and yet the gains by the bus company are maximized ? (Queuing problem)

5. How many booking counters should be provided at a station to serve in the best way the interests of both the railways and the travelling public ?.(Queuing problem)

6. What should be the strength of a dam (or a bridge) so that its cost is reasonable and yet the risk of its being swept away by the floods is minimized ?

7. How many telephone exchanges should be established in a given city so as to give the best service at a given cost ?

8. Which variety is the best out of given varieties of wheat, on the basis of yields from experimental farms ?

9. What should be the minimum premia charged by an insurance company so that the chance of its running into loss is minimized ?

10. How to decide whether a given batch of items is defective when only a sample of the batch can be examined ?

Answers for all such questions are based upon certain facts and they try to measure the uncertainty associated with some events which may or may not materialize. The theory of probability deals with the problem of measuring the uncertainty associated with various events rather pr cisely, making it these by possible today, to a certain extent of course, to control phenomena depending upon chance.

The 'measure of uncertainty' is known as probability.

In our day-to-day vocabulary we use words such as 'probably', 'likely', 'fairly good chances' etc. to express the uncertainty as indicated in the following example. Suppose a father of a X class student wants to know his son's progress in the studies and asks the concerned teacher about his son. Teacher may express to the father about the students' progress in any one of the following sentences.

1. It is certain that he will get a first class.
2. He is sure to get a first class.
3. I believe he will get a first class.
4. It is quite likely that he will get a first class.
5. Perhaps he may get a first class.
6. He may or he may not get a first class.
7. I believe he will not get a first class.
8. I am sure he will not get a first class.
9. I am certain he will not get a first class.

Instead of expressing uncertainty associated with any event with such phrases, it is better and exact if we express uncertainty mathematically. The measure of uncertainty or probability can be measured in three ways and these are known as the three definitions of probability. These methods are

i) Mathematical or Classical or Priori Probability
ii) Statistical or Empirical probability and
iii) Axiomatic probability.

Before discussing those methods, we define some of the terms which are useful in the definition of probability.

i) Experiment : An act of doing something to verify some fact or to obtain some result.
Ex: Throwing a die to observe which number will come up (Die is a six-faced cube).

ii) Trial : Conducting experiment once is known as the trial of that experiment.
Ex: Throwing a die once.

iii) Outcomes : The results of an experiment are known as outcomes.
Ex: In throwing a die, getting '1' or '2' or '6' are the outcomes.

iv) Events : Any single outcome or set of outcomes in an experiment is known as an event.
Ex: 1. Getting '1' in throwing of a die is an event.
Also getting an even number in throwing a die is also an event.
Ex: 2. Drawing two cards from a well shuffled pack of cards is a trial and getting of a king and a queen is an event.

v) Exhaustive Events : The total number of possible outcomes in any trial are known as exhaustive events.
Ex: 1. In tossing a coin there are two exhaustive events.
2. In throwing a die, there are six exhaustive

cases viz. (1,2,3,4,5,6).

vi) Favourable Events (cases) : The number of outcomes which entail the happening of an event are known as the favourable cases (events) of that event.

Ex: In throwing two dice, the number of cases favourable for getting a sum 5 are (1,4), (2,3), (3,2) and (4,1).

vii) Mutually Exclusive Events : Events are said to be mutually exclusive or incompatible if the happening of any one of them precludes or excludes the happening of all others.

Ex: In tossing a coin, the events head and tail are mutually exclusive (because both cannot occur simultaneously).

Mathematical or Classical or 'a priori' probability :

If a trial results in 'n' exhaustive, mutually exclusive and equally likely cases and 'm' of them are favourable to the happening of an event E, then the probability 'p' of happening of E is given by

$$p = \frac{\text{Favourable number of outcomes}}{\text{Total number of outcomes}} = \frac{m}{n}$$

We write p = P(E).

Ex: 1. Probability of getting head in tossing of a coin once is 1/2 because the number of exhaustive cases are 2 and these are mutually exclusive and equally likely (assuming the coin is made evenly) and of these only 1 case is favourable to our event of getting head.

Ex: 2. The probability of getting a number divisible by 3 in throwing of a fair (evenly made) die is 2/6 because the favourable cases are (viz. 3 and 6) and exhaustive cases are 6.

The probability 'q' that E will not happen is given by

$$q = \frac{n-m}{n} = 1- \frac{m}{n} = 1-p$$

Always $0 \leq p \leq 1$

If P = P(E) = 1, E is called a certain event and if P(E) = 0, E is called an impossible event.

In this method, the mathematical ratio of two integers is giving the probability and therefore, this definition is known as mathematical definition. Here we are using the concept of probability in the form of 'equally likely cases' and therefore, this definition is a classical definition. Before using this definition, we should know about the nature of outcomes (viz. mutually exclusive, exhaustive and equally likely) and therefore, it is also known as 'a priori' probability definition.

The definition of mathematical or classical probability definition breaks down in the following cases : 1. If the various outcomes of the trial are not equally likely. 2. If the exhaustive number of cases in a trial is infinite.

Ex: 1. When we talk about the probability of a pass of a candidate, is not 1/2 as the two outcomes 'pass' and 'fail' are not equally likely.

Ex: 2. When we talk about the probability of a selected real number is to be divided by 10, the number of exhaustive cases are infinite.

In such above mentioned circumstances it is not possible to use mathematical probability definition. Therefore, probability is defined in the other way as below :

Statistical or Empirical Probability

If a trial is repeated a number of times under essentially homogeneous and identical conditions, then the limiting value of the ratio of the number of times an event happens to the number of trials, as the number of trials becomes indefinitely large, is called the probability of happening that event.

Mathematically, we write

$$P = P(E) = \lim_{n \to \infty} (m/n)$$

Here n is the number of trials and m is the number of times of the occurrence of event E. The above limit should be finite.

Ex: When you throw a die 10000 times and if you get 1600 times the number '1 , then the probability of getting '1' is 1600/10000. This ratio is nothing but the relative frequency of '1'.

But this definition is also not applicable always because it is very difficult to maintain the identical conditions throughout the experiment. Therefore, the probability is defined in another way by using certain axioms. This definition is known as 'Axiomatic Probability' definition.

Here we define some of the terms which are useful in the 'Axiomatic Probability' definition.

Sample Space : The set of all possible outcomes of an experiment is known as the sample space of that experiment. Usually we denote it by S. Ex: In tossing a coin, $S = \{H, T\}$.

Sample Point: Any element of a sample space is known as a sample point.
Ex: In tossing of a coin experiment, H or T is a sample point.

Event : Any subset of a sample space is an event.
Ex: In throwing a die, (1,3,5) or (2,4,6) or (5,6) are the events where $S = \{1,2,3,4,5,6\}$.

If A and B are any two events then A, B, AUB, $A \cap B$ are also events because they are also subsets of S.

The event S (entire sample space) is known as certain event and the ev nt $\emptyset$ (empty set) is known as impossible event.

Mutually Exclusive Events : Events are said to be mutually exclusive if the corresponding sets are disjoint.
Ex: In throwing of a die experiment, if A = (1,3,5) and B = (2,4,6) then A and B are mutually exclusive because we cannot get both odd number and even number simultaneously. That is, if $A \cap B = \emptyset$, then A and B are mutually exclusive events.

Axiomatic Probability :

Let S be a sample space and ξ be the class of events. Also let P be a real valued function defined on ξ. Then P is called a probability function, and P(A) is called the probability of the event A if the following axioms hold :

i) For every event A, $0 \leq P(A) \leq 1$.

ii) $P(S) = 1$

iii) If A and B are mutually exclusive events, then $P(A \cup B) = P(A) + P(B)$.

iv) If $A_1, A_2, \ldots$ is a sequence of mutually exclusive events, then $P(A_1 \cup A_2 \ldots) = P(A_1) + P(A_2) + \ldots\ldots$

In the above definition axiom (iv) may seem to be not necessary. But it is necessary to stress that axiom (iii) should be extended to more than two events.

<u>Theorem 1</u> : If $\emptyset$ is the empty set, then $P(\emptyset) = 0$.

Proof: We know that $S = S \cup \emptyset$ and $P(S) = P(S \cup \emptyset) = P(S) + P(\emptyset)$ (because S and $\emptyset$ are disjoint and according to axiom (iii)).

But $P(S) = 1$ and therefore, $1 = 1 + P(\emptyset)$.

$\therefore P(\emptyset) = 0$

<u>Theorem 2</u> : If $\bar{A}$ is the complement of an event A, then $P(\bar{A}) = 1 - P(A)$.

Proof : $\bar{A} \cup A = S$

$P(\bar{A} \cup A) = P(\bar{A}) + P(A) = P(S)$ ($\bar{A}$ and A are disjoint).

But $P(S) = 1$, therefore,

$P(\bar{A}) + P(A) = 1$

or $P(\bar{A}) = 1 - p(A)$

<u>Theorem 3</u> : If $A \subseteq B$, then $P(A) \leq P(B)$.

Proof: We know that if $A \subseteq B$, then

$B = A \cup (B - A)$ (here we may use the notation B/A)

So, $P(B) = P(A) + P(B-A)$

But from axiom i, $P(B-A) \geq 0$

$\therefore P(B) \geq P(A)$

<u>Theorem 4</u> : If A and B are any two events, then $P(A-B) = P(A) - P(A \cap B)$

Proof: We can write, $A = (A \cap B) \cup (A-B)$

But $(A \cap B)$ and $(A-B)$ are disjoint and according to axiom (iii),

$P(A) = P(A \cap B) + P(A-B)$

or $P(A-B) = P(A) - P(A \cap B)$

<u>Theorem 5</u> : (Addition Theorem)

If A and B are any two events, then

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Proof : We can write, $A \cup B = B \cup (A-B)$. But B and $(A-B)$ are disjoint and therefore, by axiom (iii),

$P(A \cup B) = P(B) + P(A-B)$.

Also from theorem 4, $P(A-B) = P(A) - P(A \cap B)$

Hence, $P(A \cup B) = P(B) + P(A-B)$

$= P(B) + P(A) - P(A \cap B)$

This theorem is known as addition theorem and it can be extended to any number of events as follows.

Theorem 6 (Addition Theorem in case of n events)

If $A_1, A_2, \ldots, A_n$ are any n events, then

$$P(A_1 \cup A_2 \cup \ldots A_n) = \sum_{\substack{i,j=1\\ i \quad j}}^{n} P(A_i) - \sum_{\substack{i,j=1\\ i<j}}^{n} P(A_i \cap A_j) + \ldots + \ldots + (-1)^{n-1} P(A_1 \cap A_2 \cap \ldots \cap A_n)$$

Proof: This theorem can be proved by the method of induction.

For the events A_1 and A_2 we have from theorem 5,

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$$

$$= \sum_{i=1}^{2} P(A_i) + (-1)^{2-1} P(A_1 \cap A_2)$$

Hence the theorem is true for n=2.

Now, suppose the theorem is true for $n = r$ (say).

Then,

$$P(A_1 \cup A_2 \cup \cdots \cup A_r) = \sum_{i=1}^{n} P(A_i) - \sum_{\substack{i,j=1\\ i<j}}^{n} P(A_i \cap A_j) + \cdots + (-1)^{r-1} P(A_1 \cap A_2 \cap \cdots \cap A_r)$$

Now,

$$P(A_1 \cup A_2 \cdots \cup A_r \cup A_{r+1}) = P((A_1 \cup A_2 \cup \cdots \cup A_r) \cup A_{r+1})$$

$$= P(A_1 \cup A_2 \cup \cdots \cup A_r) + P(A_{r+1}) - P((A_1 \cup A_2 \cup \cdots \cup A_r) \cap A_{r+1})$$

$$= P(A_1 \cup A_2 \cup \cdots \cup A_r) + P(A_{r+1}) - P((A_1 \cap A_{r+1}) \cup (A_2 \cap A_{r+1}) \cdots \cup (A_r \cap A_{r+1}))$$

$$= \sum_{i=1}^{r} P(A_i) - \sum_{\substack{i,j=1\\ i<j}}^{n} P(A_i \cap A_j) + \cdots + (-1)^{r-1} P(A_1 \cap A_2 \cap \cdots A_{r+1})$$

$$+ P(A_{r+1}) - \left\{ \sum_{i=1}^{r+1} P(A_i \cap A_{r+1}) + \sum_{\substack{i,j=1\\ i<j}}^{r+1} P(A_i \cap A_j \cap A_{r+1}) + \cdots + (-1)^{r-1} P(A_1 \cap A_2 \cap \cdots \cap A_{r+1}) \right\}$$

$$= \sum_{i=1}^{r+1} P(A_i) - \sum_{\substack{i,j=1 \\ i>j}}^{r+1} P(A_i \cap A_j) + \cdots\cdots$$
$$+ (-1)^{r} P(A_1 \cap A_2 \cap \cdots\cdots A_{r+1})$$

Hence, if the theorem is true for n=r, it is also true for n=r+1. But we have proved that the theorem is true for n=2. Hence by the method of induction, the theorem is true for all positive integer values of n.

Corollary 1 : If A and B are two mutually exclusive events, then, P(AUB) = P(A) + P(B)

Corollary 2 : If A_1, $A_2 \ldots A_n$ are n mutually exclusive events, then $P(A_1 U A_2 \ldots U A_n) = P(A_1) + P(A_2) + \ldots + P(A_n)$.

Conditional Probability :

So far, we have assumed that no information was available about the experiment other than the sample space while calculating the probabilities of events. Sometimes, however, it is known that an event A has happened. How do we use this information in making a statement concerning the outcome of another event B ?

Consider the following examples :

Ex.1: Draw a card from a well-shuffled pack of cards. Define the event A as getting a black card and the event B as getting a spade card. Here P(A) = 1/2 and P(B) = 1/4. Suppose the drawn card is a black card then what is the probability that card is a spade card ? That is, if the event A has happened then what is the probability of B given that A has already happened. This probability symbolically we write as P(B/A). In the given example,

$$P(B/A) = 1/2 = \frac{P(A \cap B)}{P(A)} = \frac{(1/4)}{(1/2)}$$

because probability of simultaneous occurence of A and B is 1/4 and probability of A is 1/2.

Ex.2: Let us toss two fair coins. Then the sample space of the experiment is S = {HH, HT, TH, TT}. Let event A = {both coins show same face} and B = {at least one coin shows H}. Then P(A) = 2/4. If B is known to have happened, this information assures that TT cannot happen, and P {A, Conditional on the information that B has happened}

$$= P(A/B) = 1/3 = \frac{1/4}{3/4}$$

$$= \frac{P(A \cap B)}{P(B)}$$

In the above two examples, we were interested to find the probability of one event given the condition that the other event has already happened. Such events based on some conditions are known as conditional events. In the above examples B/A and A/B are the conditional events. The probability of a conditional event is known as conditional probability of that event. We write the conditional probabilities as P(A/B), P(E/F) etc.

Definition of conditional probability : The conditional probability of an event A, given B, is denoted by P(A/B) and is defined by

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

where A, B and $A \cap B$ are events in a sample space S, and $P(B) \neq 0$.

From the definition of conditional probability we know that

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

Therefore, we can write from the above

$$P(A \cap B) = P(B) \cdot P(A/B)$$

Also, we know that $P(A \cap B) = P(B \cap A)$

$$P(B \cap A) = P(A) \cdot P(B/A)$$

Hence we can write

$P(A \cap B) = P(A) \cdot P(B/A)$ or $P(B) \cdot P(A/B)$.

The above result is known as multiplication law of probabilities in case of two events.

Multiplication Theorem of Probabilities: If A and B are any events of a sample space S, then
$P(A \cap B) = P(A).\ P(B/A)$ or $P(B)\ .\ P(A/B)$.

The above theorem can be extended to any n events as follows: If $A_1, A_2, \ldots, A_n$ are any n events, then

$$P(A_1 \cap A_2 \cdots \cap A_n) = P(A_1)\ P(A_2/A_1)\ .\ P(A_3/A_1 \cap A_2) \cdots \cdots P(A_n/A_1 \cap A_2 \cdots \cap A_{n-1}).$$

This theorem can be proved by method of induction or generalization.

Baye's Theorem : If $E_1, E_2, \ldots, E_n$ are mutually exclusive events with $P(E_i) \neq 0$, $(i = 1,2,\ldots,n)$ then for any arbitrary event A which is a subset of $\bigcup_{i=1}^{n} E_i$ such that $P(A) \geqslant 0$, we have

$$P(E_i/A) = \frac{P(E_i)\ P(A/E_i)}{\sum_{i=1}^{n} P(E_i)\ P(A/E_i)} \text{ for all i.}$$

Proof: Since $A \subset \bigcup_{i=1}^{n} E_i$ we have

$$A = A \cap (\bigcup_{i=1}^{n} E_i)$$
$$= \bigcup_{i=1}^{n} (A \cap E_i) \text{ by distributive law).}$$

Since $(A \cap E_i) \subset E_i$, (for $i = 1,2,\ldots n$) are mutually exclusive events, we have by additional theorem of probability

$$P(A) = P\left[\bigcup_{i=1}^{n} (A \cap E_i)\right] = \sum_{i=1}^{n} P(A \cap E_i)$$
$$= \sum_{i=1}^{n} P(E_i)\ P(A/E_i) \text{ by multiplication theorem in case of two events.}$$

Also, we have

$$P(A \cap E_i) = P(A)\ \ P(E_i/A)$$

and $P(E_i/A) = \dfrac{P(A \cap E_i)}{P(A)} = \dfrac{P(E_i)\,P(A/E_i)}{P(A)}$

Hence, $P(E_i/A) = \dfrac{P(E_i)\ P(A/E_i)}{\sum_{i=1}^{n} P(E_i)\,P(A/E_i)}$

This theorem is very useful in calculating the conditional probabilities in certain situations.

If $P(A \cap B) = P(A).P(B)$, then we see that $P(B/A) = P(B)$ and hence we say that the probability of B is not depending upon the happening of A. That is the conditional probability of B is same as the unconditional probability of B. Such events are called independent events.

Two events A and B are independent if and only if
$P(A \cap B) = P(A) . P(B)$.

Ex: Let two fair coins be tossed and let
$A = \{\text{head on first coin}\}$, $B = \{\text{head on the second coin}\}$

Then $P(A), = P\{HH, HT\} = 1/2$

$P(B) = P\{HH, TH\} = 1/2$ and

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{1/4}{1/2} = 1/2 = P(A)$$

Thus,

$$P(A \cap B) = P(A) . P(B)$$

and we know that the probability of getting head on the first coin does not depend upon the probability of getting head on the second coin. Hence A and B are independent. Also we see that the condition $P(A \cap B) = P(A) . P(B)$ is both necessary and sufficient for those events A and B to be independent.

If there are three or more than three events, we will have the situation where every pair of these events are independent or the situation where the events in every set of events are independent. In the first case, we call the events as pairwise independent and in the second case we call as complete or mutual independent events.

Geometric Probability :

Sample space can be countably finite or countably infinite or uncountably finite or uncountably infinite depending upon the situation. If the sample space is countably finite, then it is easy to calculate the probability of any event by using either mathematical probability or axiomatic probability definition. Even if the sample space is countably infinite say $S = (e_1, e_2, \ldots)$ we obtain a probability space assigning to each $e_i \in S$ a real number p_i, called its probability, such that

$$p_i \geqslant 0 \text{ and } p_1 + p_2 + \ldots = \sum_{i=1}^{\infty} P_i = 1$$

The probability P(A) of any event A is then the sum of the probabilities of its points.

Consider the sample space $S = \{1, 2, \ldots\}$ of the experiment of tossing a coin till a head appears; here n denotes the number of times the coin is tossed. A probability space is obtained by

$$p(1) = 1/2,\ p(2) = 1/4, \ldots,\ p(n) = \frac{1}{2^n}, \ldots$$

But the calculation of probability of events regarding an uncountably finite or infinite sample space is not so easy.

Consider a situation of selecting a point at random on a line segment of length '1'. Here the sample space is uncountably finite and the procedure to find the probability of any event in case of countable sample space is not applicable.

Consider another example. Suppose that two friends have agreed to meet at a certain place between 9 a.m. and 10 a.m. They also agreed that each would wait for a quarter of an hour and, if the other did not arrive, would leave. What is the probability that they meet ?

In the above example also both the sample space and the given event are uncountable and the ordinary procedures of calculation of probability are not applicable. So we need different procedure in such cases.

If the sample space is uncountably finite, we represent that sample space by some geometrical measurement $m(S)$ such as length, area or volume, and in which a point is selected at random. The probability of an event A, i.e. the selected point belongs to A, is then the ratio of $m(A)$ to $m(S)$ is

$$P(A) = \frac{\text{length of } A}{\text{length of } S} \text{ or } P(A) = \frac{\text{area of } A}{\text{area of } S} \text{ or } P(A) = \frac{\text{volume of } A}{\text{volume of } S}$$

Such probability is known as 'geometrical probability'.

Solved Problems :

1. A bag contains 5 red, 4 white and 3 blue balls. What is the probability that two balls drawn are red and blue ?

Sol: Total number of balls = 5+4+3 = 12.

The number of ways of drawing two balls out of 12 balls =

$$_{12}C_2 = \frac{12 \times 11}{2} = 66 \text{ ways.}$$

The number of ways of drawing 1 red ball out of 5 red balls = 5 ways.
The number of ways of drawing 1 blue ball out of 3 blue balls=3 ways.
The number of ways of drawing 1 red ball out of 5 red balls and 1 blue ball out of 3 blue balls = 5x3 = 15 ways.

The required probability = 15/66 = 5/22, by using Mathematical Probability definition.

2. If the letters of the word 'STATISTICS' are arranged at random to form words, what is the probability that three S's come consecutively ?

Sol : Total no. of letters in the word 'STATISTICS' = 10.
Total no. of arrangements of these 10 letters in which 3 are of one kind (viz. S), 3 are of second kind (viz. T), 2 are of third kind (viz. I), 1 of fourth kind (viz. A) and 1 of fifth kind (viz. C)

$$= \frac{\lfloor 10}{\lfloor 3 \; \lfloor 3 \; \lfloor 2 \; \lfloor 1 \; \lfloor 1}$$

Following are the 8 possible combinations of 3 S's coming consecutively.

i) in the first three places
ii) in the second, third and fourth places
....
viii) in the eighth, ninth and tenth places.

Since in each of the above cases, the total number of arrangements of the remaining 7 letters, viz. TTTIIAC of which 3 are of one kind, 2 of second kind, 1 of third kind and 1 of fourth kind

$$= \frac{\underline{|7}}{\underline{|3}\ \underline{|2}\ \underline{|1}\ \underline{|1}}$$

and the required number of favourable cases = $\frac{8 \times \underline{|7}}{\underline{|3}\ \underline{|2}\ \underline{|1}\ \underline{|1}}$

Hence the required probability

$$= \frac{\text{Favourable cases}}{\text{Total No. of cases}} = \frac{8 \times \underline{|7}}{\underline{|3}\ \underline{|2}\ \underline{|1}\ \underline{|1}} \Big/ \left[\frac{\underline{|10}}{\underline{|3}\ \underline{|3}\ \underline{|2}\ \underline{|1}\ \underline{|1}}\right]$$

$$= \frac{8 \times \underline{|7} \times \underline{|3}}{\underline{|10}} = 1/15$$

3. What is the probability that a leap year selected at random will contain 53 Sundays ?

Sol: In a leap year there are 366 days of 52 complete weeks and 2 days more. In order that a leap year selected at random should contain 53 Sundays, one of these extra 2 days must be Sunday. But there are 7 different combinations with these two extra 2 days viz. Sunday and Monday, Monday and Tuesday, etc. Out of these 7 possible ways, only in 2 ways we are having an extra Sunday.

Required probability = 2/7.

4. Two dice are thrown simultaneously. What is the probability of obtaining a total score of seven ?

Sol: Six numbers (1,2,3,4,5,6) are on the six faces of each die. Therefore, there are six possible ways of outcomes on the first die and to each of these ways, there corresponds 6 possible number of outcomes on the second die.

Hence the total number of ways, n = 6x6 = 36. Now we will find out of these, how many are favourable to the total score of 7. This may happen only in the following ways (1,6), (6,1), (2,5), (5,2), (3,4) and (4,3) that is, in six ways where first number of each ordered pair denotes the number on the first die and second number denotes the number on the second die.

$\therefore m = 6$

Hence required probability = $\frac{\text{Favourable no. of cases}}{\text{Total No. of cases}}$

$$= \frac{m}{n} = \frac{6}{36} = \frac{1}{6}.$$

5. Two digits are selected at random from the digits 1 through 9. If the sum is even find the probability, p that both numbers are odd.

Sol: If both numbers are even or if both numbers are odd then the sum is even. In this problem, there are 4 even numbers (2,4,6,8) and hence there are ${}^{4}C_{2}$ ways to choose two even numbers. There are 5 odd numbers (1,3,5,7,9) and hence there are ${}^{5}C_{2}$ ways to choose two odd numbers. Thus there are ${}^{4}C_{2} + {}^{5}C_{2} = 16$ ways to choose two numbers such that their sum is even. Since 10 of these ways occur when both numbers are odd, the required probability,

$$p = \frac{10}{16} = \frac{5}{8}.$$

6. Six boys and six girls sit in a row randomly. Find the probability that a) the six girls sit together, b) the boys and girls sit alternately.

Sol: a) Six girls and six boys can sit at random in a row in $\underline{\lvert 12}$ ways. Consider six girls as one object and the six boys as six different objects. Now these seven objects can be arranged in $\underline{\lvert 7}$ different ways. But the six girls in the first object can be arranged in $\underline{\lvert 6}$ ways. Thus the favourable number of cases to the event of sitting all girls together is $\underline{\lvert 7}\ \underline{\lvert 6}$ ways.

Therefore, the required probability $= \dfrac{\text{Favourable no. of cases}}{\text{Total no. of cases}}$

$$= \frac{\underline{\lvert 7}\ \underline{\lvert 6}}{\underline{\lvert 12}} = \frac{1}{132}$$

b) Since the boys and girls can sit alternately in $\underline{\lvert 6}\ \underline{\lvert 6}$ ways if we begin with a boy and similarly they can sit alternately in $\underline{\lvert 6}\ \underline{\lvert 6}$, ways if we begin with a girl. Thus the total number of ways sitting the boys and girls alternately $= 2\ \underline{\lvert 6}\ \underline{\lvert 6}$

$\therefore$ The required probability $= \dfrac{\text{Favourable no. of cases}}{\text{Total no. of cases}}$

$$= \frac{2\ \underline{\lvert 6}\ \underline{\lvert 6}}{\underline{\lvert 12}} = \frac{1}{462}$$

7. Out of (2n+1) tickets consecutively numbered, three are drawn at random. Find the chance that the numbers on them are in A.P.

Sol: Suppose that the smallest number among the three drawn is 1. Then the groups of three numbers in A.P. are (1,2,3), (1,3,5), (1,4,7) ...., (1, n+1, 2n+1) and they are n in number.
Similarly, if the smallest number is 2, then the possible groups are (2,3,4), (2,4,6),...., (2, n+1, 2n) and their number is n-1.
If the lowest number is 3, the groups are (3,4,5), (3,5,7),..(3,n+2,2n+1) and their number is n-1.

Similarly it can be seen that if the lowest numbers selected are 4,5,6,...,2n-2, 2n-1, the number of selections respectively are (n-2), (n-2), (n-3), (n-3),....,2,2,1,1. Thus the favourable ways for the selected three numbers are in A.P.

$$= 2\,(1 + 2 + 3 + \dots + \overline{n-1}) + n$$

$$= \frac{2(n-1)\,n}{2} + n = n^2$$

Also the total number of ways of selecting three numbers out of (2n+1) numbers

$$= \binom{2n+1}{3} = \frac{(2n+1)\,(2n)\,(2n-1)}{1.2.3} = \frac{n\,(4n^2-1)}{3}$$

Hence the required probability $= \dfrac{\text{Favourable no. of cases}}{\text{Total no. of cases}}$

$$= \frac{n^2}{n(4n^2-1)/3} = \frac{3n}{4n^2-1}$$

8. If a coin is tossed (m+n) times (m > n), then show that the probability of at least m consecutive heads is $\dfrac{n+2}{2^{m+1}}$

Sol: Let us denote by H the appearance of head and by T the appearance of tail and let X denote the appearance of head or tail.
Now P(H) = P(T) = 1/2 and P(X) = 1

Suppose the appearance of m consecutive heads starts from the first throw, we have

(H H H .....m times) (X X .....n times)

The chance of this event = (1/2. 1/2 .... m times) = $\dfrac{1}{2^m}$

If the sequence of m consecutive heads starts from the second throw, the first must be a tail and we have

T(H H ....m times) (X X .... $\overline{n-1}$ times)

The chance of this event = $\frac{1}{2}\left(\frac{1}{2} \cdot \frac{1}{2} \text{....m times}\right) = \frac{1}{2^{m+1}}$

If the sequence of m consecutive heads starts from the (r+1) th throw, the first (r-1) throws may be head or tail but r th throw must be tail and we have

(X X ....r-1 times) T (H H ....m times) (X X ...$\overline{m+n-n}$ times)

The probability of this event = $\frac{1}{2} \quad \frac{1}{2^m} = \frac{1}{2^{m+1}}$

In the above case, r can take any value from 1,2,...n.
Since all the above cases are mutually exclusive, the required probability when r takes 0,1,2,...n

$$= \frac{1}{2^m} + \left(\frac{1}{2^{m+1}} + \frac{1}{2^{m+1}} + \text{....n times}\right)$$

$$= \frac{n+2}{2^{m+1}}$$

Hence the result.

9. What is the probability that in a group of N people, at least two of them will have the same birthday ?

Sol: We first find the probability that no two persons have the same birthday and then subtract from 1 to get the required probability. Suppose there are 365 different birthdays possible in a year (excluding leap year).
Any person might have any of these 365 days of the year as a birthday. A second person may likewise have any of these 365 birthdays and so on. Hence the total number of ways of N people to have their birthdays = $(365)^N$

But the number of possible ways for none of these N birthdays to coincide is = 365. 364....(365-$\overline{N-1}$)

$$= \frac{(365)!}{(365-N)!}$$

The probability that no two birthday coincide is

$$\frac{(365)!}{(365-N)!} \Big/ (365)^N$$

Hence the required probability (for at least two people to have the same birthday) $= 1 - \frac{(365)!}{(365-N)!\,(365)^N}$

10. A and B are two independent witnesses (i.e. there is no collusion between them) in a case. The probability that A will speak the truth is x and the probability that B will speak the truth is y. A and B agree in a certain statement. Show that the probability that the statement is true is $xy/(1-x-y+2xy)$

Sol: A and B agree in a certain statement means either both of them speak truth or make false statement. But the probability that they both speak truth is xy and both of them make false statement is $(1-x)(1-y)$.

Thus the probability of their agreement in a statement =

$$xy + (1-x)(1-y) = 1-x-y-2xy$$

Therefore the conditional probability of their statement is true $= \frac{xy}{1-x-y+2xy}$

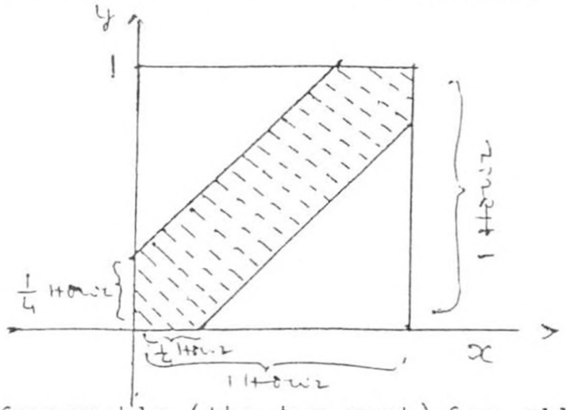
(by using the definition $P(A/B) = \frac{P(A \cap B)}{P(B)}$ where A is the event of correct statement and B is the event of common statement).

11. Two friends have agreed to meet at a certain place between nine and ten O' clock. They also agreed that each would wait for a quarter of an hour and, if the other did not arrive, would leave. What is the probability that they meet ?

Sol: Suppose x is the moment one person arrives at the appointed place, and y is the moment the other arrives.

Let us consider a point with coordinates (x,y) on a plane as an outcome of the rendezvous.

Every possible outcome is within the area of square having side corresponds to an hour as shown in the figure.



The outcome is favourable (the two meet) for all points (x,y) such that $|x-y| \leq 1/4$. These points are within the shaded part of the square in the above figure.

All the outcomes are exclusive and equally possible, and therefore, the probability of the rendezvous equals the ratio of the shaded area to the area of the square. That is, $m(A) = \frac{7}{16}$ and $m(S) = 1$.

Hence by geometric probability, the required probability

$$= \frac{m(A)}{m(S)} = \frac{7/16}{1} = \frac{7}{16}.$$

Exercises :

1. A factor of 60 is chosen at random. What is the probability that it has factors of both 2 and 5 ?

2. The numbers 3,4 and 5 are placed on three cards and then two cards are chosen at random.

a) The two cards are placed side-by side with a decimal point in front. What is the probability that the decimal is more than 3/8 ?

b) One card is placed over the other to form a fraction. What is the probability that the fraction is less than 1.5 ?

c) If there are 4 cards with numbers 3,4,5 and 6, then what are the probabilities of the above two cases ?

3. A vertex of a paper isosceles triangle is chosen at random and folded to the midpoint of the opposite side. What is the probability that a trapezoid is formed ?

4. A vertex of a paper square is folded onto another vertex chosen at random. What is the probability that a triangle is formed ?

5. Three randomly chosen vertices of a regular hexagon cut from paper are folded to the centre of the hexagon. What is the probability that an equilateral triangle is formed ?

6. A piece of string is cut at random into two pieces. What is the probability that the short piece is less than half the length of the long piece ?

7. A paper square is cut at random into rectangles. What is the probability that larger perimeter is more than 1½ times the smaller ?

8. The numbers 2,3 and 4 are substituted at random for a, b, c in the equation $ax+b=c$

a) What is the probability that the solution is negative ?

b) If c is not 4, what is the probability that the solution is negative?

9. Each coefficient in the equation $ax^2+bx+c=0$ is determined by throwing an ordinary die . Find the probability that the equation will have real roots.

10. The numbers 1, 2 and 3 are substituted at random for a, b and c in the quadratic equation $ax^2+bx + c = 0$.

a) What is the probability that $ax^2+bx+c =0$ can be factored ?

b) What is the probability that $ax^2+bx+ c=0$ has real roots ?

11. Two faces of a cube are chosen at random. What is the probability that they are in para planes ?

12. Three edges of a cube are chosen at random. What is the probability that each edge is perpendicular to the other two ?

13. A point P is chosen at random in the interior of square ABCD. What is the probability that triangle ABP is acute ?

14. Find the probability of the event of the sine of a randomly chosen angle is greater than 0.5.

15. Suppose you ask individuals for their random choices of letters of the alphabet. How many people would you need to ask so that the probability of at least one duplication becomes better than 1 in 2 ?

16. Six boys and six girls sit in a row randomly. Find the probability that i) the six girls sit together, ii) the boys and girls sit

alternately ?

17. If the letters of the word 'MATHEMATICS' are arranged at random, what is the probability that there will be exactly 3 letters between H and C ?

18. The sum of two non-negative quantities is equal to 2n. Find the probability that their product is not less than 3/4 times their greatest product.

19. If A and B are independent events then show that $\bar{A}$ and $\bar{B}$ are also independent events.

20. Cards are dealt one by one from well-shuffled pack of cards until an ace appears. Find the probability of the event that exactly n cards are dealt before the first ace appears.

21. If four squares are chosen at random on a chess-board, find the chance that they should be in a diagonal line.

22. Prove that if $P(A/B) < P(A)$ then $P(B/A) < P(B)$.

23. If n people are seated at a round table what is the chance that the two named individuals will be next to each other ?

24. A and B are two very weak students of Mathematics and their chances of solving a problem correctly are $1/8$ and $1/12$ respectively. If the probability of their making common mistake is $1/1001$ and they obtain the same answer, find the chance that their answer is correct.

25. A bag contains an unknown number of blue and red balls. If two balls are drawn at random, the probability of drawing two red balls is five times the probability of drawing two blue balls. Furthermore, the probability of drawing one ball of each colour is six times the probability of drawing two blue balls. How many red and blue balls are there in the bag ?

26. A thief has a bunch of n keys, exactly one can open a lock. If the thief tries to open the lock by trying the keys at random, what is the probability that he requires exactly k attempts, if he rejects the keys already tried ? Find the probability of the same event when he does not reject the keys already tried.

27. A problem in Mathematics is given to three students and their chances of solving it are 1/2, 1/3 and 1/4. What is the probability that the problem will be solved ?

28. A bag A contains 3 white balls and 2 black balls and other bag B contains 2 white and 4 black balls. A bag and a ball out of it are picked at random. What is the probability that the ball is white ?

29. Cards are drawn one-by-one at random from a well-shuffled pack of 52 cards until 2 aces are obtained for the first time. If N is the number of cards required to be drawn, then show that

$$P(N=n) = \frac{(n-1)\ (52-n)\ (51-n)}{50.49.\ 17.\ 13}$$

where $2 \leq n \leq 50$.

30. A,B,C are events such that
$P(A) = 0.3$, $P(B) = 0.4$, $P(C) = 0.8$, $P(A \cap B) = 0.08$
$P(A \cap C) = 0.28$ $P(A \cap B \cap C) = 0.09$
If $P(A \cup B \cup C) \geq 0.75$, then show that $P(B \cap C)$ lies in the interval $(0.23, 0.48)$.

31. A man takes a step forward with probability 0.4 and backwards with probability 0.6. Find the probability that at the end of eleven steps he is one step away from the starting point.

32. Huyghens Problem. A and B throw alternately a pair of dice in that order. A wins if he scores 6 points before B gets 7 points, in which case B wins. If A starts the game, what is his probability of winning ?

33. A Doctor goes to work following one of three routes A, B, C. His choice of route is independent of the weather. If it rains, the probabilities of arriving late, following A, B, C are 0.06, 0.15, 0.12 respectively. The corresponding probabilities, if it does not rain, are 0.05, 0.10, 0.15.

a) Given that on a sunny day he arrives late, what is the probability that he took route C? Assume that, on average, one in every four days is rainy.

b) Given that on a day he arrives late, what is the probability that it is a rainy day.

34. Bonferroni's Inequality. Given $n(>1)$ events $A_1, A_2, \ldots, A_n$
Show that

$$\sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j) \leq P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i)$$

35. Show that for any n events $A_1, A_2, \ldots, A_n$

i) $P\left(\bigcap_{i=1}^{n} A_i\right) \geq 1 - \sum_{i=1}^{n} P(\bar{A}_i)$

ii) $P\left(\bigcap_{i=1}^{n} A_i\right) \geq \sum_{i=1}^{n} P(A_i) - (n-1)$

36. If A and B are mutually exclusive and $P(A \cup B) \neq 0$, then prove that

$$P(A \mid A \cup B) = \frac{P(A)}{P(A)+P(B)}$$

37. If 2n boys are divided into two equal groups, find the probability that the two tallest boys will be a) in different subgroups and, b) in the same subgroup.

38. A small boy is playing with a set of 10 coloured cubes and 3 empty boxes. If he puts the 10 cubes into the 3 boxes at random what is the probability that he puts 3 cubes in one box, 3 in another box, and 4 in the third box ?

39. The sample space consists of the integers from 1 to 2n, which are assigned probabilities to their logarithms. a) Find the probabilities. b) Show that the conditional probability of the integer 2, given that an even integer occurs, is $\dfrac{\log 2}{n\log 2 + \log \lfloor n}$

40a. Each of n boxes contains four white and six black balls, while another box contains five white and five black balls. A box is chosen at random from the (n+1) boxes, and two balls are drawn from it, both being black. The probability that five white and three black balls remain in the chosen box is 1/7. Find n.

40b. A point is selected at random inside a circle. Find the probability p that the point is closer to the centre of the circle than to its circumference.

41. What is the probability that two numbers chosen at random will be prime to each other ?

42. In throwing n dice at a time, what is the probability of having each of the points 1,2,3,4,5,6 appears at least once ?

43. A bag contains 50 tickets numbered 1,2,3,...,50 of which five are drawn at random and arranged in ascending order of magnitude ($x_1 < x_2 < x_3 < x_4 < x_5$), what is the probability that $x_3 = 30$?

44. Of the three independent events, the probability that the first only to happen is 1/4, the probability that the second only to happen is 1/8 and the third only to happen is 1/12. Obtain the unconditional probabilities of the three events.

45. What is the least number of persons required if the probability exceeds 1/2 that two or more of them have the same birthday (year of birth need not match) ?

46. If m things are distributed among 'a' men and 'b' women, then show that the chance that the number of things received by men is

$$\frac{1}{2} \quad \frac{(b+a)^m - (b-a)^m}{(b+a)^m}$$

47. A pair of dice is rolled until either 5 or a 7 appears. Find the probability that a 5 occurs first.

48. In a certain standard tests I and II, it has been found that 5% and 10% respectively of 10th grade students earn grade A. Comment on the statement that the probability is

$$\frac{5}{100} \quad \frac{10}{100} = \frac{1}{200}$$

that a 10th grade student chosen at random will earn grade A on both tests.

49. A bag contains three coins, one of which is coined with two heads while the other two coins are fair. A coin is chosen at random from the bag and tossed four times in succession. If heads turn up each time, what is the probability that this is the two headed coin ?

50. A man stands in a certain position (which we may call the origin) and tosses a fair coin. If a head appears he moves one unit of length to the left. If a tail appears he moves one unit to the right. After 10 tosses of the coin, what are his possible positions and what are the probabilities ?

51. There are 12 compartments in a train going from Madras to Bangalore. Five friends travel by the train for some reasons could not meet each other at Madras station before getting aboard. What is the probability that the five friends will be in different compartments ?

52. The numbers 1,2,3,4,5 are written on five cards. Three cards are drawn in succession and at random from the deck, the resulting digits are written from left to right. What is the probability that the resulting three digit number will be even ?

53. Suppose n dice are thrown at a time. What is the probability of getting a sum S of points on the dice ?

54. A certain mathematician always carries two match boxes, each time he wants a match-stick he selects a box at random. Inevitably, a moment comes when he finds a box empty. Find the probability that the moment the first box is empty, the second contains exactly r match sticks (assume that each box contain N match-sticks initially).

55. There are 3 cards identical in size. The first card is red both sides, the second one is black both sides and the third one red one side and black other side. The cards are mixed up and placed flat on a table. One is picked at random and its upper (visible) side was red. What is the probability that the other side is black ?

56. n different objects 1,2,...n are distributed at random in n places marked 1,2,...n. Find the probability that none of the objects occupies the place corresponding to its number.

Answers :

1. $1/2$ 2. a) $2/3$ b) $5/6$ c) $3/4$, $3/4$ 3. $1/3$

4. $1/3$ 5. $1/10$ 6. $2/3$ 7. $2/5$ 8.a) $1/2$

8b) $3/4$ 9. $43/216$ 10.a) $1/3$ 10b). $1/3$ 11. $1/5$

12. $2/55$ 13. $1 - \frac{\pi}{8} = 0.6073$ 14. $2/3$ 15. 7

16.i) $\frac{7!\,6!}{12!}$ 16.ii) $\frac{2\,(6!)^2}{12!}$ 17. $7/55$

18. $1/2$ 20. $\frac{4(51-n)(50-n)(49-n)}{52.51.50.49}$ 21. $\frac{91}{158844}$

23. $\frac{2}{n-1}$ 24. $\frac{13}{14}$ 25. Red=6, Blue=3

26. $1/n$, $1/n(1 - \frac{1}{n})^{k-1}$ 27. $3/4$ 28. $7/15$ 31. $(0.4)^5\,(0.6)^5$

32. $30/61$ 33a) 0.5 33b) $41/131$ 37a) $\frac{n}{2n-1}$

37b) $\frac{n-1}{4n-2}$ 38. $\frac{3\;10!}{3!\,3!\,4!\;3^{10}}$ 39.a) K log 2i

39b) (log 2i) (n log 2 + log $n!$) 40a) 4 40b) $1/4$

41. $\prod(1 - \frac{1}{r^2}) = \frac{6}{\pi^2}$

42. $1 - n\left(\frac{5}{6}\right)^n + \binom{n}{2}\left(\frac{4}{6}\right)^n - \binom{n}{3}\left(\frac{3}{6}\right)^n + \binom{n}{4}\left(\frac{2}{6}\right)^n - \binom{n}{5}\left(\frac{1}{6}\right)^n$

43. $\frac{\binom{29}{2}\binom{20}{2}}{\binom{50}{2}}$ 44. $1/2$, $1/3$, $1/4$ 45. 23 47. $2/5$ 49. $8/9$

50.

Distance from origin	-10	-8	-6	-4	-2	0	2	4	6	8	10
Prob	$(\frac{1}{2})^{10}$	$\binom{10}{1}(\frac{1}{2})^{10}$	$\binom{10}{2}(\frac{1}{2})^{10}$	$\binom{10}{3}(\frac{1}{2})^{10}$	$\binom{10}{4}(\frac{1}{2})^{10}$	$\binom{10}{5}(\frac{1}{2})^{10}$	$\binom{10}{6}(\frac{1}{2})^{10}$	$\binom{10}{7}(\frac{1}{2})^{10}$	$\binom{10}{8}(\frac{1}{2})^{10}$	$\binom{10}{9}(\frac{1}{2})^{10}$	$(\frac{1}{2})^{10}$

51. $55/144$ 52. $1/5$ 53. $(-1)^k \binom{n}{k}\binom{s-6k-1}{n-1} / 6^n$

54. $\frac{\binom{2n-r}{n}}{2^{2n-r}}$ 55. $1/2$ 56. $\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots + (-1)^n \frac{1}{n!}$

RANDOM VARIABLES AND PROBABILITY DISTRIBUTIONS

Introduction: In the earlier pages, , the idea of a function, subject to certain postulates, which assigned weights called probabilities, to the points of the sample space, was introduced. We then had a probability function which allowed us to compute probabilities for events. Now we deal with the concept of Random variable.

Random Variable :

Scientific theories on models are our way of depicting and explaining how observations come about. Such theories are simplified statements containing essential features and make for easier comprehension and communication. In statistics, we use a mathematical approach since we quantify our observations. Random variable is the result of such mathematical approach dealing with the probabilities assigning to different events of a random experiment. The set of possible outcomes for a random experiment can be described with the help of a real-valued variable by assigning a single value of this variable to each outcome. For a two coin tossing experiment, the outcomes are two tails, a tail and a head, a head and a tail, or two heads. The sample space can be represented as (TT, TH, HT, HH). Here we express the outcomes by using the number of heads and so assigning the values (0, 1, 1, 2) respectively to those outcomes. Therefore, the outcomes of this experiment can be denoted by the different values of the real-valued variable viz. 0,1,2.

Any function or association that assigns a unique, real value to each sample point is called a chance or random variable. The assigned values are the values of the random variable.

Random variables are symbolized by capital letters, most often X, and their values by lower case letters. The outcome of a random experiment determines a point i.e. the sample space, called the <u>domain</u> of the random variable, and the function transform each sample point to one of a set of real numbers. This set of real numbers is called the <u>range</u> of the random variable. If the sample space is discrete, then the outcomes will be denoted by certain discrete values. The random variable associated with a discrete sample space is known as discrete random variable. Similarly the random variable associated with continuous sample space is known as continuous random variable.

<u>Probability Function</u> :

The association of probabilities with the various values of a discrete random variable is done by reference to the probabilities in the sample space and through a system of relationships or a function is called a probability set function or, simply, a probability function.

Let the discrete random variable X assume the values $X_1, X_2 \ldots X_n$. Then the system of relations can be written as

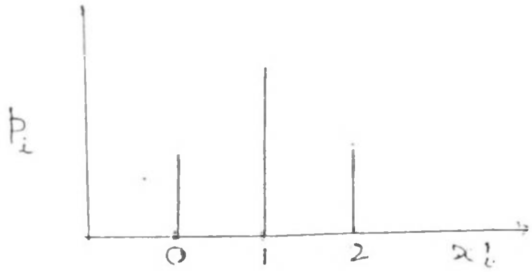
$$P\ (X = x_i) = p_i$$

This is read as 'the probability that the random variable X takes the value x_i is p_i'. The set of ordered pairs (x_i, p_i) constitutes a probability function with numerical values to be provided for the x_i's and p_i's such that $p_i \geq 0$ for all i and $\sum_i p_i = 1$.

A discrete probability function is a set of ordered pairs of values of a random variable and the corresponding probabilities.

For a two coin experiment, X takes the values 0,1,2 with the probabilities 1/4, 1/2, 1/4 respectively.

Sometimes probability function can be represented by a graph or a mathematical function. In case of above example, the X values and the corresponding probabilities can be represented with the help of the following graph.



Suppose X assume the values 1 and 0 with the probabilities p and 1-p respectively. This information can be given with the help of the following function p(x) defined as

$$p(x) = p^{x}(1-p)^{1-x}, \quad x = 0,1$$

This type of function which gives the probabilities of the different values assumed by a random variable is known as probability mass function or simply probability function. Therefore, a function p(x) is said to be a probability function of random variable or a distribution if

i) $p(x) \geq 0$ for all x

ii) $\sum_{x} p(x) = 1$

Where p(x) denotes the probability of the event that the random variable X assume the value x.

Distribution Function :

The law of probability distribution of a random variable is the rule used to find the probability of the event related to a random variables. For instance, the probability that the variable assumes a certain value or falls in a certain value. The general form of the distribution law is distribution function, which is the probability that a random variable X assumes a value smaller than a given x i.e. $F(x) = P(X \leq x)$.

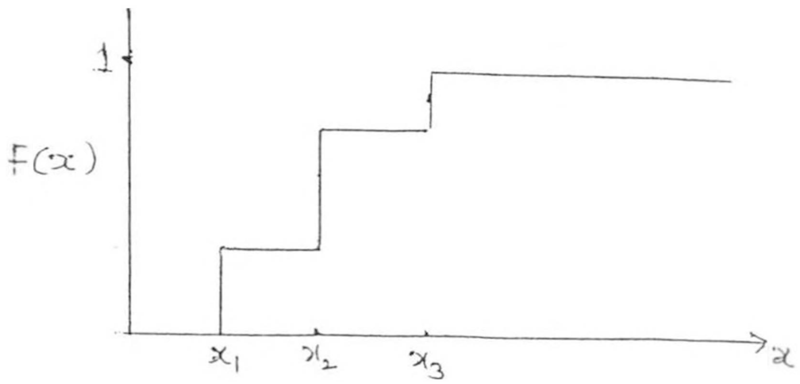
The distribution function F(x) for any random variable possesses the following properties :

i) $F(-\infty) = 0$
ii) $F(+\infty) = 1$
iii) F(x) does not decrease with an increase in x.

In case of discrete random variable

$$F(x_k) = \sum_{i=1}^{k} p(x_i)$$

where $x_1, x_2, \ldots, x_k \ldots$ are the values of the random variable. The graph of F(x) in discrete random variable case is generally as shown below :



It is seen from the above figure that the graph of F(x) is a 'step function' having jump $p(x_i)$ at $x = x_i$ and is constant between each pair of values of x. It can also be proved that

$$F(x_i) - F(x_{i-1}) = p(x_i)$$

Therefore, distribution function can also be used to indicate the distribution of the random variable instead of probability function.

Example :

A student is to match three historical events (Mahatma Gandhi's birth year, India's freedom, and first World War) with three years (1947, 1914, 1869). If he guess with no knowledge of the correct answers, what is the probability distribution of the number of answers he gets correctly ?

Solution: Here the number of correct answers is the random variable, say X. Therefore, X assumes the values 0,1,2,3 because there are three events to match with only three years. Suppose the events are E_1, E_2, E_3 and the corresponding correct years are Y_1, Y_2, Y_3. Student gets the correct answers when he/she matches E_1 to Y_1, E_2 to Y_2 and E_3 to Y_3.

All matchings are wrong only when he/she matches E to Y_2, E_2 to Y_3, E_3 to Y_1 or E_1 to Y_3, E_2 to Y_1, E_3 to Y_2. But the total possible matchings are 6. Therefore, the probability of all matchings to go wrong is 2/6 = 1/3. That is, the probability that X to take the value '0' is 1/3.

Similarly X assumes the value '1' with probability $\frac{3}{6}$ $(=\frac{1}{2})$ the value '2' with 0 probability and the value '3' with 1/6 probability.

So the probability distribution of the correct answers in the given matching is

No. of correct answers(x)	0	1	2	3
Probability	1/3	1/2	0	1/6

Exercises :

1. One cube with faces numbers 1,2,3,4,5 and 6 is tossed twice, and the recorded outcome consists of the ordered pair of numbers on the hidden faces at the first and second tosses.

a) Let the random variable X take on the value 0 if the sum of the numbers in the ordered pair is even and 1 if odd.
What is the probability function for this random variable ?

b) Let the random variable X takes on the value 2 if both numbers in the ordered pair are even, 1 if exactly one is even, and 0 if neither is even. What is the probability distribution of this random variable ?

c) Let the random variable X be the number of divisors in the sum of the two faces. What is the probability function of X ?

2. Of six balls in a bag, two are known to be black. The balls are drawn one at a time from the bag and observed until both black balls are drawn. If X is the number of trials (draws) required to get the two black balls. Obtain the probability distribution of X.

3. Suppose that the random variable X has possible values 1,2,3,... and $P(x = j) = \frac{1}{2^j}$, j = 1,2,....

   i) compute P(x is even), ii) Compute P(x is divisible by 3).

4. The probability mass function of a random variable X is zero except at the points x = 0,1,2. At these points has the values $p(0) = 3c^3$, $p(1) = 4c-10c^2$ and $p(2) = 5c-1$ for some $c > 0$.

   i) Determine the value of c.

   ii) Compute $P(1 < X \leq 2)$

   iii) Describe the distribution function and draw its graph.

   iv) Find the largest x such that $F(x) < 1/2$.

Answers :

1. a)

X	Prob
0	1/2
1	1/2

b)

X	Prob
0	1/4
1	1/2
2	1/4

c)

X	Probl
2	15/36
3	12/36
4	8/36
6	1/36

2.

X	Prob
2	1/15
3	2/15
4	3/15
5	4/15
6	5/15

3. i) 1/3 ii) 1/7

4. i) 1/3 ii) 2/3 iii) 1

DISCRETE DISTRIBUTIONS

Introduction: In the previous pages, we discussed about 'random variable', 'probability function', etc. Here we discuss some theoretical discrete distributions in which variables are distributed according to some definite law which can be expressed mathematically.

Bernoulli Distribution : Suppose you want to study the probability of different events corresponding to tossing of a single coin experiment. The two possible events are getting a head or getting a tail. Define a random variable x assuming the values 1 and 0 corresponding to these two events viz. head and tail respectively. If the probability of getting a head in tossing that coin is 'p' then the probability that the random variable to take '1' is p and the probability that the random variable to take '0' is 1-p. Therefore, the distribution of the random variable X becomes

X	Prob
1	p
0	1-p

Any experiment where there are only two possible outcomes viz. success and failure is called as Bernoulli experiment. A single trial of a Bernoulli experiment is known as Bernoulli trial.

Corresponding to any Bernoulli Experiment, it is possible to define a random variable X as given above.

A random variable X which takes two values 0 and 1, with probability q(=1-p) and p respectively is called Bernoulli variate and is said to have a Bernoulli distribution.

Binomial Distribution :

Let a Bernoulli experiment be performed repeatedly and let the occurrence of an event in any trial be called a success and its non-occurrence a failure. Consider a series of n independent Bernoulli trials (n being finite), in which the probability 'p' of success in any trial is constant for each trial. Then q=1-p is the probability of failure in any trial. Let the random variable X is the number of successes in these trials.

The probability of x successes and consequently $(n-x)$ failures in n independent trials, in a specified order (say) SS FF SSS .... FSFF (where S represents success and F failure) is given by compound probability as given below :

$$P(SSFF,....FSFF) = P(S)\,P(S)\,P(F)\,P(F)\,...P(F)\,P(S)\,P(F)\,P(F)$$
$$= p.p.qq....q\;p\;q\;q$$
$$= pp....p\;qq....q \quad (x \quad p\text{'s and } (n-x)\; q\text{'s})$$
$$= p^{x}\;q^{n-x}$$

But x successes in n trials can occur in $\binom{n}{x}$ ways and the probability for each of these ways is $p^{x}\,q^{n-x}$. Hence the probability of x successes in n trials in any order whatsoever is given by the addition of individual probabilities and is given by $\binom{n}{x}\,p^{x}q^{n-x}$. The number of successes in n trials will be either 0 or 1 or 2... or n in any experiment.

$$P(X = x) = \binom{n}{x}\,p^{x}q^{n-x}$$

is true for all $x = 0,1,2,...n$.

This function $p(x) = \binom{n}{x}\,p^{x}q^{n-x}$, $x = 0,1...,\,n$ is called the probability mass function of the Binomial distribution, for the obvious reason that the probabilities of 0,1,2,...n successes, viz. $q^{n},\binom{n}{1}\,q^{n-1}p,\,\binom{n}{2}\,q^{n-2}p^{2},...,p^{n}$ are the successive terms of the binomial expansion $(q+p)^{n}$. A random variable X is said to follow binomial distribution if its probability mess function is given by

$$P(X=x) = p(x) = \binom{n}{x}\,p^{x}q^{n-x}, \quad x = 0,1,2.....n;\; q = 1-p$$

The values n and p of this distribution are known as the parameters of the distribution. The mean and variance of this distribution are np and npq.

Example : The probability of a man hitting a target is $1/4$.
i) If he fires 7 times, what is the probability of his hitting the target at least twice ? ii) How many times must he fire so that the probability of his hitting the target at least once is greater than $2/3$?

Solution:

i) Consider 'firing once' as a Bernoulli trial. Firing 7 times is the Binomial experiment with 7 independent Bernoulli trials. If X is the number of hits in 7 trials, then the required probability of hitting the target at least twice = $P(X \geq 2)$.

We know,

$$P(X \geq 2) = 1-p(X < 2)$$
$$= 1 - P(X=0) - P(X=1)$$

and $P(X=x) = \binom{n}{x} p^x q^{n-x}$ where $n = 7$, $p = 1/4$ and $q = 1-p = 3/4$.

$$P(X = 0) = (3/4)^7$$
$$P(X = 1) = \binom{7}{1} (1/4) (3/4)^6 = 7 \frac{3^6}{4^7}$$

The required probability

$$= 1 - (3/4)^7 - 7 \frac{3^6}{4^7} = \frac{4547}{8192}$$

ii) $p = 1/4$, $q = 3/4$.

We want to find n such that $P(X \geq 1) > 2/3$

or $1 - P(X < 1) > 2/3$

or $1 - P(X = 0) > 2/3$

or $1 - q^n > 2/3$ when $q = 3/4$

$\Rightarrow (3/4)^n < 1/3$

$\Rightarrow n = 4$.

Exercises :

1. A random variable X has a binomial distribution with parameters $n = 4$ and $p = 1/3$.

i) Describe the probability mass function and sketch its graph.

ii) Compute the probabilities $P(1 < X \leq 2)$ and $P(1 \leq X \leq 2)$.

2. In a binomial distribution consisting of 5 independent trials, probabilities of 1 and 2 successes are 0.4096 and 0.2048 respectively. Find the parameter p of the distribution.

3. The probability of a man hitting a target is 1/3.

i) If he fires 5 times what is the probability of hitting the target at least twice ?

ii) How many times must he fire so that the probability of hitting the target at least once is more than 90% ?

4. The random variable X has a binomial distribution with $n = 4$, $p = 0.5$. Find $P\{|X-2| \geq 1\}$.

Answers :

1. ii) 8/27, 56/81.
2. 0.2
3. i) 131/243
   ii) 6
4. 5/16

L I N E A R   P R O G R A M M I N G

1. Linear Inequations and Convex Sets
2. Formulation of L.P. Problems
3. Applications of Linear Programming

by

Dr.G.RAVINDRA

LINEAR PROGRAMMING

Introduction :

Mathematical Programming constitutes one of the most important problem areas of Operational Research (OR). It encompasses a wide variety of optimization problems. The basic problem of Mathematical Programming is to find the optimum (maximum or minimum) of a non-linear/linear function (called the objective function variously known as cost function, gain, measure of efficiency, return function, performance index, utility measure, etc. depending on the context) in a domain determined by a given system of non-linear and linear inequalities and equalities (called constraints).

Linear Programming (LP) is a Mathematical Programming problem where the objective function and the constraints are all (at least approximated) Linear functions of the unknown variables.

In practical terms, mathematical programming is concerned with the allocation of scarce resources - men, materials, machines and money (commonly known as the 4 M's in OR) - for the manufacture of one or more products so that the products meet certain specifications and some objective function (cost/profit) is minimized or maximized. Whenever the objective function is a linear function of the decision variables and the restrictions on the utilization or availability of resources are expressible as a system of linear equations or inequations, we have a Linear Programming Problem (LPP). For example, in the case of manufacturing a variety of products on a group of machines, the production problem is to determine the most efficient utilization of available machine capacities to meet the required demand. The programming problem is to allocate the available machine resources to the various products so that the total production cost is minimum. To solve this problem, we need to know the unit production cost (cost for producing one item), unit production time, machine capacity and production requirements. This is an LPP (for more clarification see Section 3 on formulation of Linear Programming Problems for a similar example).

The standard technique of solving an LPP is by Simplex Method (due to George, B.Dantzig, 1947) which is quite complicated and is beyond the scope of this unit. However, LPP's involving two variables can be solved graphically. Moreover, there are certain special types of LPP's such as transportation and assignment problems which admit easier methods of solution. Recently, there have been some spectacular developments in the area of LP due to an Indian, Narendra Karmakar of Bell Telephone Labs, U.S.A, where he is able to reach the solution of an LPP considerably faster than simplex method.

In this unit, we confine our attention to formulation of LPP's and their solution by graphical method.

LINEAR INEQUATIONS AND CONVEX SETS :

The restrictions on the utilization (demand) or availability of resources in a linear programming problem (LPP) are expressed as a system of linear equations or linear inequations, and the set of feasible solutions of an LPP is convex set. Though any LPP (in any number of variables) could be solved by the famous Simplex Algorithm, the LPP in two variables can be solved in an easier way by graphical method essentially identifying the intersection of graphs of various linear inequations and testing the objective function for maximum or minimum at the vertices of such a graph. The graph of a linear inequation is essentially a convex set. Thus the concept of Linear Inequations (and their graphs) and convex sets play an important role in the study and the solution of Linear Programming Problem (especially in the two variables case).

Linear Inequation :

Consider the relation $2x=4$ in exactly one variable x on real number line. In this equation, the highest power of x is 1 and so it is a linear equation in one variable. The graph of the equation is the set of all those points on x axis (Real line, R) satisfying the condition $2x=4$. Since there is exactly one point satisfying the condition namely $x=2$, the graph of the equation consists of just one point namely $x=2$ and it divides the x-axis into exactly two parts A and B, where A is the set of points on the axis satisfying $2x \leq 4$ and B is the set of points on the axis satisfying $2x \geq 4$, $2x \leq 4$ and $2x \geq 4$ are linear inequations in one variable and their graphs are respectively A and B, which are two opposite rays with end point $x = 2$.

The following illustrates the graphs of equation $2x=4$ and inequations $2x \leq 4$ and $2x \geq 4$.

A : $x \leq 2$ $x = 2$ B : $x \geq 2$

In general, $ax= b$, where a and b are real numbers, is a linear equation in one variable and its graph is just the point $x = b/a$ on x-axis (real line). Also, the point $x = b/a$ is common to the rays $ax \leq b$ and $ax \geq b$.

Consider another relation $2x+3y = 6$ in two variables. This is a linear equation in two variables. The graph of the equation is the set of all the points (x,y) in the cartesian plane (i.e. R^2 or xy-plane) which satisfy the equation $2x+3y = 6$. (3,0) and (0,2) are respectively the points of x-axis and y-axis satisfying $2x+3y = 6$. Thus the graph of $2x+3y = 6$ intersects the x-axis and y-axis respectively at (3,0) and (0,2). We know that the equation of the line passing through (3,0) and (0,2) is $2x+3y = 6$. Thus, the graph of $2x+3y = 6$ is essentially the straight line which intersects x-axis and y-axis respectively at (3,0) and (0,2). Further, the graph of $2x+3y = 6$ is the common edge of the two regions C and D where C is the set of points satisfying the inequation $2x+3y \leq 6$ and D is the set of points satisfying the inequation $2x+3y \geq 6$. C and D are called the graphs of $2x+3y \leq 6$ and

$2x + 3y \geq 6$ respectively. More precisely, we observe that the xy-plane has the following partitions.

1. The set of points satisfying $2x+3y < 6$.
2. The set of points satisfying $2x+3y = 6$.
3. The set of points satisfying $2x+3y > 6$.

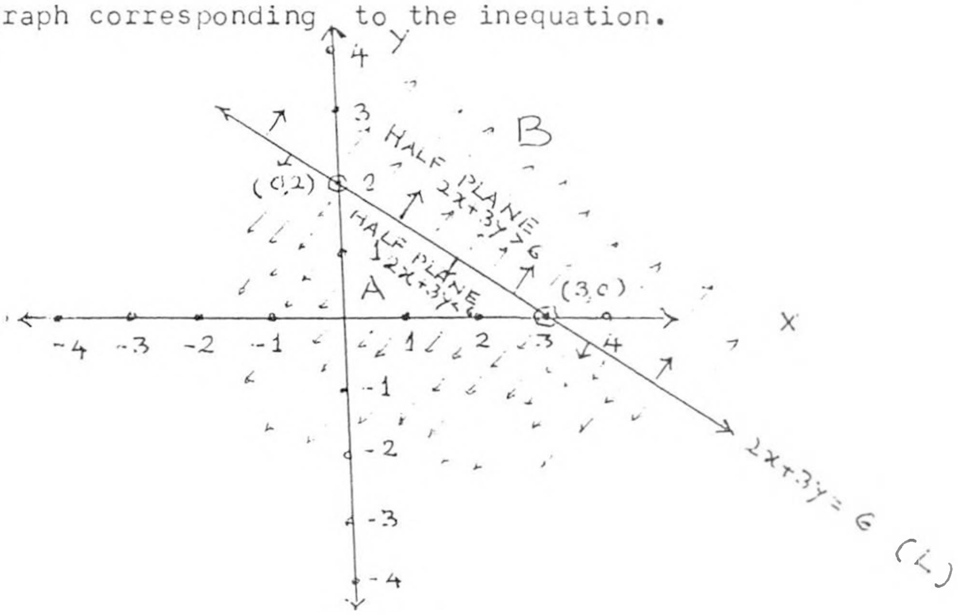
Thus, if (x,y) is a point in the xy-plane, then it belongs to <u>either</u> i) the graph of $2x+3y < 6$

<u>or</u> ii) the graph of $2x+3y = 6$

<u>or</u> iii) the graph of $2x + 3y > 6$

This is the basic philosophy in identifying the graph of an inequation. We illustrate the same as follows :

Suppose we wish to identify the graph of the inequation $2x+3y < 6$. In the following figure, L represents the graph of the line $2x+3y = 6$. The graph of $2x+3y < 6$ could be either A or B (but not a portion of both). We have to mark which one of them is the exact graph corresponding to the inequation.



Here A and B are mutually disjoint. Choose a point which does not belong to L. (0,0) is one such point. The point (0,0) satisfies the inequation $2x+3y < 6$. Hence A is the graph of the inequation.

AUL is the graph of the inequation $2x+3y \leq 6$. Suppose we wish to identify the graph $2x+3y > 6$. Since (0,0) which is in A does not satisfy the inequation, A cannot be the graph of the inequation. Therefore, B is the graph of the inequation. Also $B \cup L$ is the graph of the inequation $2x+3y \geq 6$.

In general, the graph of the linear equation $ax+by = c$ (in two variables) is the set of points on the line intersecting x-axis at $(c/a, 0)$ and y-axis at $(0,c/b)$. Further, the graph divides the xy-plane into two parts E and F, one of which is the graph of $ax+by \leq c$ and the other is the graph of $ax+by \geq c$. If a point in E (which is not on L) satisfies $ax+by \leq c$, then E is the graph of the inequation $ax+by \leq c$ and F is the graph of the inequation $ax+by \geq c$. Otherwise, E is the graph of the inequation $ax+by \geq c$ and F is the graph of the inequation $ax+by \leq c$.

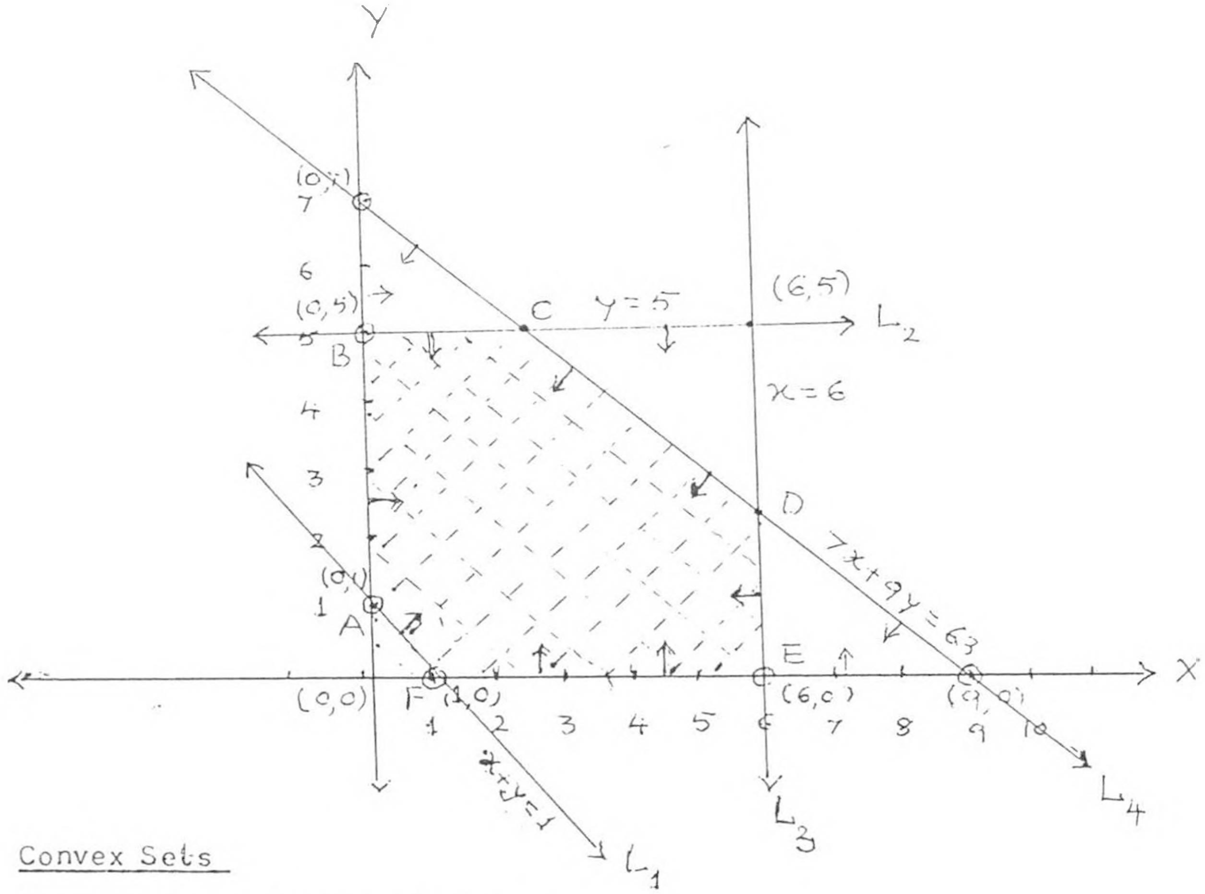
Consider the linear equation $ax+by+cz = d$ in three variables. The graph of this is a plane in the space R^3 and is common to the two parts A and B where A is a set of points (x,y,z) in R^3 satisfying $ax+by+cz \leq d$ and B is the set of the points (x,y,z) satisfying $ax+by+cz \geq d$. A and B are called <u>half planes</u>.

In general, the graph of $a_1x_1 + a_2x_2 + ... + a_nx_n = b$ is called <u>Hyper plane</u> in the space R^n (i.e. n-dimensional Euclidean space) giving rise to two parts A and B where A is the set of points $(x_1, x_2, ..., x_n)$ in R^n such that $a_1x_1 + a_2x_2 + ... + a_nx_n \leq b$ and B is the set of points such that $a_1x_1 + a_2x_2 + ... + a_nx_n \geq b$. A and B are called <u>Half spaces</u>.

In what follows, we shall mainly confine our discussion to equations and inequations in two variables only.

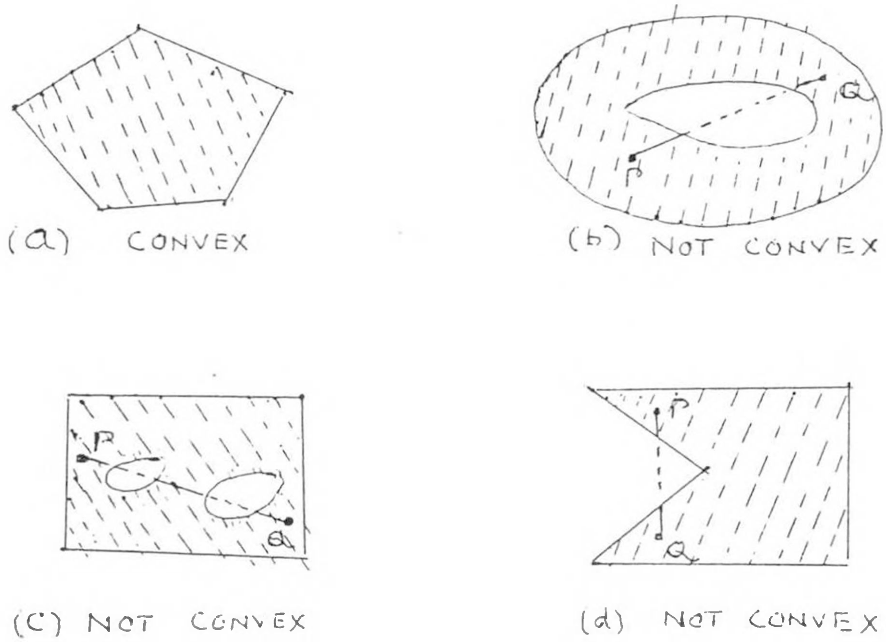
Example: Identify the intersection of graphs of the following linear inequations : $x + y \geq 1$, $y \leq 5$, $x \leq 6$, $7x + 9y \leq 63$, $x, y \geq 0$.

In the following figure, we have drawn arrow marks along the line L_1 representing $x+y = 1$ in such a way that the pointers of the arrows lie in the graph (region) of $x+y \geq 1$. The same is repeated for the rest of the inequations. The intersection of the graphs of these inequations is identified as that region which includes pointers corresponding to <u>all</u> the lines L_1, L_2, L_3, L_4, X and Y. The region enclosed by the polygon ABCDEF is such a region and hence it is the required graph satisfying all the six inequations simultaneously. Note that the region S enclosed by CDF is not the required region as no pointer corresponding to L_4 lies in it. Note that the arrows corresponding to all the lines L_1, L_2, L_3, L_4, X and Y converge in the graph satisfying all the six inequations.



Convex Sets

Examine the following figures.



The figure (a) is distinctly different from the other three. In the figure, the linear segment joining any two points is entirely within it, while the regions (b), (c) and (d) do not have the same property. For example, in (b) the line segment joining X and Y is not entirely in it, in (c) the line segment PQ is not in it and in (d) the line segment joining R and S is not entirely in it. Note that the dotted portion of the lines in (b), (c), (d) are not inside the regions. The figures like that of (a) are of special significance in the solution of LPP's and they are said to be <u>convex</u>. Speaking more precisely, a set of points C in the xy-plane (or R^n in general) is called a convex set if the line segment joining any two of its points is entirely contained in C.

<u>Examples of Convex Sets</u> :

i) xy-plane is a convex set.

ii) Circular region in xy-plane is convex but a circle is not convex. (by a circle, here we mean the set of points in xy-plane each of which is equidistant from a given point in the plane).

iii) Sphere, cube, cone, ellipsoid, paraboloid, etc. are convex sets in R^3.

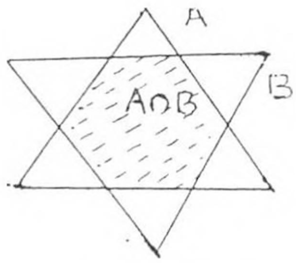
iv) Torus is not a convex set in R^3.

v) Hyperboloid is not a convex set in R^3.

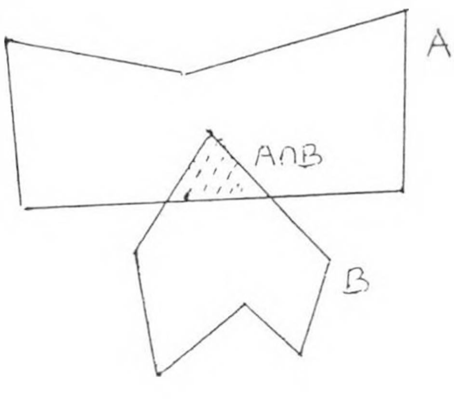
vi) The graphs of the inequations $ax+by \leq c$ and $ax+by \geq c$ are convex, i.e. half planes are convex.

vii) Half spaces in R^n are convex.

Now suppose A and B are any two sets with a given property P. The intersection of A and B may or may not have the property P, though it is part of the both. For example, if A and B are triangular regions in xy-plane their intersection is not necessarily a triangular region in the xy-plane. Similarly, if A and B are two sets in xy plane which are <u>'not convex'</u>, their intersection need not have the same property, that is, it could be convex. The following figures illustrate this.



The Intersection A B is not a triangular region.



The Intersection A B is a convex set.

If A and B are convex, will the intersection of A and B also be convex ? We will verify whether this is true or false.

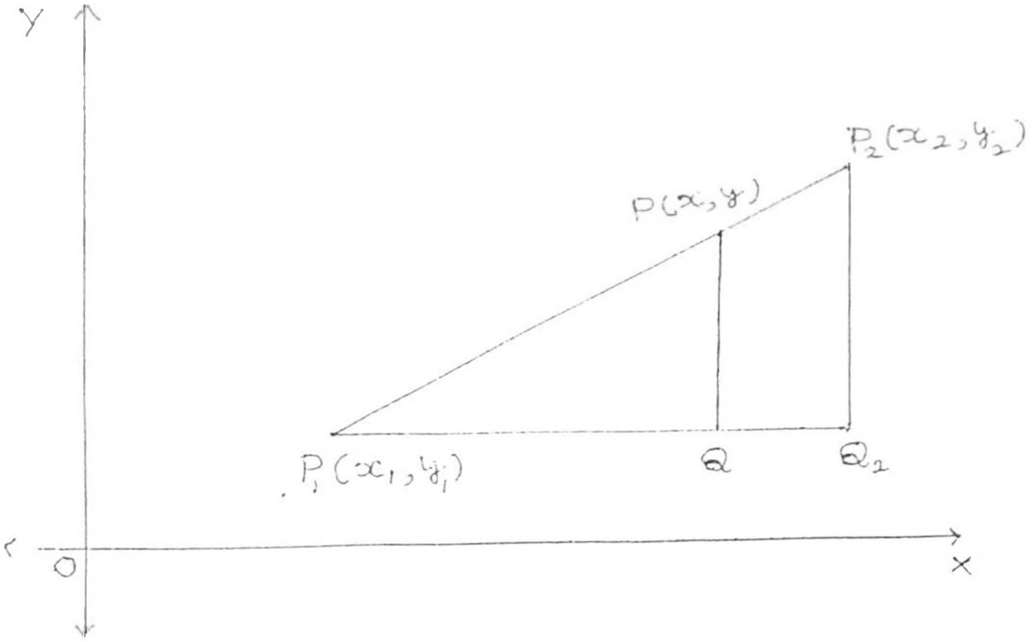
Let C be the intersection of A and B. Let P and Q be any two points in C. Let L be the line segment joining P and Q. Since A is convex, the points of L are contained in A. Since B is convex, the points of L are also contained in it. Thus the points of L are in both A and B. That is, the line segment joining any two points P and Q is entirely in C. This implies that C is a convex set. That is, the intersection of A and B is a convex set. We list the interesting result as

FACT : The intersection of any number of convex sets is also convex. Justification for this essentially follows from the above arguments, replacing sets A and B by any number of sets.

We now look for another way of defining convex sets which often helps in proving results concerning convex sets.

We know from coordinate geometry that (x,y) is a point on a line segment joining the points (x_1,y_1) and (x_2,y_2) if and only if $x = (1-t)\,x_1 + tx_2$ and $y = (1-t)\,y_1 + ty_2$, where $0 \leq t \leq 1$. Justification for the statement follows by considering the similar triangles P_1OP and $P_1P_2O_2$ and their implication viz.

$$\frac{P_1Q}{P_1Q_2} = \frac{QP}{Q_2P_2}$$



Let $X = (x,y)$, X_1 a (x_1, y_1), $X_2 = (x_2,y_2)$, $t_1 = 1-t$, $t_2=t$. Using these symbols, the above statement can be restated as follows: X is a point on the line segment joining X_1 and X_2 if and only if $X = t_1X_1 + t_2X_2$ such that $t_1 + t_2 = 1$, $t_1, t_2 \geq 0$.

(Since $X = (x,y) = ((1-t)\ x_1 + tx_2,\ (1-t)\ y_1 + ty_2)$ $= ((1-t)\ x_1,\ (1-t)y_1) + (tx_2,\ ty_2) = (1-t)\ (x_1,y_1) + t\ (x_2,\ y_2) =$ $(1-t)\ X_1 + tX_2$). The point X so expressed is said to be a <u>convex combination</u> of the points X_1 and X_2 in xy-plane.

A <u>convex combination</u> of points $X_1, X_2, \ldots, X_n$ in xy-plane (or R^n in general) is a point $X = t_1X_1 + t_2X_2 + \ldots\ldots + t_nX_n$ where t_i's are non-negative real numbers and, $t_1 + t_2 + \ldots + t_n = 1$. As seen already, a point $X = (x_1,y_1)$ belongs to the line segment joining $X_1 = (x_1,y_1)$ and $X_2 = (x_2,y_2)$ if and only if X is a convex combination of X_1 and X_2. Thus a convex set can also be defined as follows:

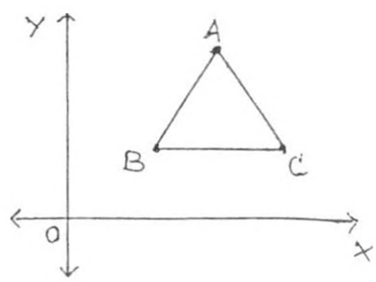
A set C in xy-plane (or R^n) is a convex set if convex combination of any two points in C is also in it.

In fact, for a given convex set C any convex combination of any number of points in C is also in C.

Not every point in C is a convex combination of some points in C. For example, consider the triangle ABC in xy-plane (The following figure). There are no two distinct points in the triangle such that the line segment joining them contains A. That is, A is not an 'intermediate' point of any line segment in the triangle. Though A is a point on the line segment AB, it is not an intermediate point but one of the extreme points. Thus, A is not a convex combination of any other two distinct points in the triangle. Similarly, the points B and C have the same property. But any other point in the triangle is an intermediate point of same line segment in C. That is any point in the triangle other than A, B and C is a convex combination of some other two distinct points. The points A,B,C are extreme points in comparison with other points in the triangle.

A point X in a convex set is called an extreme point if X cannot be expressed as a convex combination of any other two distinct points in C.

Note that in the above example, the vertices A, B and C are the only extreme points of the triangle.



Examples :

1. The end points of a line segment are extreme points.
2. Vertices or corners of a cube in R^3 are extreme points.
3. Every point of the boundary of a circular region is an extreme point.
4. All the interior points of a circular region are not extreme points.
5. No point of a xy-plane is extreme in the plane.
6. The extreme points of a polygonal region are its vertices.
7. Any point in xy-plane is an extreme point of the singleton set containing the point.
8. The point of intersection of two line segments is not an extreme point of the line segments.

The extreme points play a very significant role in the solution of a LPP. In fact, the objective function of a LPP attains its optimum at at least one of the extreme points of its feasible region which is always convex.

Exercises :

1. Which of the given points belong to the graph of the given inequations ?

i) $x + y < 5$ (0,0); (3,2)

ii) $x - y > 6$ (4,3); (11,4)

iii) $3x+y \leq 2$ (0,0); (0,4)

2. State whether the solution set of the following system of linear inequations is a null set or not.

i) $x \leq 0$ and $x \leq 2$

ii) $x < 2$ and $x > 2$

iii) $y > 1$ and $y > -1$

3. State true or false.

i) The line $y = 10x + 50$ separates the xy-plane in two half planes.

ii) A half plane is the graph of the inequation.

iii) The graph of a linear inequation is a convex set.

iv) The union of two convex sets in xy-plane is also a convex set in xy-plane.

v) The intersection of two convex sets in xy-plane is a convex set in xy-plane.

vi) If A and B are two sets in R^2 which are not convex, their intersection is also not convex in R^2.

vii) Vertices of a cube are extreme points.

viii) If m is the number of linear inequations in two variables and if the intersection of their graphs is a polygonal region with n sides then m = n.

ix) If a point (x,y) in xy-plane is a convex combination of two points (r,s) and (p,q) in the plane, then it lies on the line joining the two points (p,q) and (r,s).

x) The converse of the above statement is generally not valid.

xi) The intersection of two convex sets could possible be disjoint union of two convex sets.

xii) Union of two convex sets is convex.

xiii) Every point in a convex set is a convex combination of two other points in it.

4. Find two points in xy-plane that satisfy each of the following.

i) $y = 5x$, ii) $y < 5x$ iii) $y > 5x$

5. Mark the region which represents the graph of following inequations.

a) $x < 3$ b) $y > 3$ c) $2x + 4y \leq 8$

d) $x + y \leq 4$

6. State whether the region representing the following is bounded or unbounded.

$x \geq 0$, $y \geq 0$ and $x+y \leq 8$.

7. Let ABCD is a square in the first quadrant of xy-plane.

i) If $x + y = 1$ is the equation of the side AB, find the equations of the sides, BC, CD and DA.

ii) Write the inequations whose intersection is the <u>interior</u> of the square.

8. Let ABCDEF be a regular hexagon with length of each of its sides equal to 1 unit. Write the inequations whose intersection is the given hexagon.

9. Prove or disprove :

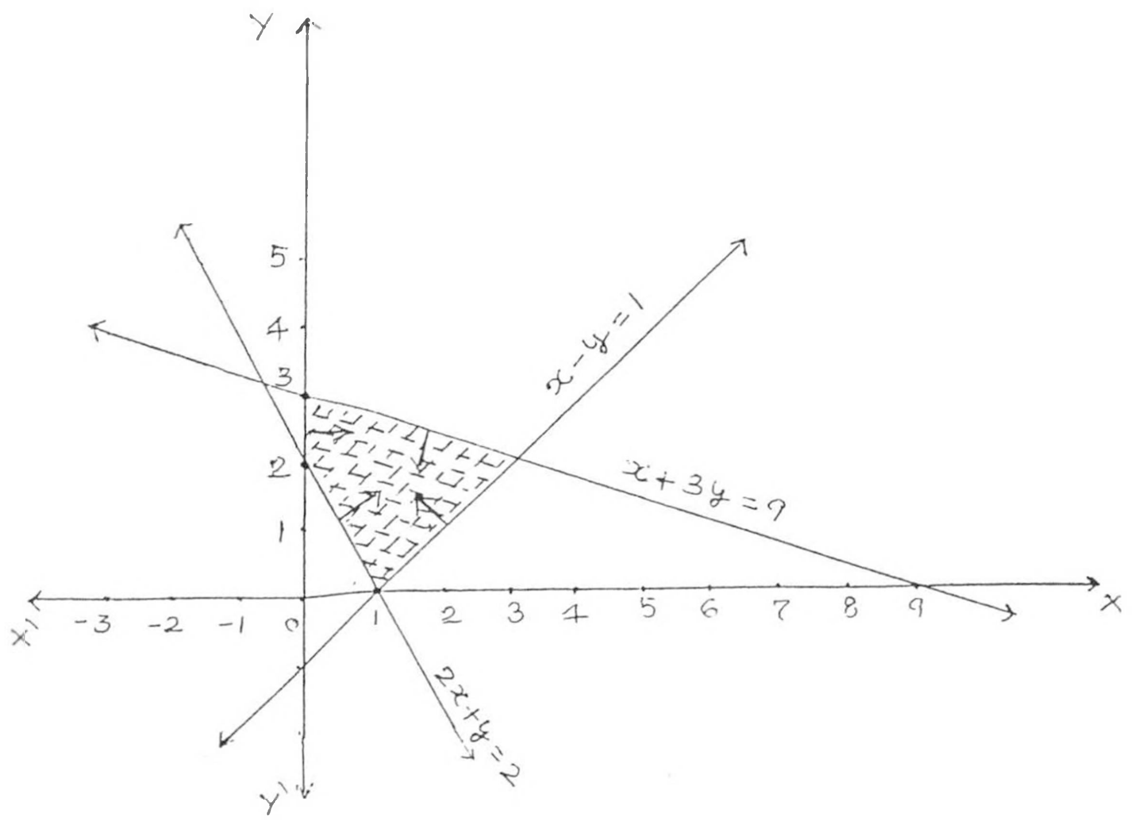
i) The circle $x^2+y^2 = a^2$ (a is a given real number) is a convex set.

ii) Every point on the boundary of a circular region is an extreme point.

iii) If G is the graph satisfying m linear inequations simultaneously, then G is a polygonal region having m sides.

iv) A set consisting of single element of R^2 is a convex set in R^2.

10. Find the linear constraints for which the shaded region in the following figure is the solution set.



FORMULATION OF LINEAR PROGRAMMING PROBLEMS

A large class of problems can be formulated as LP models. While formulating an LP model it is worth-while to remember the following 3-way rule suggested by Dantzig..

i) Identify the unknown activities to be determined and represent them by suitable algebraic symbols. Identify the inputs and outputs associated with each activity.

ii) Identify the restrictions (constraints) in the problem and express (at least approximate) them as linear algebraic equations/inequations.

iii) Identify the objective function and express it as a linear function of the unknown variables.

Proper definitions of the variables (step (i)) is a key step and will largely facilitate the rest of the work.

Let us illustrate the formulation by a few examples.

Example : Suppose we are concerned with a problem encountered by a man who sells oranges and apples in a running train. He has only Rs.120 with him and he decided to buy atleast 5 kgs of each item. One kg of apple costs Rs.10 and 1 kg of orange costs Rs.5. He can carry to the train only a maximum load of 15 kgs which his bag would hold. He expects a profit of Rs.2 per kg from applies and Rs.1 per kg from oranges. How much each of these two items should he buy (if he is wise enough) so as to get a maximum profit ?

Here, the ultimate goal or objective of the fruit seller is to get the maximum profit in his business, i.e. he wants to maximise his profit. To achieve this, he cannot purchase the items at random. The problem is to find out in what combinations should he buy apples and oranges so that the profit is maximum. Let us try to find out the possible combinations. The man can buy a total of 15 kgs of apples and oranges. Can he buy 15 kgs of oranges? Of course, not, because he has to buy at least 5 kgs of apples, i.e., he can buy a maximum of 10 kgs of oranges. Can he buy 15 kgs of apples ? He cannot because he should buy at least 5 kgs of oranges i.e. he can buy a maximum of 10 kgs of apples. He can purchase oranges from 5 kgs to 10 kgs and so also apples. We can list all the possible combinations of his purchase of apples and oranges and calculate the profit in each case. See the table below.

PURCHASE (in kgs)		COST			PROFIT		
Orange	Apple	Orange Rs.5	Apple Rs.10	Total	Orange Re.1	Apple Rs.2	Total
5	10	25.00	100.00	125.00	Not possible		
6	9	30.00	90.00	120.00	6.00	18.00	24.00
7	8	35.00	80.00	115.00	7.00	16.00	23.00
8	7	40.00	70.00	110.00	8.00	14.00	22.00
9	6	45.00	60.00	105.00	9.00	12.00	21.00
10	5	50.00	50.00	100.00	10.00	10.00	20.00

Look at the last column. The maximum profit is Rs.24. He gets this profit when he purchases 6 kgs of oranges and 9 kgs of apples.

This is the solution of the problem which maximises or optimises the profit. So we call it an optimal solution of the problem.

Optimal solution = 9 kgs of apples and 6 kgs of oranges.

Optimum profit = Rs.24.

After investigating the next example, where we maximise the profit as in this example, we will be able to see if we can arrive at the optimal solution by trial and error method. Before that let us formulate the above example in Mathematical terms (see Dantzig's 3-way rule).

i) Definition of variables

Let x be the number of kgs of oranges and y be the number of kgs of apples bought.

ii) Constraints : Since one cannot buy negative number of oranges or apples it is clear that $x \geq 0$ and $y \geq 0$.

Since one kg of orange costs Rs.5, x kgs of orange will cost Rs.5x. Similarly, y kgs of apple costs Rs.10y. Therefore, the total cost will be 5x + 10y. Since he has only Rs.120 with him we have,

$5x + 10y \leq 120.$

Since he has decided to buy atleast 5 kgs of each item,

$x \geq 5, \quad y \geq 5.$

As he cannot carry more than 15 kgs

$x + y \leq 15.$

iii) The objective function :

Since he expects a profit of Rs.2 per kg from apples and Re.1 per kg from oranges, his total profit would be x+2y which has to be maximised. The L.P. model is : Maximise $Z = x+2y$ subject to $x \geq 5$, $y \geq 5$, $5x+10y \leq 120$, $x+y \leq 15$; and $x,y \geq 0$. In this problem, the non-negativity restrictions are not necessary in view of the constraints $x,y \geq 5$.

Example : A company sells two different types of radios - 3 band types and 2 band types. Company has a profit of Rs.50 for each of the former type and Rs.30 for each of the second type. The production process has a capacity of 80,000 man hours in total. It takes 10 man hours labour to assemble 3-band type and 8 man hours for 2-band type. It is expected that a maximum of 6000 numbers of the former type and a maximum of 8000 of latter type can be sold out. How many of each type should be produced so as to maximise the profit ?

In this problem, the company aims at getting the maximum profit. i.e. profit is to be maximised. The problem is to find out in what combination should he produce 2-band radios and 3-band radios in order to achieve this objective. We know that the company gets more profit from the 3-band radios. Naturally, we can think of a possibility where all the radios produced are 3-band type. This could not be done since the maximum number of 3-band type radios should be six thousand. The other possibility is to think of another way. The man hours needed to produce a 2- band radio is smaller compared to 3-band radios. In that case, he should increase the number of 2-band radios, which should not exceed 8000. Naturally, a third question arises - can the company produce 6000, 3-band radios and 8000, 2-band radios. In that case, we have to take into consideration the man hours available. The man hours required for producing 8000, 2-band radios is 8x8000 = 64000. The total man hours required to produce 6000 3-band type and 8000 2-band type is 124000 which is greater than the man hours available. From the above discussion, we found that the number of 3-band radios can extend from 0 to 6000 and that of 2-band radios from 0 to 8000. To get a solution for this problem, we have to enumerate all the cases from 0 to 6000 and 0 to 8000, which evidently is laborious. Therefore, we have to find out an easier method to solve such problems.

We will now think of evolving an easy method to solve such problems. Before entering into the details of this method, let us explain the problem mathematically. In other words, let us try to write the LP formulation of the problem.

In the above problem, what we are expected to find is the number of 3-band radios and 2-band radios to be produced so as to get the maximum profit. Let us assume that the number of 3-band radios produced is 'x' and the number of 2-band radios produced is 'y'.

Number of 3-band radios = x
Number of 2-band radios = y

Once we know the number of each type of radios, we can calculate the total profit of the company. Profit from a 3-band radio is Rs.50 and the profit from a 2-band radio is Rs.30.

Total profit = $50x + 30y$.

The objective of the company is to get the maximum profit i.e. $50x + 30y$ should be maximum. We call this the objective function of the problem. Now the problem reduces to finding the maximum values of $50x + 30y$. In other words, we have to maximise $50x + 30y$.

What are the conditions to be satisfied ?
We know that 'x' and 'y' are the numbers of radios produced. So we can say that x and y cannot be negative. Mathematically, we put it as

$x \geq 0$ and $y \geq 0$

x is the number of 3-band radios. The maximum number of 3-band radios produced is 6000.

i.e. $x \leq 6000$

Similarly $y \leq 8000$

The total man hours available is only 80000. Man hours required to produce one 3-band radio is 10.
Man hours required for x radios = $10 \times X$
= $10X$

In a similar way, man hours needed for y, 2-band radios = 8y.
The total man hours should not exceed 80000

i.e. $10x + 8y \leq 80000$

Thus the restrictions or conditions to be satisfied are

1. $x \geq 0$
2. $y \geq 0$
3. $x \leq 6000$
4. $y \leq 8000$
5. $10x + 8y \leq 80000$

These conditions are generally called constraints of the problem. The first two viz. $x \geq 0$ and $y \geq 0$ are called non-negativity restrictions. Each of these constraints is an inequation of degree 1. Hence, they are called linear constraints.

The mathematical formulation of the problem is as follows :

Maximise $50x + 30y$

subject to $x \geq 0$

$y \geq 0$

$x \leq 6000$

$y \leq 8000$

and $10x + 8y \leq 80000$

Here the objective function as well as the constraints are all linear (first degree).

A typical LP Model :

Suppose a company with two resources (labour and material) wishes to produce two kinds of items A and B.

Let t_1, t_2 units of time (hours or minutes) be respectively time required to produce one unit of A and B, m_1 and m_2 be the amount of unit material (in Kg or pounds or any unit of weight) respectively required for one unit of A and B, and Rs.p_1 and Rs.p_2 profit per unit of A and B. Suppose the daily availability of manpower (labour) is T hours and the supply of raw material is restricted to M Kgs per day. The problem of the company is :

How many items of kind A and how many items of kind B be produced everyday, so that the total profit is maximum ?

This kind of problem is generally known as Product-Mix Problem.

The entire information of the problem can be stored in matrix (tabular) form as follows :

Resources	Kinds of Items A	B	Supply/availability
Labour (hours/unit)	t_1	t_2	T
Material (Kgs/unit)	m_1	m_2	M
Profit (Rs./unit)	p_1	p_2	

In view of the 3-way rule suggested earlier we have

Step 1 : Let x = Daily production of kind A

y = Daily production of kind B

Step 2 : Constraint corresponding to the first row :

$$t_1x + t_2y \leq T$$

Constraint corresponding to second row :

$$m_1x + m_2y \leq M$$

Non negativity conditions :

$$x, y \geq 0$$

Step 3 :

The third row corresponds to the objective function and is given by

$$Z = p_1x + p_2y$$

Thus the mathematical formulation of the problem is :

(I) - Find numbers x, y which will maximize

$$Z = p_1x + p_2y$$

subject to the constraints

$$t_1x + t_2y \leq T$$

$$m_1x + m_2y \leq M$$

and $x, y \geq 0$

Note that in the mathematical formulation (I) above we deal only with numbers, equations, inequations and the given situation (that is company's problem) is no longer under consideration.

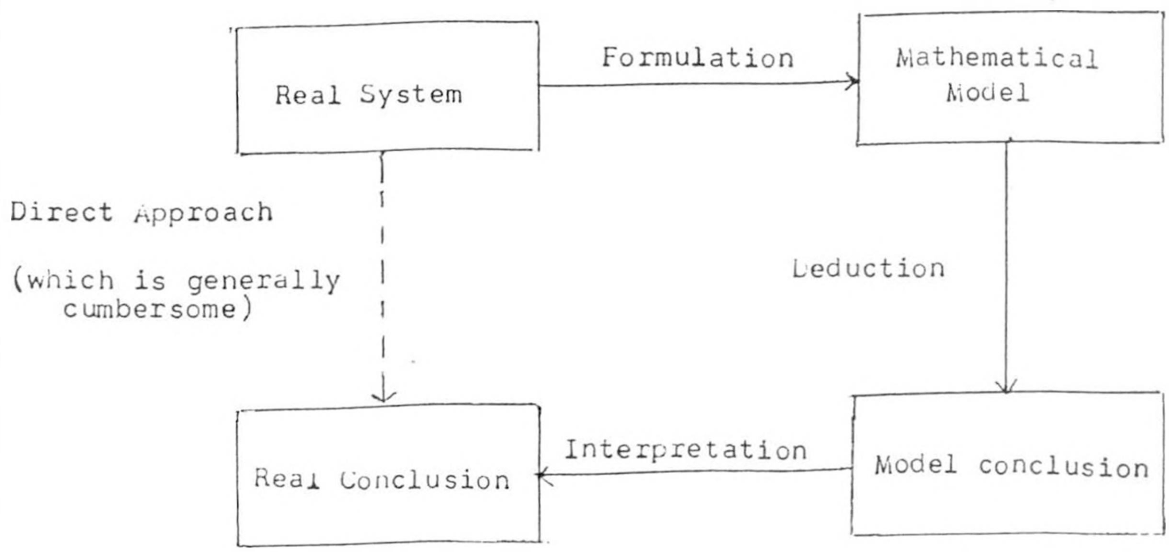
The above typical problem can be adopted in many real life situations and thus a teacher can find a problem of linear programming according to the nature of the students (urban, rural, etc). For example, if the 'company' is an industry like "ORKAY" A could be taken as Idly mix and B could be taken as Dosa mix. The relevant information concerning resources and profit (possibly in terms of cost price and selling price) can be obtained in the form of a matrix. Such matrix will help in identification of the problem as well as in its mathematical formulation. If we consider a comfy in kitchen appliance,

A could be considered as a pressure cooker
B could be considered as pressure pan.

If we want to have a farmer's problem, we can take A and B respectively to be areas of a given field for production of wheat and gram. The resource corresponding to material could be fertilizer. Here we will have an extra constraint viz. $x+y \leq a$ where 'a' is the area of the given field. Note that there could be any number of resources (and hence constraints) depending upon the situations.

Linear Programme, a Mathematical Model :

A mathematical model is a symbolic representation of a real situation. The process of mathematical modelling is depicted in the following figure.



In example 1, the real situation is 'selling of oranges and apples'. In example 2, the real problem (situation) is 'to evolve a selling policy of two kinds of radios' and in the product mix problem the real situation is 'productive scheduling'. In all these problems, mathematical formulation is mathematical model. The mathematical models in the above examples consist of objective function and constraints which are expressed quantitatively or mathematically as functions of decision variables 'Mathematical conclusion' and 'Real conclusion' constitute the solution of a linear programming problem, which we would be dealing within the next section.

EXERCISES :

1. A company makes two kinds of leather belts A, B. Belt A is of higher quality and belt B is of lower quality. The respective profits are Rs.4 and Rs.3 per belt. Each belt of type A requires twice as much time as a belt of type B, and, if all belts were of type B, the company, could make 1000 per day. The supply of leather is sufficient for only 800 belts per day (both A and B combined). Belt A requires a fancy buckle and only 400 per day are available. There are only 700 buckles a day are available for belt B. Formulate this as a linear programming model.

2. Give an example of a real situation (other than those mentioned in this lesson) whose mathematical model is a linear programming model.

3. Give an example of a mathematical model which is not a linear programming model.

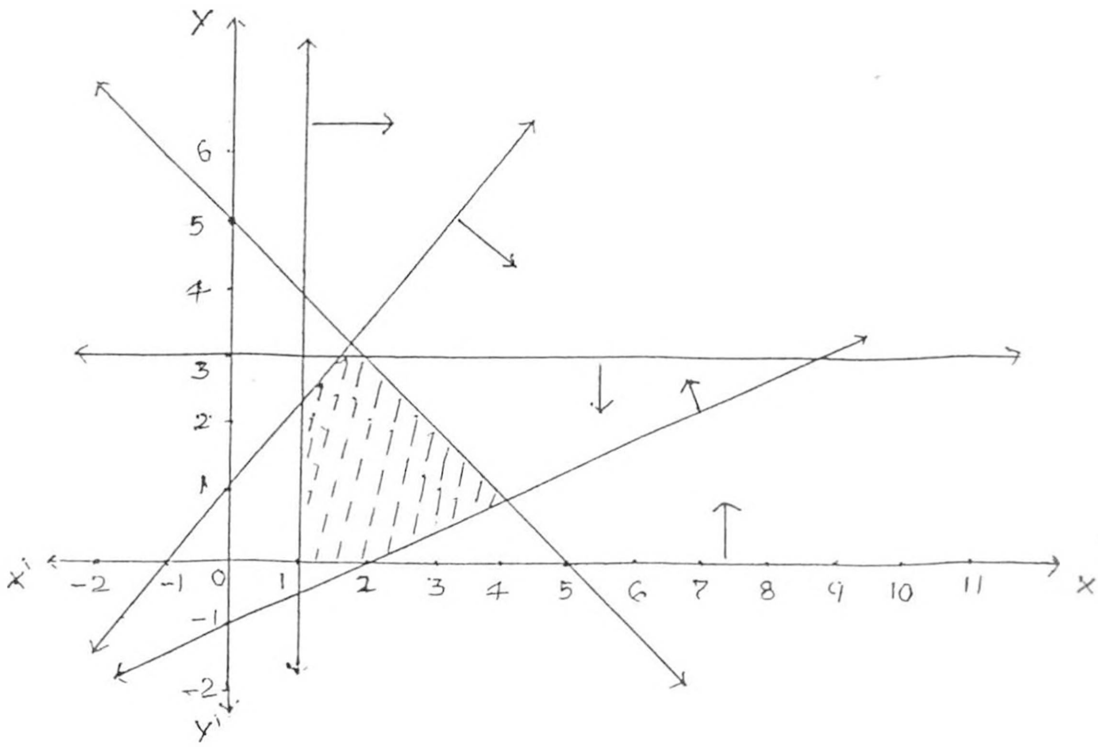
4. An Advertising company wishes to plan an advertising campaign in three different media - television, radio and magazines. The purpose of the advertising company is to reach as many potential customers as possible. Results of the market study are given below :

	Television		Radio	Magazine
	Day Time	Prime Time		
	Rs.	Rs.	Rs.	Rs.
Cost of an advertising unit	40,000	75,000	30,000	15,000
Number of potential customers reached per unit.	400,000	900,000	500,000	200,000
Number of women customers reached per unit	300,000	400,000	200,000	100,000

The company does not want to spend more than Rs.800,000 on advertising. It further requires that (i) atleast 2 million exposures take place among women, (ii) advertising on television be limited to Rs.500,000, (iii) atleast 3 advertising units be bought on day time television and two units during prime time; and (iv) the number of advertising units on radio and magazine should each be between 5 and 10.

Find different types of advertising units which minimize the total number of potential customers reached is maximum.
(Note: The problem involves four decision variables).

5. Write the constraints associated with the solution space shown in the following figure and identify all redundant constraints.



SOLUTION OF LINEAR PROGRAMMING PROBLEM BY GRAPHICAL METHOD :

Let us consider another example of an optimisation problem. We can examine whether this is a linear programming problem by formulating a mathematical model of the problem. We can also try to find the solution of the problem by graphical method.

Example 1 : A contractor has 30 men and 40 women working under him. He has contracted to move at least 700 bags of cement to a work site. Due to the peculiar nature of the work site he could employ at the maximum of 50 workers at a time. A man will carry 25 bags in a day and a woman will carry 20 bags in a day. A man demands Rs.45 a day and a woman demands Rs.35 a day as their wages. In what ratio should the contractor employ men and women so that the cost of moving the cement to the work site is minimum ?

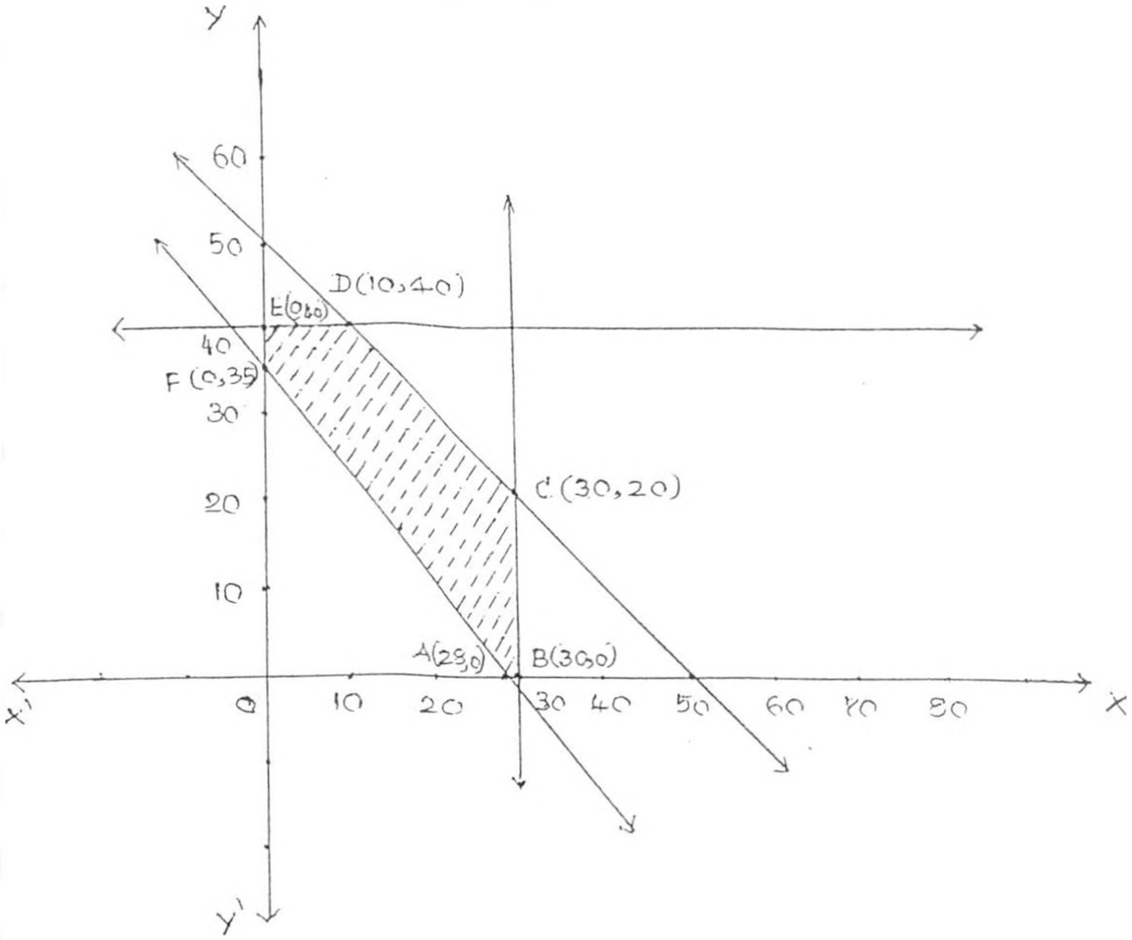
Now the mathematical model of the problem is :
Minimize $Z = 45x + 35y$

subject to the conditions
$x \geq 0$
$y \geq 0$
$x \leq 30$
$y \leq 40$
and $5x + 4y \geq 140$
(x and y are respectively the number of men and women employed and z is the total wage for them).

The above problem is an optimisation problem. The objective function as well as the constraints are linear. Hence, it is a LPP.

The next step is to find a value for x and a value for y such that $45x + 35y$ is minimum subject to the conditions laid down in the problem.

We first draw the graph of the inequations and see how the graph will give the solution of the problem.



The intersection of the graphs of the inequations is the region of the polygon ABCDEF, called the feasible region. Any point p (x,y) in the feasible region is a feasible solution of the LPP. The coordinates of such a point will satisfy all the inequations. Let us consider a point p(20, 20) in this region. We can easily verify that it satisfies all inequations. So we can consider the x-coordinate of P as a value of x and y - coordinate of P as a value of y. i.e. x = 20 (x- coordinate of P) and y = 20 (y - coordinate of P) is a feasible solution of the LPP. If we select another point say D (10,40) in the region, x = 10 and y = 40 is another feasible solution of the problem. We know that there are infinite number of points in the region ABCDEF. The coordinates of each point will give a feasible solution of the problem i.e.

the number of feasible solutions are infinite. The problem is to decide which one of these is optimal. For this, we make use of the following key result.

THEOREM : If there exists an optimal solution to an LPP, the objective function of the LPP always attains its optimum (minimum or maximum) at at least one of the corners (extreme points) of the feasible region.

Proof: We prove the validity of the theorem for two variables (coordinates) and in fact the same arguments can be extended to prove the theorem for any number of variables.

Let K be the set of feasible solutions of a linear programming problem. Let $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ be the extreme points (corners), of the feasible region corresponding to K. Let $Z(x,y) = C_1 x + C_2 y$ be the objective function of the linear programming problem.

Suppose for $x = x_0$ and $y = y_0$ the objective function attains its minimum.

That is, $Z(x_0, y_0) = C_1 x_0 + C_2 y_0$ is the minimum value of the objective function. Let $m = Z(x_0, y_0)$.

If (x_0, y_0) is one of the extreme points (corners) of the region representing K, the theorem is true. Therefore, we assume that (x_0, y_0) is not an extreme point. Hence, by the definition of extreme point, (x_0, y_0) can be expressed as a convex combination of extreme points of K.

That is,

$$(x_0, y_0) = t_1 (x_1, y_1) + t_2 (x_2, y_2) + \ldots + t_n (x_n, y_n) \qquad (1)$$

where $t_1 + t_2 + \ldots t_n = 1$ and $t_i \geq 0$.

This implies that

$$m = Z(x_0, y_0) = t_1 z(x_1, y_1) + t_2 z(x_2, y_2) + \ldots + t_n z(x_n, y_n)$$

Suppose $Z(x_r, y_r)$ be minimum among $Z(x_1, y_1), \ldots, z(x_n, y_n)$ so that

$$Z(x_i, y_i) \geq Z(x_r, y_r), \quad 1 \leq i \leq n \qquad (2)$$

Now (1) and (2) together imply that

$m \geq t_1 z(x_r, y_r) + t_2 z(x_r, y_r) + \dots + t_n z(x_r, y_r)$

(Since t_i' s are non negative).

That is,

$m \geq (t_1 + t_2 + \dots + t_n) \; Z(x_r, y_r)$

or $m \geq Z(x_r, y_r)$ (Since $t_1 + t_2 + \dots + t_n = 1$) (3)

By definition of minimum

$m \leq Z(x,y)$ for every (x,y) in K and in particular

$m \leq Z(x_r, y_r)$. (4)

(3) and (4) together imply that

$m = Z(x_r, y_r)$ where (x_r, y_r) is an extreme point. Thus Z (the objective function) attains its minimum at an extreme point of the feasibility region.

Remark :

Let for $x = x^1$ and $y = y^1$, $z(x,y)$ (the objective function) attain its maximum. Then by definition of maximum

$z^1 \geq z(x,y)$ for every x,y in K (where $z^1 = z(x^1, y^1)$)

$\Rightarrow -z^1 \leq -z(x,y)$

$\Rightarrow -z^1$ is the minimum value of $-z(x,y)$

That is, $-z^1 = \min(-z(x,y))$

or $-(\max z(x,y)) = \min(-z(x,y))$

or $\max z(x,y) = -\min(-z(x,y))$

Thus minimisation problem can be converted to maximization problems by considering negative of the objective function z(x,y). And accordingly, the above theorem is true in the case of maximisation problems also.

In view of the above theorem, it is sufficient to concentrate our attention only on the corner points of the polygon ABCDEF. Evaluating the objective function at each of the vertices of ABCDEF and selecting the minimum of these values, we get the minimum value of the objective function. The coordinates of the corresponding vertices will constitute an optimal solution. The details are shown in the table given below :

Corner Point	Value of the objective function $Z = 45x + 35y$	
A (28, 0)	45x28 + 35x0	= 1260
B (30,0)	45x30 + 35x0	= 1350
C (30,20)	45x30 + 35x20	= 2050
D (10,40)	45x10 + 35x40	= 1850
E (0,40)	45x0 + 35x40	= 1400
F (0,35)	45x0 + 35x35	= 1225

Thus, it is clear that when the contractor employs 35 women and no men the cost of moving cement to work-spot is minimum and the minimum cost is Rs.1225. Now let us solve a maximisation problem by graphical method.

Example2: If a young man rides his motor cycle at 25 km per hour, he has to spend Rs.2 per km on petrol; if he rides at faster speed of 40 km per hour, the cost increases to Rs.5 per km. He has Rs.100 to spend on petrol. What is the maximum distance he can travel within one hour ?

Let x = distance travelled by the young man in one day at the speed of 2 km/hour,

and y = distance travelled by the young man in one day at the speed of 40 km/hour.

Let Z = X+Y

Objective Function : $Z = x+y$ (with the objective to maximize Z)

Constraints:

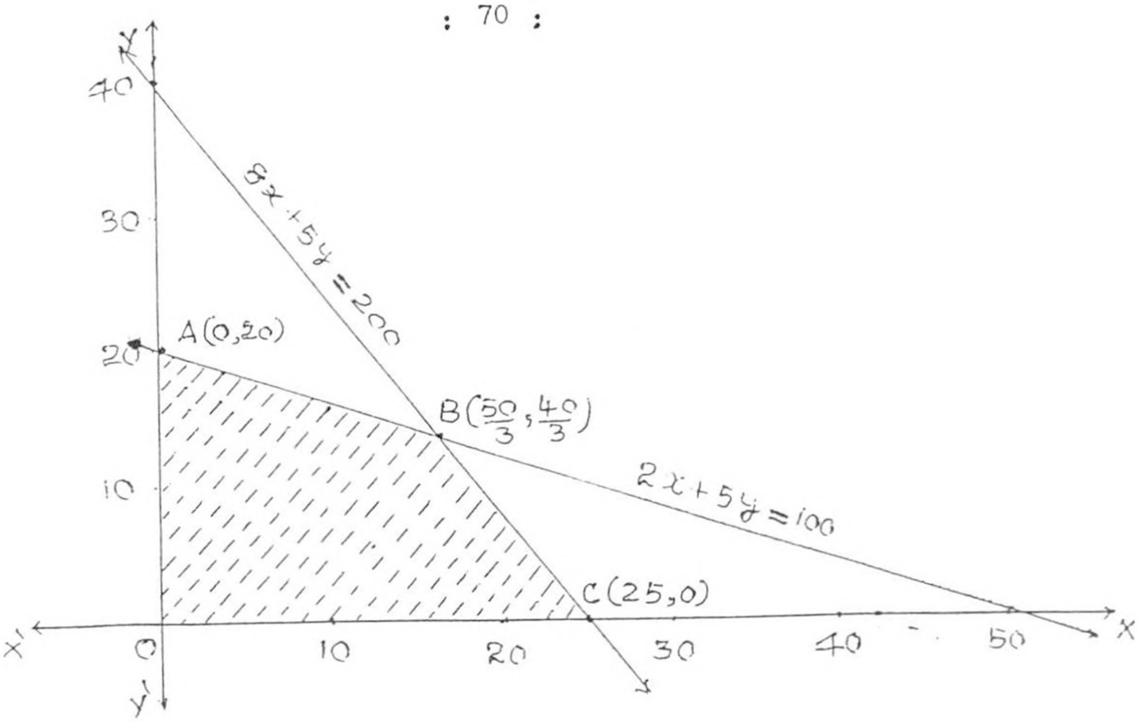
i) money spent on petrol = $2x+5y \leq 100$ (constraint due to money)

ii) total time of travel = $\frac{x}{25} + \frac{y}{40} \leq 1$ (constraint due to time)

or $8x + 5y \leq 200$

iii) non negativity conditions : $x \geq 0$, $y \geq 0$

We now draw the graph corresponding to the constraints.



The feasible region is the shaded region of the polygon OABC.

Corner point	Value of z = x+y
O (0,0)	0
A(0,20)	20
B ($\frac{50}{3}$, $\frac{40}{3}$)	30
C (25,0)	25

Therefore, 30 = Max z = the maximum distance the young man can travel in one day.

The procedures that we follow in solving a LPP (in two variables) by graphical method is summarised below :

1. Mark the feasible region. (This is the intersection of the graphs of constraints).

2. Evaluate the objective function at each of the corner points of the feasible region and pick out the point which gives the minimum (maximum) value for the objective function as the case may be.

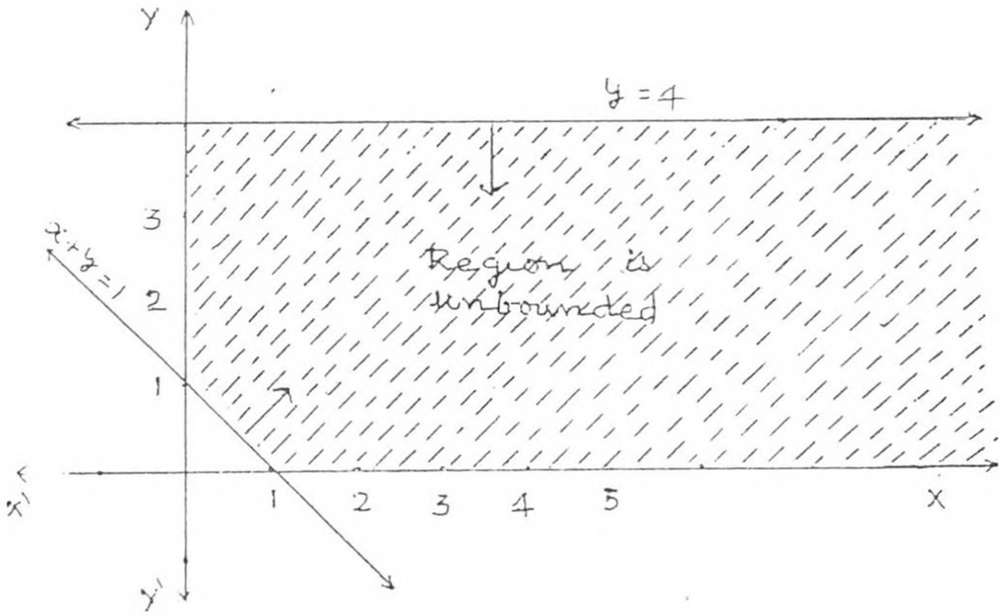
Theorem holds true if there exists an optimal solution to a LPP. There may be cases where the objective function has no finite optimal value. For example, Maximise $Z = x + 2y$

subject to $x + y \geq 1$

$x \geq 0$, $y \geq 0$

$y \leq 4$

The shaded region in the following figure is the feasible region of the problem. Note that the feasible region is not a polygonal region, but is unbounded.



In this case, moving farther away from the origin increases the value of the objective function $Z = x + 2y$ and the maximum value of Z would tend to $+\infty$ i.e., Z has no finite maximum. Whenever a LPP has no finite optimal value (maximum or minimum), we say that it has an unbounded solution. Further, there could be a linear programming problem such that it has no feasible solution.

For example,

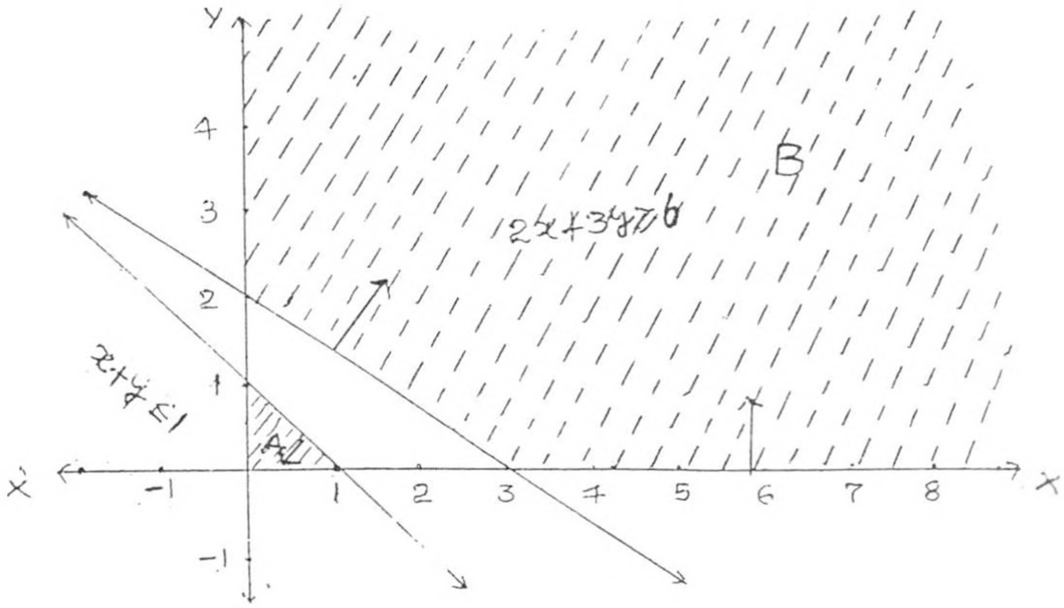
Maximise $Z = 4x + 3y$

subject to $x + y \leq 1$

$2x + 3y \geq 6$

$x \geq 0,\ y \geq 0$

The shaded regions A and B in the following figure indicate the graphs of the inequation $x+y \leq 1$ and the graph of the inequation $2x + 3y \geq 6$ respectively.



Obviously, the intersection of A and B is empty. Hence the LPP has no feasible solution.

The following LPP has or does not have a feasible solution depending upon the value of L.

Maximise $Z = x$

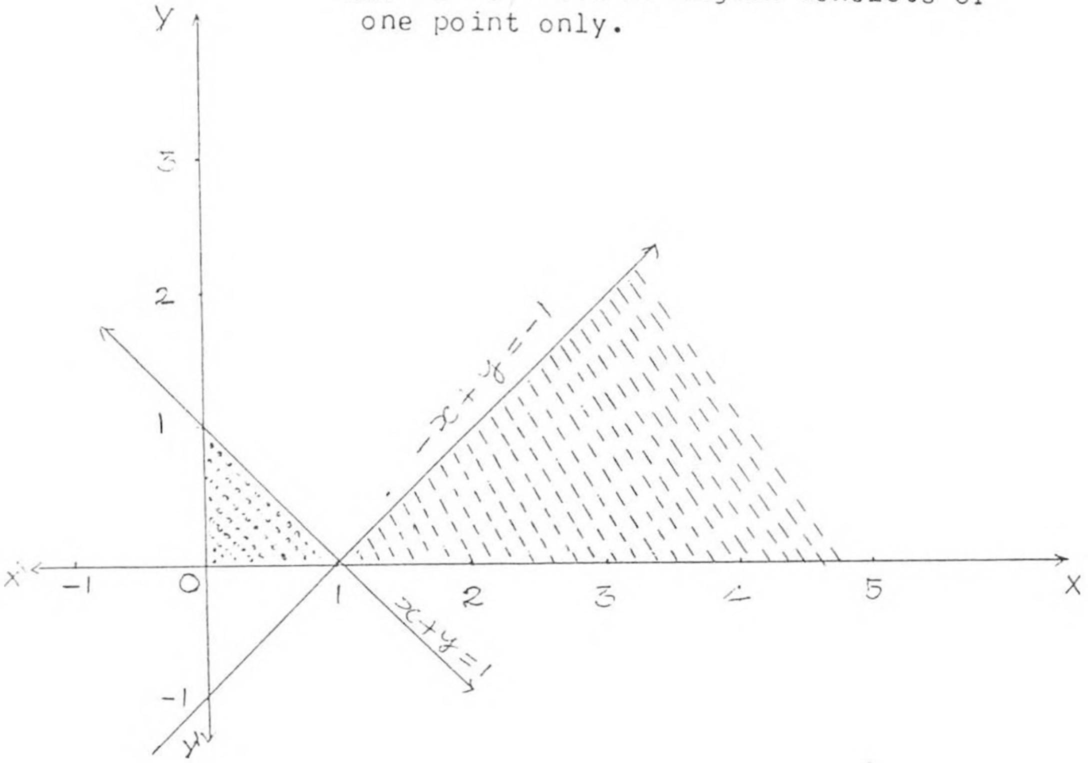
subject to $x+y \leq L$

$-x + y \leq -1$

$x \geq 0\ ,\ y \geq 0$

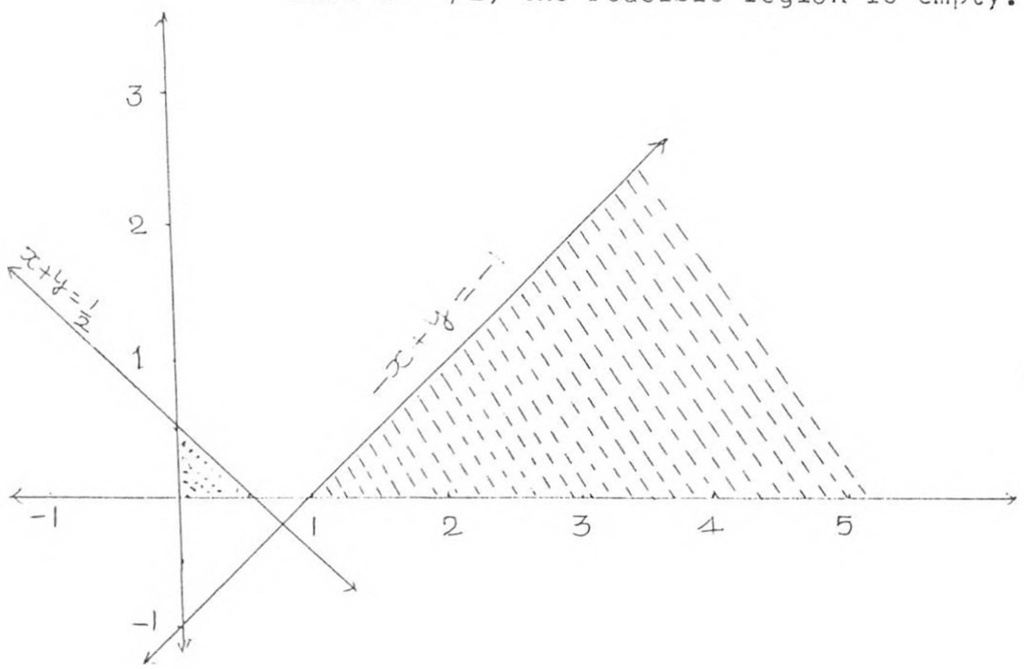
If $L = 1$, the feasible region of the problem consists of just one point (1,0) (See figure shown below).

when L=1, feasible region consists of one point only.



If L = 1/2, the feasible region is empty since there are no points satisfying the non-negativity restrictions.

when L = 1/2, the feasible region is empty.



In fact, for all values of $L < 1$ the feasible region corresponding to the given constraints is empty.

The above fact can also be verified analytically. For $L < 1$, suppose there exists a point (x_1, y_1) satisfying the constraints of the problem.

That is $x_1 + y_1 < 1$, since L is strictly less than 1

$-x_1 + y_1 \leq -1$

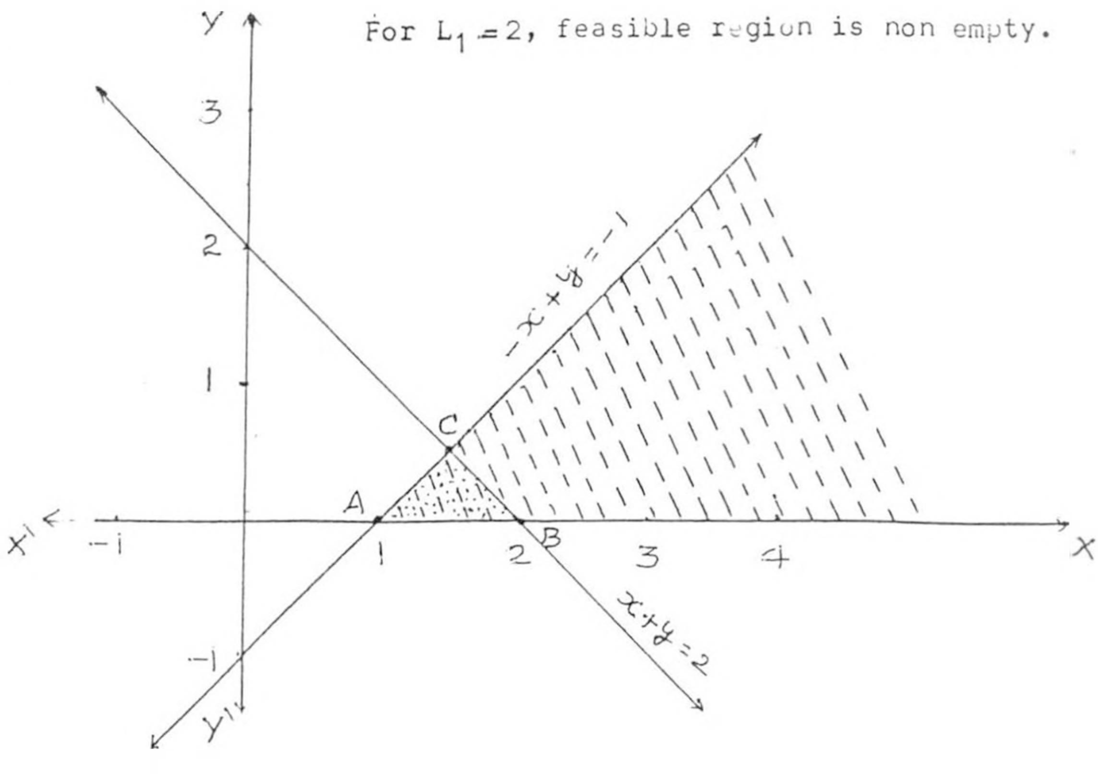
and $x_1 \geq 0$, $y_1 \geq 0$

The first two inequalities imply (by adding them), that $2y_1 < 0$.

In other words, $y_1 < 0$ which contradicts the fact that $y_1 \geq 0$.

Thus we conclude that there is no point (x,y) which satisfies the given constraints whenever $L < 1$.

If $L = 2$, the feasible region is the shaded region ABC of the figure which is non empty.



From the foregoing discussion, it is clear that the feasible region is non-empty for all values of $L \geq 1$.

If $L = 1$, it consists of just one point. If $L > 1$ it consists of infinitely many points.

We can verify this analytically also. Given constraints are

$x + y \leq L$

$-x + y \leq -1$

$x \geq 0, \ y \geq 0$

First two inequalities (by adding them) imply that

$2y \leq L-1$ or $L-1 \geq 2y$

This implies that

$L-1 \geq 0$, (since $y \geq 0$)

In other words, $L \geq 1$

If $L \geq 1$, choose non negative numbers x_1 and y_1 such that

$2x_1 = L + 1$

and $2y_1 = L - 1$

(This is possible since $L-1 \geq 0$)

These equations imply that

$2x_1 + 2y_1 = 2L$ and $-2x_1 + 2y_1 = -2$

That is, $x_1 + y_1 = L$ and $-x_1 + y_1 = -1$ obviously, such x_1 and y_1 satisfy the given constraints.

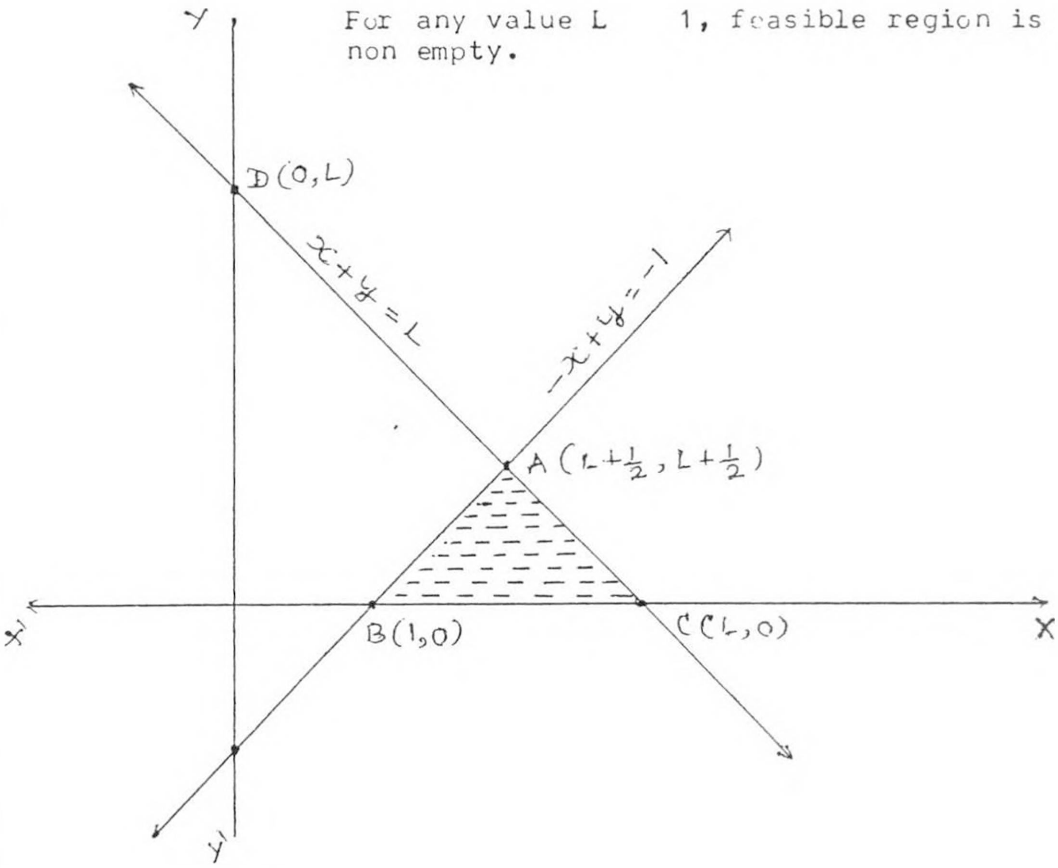
Thus we conclude that there exist numbers $x=x_1$ and $y=y_1$ satisfying the given constraints <u>if and only if</u> $L \geq 1$. That is the given L.P.P. has a feasible solution if and only if $L \geq 1$.

We now solve the given L.P.P.

If $L < 1$, then the given problem has no feasible solution. Therefore, let $L \geq 1$.

If $L = 1$, the feasible solution has just one point (1,0) and so the maximum value of Z is 1. The feasible region for any value of $L > 1$ will look like the shaded region ABC of the following figure.

For any value L   1, feasible region is non empty.



The coordinates of A are obtained by solving x+y = L and -x+y = -1.

i.e., $x = \frac{L+1}{2}$, $y = \frac{L-1}{2}$

Now,

the value of Z at A = $\frac{L+1}{2}$

The value of Z at B = 1

The value of Z at C = L

Since $L \geq 1$, $L+1 \geq 2$ and so $\frac{L+1}{2} \geq 1$

Also, since $L \geqslant 1$, $2L \geqslant L+1$ and so $L \geqslant \frac{L+1}{2}$

Thus, we have

$$1 \leq \frac{L+1}{2} \leqslant L$$

Therefore, max $\left\{ 1, \frac{L+1}{2}, L \right\} = L$

That is maximum value of Z is L, and Z attains the maximum at C.

If the problem is to minimize Z=x with the same constraints, minimum value of Z is 1 and it is attained at B.

Exercises :

1. Choose the most appropriate answer.

i) The set of feasible solutions of a linear programming problem is

a) convex b) not a convex set c) convex or concave

b) bounded and convex

ii) The minimum number of inequations needed to find a feasible region in a linear programming problem is

a) 1, b) 2, c) 3, d) 4

iii) The maximum value of the objective function of a linear programming problem always occurs

a) exactly at one vertex of the feasibility region.

b) everywhere in the feasibility region.

c) at all the vertices of the feasibility region.

d) at some vertices of the feasibility region.

iv) The feasible region of a linear programming problem intersects

a) first quadrant b) second quadrant

c) third quadrant d) fourth quadrant

v) A factory has an auto lathe which when used to produce screws of larger size produces 400 items per week and when used to produce screws of smaller size produces 300 items per week. Supply of rods used in making these screws limits the total production of both types/week to 380 items in all. The factory makes a profit of 25 paise per large screw and 10 paise per small screw. How much of each type should be produced to get a maximum profit ? (Ans. 80,300)

vi) Using graphical method maximise $Z = 3x + 4y$

subject to $4x + 2y \leq 80$

$2x + 5y \leq 180$

$x \geq 0, \quad y \geq 0,$

(Ans: $x = 2.5$; $y = 35$; maximum value = 147.5)

vii) Using graphical method

minimise $Z = 4x + 2y$

subject to $x + 2y \geq 2$

$3x + y \geq 3$

$4x + 3y \geq 6$

$x \geq 0, \quad y \geq 0$

(Ans: $x = .6$, $y = 1.2$, minimum value = 4.8)

viii) Consider the following problem :

Maximize $Z = 6x_1 - 2x_2$

subject to $x_1 - x_2 \leq 1$; $3x_1 - x_2 \leq 6$; $x_1, x_2 \geq 0$.

Show graphically that at the optimal solution the variables x_1, x_2 can be increased indefinitely, while the value of the objective function remains constant.

ix) Consider the following LPP :

Maximize $Z = 4x + 4y$

subject to $2x + 7y \leq 21$; $7x + 2y \leq 49$; $x, y \geq 0$.

Find the optimal solution (x,y) graphically. What are the ranges of variation of the coefficients of the objective function that will keep (x , y) optimal ?

x) Consider the following problem.

Maximize $Z = 3x + 2y$ subject to $2x + y \leq 2$, $3x + 4y \geq 12$, $x, y \geq 0$.

Show graphically that the problem has no feasible extreme points. What can one conclude concerning the solution of the problem ?

xi) Prove or disprove :

a) For some LPP, the set of feasible solutions is a disjoint union of convex sets.

b) The set of feasible solutions of every LPP is non empty.

c) Every L.P.P. is a mathematical model.

Applications of L.P.

LP is a powerful and widely applied technique to solve problems related to decision making. It was employed formally in three major categories - military applications, inter industry economics and zero sum two-person games. But, now the emphasis has been shifted to the industrial area. The following are a few of the applications of L.P.

1. Agricultural applications :
   Farm economics and Farm management - the first is related to the economy of a region whereas the second is related to individual farm.
2. Industrial applications :
   a) Chemical Industry - Production and Inventory control - chemical equilibrium problem.
   b) Coal industry
   c) Airline operations
   d) Communication industry - optical design and utilisation of communication network
   e) Iron and steel industry
   f) Paper industry - for optimum newsprint production
   g) Petroleum industry
   h) Rail road industry
3. Economic analysis - Capital budgeting
4. Military - Weapon Selection and Target analysis
5. Personal assignment
6. Production scheduling - inventory control and planning cost controlled production
7. Structural designs
8. Traffic analysis
9. Transportation problem and network theory
10. Travelling salesman problem
11. Logical design of electrical network
12. Efficiency in the operations of a system of Dams

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N U M E R I C A L   M E T H O D S

1. The solution of Algebraic Equations

2. Numerical Integration

by

Dr.B.S.P.RAJU

NUMERICAL METHODS :

Introduction :

Numerical Methods are the methods or procedures that explains how to find approximate solution of problems from a given numerical data. These procedures use only basic arithmetic operations like addition , subtraction, multiplication, division and exponentiation and certain other logical operations such as algebraic comparisons unlike other methods where we use complicated techniques like differentiation and integration. This is because the procedures are so designed for the execution on a modern high speed digital computer.

The numerical data used in solving the problems of every day life are usually not exact, and the numbers expressing such data are therefore not exact. Not only the data, sometimes the methods and processes by which the desired result is to be found are also approximate. The numerical data is not exact because of one or many of the following reasons.

1. The imperfections in the instrument such as faulty graduation marks, warping of a wooden yardstick due to moisture, precision of the instrument.

2. Personal errors arising due to personal bias or judgement; lack of knowledge of the use of instruments.

3. Theoretical errors arising due to the use of the instruments other than those for which the instrument is designed or calibrated.
4. Accidental errors over which the observer has no control.

So all numerical calculations are approximate and so we study what are approximate numbers.

1.2 Approximate Numbers :

Approximate numbers will arise from measurement, from estimates, from rounding exact numbers or from computations with exact numbers. The rational numbers 5/6 and $0.83\frac{1}{3}$ are both exact but the natural numbers .83 is an approximate number for $\frac{5}{6}$. The irrational number $\sqrt{10}$ is an exact number, but the rational number 3·16 is an approximation of the irrational number $\sqrt{10}$.

Since approximate number implies the existence of an exact number, we naturally wish to know how "good" or how "close". The approximation is. This we read in the next two sections.

Significant figures :

A significant figure is any one of the digits 1,2,3,...,9 and 0 is a significant figure except when it is used to fix the decimal point or to fill the places of unknown or discarded digits.

It can be split it into the following ways :

1. All non zero figures are always significant wherever used.
2. Zeros occuring between non-zero figures are always significant.
3. Terminal zeros following the decimal point are always significant.
4. In a number less than 1, zeros immediately following the decimal point are not significant.

Ex: 48.3 all are significant figures.
46.05 -all are significant figures.
.002-two is the only significant figure.
0.20-two and zero are significant figures.

Rounding off numbers :

To round off a number to n significant figures discard all digits to the right of the nth place.

If the digit in the (n+1) th place is less than 5 leave the nth digit unchanged. If the digit in (n+1) th place is greater than 5 add 1 to the nth digit. If the digit in the n+1 th place is equal to 5 leave the nth digit unaltered if it is an even number, but increase it by 1 if it is an odd number.

We follow the above rule, to reduce the errors to a minimum.

Nos.	Rounded to 4 significant figures
65.634	65.63
65.63618	65.64
65.63501	65.64
65.68501	65.68

The solution of Numerical Algebraic Equations

Introduction : Finding the roots of an algebraic equation is one of the challenging, interesting and is fascinating to many since a long time. We have a well defined formula to find the roots of any quadratic equation and beyond this we do not have definite formula that helps to solve any equation. Well; though there is a method to solve a cubic equation, it is not that simple. There are methods to solve when the equations are of particular type say reciprocal equations etc. but no general method occurs beyond the degree 3.

Here we will discuss some methods in the following sections.

Equation: If n is a positive integer, and a_0, a_1, a_2,..., a_n are constants and $a_0 \neq 0$ an expression of the form

$$a_0 x^n + a_1 x^{n-1} + \cdots + a_n$$

is called a polynomial in x of nth degree.

The equation obtained by putting the polynomial equal to zero is called an algebraic equation.

$$a_0 x^n + a_1 x^{n-1} + \cdots + a_n = 0 .$$

of nth degree.

Root : Any value of x for which the polynomial f(x) vanishes is called a root of the equation

$$f(x) = 0$$

Relation between the roots and coefficients of equation :

Let the equation be

$$a_0 x^n + a_1 x^{n-1} + \cdots + a_n = 0$$

If $\alpha_1, \alpha_2, \cdots \alpha_n$ are the roots of this equation, then we have

$$\sum_{i=1}^{n} \alpha_i = -\frac{a_1}{a_0}$$

$$\sum_{i,j=1}^{n} \alpha_i \alpha_j = \frac{a_2}{a_0}$$

$$\alpha_1 . \alpha_2 \alpha_3 . . x_n = (-1)^n \frac{a_n}{a_0}$$

Here we give some results which are useful in the coming sections :

1. f(x) defined above is a continuous function in x for all values of x.
2. Every equation f(x) = 0 of the nth degree has n and only n roots.
3. In an equation with real coefficients, imaginary roots occur in pairs.
4. In an equation with rational coefficients irrational roots occur in pairs.

Finding an approximate value of the roots (Initial approximation) :

All the methods in this chapter require one or two approximate values of a root (initial values) to begin with, which can be found in one of the following ways :

Given f(x) = 0;
We substitute at random successive values for x till for two successive values of x, f(x) changes sign.
This is based on the following theorem :

If f(x) is continuous from x = a to x = b and if f(a) and f(b) have opposite signs, then there is atleast one real root between a and b.

Let $f(x) = a_0x^n + a_1x^{n-1} + \dots + a_n = 0$

The largest root may frequently be approximated by the root of the linear equation.

$$a_0x + a_1 = 0$$

or by the root larger in absolute value of the quadrative equation

$$a_0x^2 + a_1x + a_2 = 0$$

The smallest root may similarly be approximated by the root of the equation

$a_{n-1}x + a_n = 0$

or by the smaller root in absolute value of the quadratic equation

$a_{n-2}x^2 + a_{n-1}x + a_n = 0$

BISECTION METHOD OR HALF INTERVAL METHOD

The method has been explained in the text book. But here we give an algorithm.

Step 1 : Find out two values x_1 and x_u for the roots of the given equation so that f(x) have opposite signs

i.e. $f(x_1)\ f(x_u) < 0$

Step 2 : Compute $x_r = \frac{x_1 + x_u}{2}$

If this x_r is accurate enough to meet your requirement, go to Step 4.

Step 3 : If $f(x_1)\ f(x_r) < 0$, then the root lies in

$[x_1, x_r]$ and set $x_u = x_r$ and to go to step 2.

If $f(x_1)\ f(x_r) > 0$; then the root lies in $[x_r, x_u]$

and set $x_1 = x_r$ and go to step 2.

If $f(x_1)\ f(x_r) = 0$; then the root is equal to x_r and go to step 4.

Step 4: x_r is one of the root of the equation.

Since each application of the iterative scheme reduces the length of the interval in x, by half, known to contain α. Where α is the root of the equation, this procedure is called the half interval method.

2.7.1 To find the number of iterations or operations to carry out to get the root within a prescribed tolerance :

Let $[a_0, b_0]$ be the initial interval in which the root lies, and be the prescribed tolerance and so

$$\frac{b_0 - a_0}{2^n} \leq \varepsilon$$ where n is the number of iterations

Taking logarithms on both sides

$$\log \frac{b_0 - a_0}{2^n} \leq \log \varepsilon$$

i.e., $\log (b_0 - a_0) - \log 2^n \leq \log \varepsilon$

i.e., $\log (b_0 - a_0) - \log \leq n \log n$

i.e. $n \log 2 \geq \log\left(\frac{b_0 - a_0}{\varepsilon}\right)$

$$n \geq \frac{\log\left(\frac{b_0 - a_0}{\varepsilon}\right)}{\log 2}$$

Newton - Raphson Method :

We give here a method which is different from that of text book.

Let $y = f(x)$ be the equation whose roots are to be found. Let α be one of theroot, x_0 be the approximate value of the root and h denote the correction which must be applied to give the exact value of the root, so that

$$f(x_0 + h) = f(\alpha) = 0 \qquad (1)$$

By Taylor's Theorem,

$$f(x_0 + h) = f(x_0) + h\frac{f'(x_0)}{1} + \frac{h^2 f''(x_0)}{2} + \cdots$$

By neglecting the higher order terms from h^2 and by (1)

$$0 = f(x_0) + \frac{h\,f^1(x_0)}{1}$$

$$h = \frac{-f(x_0)}{f^2(x_0)}$$

If $x_1 = x_0 + h$, then $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

Similarly, $x_2 = x_1 + h = x_1 - \frac{f(x_1)}{f'(x_1)}$

$$\vdots \qquad \vdots$$

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

To find the square root of a number by Newton Raphson Method :

Let K be the number whose square root is to be found.

i.e. Let $\sqrt{k} = x$

Squaring on both sides

$$K = x^2$$

i.e. $x^2 - k = 0$

If we solve the equation we will get the square root of the given number k.

Let $f(x) = x^2 - k$

$f^1(x) = 2x$

By applying the Newton Raphson method

$$x_{n+1} = x_n - \frac{x^2_n - k}{2x_n}$$

$$= \frac{x_n^2 + K}{2x_n}$$

$$= \frac{1}{2}\left(x_n + \frac{k}{x_n}\right) \text{ for } n = 0,1,2,\ldots.$$

To find the reciprocals of numbers without division by Newton Raphson Method.

For a given number $k > 0$, we want to find the value of $1/k$ (we are not considering $K = 0$, because in that case $\frac{1}{k} = \frac{1}{0}$ is ∞ - undefined and $k < 0$; i.e. $k =$ -ve i.e. $\frac{1}{-k} = -\frac{1}{k}$ so we can add - sign to $+\frac{1}{k}$ and hence it is unnecessary).

Let $x = \frac{1}{k}$

i.e. $\frac{1}{x} = \frac{1}{1/k}$

(we are not considering $x - \frac{1}{k} = 0$ in which case

$$f(x) = x - \frac{1}{k}$$

and $f^1(x)$ becomes 1

$$x_{n+1} = x_n - \frac{(x_n - 1/k)}{1} \quad \text{i.e.,} \quad x_{n+1} = \frac{1}{k})$$

i.e. $\frac{1}{x} - k = 0$

Let $f(x) = \frac{1}{x} - k$

$$f^1(x) = -\frac{1}{x^2}$$

Substituting $f(x)$ and $f^1(x)$ in Newton Raphson's formula

$$x_{n+1} = X_n - \frac{f(x_n)}{f^1(x_n)}$$

$$x_{n+1} = x_n - \frac{\frac{1}{x_n} - k}{-\frac{1}{x_n^2}}$$

$$= x_n + \left(\frac{1}{x_n} - k\right) \times x_n^2$$

$$= x_n + x_n - K x_n^2$$

$$= 2x_n - k x_n^2 \qquad = x_n (2 - kx_n) \text{ for } n = 0,1,2,\ldots$$

By Newton Raphson method, we get the root in a few iterations that the methods we discussed earlier. But it is not always guaranteed that as the iterations increases it gives convergent root. It depends on the initial value chosen. So here we give the bounds between which the initial value should lie so that it gives a convergent root in the case of finding the reciprocal of a number by Newton Raphson method.

We have $x_{n+1} = x_n\,(2-kx_n)$

Let $F(x) = x\,(2-kx)$

$$= 2x-kx^2 \text{ so } F^1(x) = 2-2kx$$

$$-1 < F^1(x) < 1$$

i.e. $-1 < 2\,(1-kx) < 1$

i.e. $-1 < 2\,(1-kx)$	$2\,(1-kx) < 1$
$-1/2 < 1-kx$	$1-kx < 1/2$
$-3/2 < -kx$	$-kx < 1/2$
$3/2 > kx$	$kx > 1/2$
$kx < 3/2$	$x > 1/2\,K$
$x < 3/2K$	

i.e. $\dfrac{1}{2k} < x < \dfrac{3}{2k}$

k is a positive integer

$$0 < x < \frac{3}{2k} < \frac{2}{k}$$

i.e. $0 < x < 2\,k^{-1}$

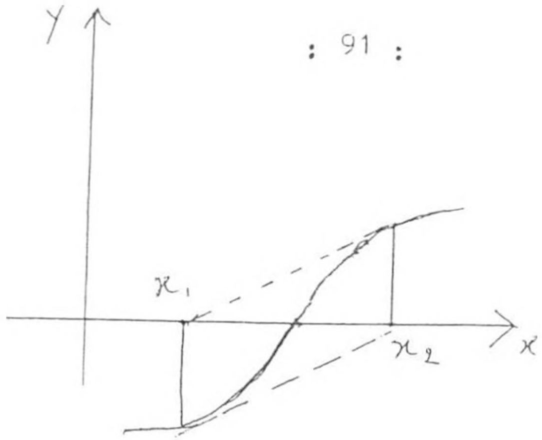
In order to get a convergent root it is better if we choose the initial value of x between 0 and 2/k.

As an example for the notes given above

If $f(x) = \begin{cases} \sqrt{x-r} & \text{for } x \geq r \\ -\sqrt{r-x} & x < r \end{cases}$

and $x_1 = r-h$; then $x_2 = r+h$

and successive approximations will move back and forth between the two values.



Remark: The method will fail at any time when $f^1(x_n) = 0$ for some n. Then in that case, choose a new starting value.

Assignment and Self Test

1. Find a root of the equation given below by Bisection method, false position method and Newton Raphson method.

1. $x = \frac{1}{(x+1)^2}$
2. $x = (5-x)^{1/3}$
3. $x^3-x-4 = 0$
4. $x^4-x - 10 = 0$
5. $x^3-100 = 0$
6. $-0.874x^2 + 1.75x + 2.627 = 0$
7. $t^3-4x^2 - 6t + 4 = 0$
8. $x^3-2x-5 = 0$
9. $x^3 - 5x + 3 = 0$

Find the square root of the following by Newton-Raphson method to four significant figures.

1. 3 2. 5 3. 7 4. 11

Find the reciprocals of the following numbers by Newton Raphson method without actual division.

1. 3 2. 6 3. e

NUMERICAL INTEGRATION

Introduction:

Given a polynomial we can be able to write the values of y for a given value of x. But in scientific applications generally we get the tabulated data. From the tabulated data we fit a polynomial using finite differences. Such polynomials are called inter-polating polynomials. There are several inter-polation formulae like Newton's gregory forward and backward formulae, Bessel's Sterling's formula etc. which are very much useful in numerical differentiation and integration. Here in the following section we derive Newton-Cotes Integration formula, so that we can deduce the Simpson's 1/3 rule, Simpson's 3/8th rule, trapezoidal rule, Weddle's rule etc.

Newton-Cotes Integration Formula :

Let $x_0, x_1, x_2, \ldots, x_n$ are the n+1 evenly spaced base points from which an interpolating polynomial of degree n has been obtained with the help of the functional values at the base points. (Here we assume the polynomial. For an understanding of inter-polating polynomials, readers can refer the book Numerical Analysis by SCARBOROUGH). Let a be the lower limits of integration coincides with the base point x_0. Let b, the upper limit of integration, be arbitrary for the moment.

Then,

$$\int_a^b f(x)\, dx = \int_{x_0}^{b} P_n(x)\, dx$$

$$\alpha = \frac{x - x_0}{h} \qquad \therefore \text{ Lower limit } = \frac{x_0 - x_0}{h} = 0$$

$$x = x_0 + h \qquad \therefore \text{ Upper limit } = \frac{b - x_0}{h}$$

$$dx = h.d\alpha \qquad \text{Let this be equal to } \bar{\alpha}$$

$$= h \int_0^{\bar{\alpha}} P_n(x_0 + h)\, d\alpha$$

$$\approx h \int_0^{\bar{\alpha}_0} \left[f(x_0) + \alpha \Delta f(x_0) + \frac{\alpha(\alpha-1)}{\lfloor 2} \Delta^2 f(x_0)) + \cdots \right] d\alpha$$

$$\approx h \left[\alpha f(x_0) + \frac{\alpha^2}{2} \Delta f(x_0) + \left(\frac{\alpha^3}{6} - \frac{\alpha^2}{4}\right) \Delta^2 f(x_0) + \right.$$
$$\left. + \left(\frac{\alpha^4}{24} - \frac{\alpha^3}{6} + \frac{\alpha^2}{4}\right) \Delta^3 f(x_0) + \cdots \right]_0^{\bar{\alpha}}$$

All the terms vanish at the lower limit and so

$$\int_0^b f(x) dx \simeq h \left[\bar{\alpha} f(x_0) + \frac{\bar{\alpha}^2}{2} \Delta f(x_0) + \left(\frac{\bar{\alpha}^3}{6} - \frac{\bar{\alpha}^2}{4}\right) \Delta^2 f(x_0) \right.$$
$$\left. + \left(\frac{\bar{\alpha}^4}{24} - \frac{\bar{\alpha}^3}{6} + \frac{\bar{\alpha}^2}{4}\right) \Delta^3 f(x_0) + \cdots \right] \rightarrow (I)$$

If the upper limit b is chosen to coincide with one of the base points so that $b = x_m$ (say) then

$$\bar{\alpha} = \frac{x_m - x_o}{h} = \frac{x_o + mh - x_o}{h}$$

$$= \frac{mh}{h} = m$$

i.e. assumes the integral value m.

By giving various values to $\bar{\alpha}$ we get various formulas i.e.
by putting $\bar{\alpha} = 1$, we get trapezoidal rule
by putting $\bar{\alpha} = 2$, we get Simpson's 1/3rd rule
by putting $\bar{\alpha} = 3$, we get Simpson's 3/8th rule
by putting $\bar{\alpha} = 6$, we get Weddle's rule and so on.

Trapezoidal Rule :

Put $\bar{\alpha} = m = I$, in(I), $a = x_o$

$$\int_{x_0}^{x_0+h} f(x)\, dx = h \left[f(x_o) + \frac{1}{2} \Delta f(x_o) \right]$$

$$= h \left[f(x_o) + \left[\frac{f(x_o+h) - f(x_o)}{2} \right] \right]$$

$$= \frac{h}{2} \left[2 + (x_o) + f(x_o+h) - f(x_o) \right]$$

$$= \frac{h}{2} \left[f(x_o) + f(x_o+h) \right]$$

$$= \frac{h}{2}\left[f(x_0) + f(x_1)\right] \qquad \text{where } x_1 = x_0 + h$$

$$\int_{x_0+h}^{x_0+2h} f(x)\,dx = \frac{h}{2}\left[f(x_1) + f(x_2)\right]$$

$$\vdots$$

$$\int_{x_0+(n-1)h}^{x_0+nh} f(x)\,dx = \frac{h}{2}\left[f(x_{n-1}) + f(x_n)\right]$$

Adding all these integrals we get

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n)\right]$$

$$= \frac{h}{2}\left[f(x_0) + 2\left\{f(x_1) + f(x_2) + \ldots + f(x_{n-1})\right\} + f(x_n)\right]$$

This is called the trapezoidal rule.

Note: If we increase the number of intervals then the trapezoidal rule gives a better value.

SIMPSON'S 1/3rd RULE :

Let $\alpha = m = z$ in (I) (derived earlier, then $b = x_2$; $a = x_0$)

$$\int_{x_0}^{x_2} f(x)\,dx = h\left[2f(x_0) + \frac{4}{2}\Delta f(x_0) + \left(\frac{2^3}{6} - \frac{2^2}{4}\right)\Delta^2 f(x_0)\right]$$

$$= h\left[2f(x_0) + 2\,\Delta f(x_0) + \frac{1}{2}\Delta^2 f(x_0)\right]$$

$$= h\left[2f(x_0) + 2\left\{f(x_1)-f(x_0)\right\} + 1/3\left\{f(x_2)-2f(x_1)+f(x_0)\right\}\right]$$

$$= h\left[2f(x_0) + 2f(x_1)-2f(x_0) + 1/3\, f(x_2) - 2/3\, f(x_1) + 1/3 f(x_0)\right]$$

$$= \frac{h}{3}\left[6f(x_1) + f(x_2) - 2f(x_1) + f(x_0)\right]$$

$$= \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]$$

III $$\int_{x_2}^{x_4} f(x)\,dx = \int_{x_0+2h}^{x_0+4h} f(x)\,dx$$

$$= \frac{h}{3}\left[f(x_2) + 4f(x_3) + f(x_4)\right]$$

$$\int_{x_0+(n-2)h}^{x_0+nh} f(x)\,dx = \frac{h}{3}\left[f(x_{n-2}) + 4(x_{n-1}) + f(x_n)\right]$$

Adding all the above we get

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{3}\Big[f(x_o) + 4\left\{f(x_1)+f(x_3) + \ldots + f(x_{n-1})\right\} + 2\left\{f(x_2) + f(x_4) + \ldots + f(x_{n-2})\right\} + f(x_n)\Big]$$

This formula is known as Simpson's 1/3rd rule.

Note: The interval of integration should contain an even number of steps of width h.

By means of an example we explain how to choose the interval h so that the value obtained by Trapezoidal rule is correct to any number of given places.

Ex: Predict how tightly values of f(x) must be packed (what interval h) for the trapezoidal rule itself to achieve a correct result to four places,

for $\int_1^2 \frac{dx}{x}$

Ans: Given $f(x) = \frac{1}{x}$

$$f^1(x) = -\frac{1}{x^2}$$

$$f^{11}(x) = \frac{2}{x^3}$$

$$\text{Max. } f^{11}(x) = 2 \quad \text{for } x = (1,2)$$

Truncation error $-\frac{(b-a)}{12}\, h^2 \quad \underset{\text{Max}}{f^{11}(x)}$ (Reader can get it from any standard book on Numerical Analysis)

$$= -\frac{(2-1)}{12}\, h^2 \qquad (2)$$

$$= -\frac{h^2}{6} < .00005 \quad \text{(given in problem)}$$

i.e. $h^2 < .0003$

i.e. $h < \frac{\sqrt{3}}{100}$

Evaluate the following integrals by using trapezoidal rule and Simpson's 1/3rd rule.

1. $\int_{1}^{3} (x^3-2x^2+x-5)\,dx$

2. $\int_{1}^{1.3} \sqrt{x}\,dx$

3. $\int_{1}^{2} \frac{dx}{x}$

4. $\int^{1.6} \frac{x}{\sinh x}\,dx$

5. $\int_{-3}^{3} x^4\,dx$

6. $\int_{0}^{1} \frac{dx}{1+x}$

7. Predict how many values of $f(x)$, or how small an interval h will be needed for Trapezoidal rule to produce $\log_e 2$ connect to four places.

A L G O R I T H M S & F L O W C H A R T S

by
Dr.D.BASAVAYYA

ALGORITHMS AND FLOWCHARTS

Computers cannot think for themselves. They are automatically controlled and do the work of many human beings incredibly at high speeds. The really important thinking is done by the human who feed them with information and the desired instructions. A set of instructions given to a computer to solve a problem is known as a computer program. Before writing such computer programs, it is useful and necessary to list out the steps involved in solving the problem sequentially. A step-by-step list of instructions for solving a particular problem is known as the algorithm of that problem.

The concept of algorithm is basic to any computational scheme, numerical or non-numerical. The word 'algorithm' originated from the word 'algorism' - means the art of computing Arabic numerals. Some scholars relate the origin with the name of a famous Arabic mathematician, Abu Jafar Muhammed ibu Musa al-Khwarizmi (825 A.D.) who first suggested the method of adding decimal numbers by taking one digit from each of the operands and a previous carry digit.

Algorithm is a logical process of analysing a mathematical problem and data step by step so as to make it vulnerable for easy computing or for conversion of data into information. In our day-to-day work, the brain automatically performs the algorithm by experience but not in a systematic and error free way. Algorithm represented in many forms such as

i) step-by-step method
ii) flowcharting and
iii) by programming language.

Step by Step Method :

Example 1 : Suppose we want to find the largest of any three given numbers.

Assume that A, B and C are the given numbers. First compare A and B and identify the larger. Then compare this bigger number with C and identify the bigger among these two as the largest of all the three given numbers.

This method of solving this problem may be expressed as the following series of instruction.

Step 1 : Note the three given values as A, B, C.

Step 2 : Compare A and B.

Step 3 : If A is greater than B, then do the step 4 otherwise do the step 5.

Step 4 : Compare A and C and write the greater number and stop.

Step 5 : Compare B and C and write the greater number and stop.

Example 2 : Suppose you want to solve the following pair of equations for x and y given the values a,b,c,d,e,f.

$$ax + by = c \quad \ldots\ldots\ldots \quad (1)$$

$$dx + ey = f \quad \ldots\ldots\ldots \quad (2)$$

Assume that a,b,c,d,e,f are non zero. The procedure to solve these equations is as follows : From equation (1) we have

$$x = (c-by)/a \quad \ldots \quad (3)$$

Substituting x in (2) we get

$$\frac{d\ (c - by)}{a} + ey = f$$

or $(e - db/a)\ y = f - \frac{dc}{a}$

Thus using this value of y in (3) we get the value of x. This method of solving the equations may be expressed as follows :

Step 1 : Note the values of a,b,c,d,e,f.

Step 2 : Calculate the value of $(e - \frac{db}{a})$ and call it as s.

Step 3 : Calculate the value of $(f - \frac{dc}{a})$ and call it as t.

Step 4 : If s = 0 then write 'No solution' and stop.

Step 5 : Set y = t/s.

Step 6 : Set $x = \frac{(c - by)}{a}$

Step 7 : Write the values of x and y as answers.

Step 8 : Stop.

The above set of 8 instructions is known as an algorithm of that problem.

A good algorithm should possess the following characteristics.

Algorithm 1. should be simple (simplicity).

2. should be clear without ambiguity (definiteness).

3. should lead to unique solution of the problem (uniqueness).

4. should involve a finite number of steps to arrive a solution (finiteness).

5. should have capability to handle some unexpected situations which may arise during the solution problem like division by zero (completeness).

6. should be effective in saving time (effectiveness) - using that particular algorithm, processing should be quicker.

The description of an algorithm (Step-by-step method) requires a suitable language. Although we can describe an algorithm in a combination of natural language (say, English) and mathematical notations, there are some drawbacks. They are

1. The algorithm is not usually concise.
2. A natural language is ambiguous.
3. The manner of expression does not reveal the basic structure of the algorithm.
4. Certain operations cannot be expressed by existing mathematical notations.
   Ex : Replace B by A.

Because of these drawbacks, sometimes, we have to prefer other methods. One of the most convenient languages which is effective for communication and description of an algorithm is the language of flow-charting.

Flowcharts :

Flowcharting is a technique for representing a succession of events in symbolic form. Flowchart is the diagrammatic representation of a sequence of events, usually drawn with conventional symbols (geometrical figures) representing different types of events and their interconnection. Flowchart is also known as flowdiagram.

In flowchart the essential steps of the algorithm are pictured by boxes of various geometrical shapes and the flow of data between steps is indicated by arrows, or flowlines. The flowchart is usually drawn so that the flow direction is downward or from left to right. The symbols themselves are of standardized shapes that indicate the type of action taking place at that step of the algorithm. In fact, each symbol is labelled by its algorithm step, written within the symbol. Flowcharts may be divided into systems flowcharts and program flowcharts. A chart that depicts the flow of data in the over -all data processing system, or phase of the system, is called a system flow-chart or process chart. A chart that depicts the operations and logical decisions in a problem solving is called a program flowchart or logical flowchart. In our discussion we are concerned with program flowchart. The preparation of either type of chart is called flowcharting. The different symbols used in any flowchart (standard) are as follows :

Terminal Symbol : The oval symbol is used to indicate the beginning or end of an algorithm by (Start) or (Stop) respectively. Clearly, a flowchart can contain only one start symbol; however it can contain more than one stop symbol, since the algorithm may contain alternative branches. Sometimes, we omit the start/stop symbols if it is clear where the chart begins/ends.

Input/Output Symbol : The parallelogram symbol is used to indicate an input or an output operation. Specifically, we write

Read A,B,C,D

to indicate that data are to be inputted into the memory locations A, B, C and D, in that order. Similarly,

Write X,Y,Z

indicates that the data in the memory locations X,Y,Z are to be outputted. We can also output messages by including the message in quotation. e.g.

Write 'No solution'

indicates that the message 'No solution' is to be outputted.

Process Symbol : The rectangle symbol is used to indicate a processing operation. This can be an assignment statement, defined below or it can be macroinstruction, whose programming language translation would otherwise require an entire list of computer statements; e.g.

Alphabetize Names

Find mean and standard deviation of marks

are macroinstructions.

The assignment statement has the form

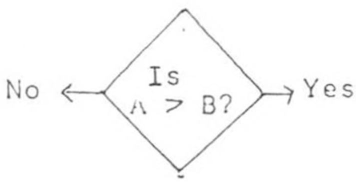
Variate = arithmetic expression

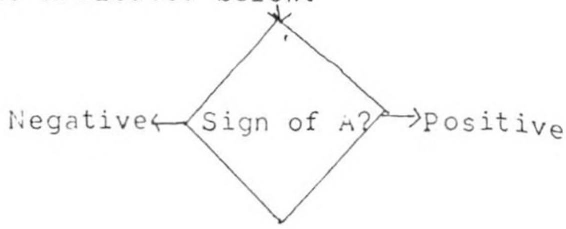
Another acceptable form for an assignment statement is

Variable ← arithmetic expression

in which the backward arrow ← is used to instead of the equals sign.

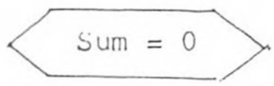
Decision Symbol : The diamond-shaped (Rhombus) symbol is used to indicate a decision. This represents a point at which different process paths may be selected. The question is placed inside the symbol, and each alternative answer to the question is used to label the exist arrow which leads to the appropriate next step of the chart. We note that the decision symbol is the only symbol that may have more than one exist. Some of the ways are as indicated below.





Connector Symbol : A flow chart that is long and complex may require more than one sheet of paper, which means that certain flow lines cannot be drawn or the flow chart may contain crossing flow lines that could cause confusion. The connector symbol, a small circle ◯ , is used to remedy such situations. Specifically, one assumes that a flow line exists between any part of identically labelled connector symbols such that the flow is out of the flow chart at one of the connectors (exist connector) and into the flow chart at the other connector (entry connector).

Preparation Symbol : This symbol indicates the preparation for some procedure by initializing certain variables, e.g.



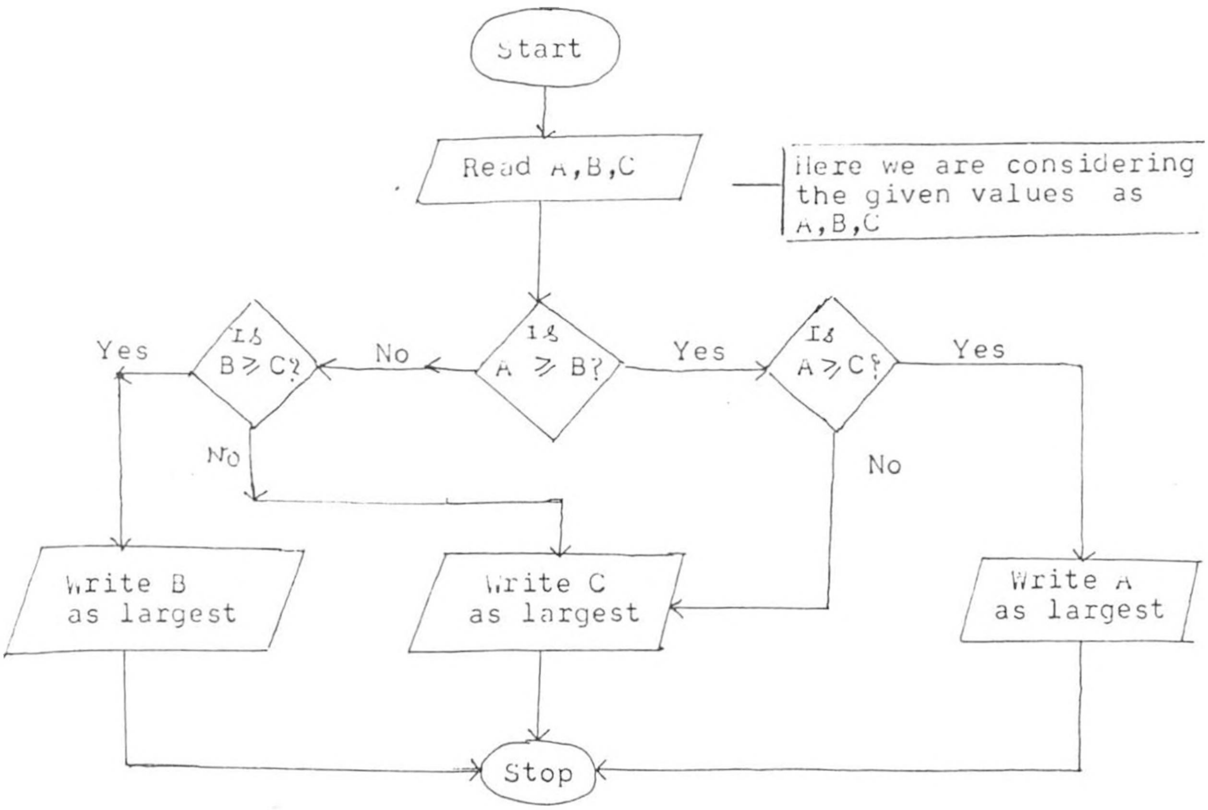
Many programmers indicate a preparation by means of the process symbol only.

Comment Symbol : The following symbol used to indicate comments on the contents of a symbol.

Consider the following examples to explain the use of different symbols in writing a flowchart.

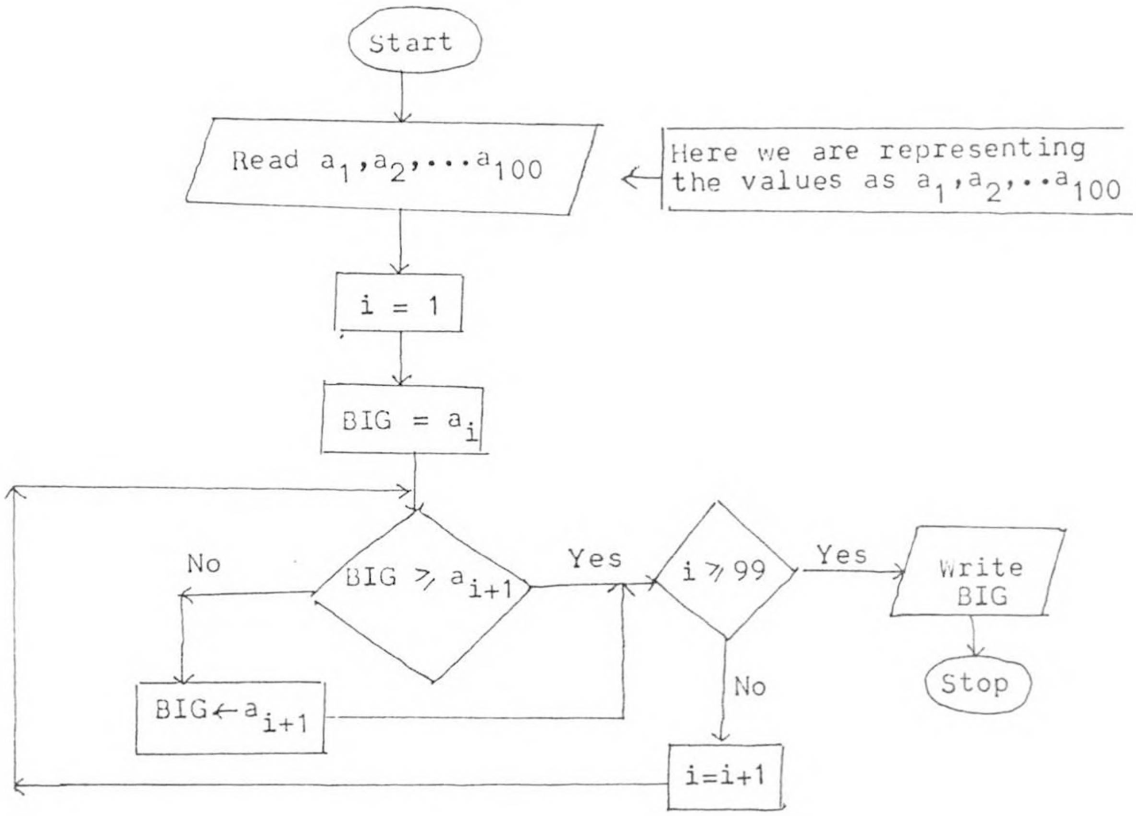
Example 1 : Pick up the largest of any three given numbers.

The flowchart for this problem is as follows :



Example 2 : Pick up the largest of any given hundred numbers.

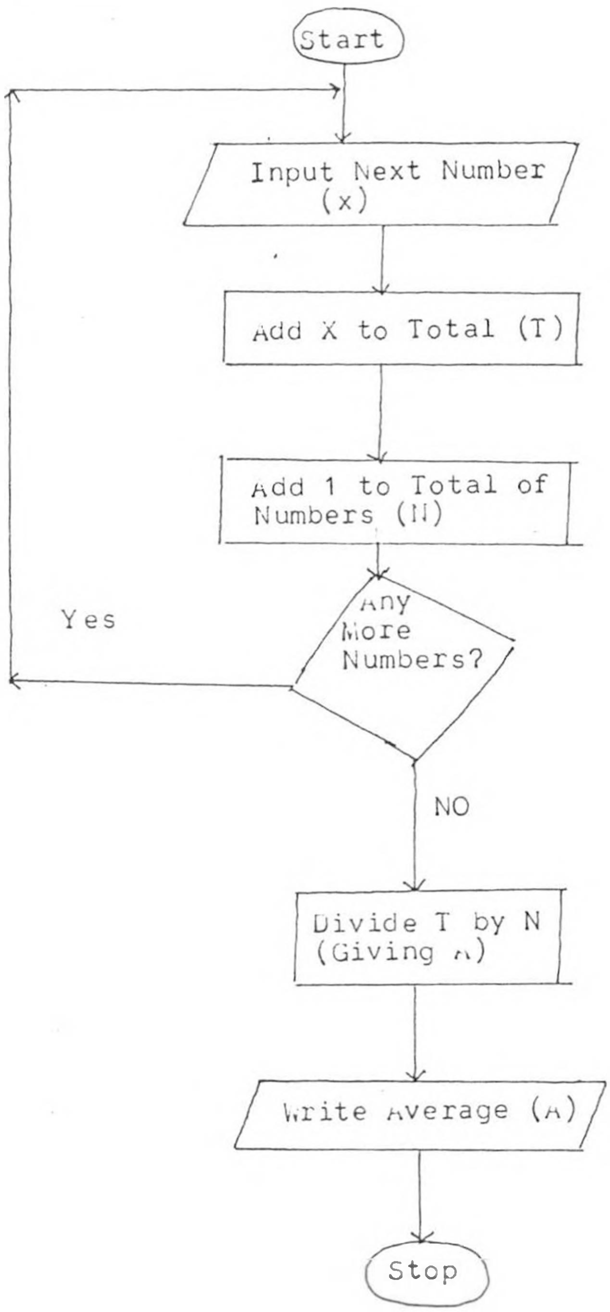
The required flowchart is



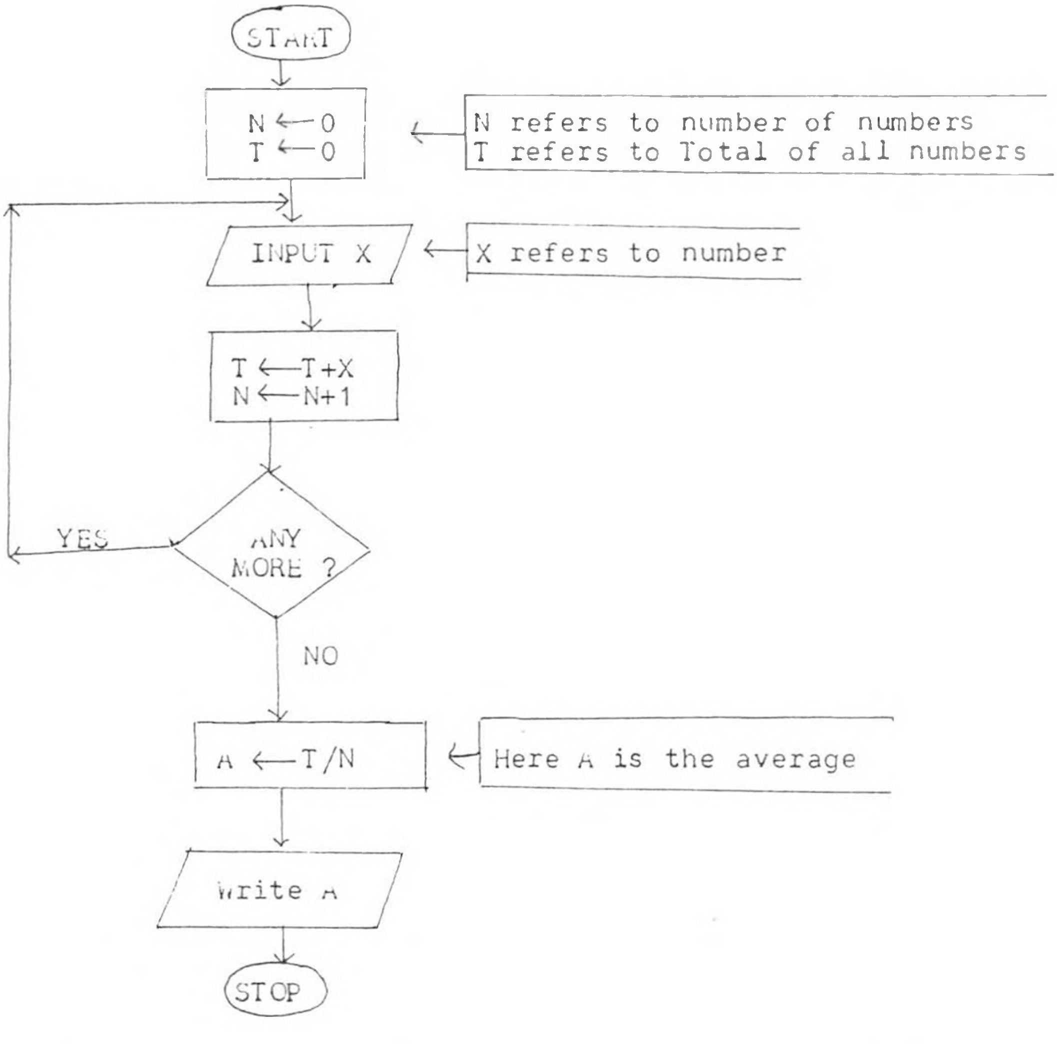
Example 3 : Draw a flowchart to find the average of any number of numbers.

Here, we write the flowchart in two ways : i) by using English words and ii) by defining variables.

First type :


Start
Input Next Number (x)
Add X to Total (T)
Add 1 to Total of Numbers (N)
Any More Numbers?
Yes
NO
Divide T by N (Giving A)
Write Average (A)
Stop

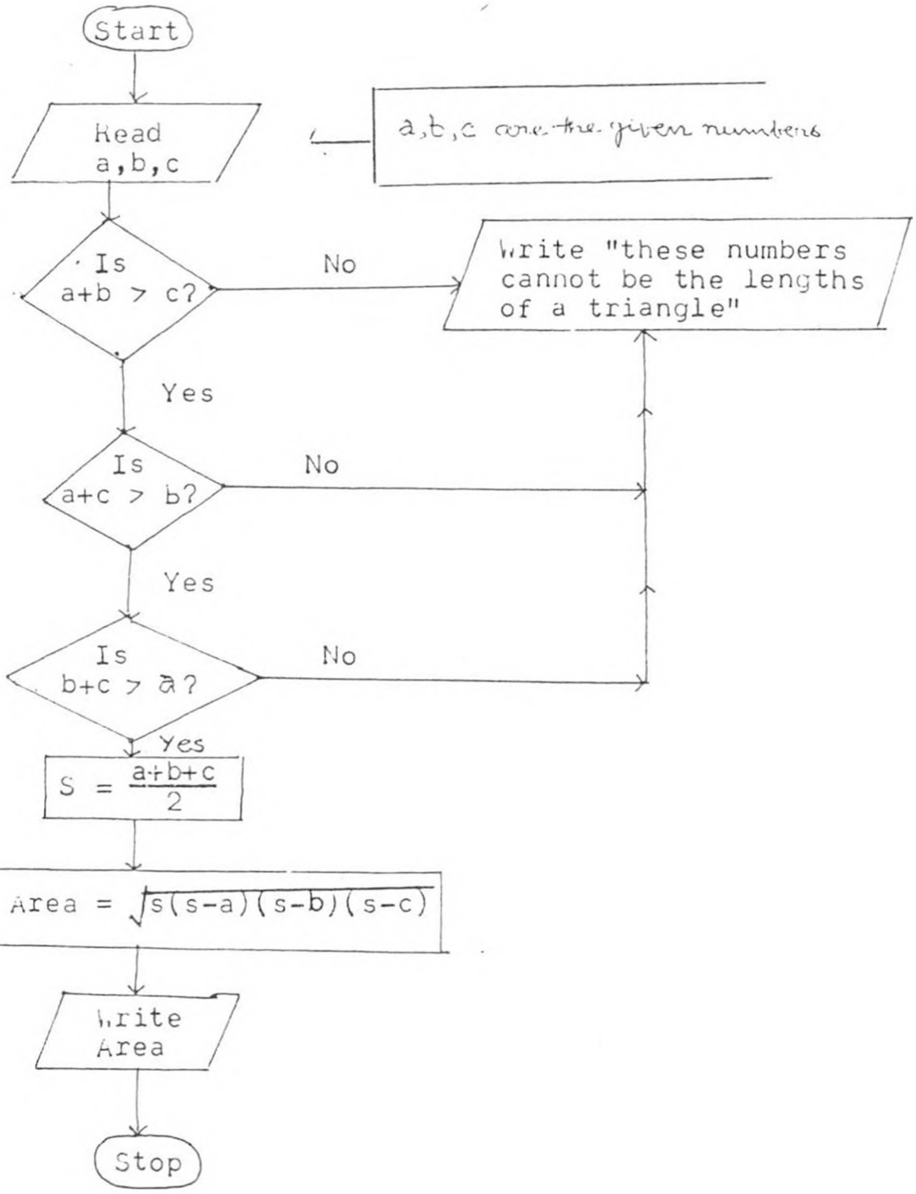
Second Type :



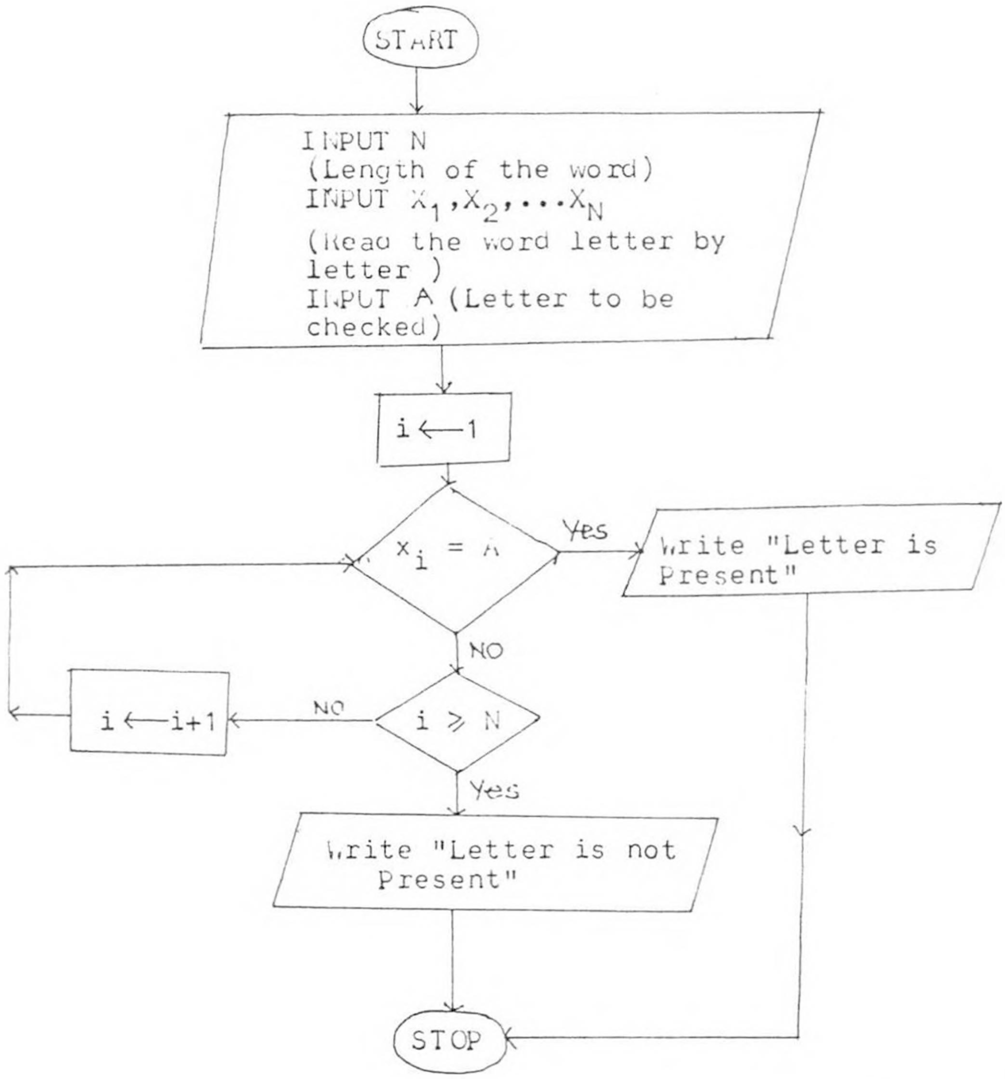
Example 4 : Prepare a flowchart to check the three given numbers a,b,c can be the lengths of the three sides of a triangle and if so to find the area of that triangle using the formula

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)} \quad \text{where } s = \frac{a+b+c}{2}$$

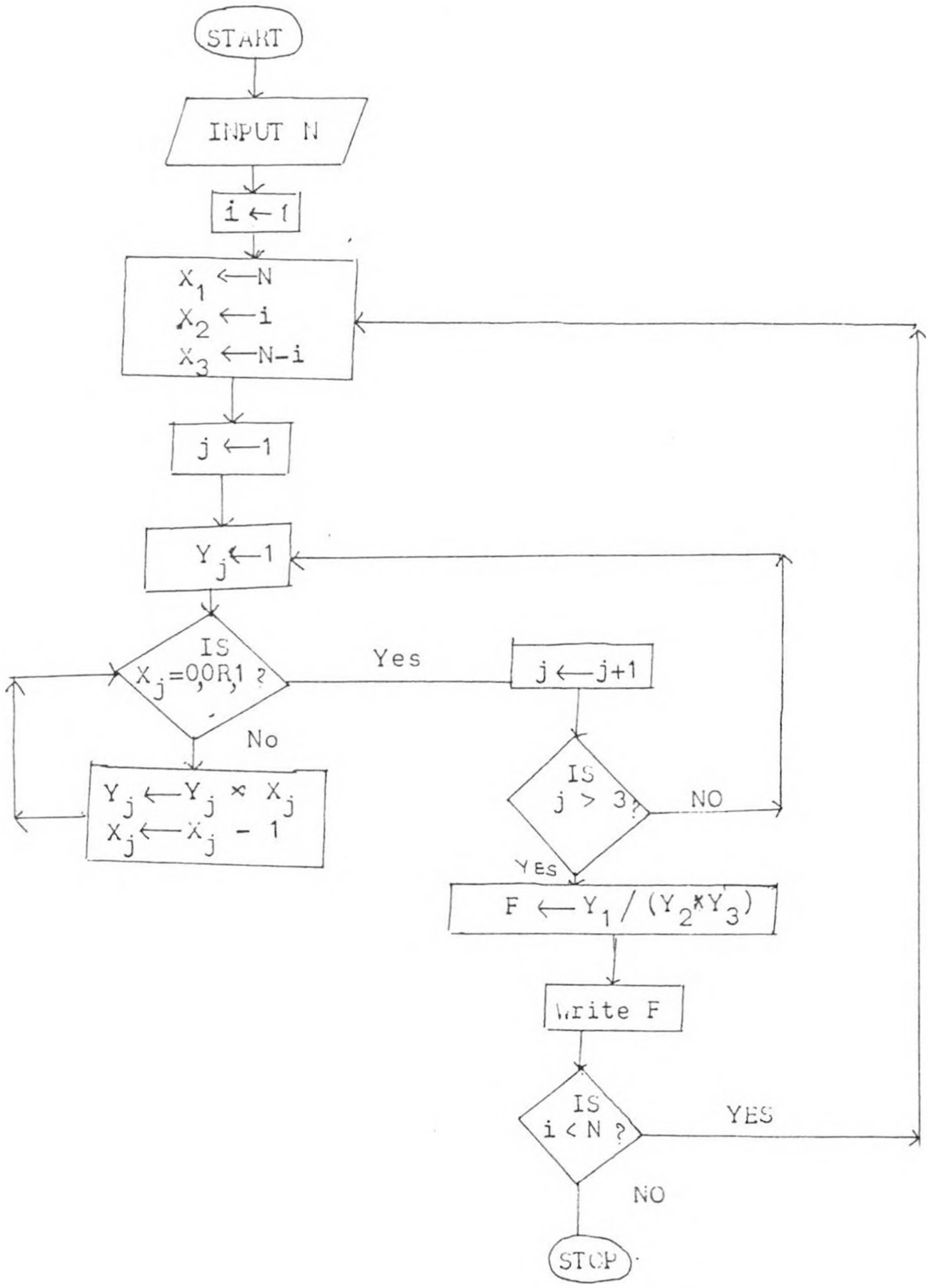
Solution :


Start
Read
a,b,c
a,b,c are the given numbers
Is
a+b > c?
No
Write "these numbers
cannot be the lengths
of a triangle"
Yes
Is
a+c > b?
No
Yes
Is
b+c > a?
No
Yes
S = $\frac{a+b+c}{2}$
Area = $\sqrt{s(s-a)(s-b)(s-c)}$
Write
Area
Stop

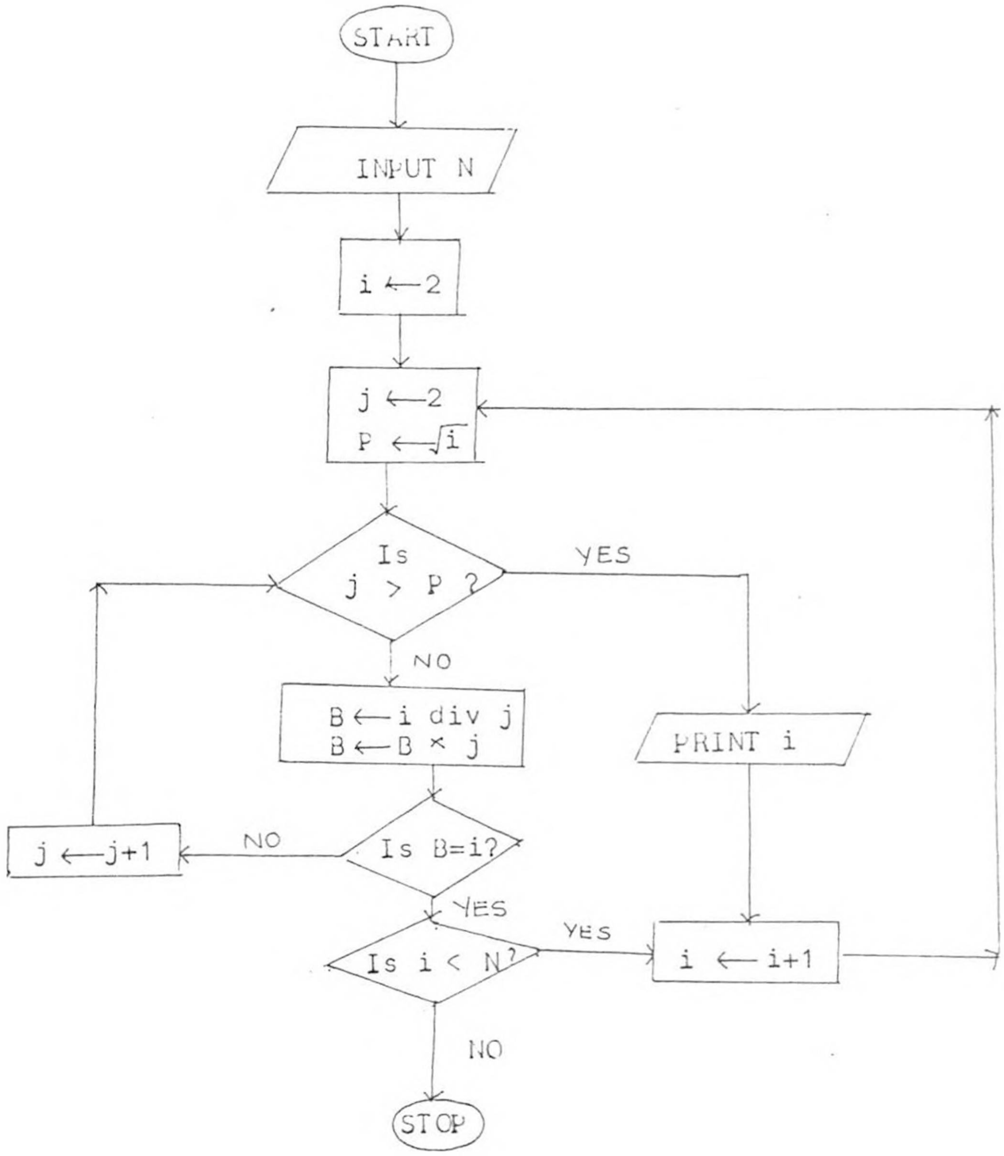
Flowchart to verify whether a given letter is existing in the given word or not.



Flowchart for the list of Binomial Coefficients.

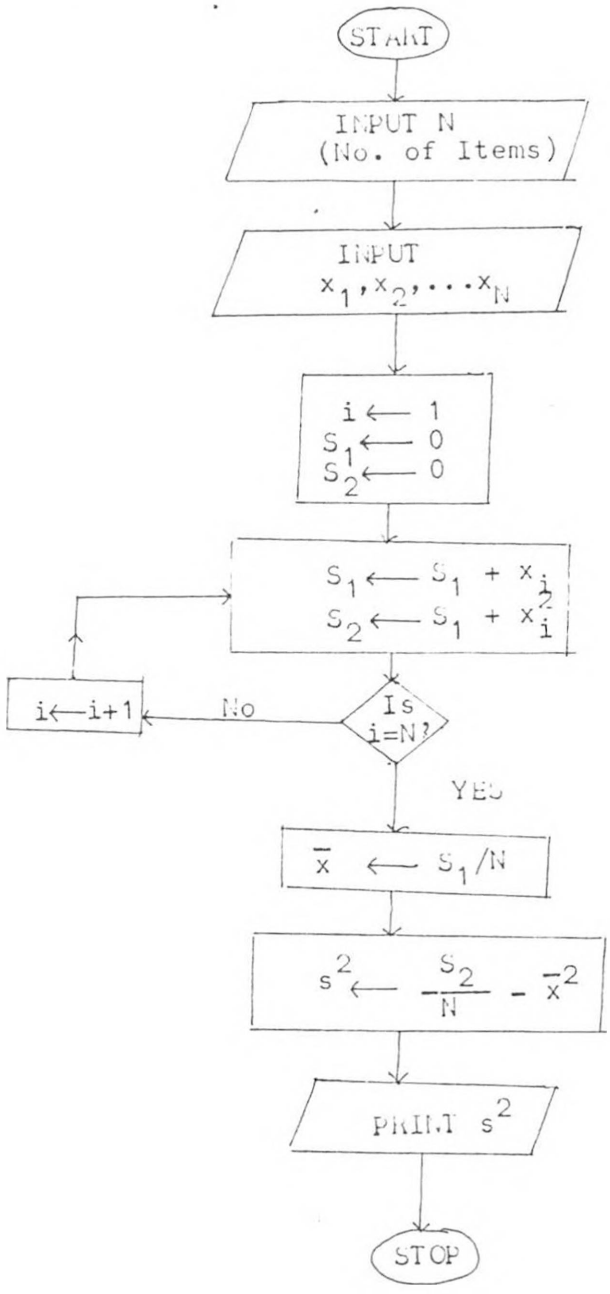


Flowchart for listing of Prime number below any given number N.

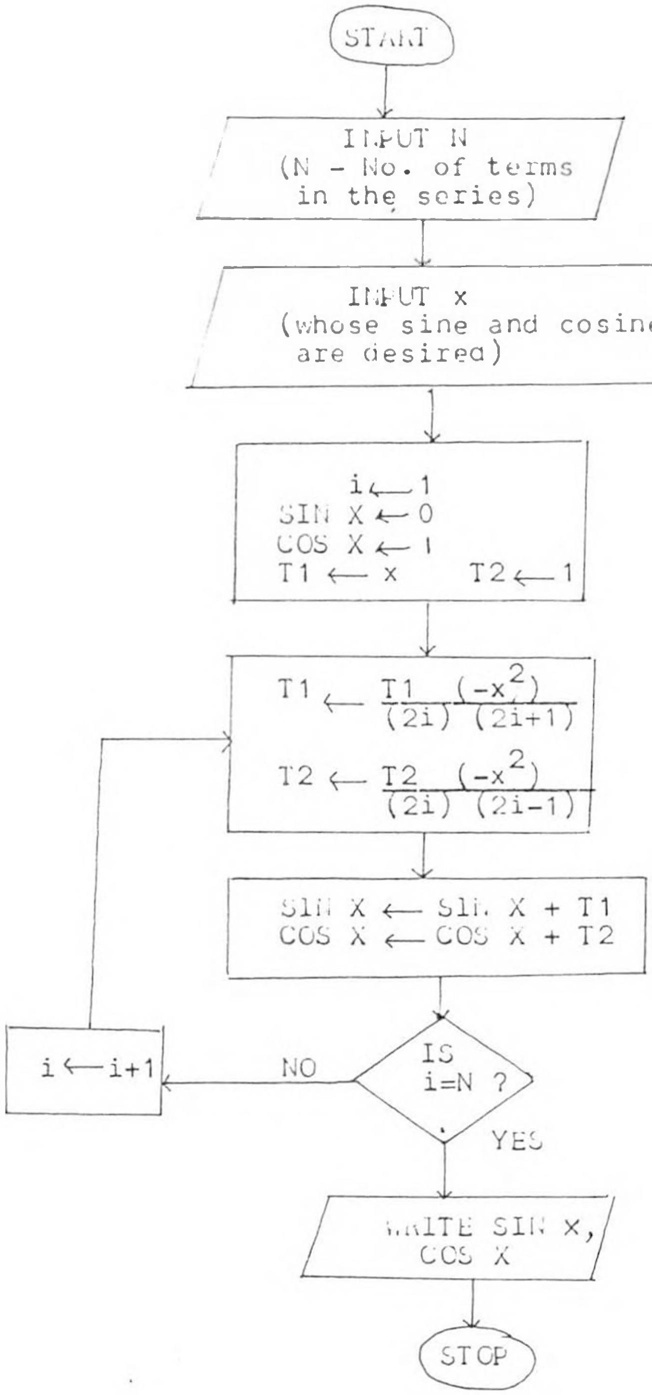


Flowchart for calculation of variance (s^2) using the formula

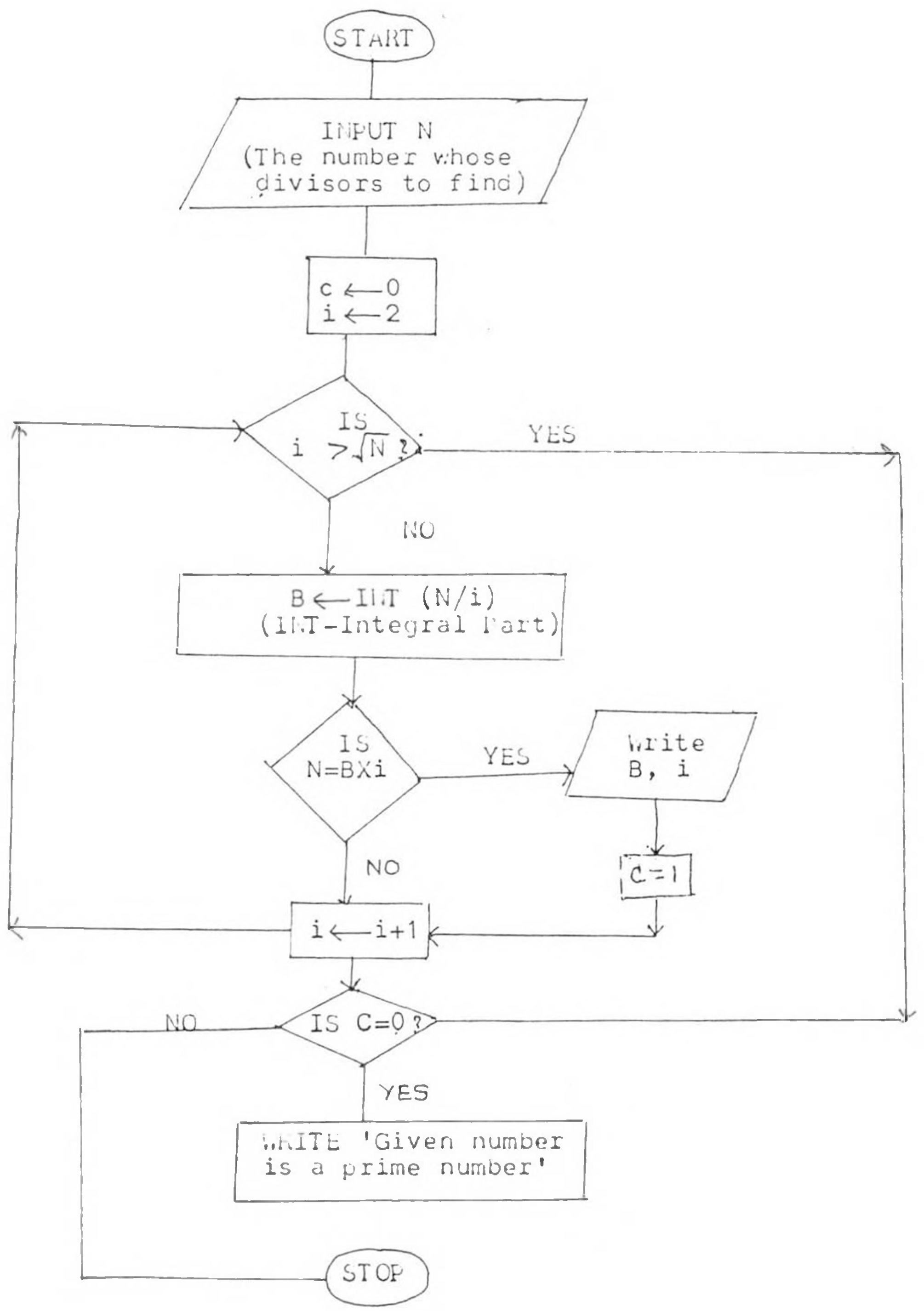
$$s^2 = \frac{\sum x_i^2}{n} - (\bar{x})^2.$$



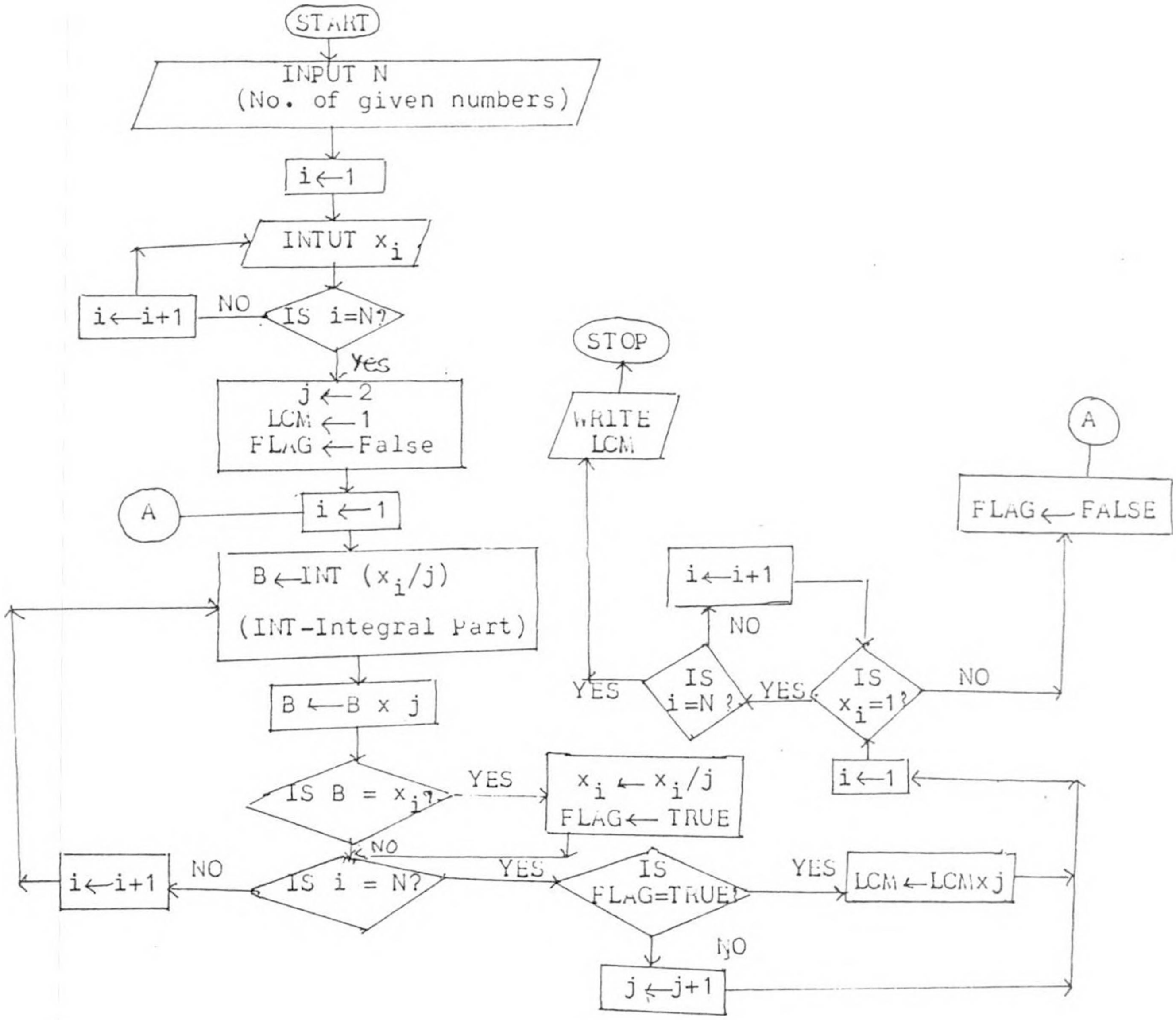
Flowchart to find the values of Sin(x) and Cos(x) for any given value of x.



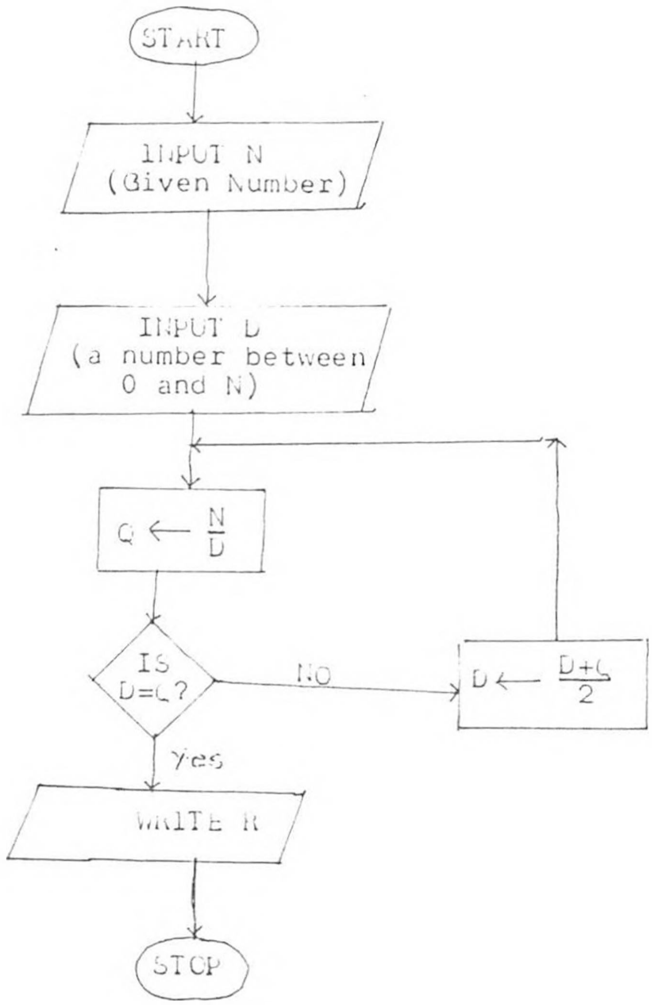
Flowchart to find all divisors of a given number.



Flowchart to find LCM of given numbers.



Flowchart to find square root of a given number.



Exercises :

1. What is a flowchart ? Why is it called a flowchart ?

2. What are the various symbols used in a flow chart ?
What does each symbol represent ?

3. Prepare a flow chart to add digits from 1 to 100.

4. Prepare a flow chart to pick the largest of three given numbers.

5. What is a program flow chart ?

6. Prepare a flow chart/algorithm in each of the following situations.

a) Preparation of a multiplication table.
b) Preparation of a bill in any consumer shop.
c) To verify whether there exists any given letter in any given word or not.
d) Calculation of compound interest for any given principle, rate and time.
e) To verify whether the given triangle is an equilateral or not.
f) To calculate the possible number of linear arrangements of 8 students so that two particular students sit together.
g) To obtain the transpose of any given matrix.
h) To show that the opposite angles in a cyclic quadrilateral are supplementary to one another.
i) Calculation of sin(x) and cos(x) for different values of x.
j) To verify whether the given two straight lines are parallel or not.
k) To calculate the value of the following polynomial for a given value of x.

$$a_o x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_n$$

l) To find the product of any two given polynomials of degree 'n'.
m) To convert the given angle in radians to degrees.
n) To find divisors of any given number.
o) To write a given number in reverse order.
p) To find the d ay of a week for a given date.
q) To categorize a given triangle as an obtuse, acute or right angled triangle with given sides.
r) Calculation of square root of any given number.
s) Calculation of LCM of any given numbers.
t) To list the binomial coefficients for any given 'n'.

u) To calculate the variance (s^2) of any given set of values $x_1, x_2, \ldots, x_n$ using the formula

$$s^2 = \quad (x_i - \bar{x})^2 \quad n \text{ where } \bar{x} = \quad x_i/n$$

v) To calculate the area of trapezium with given sides.

w) Printing of all prime numbers below a given number.

x) To find the roots of any given quadratic equation.

y) To convert a decimal number into a binary number.

z) To solve the following differential equation

$$\frac{dy}{dx} = 2x+3$$

given $y = 4$ when $x = 1$.