# Enrichment Materials in Mathematics for class IX of Karnataka

Prepared by

Mathematics Faculty, RIEM

NCERT

Regional Institute of Education, Mysore
[National Council of Educational Research and Training]

# Enrichment Materials in Mathematics for class IX of Karnataka


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# NUMBER SYSTEM

Students are familiar with the numbers 1,2,3,.... which are used in counting objects. Hence, these numbers are called counting numbers. They are also called natural numbers. The set of all natural numbers is denoted by "N". Thus,

N = { 1,2,3,.... }

The three dots after the number 3 stand for the natural numbers which come after 3, namely, the numbers 4,5,6 and so on. We know that two natural numbers can be added, and their sum is again a natural number. This property of natural number is described by saying that the set of natural numbers is closed with respect to the operation of addition. As regards the operation of subtraction, this property does not hold good. Subtracting one natural number from another does not always lead to a natural number. For example, if you subtract 8 from 5, you do not get a natural number. This is so because there is no natural number which can be added to 8 to get 5. The introduction of negative integers -1, -2, -3,... and the number 0 (zero) enabled us to remedy this defect. The negative integers, the number zero and the natural numbers (also called positive integers), taken together form a set which is called "the set of integers", and is denoted by "Z". Thus,

Z = { ...., -3, -2, -1, 0, 1,2,3,... }

The sum of any two integers (positive, negative or zero) is always an integer, and if you subtract an integer from any other integer, the result is always an integer. Thus we see that the set Z is closed with respect to the operations of addition and subtraction.

**Rational Numbers :**

The set Z of all integers has another property. If one integer (positive, negative or zero), is multiplied by another integer (positive, negative or zero), the product is again an integer. Thus the set Z is closed with respect to the operation of multiplication. In particular, given any two integers say 3 and 2, their product 3 x 2 always is an integer 6. But there is no integer by which you can multiply 3 to get 2. Z is not closed w.r.t. division. Such a situation necessitates the introduction of new numbers. A new number, denoted by the symbol 2/3, is introduced with the property that the new number 2/3 multiplied by 3, gives 2. The number 2/3 is not an integer and is called a rational number, which is read as "2 divided by 3", "2 by 3" or "2 upon 3".

In general, if p is any integer and q is a non-zero integer, there may not exist any integer r such that q x r = p i.e. it may not be possible to divide the integer p by the integer q. So we introduce a new number denoted by the symbol p/q called a rational number, which has the property that its products with q is the integer p. In this manner, with every pair of integers p and q, $q \neq 0$, a rational number p/q can be associated. However, when we consider integers p and 0, if it is possible to define the rational number p/0, then its product with 0 must be p which is impossible since any number multiplied by 0 has to be 0. Hence the condition $q \neq 0$ for a rational number p/q is necessary.

Two rational numbers p/q and r/s are said to be equal to each other if and only if ps = rq. Stated differently

p/q = r/s, if and only if ps - rq = 0

Hence $\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{4}{8} = \ldots.\ etc.$

Also,

p/q > r/s, if and only if ps - rq > 0

and

p/q < r/s if and only if ps - rq < 0

The set of all rational numbers is denoted by Q. Thus,

$$Q = \left\{ \frac{p}{q} : p \ and \ q \in Z \ and \ q \neq 0 \right\}$$

When we write a rational number in the form p/q, we can always take q > 0, whereas p may be positive, negative or zero. We will follow this convention.

Every integer (positive, negative or zero) can be written in the form p/q, where q = 1. For example, 3 = 3/1, -5 = -5/1 or 0 = 0/1 and so forth. Hence it is clear that every integer is a rational number and that the set Z of all integers is a subset of the set Q of all rational numbers. In symbols, $Z \subset Q$.

Just as, corresponding to every positive integer n there is a negative integer -n, similarly, corresponding to every positive rational number p/q, there exists a negative rational number -p/q.

Students have learnt in lower classes the methods of adding and multiplying two rational numbers. They also learnt as to how to subtract a rational number from another and how to divide one rational number by another non-zero rational number. These operations, as defined for rational numbers, are quite consistent with the corresponding operations defined for integers. This means that performing an operation with two integers, treating them as rational numbers, will give the same result, as is given by performing the corresponding operation with them as defined for integers. For example, treated as integers 2 + 3 = 5. Now treating 2 and 3 as rational numbers $\frac{2}{1}$ *and* $\frac{3}{1}$, $\frac{2}{1}+\frac{3}{1}=\frac{2\times 1+3\times 1}{1}=\frac{2\times 3}{1}=\frac{5}{1}=5$.

If p = k m and q = k n, where q and n are positive integers, then p/q = m/n. For pn = km . n = m. kn = mq. Hence p/q = m/n. For example, 4/6 = 2/3, since 4 = 2 x 2 and 6 = 2 x 3.

**Properties of Rational Numbers**

In the set Q of all rational numbers, there are two basic operations, namely, the operation of addition (+) and the operation of multiplication (.). Subtraction and division are respectively inverse operations of addition and multiplication. We know that the sum of two rational numbers is a rational number and the product of two rational numbers is also a rational number. This means that the set Q is closed with respect to the two operations '+ ' and '.'. There are certain familiar properties, satisfied by Q in relation to the two operations which we list below. In what follows, we use the letters a,b, c for rational numbers.

1. $a + b = b + a$ for all $a, b \in Q$ (Commutative law for addition)
2. $(a + b) + c = a + (b + c)$ for all $a,b, c \in Q$ (Associative law for addition)
3. The rational number 0 is such that
   $a + 0 = 0 + a = a$ for all $a \in Q$ (Existence of Additive Identity).
4. To each $a \in Q$, there is a number $- a \in Q$ such that $a + (-a) = (-a) + a = 0$
   (Existence of Additive inverse)
5. $a.b = b. a$ for all $a,b \in Q$ (Commutative law for multiplication)
6. $(a.b).c = a.(b.c)$ for all $a,b,c \in Q$ (Associative law for multiplication)
7. The rational number 1 (unity), is such that $1.a = a.1$ a for all $a \in Q$ (Existence of Multiplicative identity)
8. To every non-zero $a \in Q$ there corresponds a rational number 1/a such that a . 1/a = 1/a . a = 1, (Existence of Multiplicative inverse)
9. For all $a,b,c, \in Q$,
   $a . (b + c) = a.b + a.c$ and $(a + b) . c = a.c. + b. c$ . (Distributive law)

The system { Q, +, . } is said to be a Field because of the nine properties listed above. Simply speaking, the rational numbers under the usual operations of + and form a field. It is called the field of rational numbers.

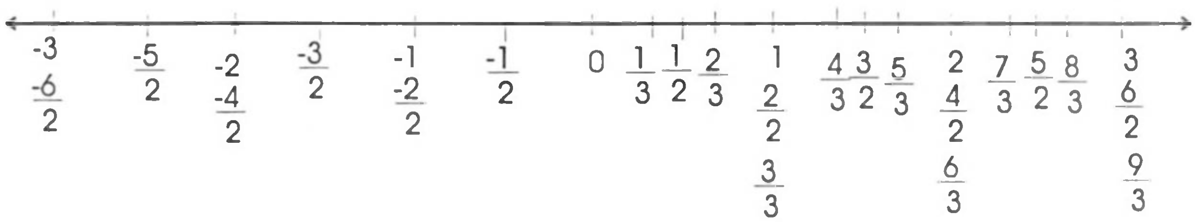
**The Number Line**

We will now consider the representation of rational numbers on a straight line, which extends endlessly in both the directions, as shown in the figure below.


The straight line l extends in both the directions endlessly as indicated by the arrowheads. Choose a point on l and label it 0 (zero). Next, choose another point on the line l to the right of 0 and label it 1 (one). We agree that the point 0 represents the number zero and the point labeled 1 represents the number 1. The length of the line segment between the points 0 and 1 is the unit length for our purpose. Again, mark on l, to the right of the point 1, points labeled 2,3,4,5,... in such a manner that the line segment between any two consecutive points, is of unit length. The points labeled 2,3,4 represent respectively the numbers 2,3,4. Since the line l extends endlessly on the right of the point 0, to every positive integer, there will correspond a point on l. Similarly, mark points -1, -2, -3, and so on, on the line l to the left of the point 0. In this way to every negative integer there will correspond a point on l to the left of 0, as the line l extends endlessly to the left of 0.

Next, we consider the representation of the rational numbers on the number line l. The mid point of the segment between 0 and 1 represents the number ½. Mark on l to the right of the point ½ new points such that the length of the line segment between two consecutive new points has half the unit length. These points will successively represent the numbers 2/2, 3/2, 4/2, 5/2,... Thus all those positive rational numbers which have 2 as their denominators will be represented by points on l. Repeating the same process to left of 0, we get all negative rational numbers which have 2 as their denominators, e.g. the numbers -1/2, -2/2, -3/2,... Next, taking one-third of the unit length and marking points on l to the right of 0 such that the length of the line segment between any two successive points is one-third of the unit length, we get points on l which respectively represent rational numbers 1/3, 2/3, 3/3, 4/3, 5/3,... Thus all the positive rational numbers having 3 as the denominator will be represented on l. In the same way, we mark points on the left of 0, to represent negative rational numbers -1/3, -2/3, -3/3, -4/3, -5/3,... Similarly, we mark points on l to the right of 0 corresponding to positive rational numbers having denominators 4, 5, 6,7,8 ,... and repeating the same process to the left of 0, we get points representing the corresponding negative rational numbers. In this way, all the rational numbers will be represented by points on the number line l.

It will be noticed that certain points on the line l appear to represent more than one rational number. For example, the point, representing the positive integer 2, also represents the numbers 4/2, 6/3, 8/4 and so on. There is nothing unnatural about it, since 4/2 = 6/3 = 8/4 = 2. Hence there is no ambiguity about this representation. Thus every rational number is represented by one and only one point on the number line. A question naturally arises here. Does every point on the number line represent a rational number? The answer to this question is in the negative. We will discuss this after a while.




First we prove that there is no rational number whose square is 2.

Proof: Since $1^2 = 1$ and $2^2 = 4$, it follows that if 2 is the square of a positive rational number, it cannot be an integer and it must be greater than 1. Suppose that there is a rational number p/q such that its square is 2. Since p/q is not an integer $q \neq 1$. Hence, we can suppose that the integer q is greater than 1 and that p and q have no common factors. Then,

$$2 = \frac{p^2}{q^2} \quad \therefore\ p^2 = 2q^2 \quad \therefore\ p^2 \text{ is an even integer.}$$

But square of an odd integer is odd. Hence p is an even integer. Let p = 2m where m is an integer.

$\therefore\ p^2 = 4m^2$.

$\therefore\ 2q^2 = 4m^2$.

$\therefore\ q^2 = 2m^2$ .

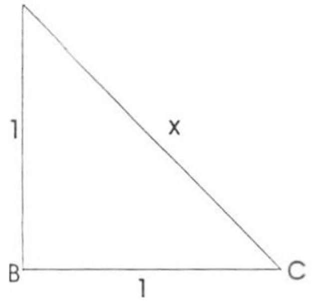
$\therefore\ q^2$ is even.

$\therefore$ q is an even integer. Hence p and q are both even integers and hence have a common factor 2, contradicting our assumption. Therefore, 2 is not the square of a rational number.

We use rational numbers very often in everyday life. When we measure lengths, or distances, or weights, we use rational numbers, e.g. 2 ½ meters, 5 ¾ kilometers or 3 ¼ kilograms, etc. But as we will now show, there are lengths which cannot be measured in terms of rational numbers.

Let ABC be a right-angled triangle, right angled at B, such that AB = BC = 1 unit. Suppose that AC = x units. Then by the Pythagoras theorem

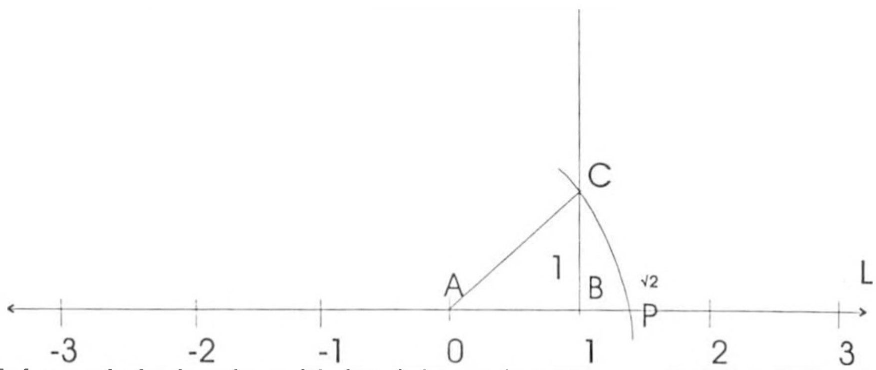
$AC^2 = AB^2 + BC^2$
i.e. $x^2 = 1 + 1 = 2$




Since $x^2 = 2$, x cannot be a rational number, as we saw earlier. Thus the length of AC cannot be measured in terms of rational numbers.

This shows the inadequacy of rational numbers in measuring lengths. However, the segment AC has a finite length and we have to express it in numbers using the unit of length. It can be said that there is a non-rational number whose square is 2 and therefore, this number is denoted by $\sqrt{2}$ units. Instead of using the term "non-rational", we say that $\sqrt{2}$ is an irrational number. Similarly, we define $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$ and say that they are irrational numbers.

Let us try to represent the number $\sqrt{2}$ on the number line l. Denote the point 0 on the line l by A, and the point 1 by B. Then the line segment AB is of unit length. At the point B, draw a straight line perpendicular to the line l and cut off a segment BC of unit length from this perpendicular line. With A as centre, and AC as radius draw a circular arc. Let this arc intersect the line l in P as shown in the figure. Then AP = AC. Since




$\Delta ABC$ is a right-angled triangle, with its right angle at B, and AB = BC = 1, it follows from the Pythagoras theorem that AC = $\sqrt{2}$. Hence AP = $\sqrt{2}$.

Thus the point P on the number line corresponds to the irrational number $\sqrt{2}$. We have now discovered a point on the number line which does not represent any rational number. Geometrical constructions can be devised to identify the points on the number line, which correspond to the irrational numbers $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$ and so on. It may be pointed out here that irrational numbers are not obtained only by the method of root extraction as is the case with $\sqrt{2}$, $\sqrt{3}$ and so forth. We shall discuss a general method of identifying irrational numbers. It may, however, be mentioned here that there are infinitely many irrational numbers, and in a sense, there are "more" irrational numbers than rational numbers.

**Decimal Representation of Rational Numbers :**

We know that every rational number can be represented either as a terminating decimal or as a non-terminating repeating decimal. For example, ½ = 0.5, 7/5 = 1.4 and 1/3 = 0.333…., 7/6 = 1.16666. In the decimal representation of 1/3, the digit 3 goes on repeating, and in the representation of 7/6, the first digit before the decimal point is 1, and after 1 the digit 6 goes on repeating. You are perhaps familiar with the following result. If p and q are positive integers, having no common factors the rational number p/q will have a terminating decimal only when the prime factors of q are only twos and five's, i.e. $q = 2^m \times 5^n$, m, n = 0,1,2,3,… For example, 7/8, 3/20, 11/25 and 99/200 will all have terminating decimal representations. (The proof is not difficult and can be given in the classroom by the teachers). Repeating decimals are also called 'periodic decimals' or 'recurring decimals'. Repeating decimals which consist of only one repeating digit, are written simply by putting a dot (.) above the repeating digit, e.g. 7/6 = 1.16666 is written as $1.1\dot{6}$. If, however, the number of digits in the repeating part is more than one, then a dot is put on the first digit and another on the last digit of the repeating part; e.g. 15/7 = 2.142857142857…., which is simply written as $2.\dot{1}4285\dot{7}$, meaning thereby that the entire block of six digits 142857 is repeating. Sometimes a line, called vinculum, is drawn covering the entire block of repeating digits, eg. 15/7 = $2.\overline{142857}$.

Conversely, every terminating decimal and every repeating decimal can be converted into a rational number of the form p/q, where p and q are integers, q > 0. For example, 0.25 = 25/100 = ¼. A different method is needed to convert a repeating decimal into a fraction p/q. Take the decimal 0.333…. Let x = .333…. Multiplying both the sides by 10, we get 10 x = 3.333… = 3 + x.

$\therefore$ 10 x - x = 3

i.e. 9 x = 3, and x = 3/9 = 1/3

This method, however, needs justification, which cannot be given here. But the method is correct.

A decimal with a repeating 9 can be converted into a terminating decimal by increasing the last digit before 9 by one. A terminating decimal can be converted into a repeating decimal either by adding a repeating zero to the right of the last digit after the decimal point, or by reducing the last digit by 1 and adding a repeating 9. For example,
0.1999... = 0.2
and 0.1 = 0.10000....
or 0.1 = 0.099999

It is, thus, clear that the same number can have three types of decimal representations. In order to make the decimal representation unique, we adopt the convention that a terminating decimal will be represented as a decimal with a repeating zero and a decimal with a repeating 9 will be converted into a decimal with a repeating zero. This convention enables us to assert that every rational number has a unique (non-terminating) repeating decimal representation and conversely, every repeating decimal represents a rational number.

**Irrational Numbers**

Irrational numbers are not obtained only by extracting square roots or cube roots of positive integers which are not perfect squares or perfect cubes. As we will see below, irrational numbers are represented by a special type of decimals.

Every irrational number is represented by a non-terminating non repeating decimal, and conversely, every non-terminating non repeating decimal represents an irrational number. This representation is unique. Now consider the following decimal expression :

$$0.1010010001000001.... \qquad (1)$$

Observe that in the above decimal expression (1), on the right of the decimal point there are either 1's or zeros and that the 1's are separated respectively by one zero, then two zeros, then three zeros and so on. Thus the number of zeros separating two successive 1's goes on increasing by 1 successively. This shows that the decimal expression (1) is non-terminating and non-repeating. Hence it cannot represent a rational number. We say that the decimal expression in (1), by definition, represents an "irrational number". We will see later that irrational numbers like $\sqrt{2}$, $\sqrt{3}$ also have non-terminating and non-repeating decimal representations.

Denoting the irrational number, given in (1), provisionally by a, let us examine as to where this irrational number a stands in relation to rational numbers, and also see whether there is a point corresponding to a on the number line l. It is easily seen that

$0.1 < a < 0.2$, where 0.1 and0.2 are rational numbers.
Further, $0.101 < a < 0.102$.
Again, $0.101001 < a < 0.101002$
and so on.

Continuing in this manner, we find closer and closer approximations of the irrational number a by rational numbers.

**Decimal Representation of Irrational Numbers**

We saw in an earlier section that $\sqrt{2}$ is an irrational number. Let us find out whether $\sqrt{2}$ has a decimal representation, and if so, examine the nature of the decimal representation. The process of finding the square root of 2 by the division method can give us the decimal representation of $\sqrt{2}$. We will however, follow here a more elementary method. It is easy seen that

$$1^2 = 1 < 2 < 4 = 2^2$$

Taking positive square roots, we get

$$1 < \sqrt{2} < 2$$

Next,

$$(1.4)^2 = 1.96 < 2 < 2.25 = (1.5)^2$$

Taking positive square roots again, we have
$1.4 < \sqrt{2} < 1$

Further,

$$(1.41)^2 = 1.9881 < 2 < 2.0164 = (1.42)^2$$

Again, taking the positive square roots, we obtain

$$1.41 < \sqrt{2} < 1.42$$

If we continue this process the next step will lead to the following inequalities.

$$1.414 < \sqrt{2} < 1.415$$

Proceeding in this manner, every new step will give a closer decimal approximation of $\sqrt{2}$ than the previous step. The eighth step will give the following inequalities :

$(1.4142135)^2 = 1.9999992358225 < 2$
$< 2.0000001 0642496 = (1.4142136)^2$

Hence $1.4142135 < \sqrt{2} < 1.4142136$

This is a very close approximation of $\sqrt{2}$.

Since $\sqrt{2}$ is not a rational number, this process will not terminate and will lead to a decimal expansion which will not terminate, nor will it be repeating. Hence, the non-terminating and non-repeating decimal expansion of $\sqrt{2}$ will be given by

$$\sqrt{2} = 1.4142135 \ldots$$

Where the dots indicate that this decimal representation will not terminate. Similarly, we can show that the decimal representations of $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$, and other non-rational numbers obtained by processes of root extractions, will not terminate nor will they be repeating. Hence we conclude that

> A number is irrational if and only if its decimal representation is non-terminating and non-repeating.

We can find numerous decimal expressions, like what is given in (1) by changing the digits and by varying the frequencies of a digit's occurrence in the decimal expression. For example, consider the following three decimal expressions :

i. 0.121121112111112...
ii. 0.020020002....
iii. 0.300030000030000003...

All the three decimal expressions represent irrational numbers. We can go on multiplying such examples endlessly. Hence we conclude that :

> There are infinitely many irrational numbers. There is a point on the line l corresponding to every irrational number.

We can locate these points on the Number Line. Thus we see that on the number line l there are infinitely many points which do not correspond to rational numbers.

**Real Numbers :**

Between two distinct rational numbers, howsoever close to each other they may be, there are infinitely many distinct rational numbers. This result gives the impression that rational numbers are very 'densely' located on the number line l. Now we find that, even through the rational numbers are densely located on the number line l, infinitely many points on l are left out and these points correspond to irrational numbers. Rational numbers and irrational numbers taken together form the set of real numbers. This set is denoted by R. Thus every real number is either a rational number or an irrational number. In either case, it has a non-terminating decimal representation. If this representation is repeating (including repeating zeros) it is a rational number, and if it is non-repeating it is an irrational number. It is clear that $Q \subset R$. Also, corresponding to every real number, there is a unique point on the number line l. It can also be shown that every point on the line l corresponds to a real number (rational or irrational). Let us now state the final conclusion.

> To every real number, there corresponds a unique point
> on the number line, and conversely, to every point on
> the number line there corresponds a real number.

It may be noted that this correspondence is one-to-one, and for this reason the number line is called the Real Number Line.

Now that we have extended the number concept from rational numbers to real numbers, it is natural to ask whether the fundamental operations of addition, subtraction, multiplication and division can be extended to the real numbers also. The answer is : "yes, the operations can be extended to the real numbers". In the first place we observe that the order relation "greater than" holds in the case of real numbers also. Given two distinct real numbers, one of them is greater than the other. The greater real number lies to the right of the smaller one on the real number line. Every real number on the right of 0 on the number line is positive and every real number on the left of 0 is negative. Corresponding to every positive real number a, there is a negative real number -a. As regards the operation of addition, two real numbers can be added on the number line. When we add two rational numbers, we get the answer in a compact form. For example, $\frac{2}{3} + \frac{3}{4} = \frac{7}{12} = 1\frac{5}{12}$.n On the other hand, although the sum of two real numbers is a real number, it is not always written in a simplified form. For example, the sum of 2 and $\sqrt{3}$ is just written as $2 + \sqrt{3}$;

also the sum of $\sqrt{2}$ and $\sqrt{3}$ is just written as $\sqrt{2} + \sqrt{3}$. As regards multiplication, defining the product of two real numbers is not easy on the number line. It is however, true that the product of two real numbers is always a real number. The product of 2 with $\sqrt{3}$ is simply written as 2 $\sqrt{3}$, and the product of $\sqrt{2}$ and $\sqrt{3}$ is written as $\sqrt{2} . \sqrt{3}$ or = $\sqrt{2.3} = \sqrt{6}$. We similarly deal with the operations of subtraction and division. Division by zero is not defined.

Real numbers also possess all the properties of rational numbers listed earlier. You have only to replace there the word 'rational' by 'real' and the set Q by the set R. Thus the set R of real numbers is also a field under the operations of addition (+) and multiplication(.). Real numbers are also ordered. There is however, an important difference :

Every point of the number line corresponds to a real number.
This is not true of rational numbers.

**Worked Examples :**
**Example 1 :**

i. If a is irrational, then -a is also irrational.
ii. The sum of a rational number with an irrational number is always irrational.
iii. The product of non-zero rational number with an irrational number is always irrational.

**Proof:**
i.. Let a be irrational. If -a is not irrational, then -a must be rational, since it is a real number. We know that the negative of a rational number is always rational, hence -(-a) must be rational. But -(-a) = a, which is irrational. This contradiction proves that our supposition that -a is not irrational is false. This proves the result.

ii. Suppose that a is rational and b is irrational. Now if a + b is not irrational, then a + b must be rational. If we subtract the rational number a from the rational number (a + b), then the remainder must be rational. But (a + b) - a = b, which is irrational. This contradiction proves that a + b must be irrational.
iii. Let a be a non-zero rational number, and b an irrational number. If a . b is not irrational, a.b must be rational. Now dividing the rational number a.b by the non-zero rational number a will give a rational number, but when we divide a.b by a we get b which is supposed to be irrational. This contradiction proves the result.

**Example 2 :** The sum and product of two rational numbers are always rational, but neither the sum nor the product of two irrational numbers is always an irrational number.

Solution : We know that $\sqrt{2}$ is an irrational number. Hence $-\sqrt{2}$ is also an irrational number. Their sum is $\sqrt{2} + (-\sqrt{2})$ which is equal to 0, is a rational number.
Similarly $\sqrt{2} \cdot \sqrt{2} = (\sqrt{2})^2 = 2$, which is, again, a rational number.

In the first case the sum of two irrational numbers is a rational number and in the second case, the product of two irrational numbers is rational.

On the other hand, the sum of the two irrational numbers $\sqrt{2}$ and $\sqrt{3}$ is irrational. To prove this suppose that $\sqrt{2} + \sqrt{3}$ is not irrational. Then $\sqrt{2} + \sqrt{3}$ must be rational. Since the square of a rational number is rational $(\sqrt{2} + \sqrt{3})^2$ should be rational.

Thus $(\sqrt{2} + \sqrt{3})^2 = 2 + 3 + 2\sqrt{2}.\sqrt{3}$ is a rational number; i.e. $5 + 2\sqrt{6}$ is a rational number. Now, $\sqrt{6}$ is irrational, hence $2\sqrt{6}$ is irrational. Since the sum of a rational number with an irrational number is irrational [Example 4(ii)], it follows that $5 + 2\sqrt{6}$ is irrational, which implies that $(\sqrt{2} + \sqrt{3})^2$ is irrational. This contradiction proves that $\sqrt{2} + \sqrt{3}$ is an irrational number.

Again, the product of two irrational numbers $\sqrt{2}$ and $\sqrt{3}$ is irrational, since $\sqrt{2} \cdot \sqrt{3} = \sqrt{6}$ which is irrational.

Thus, we see that the sum and product of two irrational numbers may be rational in some cases and irrational in some other cases.

**Example 3 :** Give a rational approximation to $\sqrt{3}$ correct to two places of decimals.
Solution: We know that $1^2 = 1 < 3 < 4 = 2^2$
Taking positive square roots we get
$1 < \sqrt{3} < 2$
Next, $(1.7)^2 = 2.89 < 3 < 3.24 = (1.8)^2$
Taking positive square roots, we have
$1.7 < \sqrt{3} < 1.8$
Again, $(1.73)^2 = 2.9929 < 3 < 3.0276 = (1.74)^2$

Taking positive square roots, we obtain
$1.73 < \sqrt{3} < 1.74$
Hence the required approximation is 1.73.
Note : $\sqrt{3} = 1.7320508.....$

8. Match the lists.

| **A (elements)** | **B (Sets)** |
|---|---|
| 1. -5 | { x : x is a square number} |
| 2. 5 | {x : x is a -ve integer} |
| 3. 14 | {x : x is a divisor of 60 } |
| 4. 36 | { x : x is an even no < 20 } |
| 5. 19 | {x : x is a prime no > 17 } |
| | {x : x is not an integer } |

9. Which of the following sets are equivalent to the given set A = { 4,7,11,17,20 }
i. {e, b, c },
ii. {a, e,i, o, u }
iii. {Prime numbers which lie between 10 and 25 }
iv. {all odd numbers < 10 }
v. {All months of the year with 30 days }

10. From the given sets identify equal sets and equivalent sets.
A = { o, a }, B = {1,2,3,4}, C = {4,8,12}, D = {3,1,2,4}, E = {1, 0}
F = {2 + 2, 6 + 2, 3 x 4 }, G = {1,5,7,11}, H = {a , b }

11. **If two sets are equal, then they are also equivalent sets but if two sets are equivalent they may not be equal.** Illustrate this using examples of (10).

12. Which of the following sets are empty sets ?
i. {x : x is an integer between 0 and 1 }
ii. {x : x is an odd number between 3 and 5 }
iii. Set of all students who study for 28 hours in a day.
iv. Set of all states of India

13. Which of the sets are finite/not finite ?
i. The set of all lines passing through a point.
ii. The set of all points common to two given parallel lines.
iii. Set of all natural numbers which are divisible by 7.
iv. Set of all natural numbers which are factors of 1024.
v. Set of people living in Karnataka State.

14. State whether the following statements are true or false. Give reasons.
i. {0} is a subset of the set of all whole numbers.
ii. $\phi$ is a subset of every set.
iii. The number of subsets of the set {0} is 2.
iv. Subset of every finite set is finite.
v. {0} is a subset of every set.
vi. The number of elements in an $\phi$ is 0.

15. State whether the following statements are true or false ?

i. $\{1,3,2\} \subset \{1,2,3\}$

ii. $\{1\} \in \{1,2,3\}$

iii. $\{2\} \in \{2,5,6\}$

iv. $\{2\} \subset \{2,5,6\}$

v. $8 \subset \{a,b,8,10\}$

vi. $\phi \subset \{0\}$

16. Write examples of three non empty sets A, B, C satisfying the following conditions.

i. $A \subset B \subset C$

ii. A is disjoint with B and C

iii. B is disjoint with A and C

iv. $C \subset A$, $B \subset A$ but A and C are disjoint.

17. Write a universal set related to following sets.

1. a) {x : x is an integer }, b) {x : x is an odd number }
   c) { x : x is a multiple of 5 }, d) {x : x is a counting number }

2. If U = { x : x is a quadrilateral } is the universal set. Write any 5 subsets of U.
3. If U = { x : x is a book in the Regional College library} is the universal set. Write any 5 subsets of U.

18.

i. $P \subset Q$ find P U Q and $P \cap Q$.

ii. If U = {4,7,10,12,15,16,20}; A = {4,10,15,20}, B = { 7,10,15,12 }, find A', A U A', (AUB)', B', AUB'.
Illustrate these sets using Venn diagrams.

iii. If A = {1,2,3,4,5}, B = { 4,5,6,7,8 }, find A - B, B - A, A - A.

19. Find the set of all subsets of the set { R, I, E }

## PROPERTIES OF OPERATIONS ON SETS

The operations of 'Union', 'intersection' and 'complementation' defined on the sets possess certain properties.

For sets A, B and C.

1. i. $(A \cup B) \cup C = A \cup (B \cup C)$ — Associativity
   ii. $(A \cap B) \cap C = A \cap (B \cap C)$

2. i. $A \cup B = B \cup A$ — Commutativity
   ii. $A \cap B = B \cap A$

3. i. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ — Distributivity
   ii. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

4. i. $(A \cup B)' = A' \cap B'$ — Demorgan's laws
   ii. $(A \cap B)' = A' \cup B'$

5. i. $(A')' = A$
   ii. $A \cap A' = \phi$
   iii. $A \cup A' = U$

6. i. $A - B = A \cap B'$
   ii. $B - A = A' \cap B$

7. i. $A \cup U = V$
   ii. $A \cap U = A$

Each of the properties listed above is an identity which can be proved.

**Proving identities : An illustration**

In proving the equality of sets, we use the technique of showing that each is a subset of the other. From this, equality of sets follows, i.e. If A and B are two given sets, then $A \subset B$ and $B \subset A$ together imply that $A = B$.

Prove : $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.................. (*)

Proof: Let x be any element belonging to the set $A \cap (B \cup C)$ of (*)
i.e. $x \in A \cap (B \cup C)$
$\Rightarrow$ $x \in A$ and $x \in (B \cup C)$
$\Rightarrow$ $x \in A$ and ($x \in B$ or $x \in C$)
$\Rightarrow$ $x \in A$ and $x \in B$ or $x \in A$ and $x \in C$.

$\Rightarrow$ $x \in (A \cap B)$ or $x \in (A \cap C)$.
$\Rightarrow$ If $x \in (A \cap B) \cup (A \cap C)$ which is the RHS set of (*)
$\therefore$ If $x \in$ Set on LHS of (*) then $x \in$ set on the RHS of (*)

Hence, $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$... (1)

Next, let y be *any* element.

$\in$ $(A \cap B) \cup (A \cap C)$ [ the RHS set of * ]
$y \in (A \cap B) \cup (A \cap C)$.
$\Rightarrow$ $y \in (A \cap B)$ or $y$ $(A \cap C)$
$\Rightarrow$ $y \in A$ and $y \in B$ or $y \in A$ and $y \in C$
$\Rightarrow$ $y \in A$ and $y \in B$ or $y \in C$
$\Rightarrow$ $y \in A$ and $y \in B \cup C$
$\Rightarrow$ $y \in A \cap (B \cup C)$

$\therefore$ we have proved that
$y \in (A \cap B) \cup (A \cap C) \Rightarrow y \in A \cap (B \cup C)$
or $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$ .... (2)

From (1) and (2), we have

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Examples:

1. Verify distributive laws when
   $A = \{ 1,2,3,4,5\}$, $B = \{ 2,4,6 \}$, $C = \{ 1,3,4 \}$

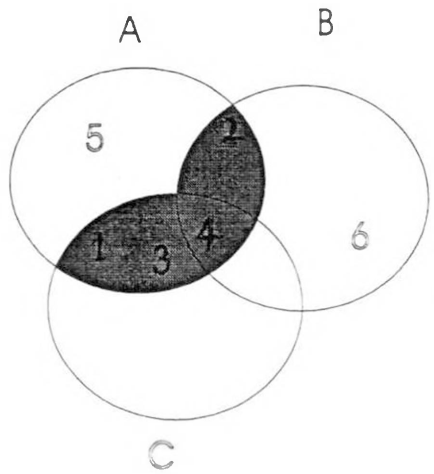
First let us verify the law $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

$B \cup C = \{ 1,2,3,4,6 \}$
$A \cap (B \cup C) = \{ 1,2,3,4,5\} \cap \{1,2,3,4,6 \}$
$= \{1,2,3,4\}$ .... (1)

Again,

$A \cap B = \{ 2,4 \}$
$A \cap C = \{ 1,3,4 \}$
$(A \cap B) \cup (A \cap C) = \{2,4 \} \cup \{ 1,3,4 \} = \{ 1,2,3,4\}$ ... (2)

From 1 and 2,
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (see the shaded region).




Next let us verify the distributive property.
A U (B $\cap$ C) = (A U B) $\cap$ (A U B)

B $\cap$ C = {2,4,6} $\cap$ {1,3,4} = $\phi$ (null set).

A U (B $\cap$ C) = A U $\phi$ = A = {1,2,3,4,5,}... (1)

A U B = { 1,2,3,4,5,6}
A U C = {1,2,3,4,5}

$\therefore$ (A U B) $\cap$ (A U C) = { 1,2,3,4,5} (2)

From 1 and 2,

A U (B $\cap$ C) = (A U B) $\cap$ (A U C ) = { 1,2,3,4,5 }

Hence both the distributive laws are verified.

2. Verify De Morgan's laws for the given sets

U = { 1,2,3,4,5,6,7,8,0 }
A = { 1,2,3,4,5 }, B = { 0,1,3,7 }

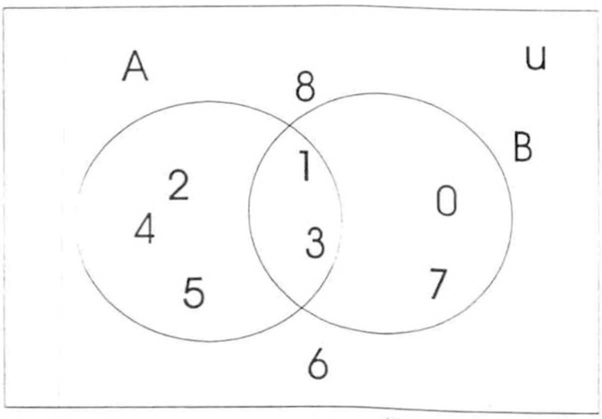
Firstly, let us verify the law (A $\cap$ B)' = A' U B'.
A $\cap$ B = { 1,3 }; (A $\cap$ B)' = {0,2,4,5,6,7,8 } (1)

A' = { 6,7,8,0 } ; B' = { 2,4,5,6,8 }

A' U B' = { 0,2,4,5,6,7,8 } (2)

From 1 and 2

(A ∩ B)' = A' U B' = { 0,2,4,5,6,7,8 }
See Fig.




(2)

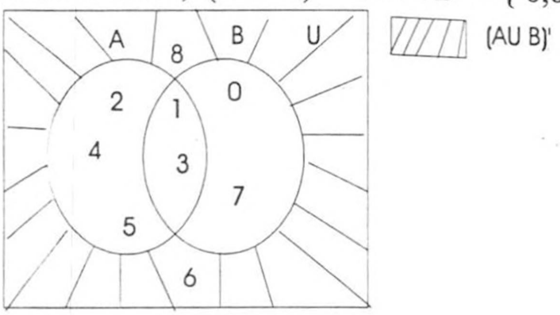
Next we shall verify the law.
(A U B)' = A' ∩ B'

A U B = { 0,1,2,3,4,5,7 }
(A U B) ' = { 6,8 }.. (1)

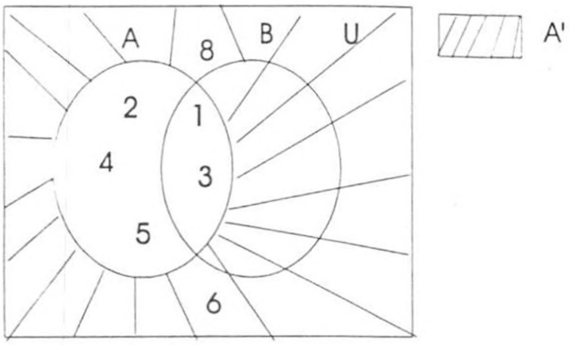
A' = { 0, 6,7,8 } ; B' {2,4,5,6, 8 }

A' ∩ B' = { 6,8 }

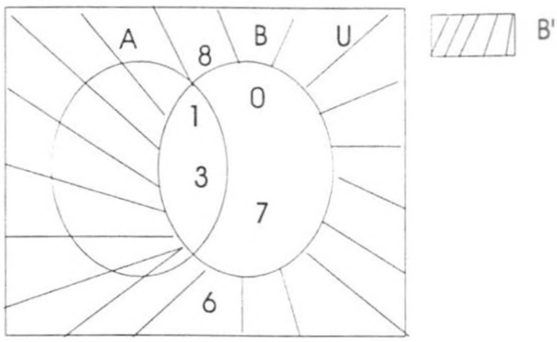
From 1 and 2, (A U B)' = A' ∩ B' = { 6,8 }




2(a)




2(b)




2(c)

Note: Figs 2(b) and 2( c) may be drawn on transparent sheets. If these figures are placed one above the other, we can recognize the set A' ∩ B' as the same as the set (A ∪ B)' given in fig 2(a).

**Example 3**

Let the universal set be the set of all odd numbers less than 20.

$\therefore$ U = { 1,3,5,7,9,11,13,15,17,19 }

Let A = {x | x $\in$ U and x is a multiple of 3 }

$\therefore$ A = { 3, 9, 15 }

B = { x : x $\in$ U and x is a prime number }

$\therefore$ B = { 3,5,7,11,13,17,19 }

A' = { 1,5,7,11,13,17, 19 }
B' = { 1, 9, 15 }
A' U B' = { 1,5,7,9,11,13,15,17, 19 }
A $\cap$ B = { 3 }

(A $\cap$ B)' {1,5,7,9,11,13,15,17,19 } (2)

Hence the De Morgan's law

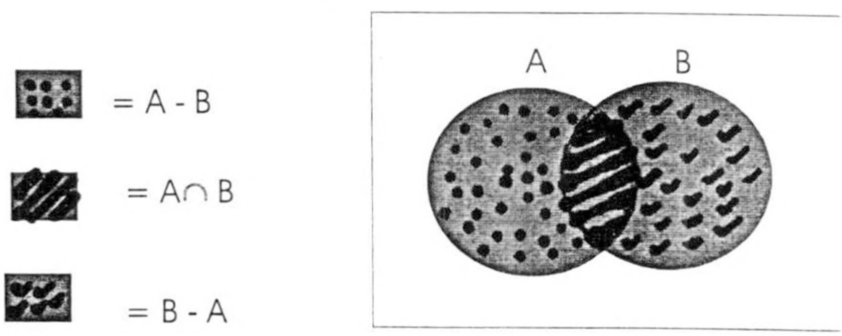
$\therefore$ (A' U B') = (A $\cap$ B)' is verified.

Next, A U B = { 3,5,7,9,11,13,15,17,19 }
(A U B)' = { 1 } (1)
A' $\cap$ B' = { 1 } (2)

$\therefore$ (A U B)' = A' $\cap$ B' thus verifying the $2^{nd}$ De Morgan's Law.

**An Addition Rule** (Ref to Page 17, Sec. 3.2)

If A and B are finite sets, then
n(A) + n (B) = n (AUB) + n (A $\cap$ B)
(A - B) U (A $\cap$ B) = A (See fig.)
and (B - A) U (A $\cap$ B) = B.



Also A-B, $A \cap B$ and B-A are mutually disjoint sets

Further, $A \cup B = (A - B) \cup (A \cap B) \cup (B - A)$
$\therefore\ n(A \cup B) = n(A - B) + n(A \cap B) + n(B - A)$
But $n(A) = n(A - B) + n(A \cap B)$
and $n(B) = n(B - A) + n(A \cap B)$

$\therefore\ n(A) + n(B) = [\ n(A - B) + n(A \cap B) + n(B - A) + n(A \cap B)$

$= n(A \cup B) + n(A \cap B)$.

Hence, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

**Corollary:** If A and B are disjoint sets, then $A \cap B = \phi$ so that $n(A \cap B) = 0$. The above formula in this case becomes

$$n(A \cup B) = n(A) + n(B).$$

Problem 1: In the school out of 25 teachers, 18 teach English and 15 teach Mathematics. Find how many teach

a) both English and Mathematics
b) Only English
c) Only Mathematics

Let A = Set of teachers teaching English
B = Set of those teaching Mathematics
Then, $A \cup B$ = Set of teachers teaching English or Mathematics
and $A \cap B$ = Set of teachers teaching English and Mathematics

$n(A) = 18,\ n(B) = 15,\ n(A \cup B) = 25$

Using the rule

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

$25 = 18 + 15 - n(A \cap B)$

$\therefore\ n(A \cap B) = 8.$

a) 8 teachers teach both English and Maths
b) $A - (A \cap B) =$

The set of teachers teaching only English

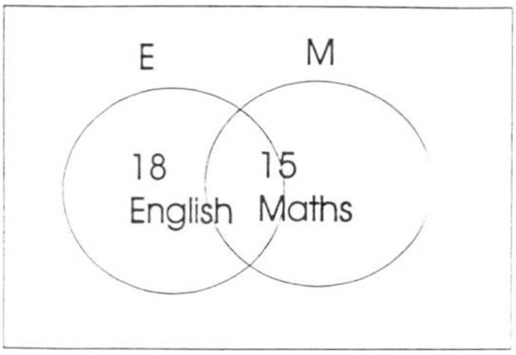
Number of such teachers =

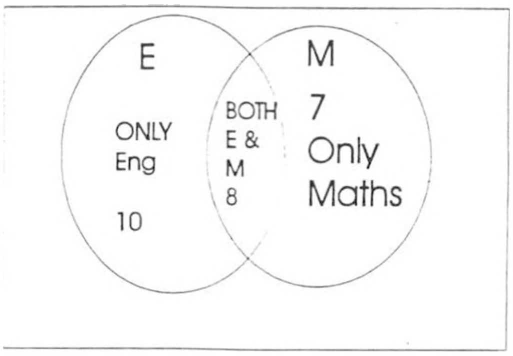
$n(A) - n(A \cap B) = 18 - 8 = 10$

c) $B - (A \cap B)$ gives the set of teachers teaching only Maths

$\therefore$ $n(B) - n(A \cap B) = 15 - 8 = 7$

$\therefore$ 7 teachers teach only Maths (See Fig).




2. Of the 80 students in class IX, 45 study Kannada, 35 study Hindi. 15 students study both Kannada and Hindi. Find the number of students who study (a) Kannada only, (b) Hindi only, ( c ) Kannada or Hindi, (d) None of the two languages.

Let K = set of students who study Kannada; $n(K) = 45$

H = set of students who study Hindi, $n(H) = 35$

$K \cap H$ = set of students who study Kannada and Hindi

$\therefore$ $n(K \cap H) = 15$.

a) No. of students who study Kannada only

$= n(K) - n(K \cap H) = 45 - 15 = 30$

b) Number of students studying Hindi only

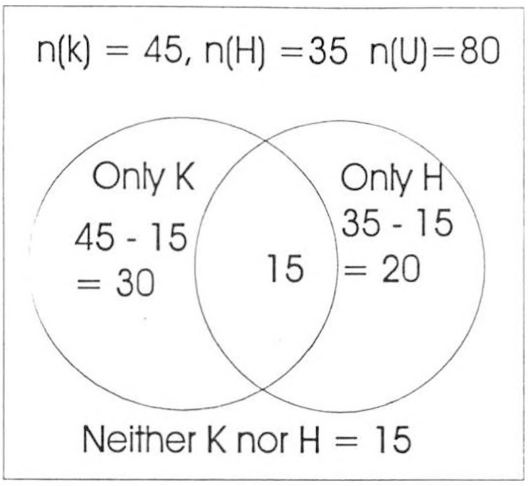
$= n(H) - n(K \cap H) = 35 - 15 = 20$

c) $n(K \cup H) = n(K) + n(H) - n(K \cap H)$

$65 = 45 + 35 - 15$

$\therefore$ 65 students study Kannada or Hindi

d) Number of students who study neither Kannada nor Hindi = 80 - 65 = 15.




**Ordered pair and Product Sets (Cartesian Products)**

1. Ordered pairs and their graphical representation.

Consider non empty sets A and B. Let $a \in A$ and $b \in B$. We form a pair (a,b) and call it an ordered pair. The pair is said to be ordered since the order in which we write elements a and b in the pair is important (should not be ignored).

A is called the first element (entry/coordinate) and b is called the second element (" / ").
This means ordered pair (a,b) is different from the ordered pair (b,a).

Two ordered pairs (a,b) and (c,d) are equal if and only if a = c and b = d.
**Examples :**

1. Heights and weights of three boys are given below.

| | | |
|---|---|---|
| Height (in cms): | 128, 140, 145 | Here Set A ` { 128, 140, 145 } |
| Weight (in kg) : | 45, 53, 58 | and B = {45, 53, 58} |

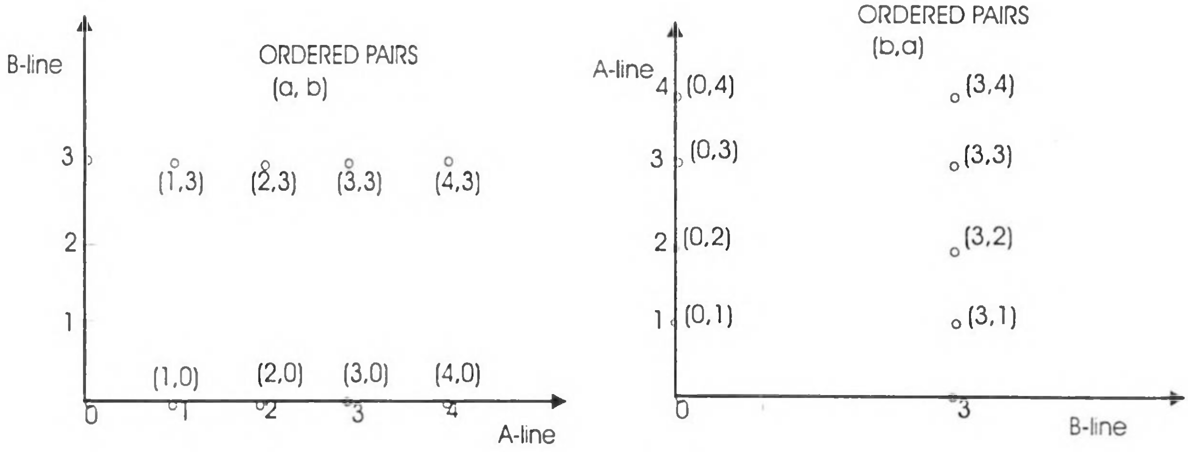
The above information is written in the form of ordered pairs as (128, 45), (140, 53), (145, 58). In each ordered pair, the first element belongs to A and the second element belongs to B.

We can also write the ordered pairs in which first element comes from B and the second element comes from A. They are
(45, 128), (53, 140), (58, 145).

2. Let A = { 1,2,3,4 } and B = { 3, 0 }. Let us list all the ordered pairs (a,b) such that a ∈ A and b ∈ B.
3. { (1,3), (1,0), (2,3), (2,0), (3,3), (3,0), (4,3), (4,0) }

The possible ordered pairs (b,a) such that b ∈ B and a ∈ A are
(3,0), (0,1), (3,2), (0,2), (3,3), (0,3), (3,4), (0,4)

**Graphical Representation:** In the above illustration, let the elements of the set A be denoted by points on a horizontal number line and those of B by points on a vertical number line as shown.




3. Given the ordered pairs (H, T), (T, H) and (T, R). Let us list the first and the second elements in the ordered pairs.

| **Ordered Pair** | **1st element** | **2nd element** |
|---|---|---|
| (H, T) | H | T |
| (T, H) | T | H |
| (T, R) | T | R |

**Exercises :**

1. List all the ordered pairs of the type (a,b), a ∈ A and b ∈ B where A = { 0,1,2,3}; B = { 1,3,4} and plot then on a graph.

2. A coin is tossed twice in succession. Write down all possible outcomes as ordered pairs. For Ex: (h,t) means head in the first toss and tail in the second toss.

**Product Sets**

Let A and B be two given non empty sets. We form a new set, namely, the set of all possible ordered pairs (a,b) such that $a \in A$ and $b \in B$. The new set of ordered pairs so formed, is called the product set of A and B and is denoted by

A X B. Thus, A x B = $\{ (a, b) : b \in B \text{ and } a \in A\}$

Similarly, B x A = $\{ (b, a) : b \in B \text{ and } a \in A \}$ .

Example : Let A = (a, i, o } and B = {n, m}

Then A x B = { (a,n), (a,m), (i,n), (i,m), (o,n), (o,m) }
and B x A = { (n,a), (m,a), (n,i), (m,i), (n,o), (m,o) }

Evidently A x B $\neq$ B x A. (a,n) $\in$ A x B but (a,n) $\notin$ B x A.

**Product Set of A with itself**

Instead of forming the product set A x B of two different sets A and B, we can form A x A, the product set of A with itself; then

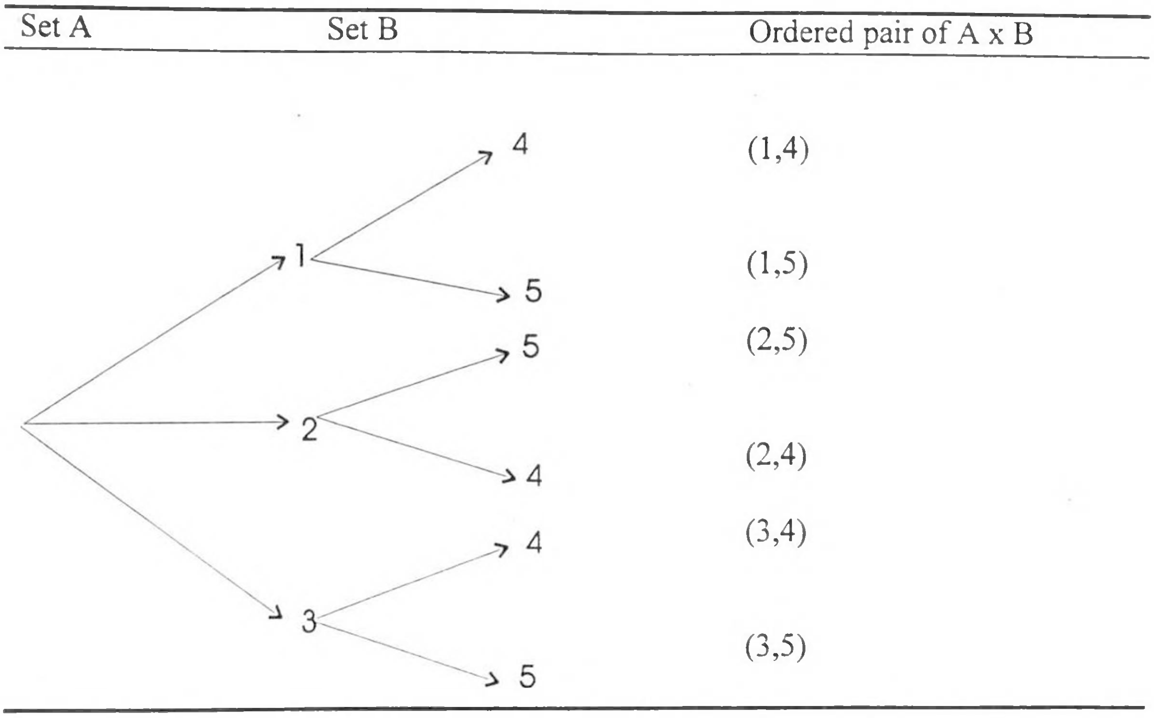
A x A = $\{ (a,b) : a \in A, b \in A\}$

Example Let A = { 2,3,0 }

Then A x A = { (2,2), (2,3), (2,0), (3,3), (3,0), (0,0), (3,2), (0,2) }

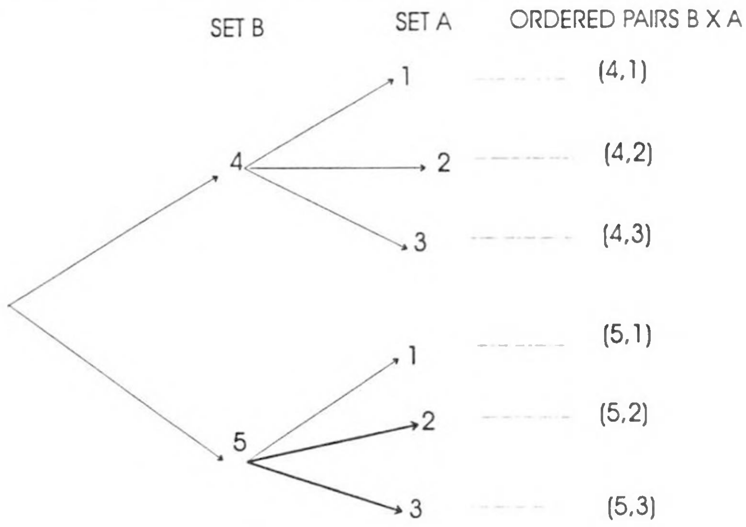
**Tree diagrams to find Cartesian product A x B :**

Let Set A = { 1,2,3 } and B = { 4,5}. Tree diagram to find the ordered pairs of A x B is illustrated below.




The tree is constructed from left to right. From the last column,
A x B = {(1,4), (1,5), (2,4), (2,5), (1,4), (1,5) }

Here is the tree diagram for the set B x A (above example).




$\therefore$ B x A = { (4,1), (4,2), (4,3), (5,1), (5,2), (5,3) }

A x B and B x A of this example are graphically represented below.

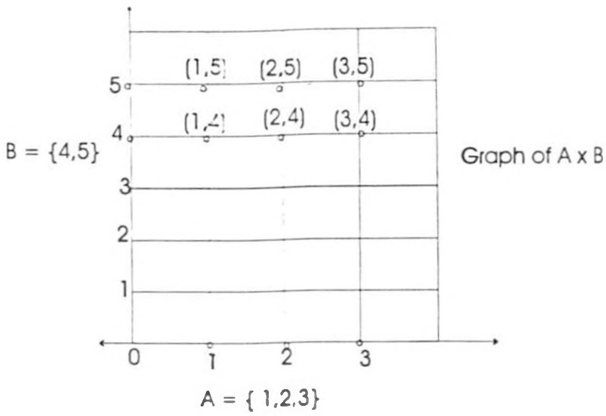



Fig. 1

Graph of A x B of sets A and B of numbers is the set of cell points in the plane represented by the ordered pairs of A x B.

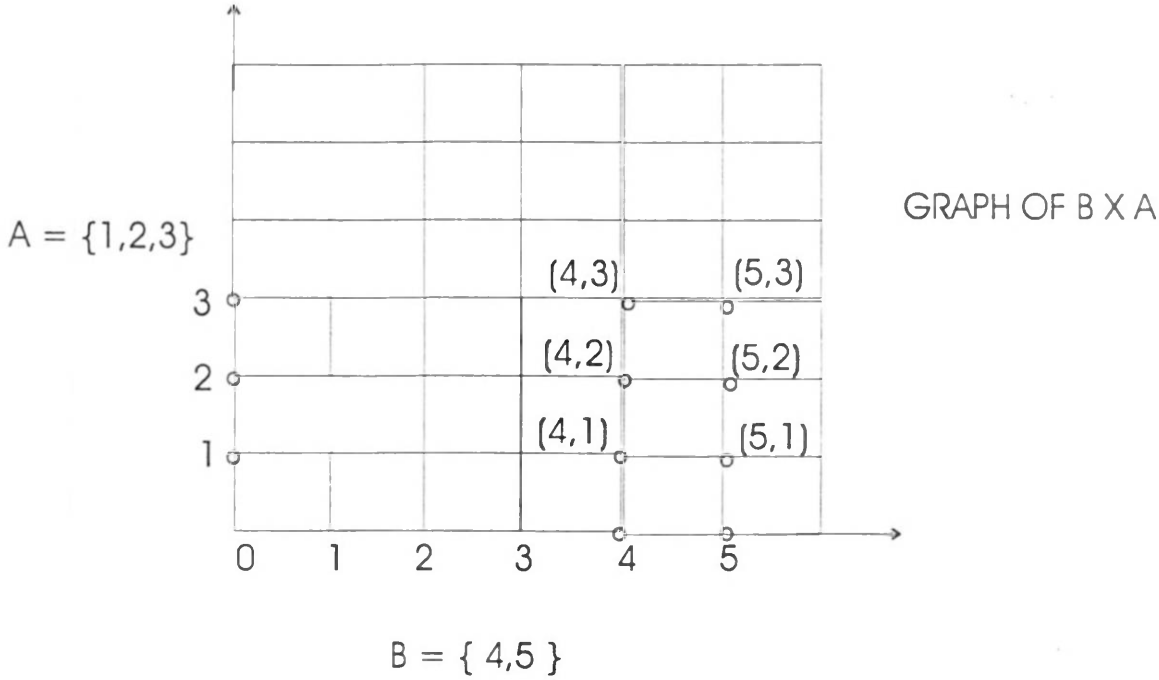


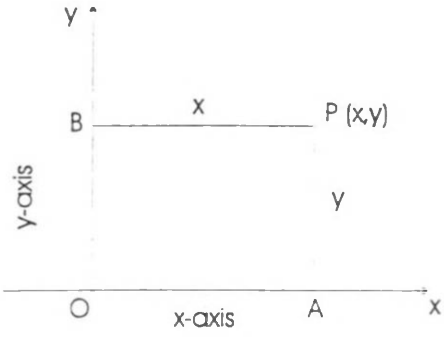

FIG. 2

Two important product sets :

1. Let N be the set of natural numbers. The cartesian product N x N is an infinite set of ordered pairs. N x N = { (a,b) : a $\in$ N, b $\in$ N ) } .

2. Let R be the set of real numbers. Then the product set $R^2$ = R x R = { (x,y) : x $\in$ R, y $\in$ R } consisting of all ordered pairs of real numbers x,y.

When $R^2$ is represented graphically, the picture we get is that of all points of the plane determined by two number lines taken at right angles.




Calling the horizontal number line as x axis (see Fig ) and the vertical number line as Y axis. The graph of the set $R^2$ consists of all points of the plane determined by x and y axes. The plane so formed is called the Cartesian plane (after the French Mathematician Rene Descartes).

Recall that for every point on the number line, a real number is associated, and with every real number a unique point on the number line is associated. An immediate consequence of this is that for every ordered pair (x,y) of real numbers there corresponds a unique point P in the cartesian plane and to every point P in plane there is a unique ordered pair (x,y) associated with it. x and y are called coordinates of the point P.

**Relation from Set A to Set B**

**1.** Consider a list of names of girls and their brothers.

| A | B |
|---|---|
| **Name of the girl** | **Name of her brother** |
| Sushma | Krishna |
| Gowri | Raghu |
| Shanti | Amar |

Here, the set A { Sushma, Gowri, Shanti }

and set B = { Krishna, Raghu, Amar }

We can express the above information using ordered pairs as (Sushma, Krishna), (Gowri, Raghu), (Shanti, Amar).

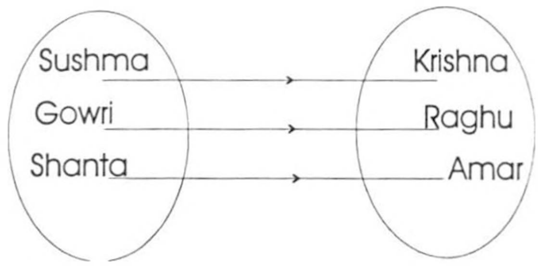
In each ordered pair first element is related to the second element. This is expressed as Sushma R Krishna, Gowri R Raghu, Shanti R Amar

Here R is a relation 'is the sister of '.This relation R from A to B is written as R : A → B and is R is a set of ordered pairs.

R = { {Sushma, Krishna) , (Gowri, Raghu), (Shanti, Amar) }.

R = { (a,b), : a $\in$ A, b $\in$ B and a R b }

Comparing set R with the set A x B. We see that R is a proper subset of A x B. i.e. R $\subset$ A x B.




2. Consider the names of men and then respective native places :

A = { Rama, Rahim, Govinda Srihari }

B = { Mysore, Gulbarga, Hassan, Melkote }

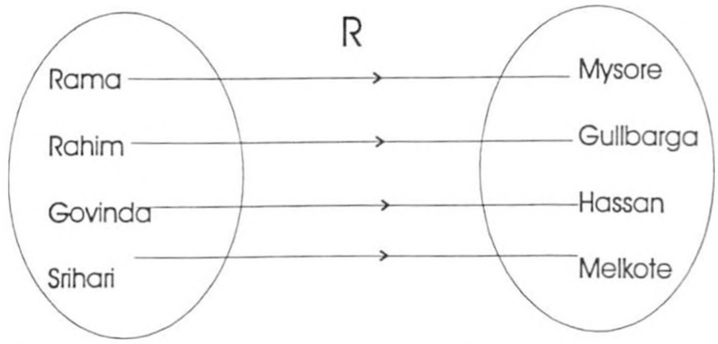
Here the relation R from A to B is given by "is the native of".

Related elements are expressed as ordered pairs and the relation R : A → B is the set of all these ordered pairs.

R = { (Rama, Mysore), (Rahim, Gulbarga), (Srihari, Melkote) }.

Note that R $\subset$ A x B. R = {(a,b) : a $\in$ A, b $\in$ B, a R b }.

**Arrow diagram of R**




Ex: 3 Let A = { 1,4,9,16,25 }. Apply the relation R "is the square of" to the elements of A. Then we get the set B = {1,2,3,4,5} where
1R1, 4R2, 9R3, 16R4 and 25R5.

R is a relation from A to B. (Write as $R : A \rightarrow B$).

$R = \{ (1,1), (4,2), (9,3), (16,4), (25,5) \}$

$= \{ (a,b) : a \in A, b \in B \text{ and } a = b^2 \}$

$R \subset A \times B$.

On the basis of the above examples, we have

1. A relation R from set A to set B is a subset of A x B.
2. R consists of ordered pairs (a,b) where $a \in A$ and $b \in B$ and a is related to b.

Any subset of A x B is called a relation from A to B.

Given a relation R from A to B $(a,b) \in R$ then we write aRb to mean a is related to b under R.

A is called the domain and B is called the Codomain of R.

Set of all $\{ b : (a,b) \in R \}$ is called the range of R.

**Arrow diagram of relation R from A to B.**

Venn diagrams of the sets A and B are first written. Elements of A are then connected to the corresponding or related elements of B by arrows directed towards B.

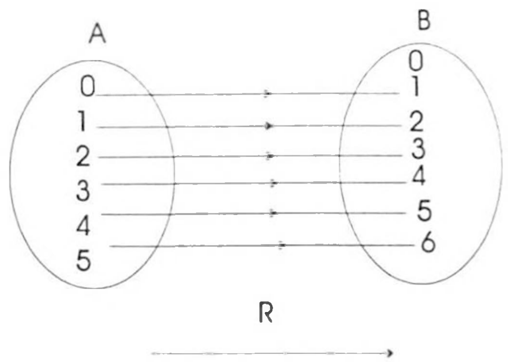
Example :

$A = \{ 0,1,2,3,4,5 \}$
$B = \{ 0,1,2,3,4,5,6 \}$

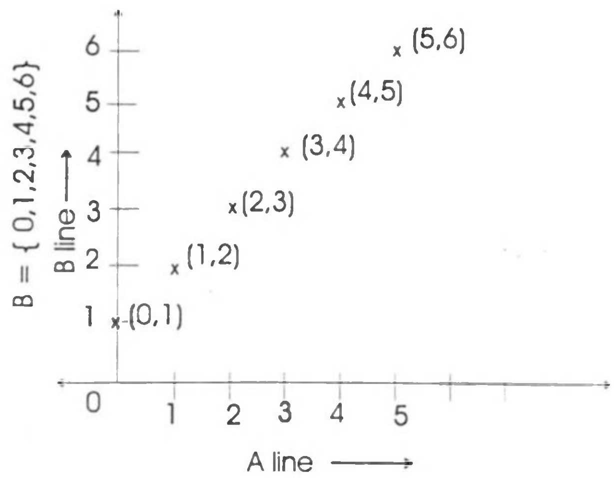
Let a R b be defined by $b = a + 1$.

Then $R = \{ (0,1), (1,2), (2,3), (3,4), (4,5), (5,6) \}$

The arrow diagram of R is




b) Graph of R : A → B (R from A to B). Graph of R consists of points (a,b) with $a \in A$ and $b \in B$ in the plane. Sets A and B are taken on the number lines along x and y axis respectively. Graph of the (above example) relation R defined by $b = a+1$ is

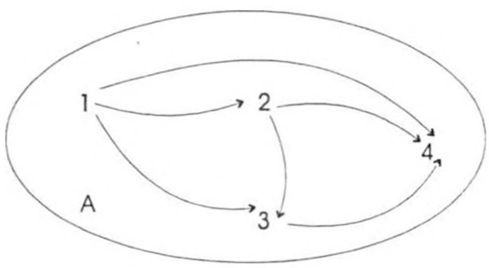



**Relation on a Set**

Given a non empty set A, a relation R from A to A itself (R : A → A) is called a relation on the set A. Clearly $R \subset A \times A$ for all elements $a, b \in A$ such that aRb.

For instance, Let A = { 1,2,3,4 } and R be given by $a < b$, $a . b \in A$.
R = { (1,2), (1,3), (1,4), (2,3), (2,4), (3,4) }

Here a R b means "a is less than b". Arrow diagram of R is (See fig)



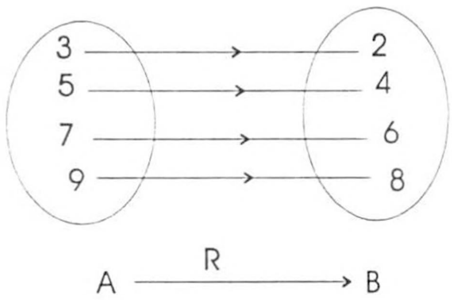
**Inverse of a relation :**

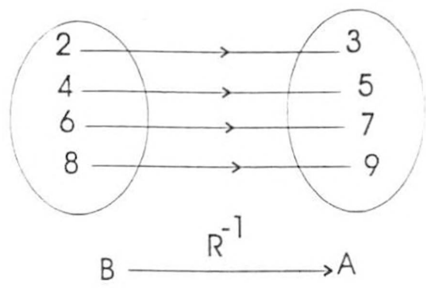
Given a relation R from A to B (written as $R : A \rightarrow B$ ). The inverse of R is denoted by $R^{-1}$ - the relation. $R^{1} : B \rightarrow A$ is such that $bRa^{-1}$ if and only if a Rb. Equivalently,

| $(b,a) \in R^{-1}$ if and only if $(a,b) \in R$, $a,b \in A$. |
|---|

**Examples :**

Let A = { 3,5,7,9 } , B = {2,4,6,8 }. Let us define a relation R by the set,
$R = \{ (3,2), (5,4), (7,6) \} \subset A \times B$, then $R^{-1} = \{ (2,3), (4,5), (6,7) \} \subset B \times A$

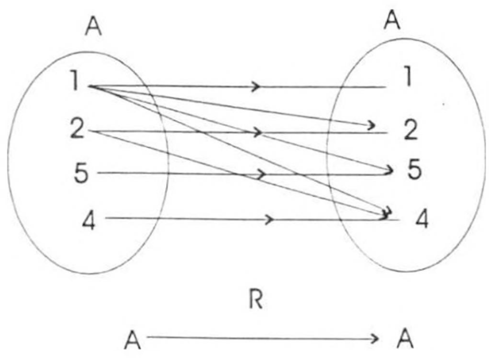


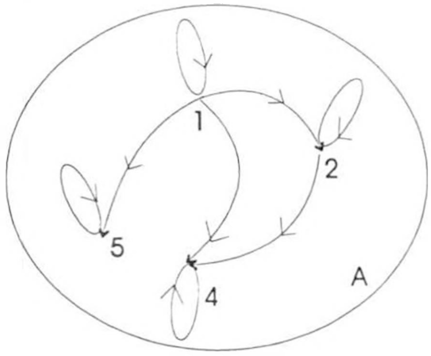

2. Let A = { 1,2,5,4,}

R is a relation on A defined by
aRb iff a' is a factor of b'. $a, b \in A$
i.e. R = { (a,b) : a is a factor of b }
R = { (1,1), (1,2), (1,5), (1,4), (2,4), (2,2) , (4,4), (5,5) }

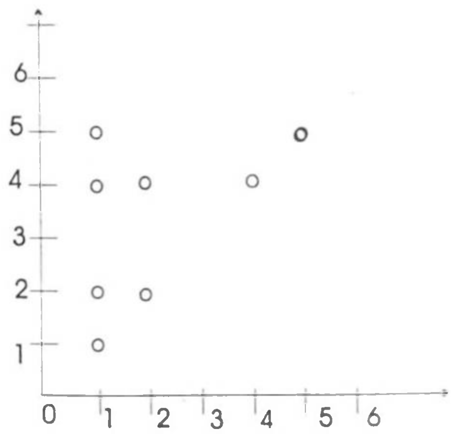
**Arrow diagram of R :**




GRAPH OF R

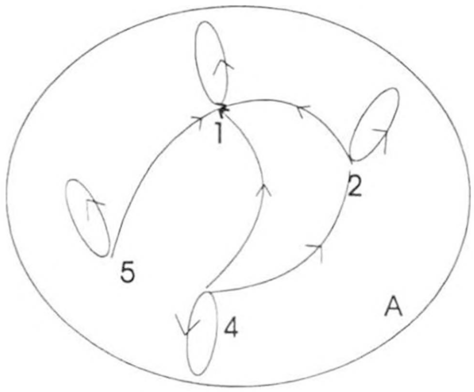



GRAPH OF R




The relation $R^{-1}$ = { (1,1), (2,1), (4,1), (5,1), (2,2), (4,4), (5,5), (4,2), (4,4), (5,5), (4,2)}

**Arrow diagram of $R^{-1}$.**

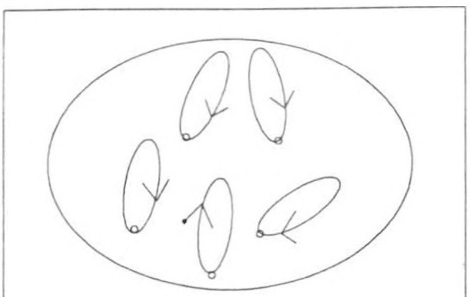



$R^{-1} : A \longrightarrow A$

**Reflexive Relation :**

Let R be a relation on a non empty set A. we know that R is a subset of A x A. R is said to be reflexive if for all a $\in$ A, (a,a), $\in$ R.

The arrow diagram of a reflexive relation has a loop at every point.



Reflexive - R

Examples :

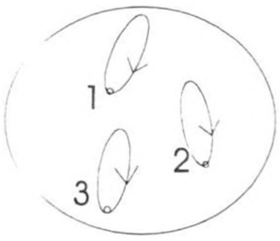
1. The relation of equality on a set of numbers.

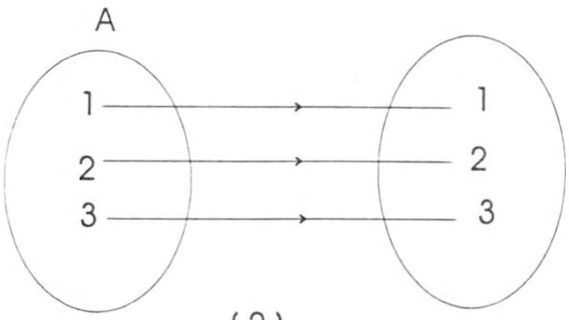
Let R = { 1,2,3 }

A

Define the relation R on A by aRb to mean "a is equal to b" $a, b \in A$.

Then $R = \{ (1,1), (2,2), (3,3) \}$ ; $R \subset A \times A$
R is represented by arrow diagrams 1 and 2.




(1) (2)

Here R is an identity relation.

2. Let $A = \{1,2,3\}$
Define the relation "greater than or equal to " ( $\geq$ on the set A. aRb means a is greater than or equal to b.

$$R = \{ (1,1), (2,2), (3,3), (2,1), (3,2), (3,1) \}$$

Here R is reflexive because for every $a \in A$, $(a,a) \in R$ is true. R contains as a subset $\{ (1,1), (2,2), (3,3) \}$ which is the identity relation on A. (Ref. Ex.1 )
R itself is not an identity relation.

$\therefore$ Every reflexive relation need not be an identity relation. But every identity relation is reflexive.
More examples :

1. Congruency of triangles is a reflexive relation as every triangle is congruent to itself.
2. 'Similarity' of triangles. Every triangle is similar to itself.
3. The relation "is concentric to" or "has the same centre as " on the set of all circles in a plane. Any circle C is "concentric to" itself.

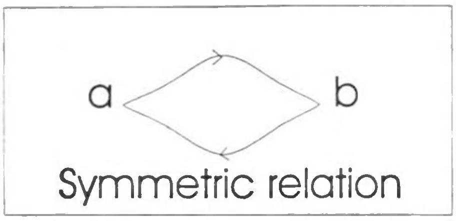
Non examples :

1. The relation 'less than' ( $<$ ) on a set of numbers is not reflexive.
2. The relation 'greater than' ($>$) on a set of numbers is not reflexive.
3. The relation 'is perpendicular to' on the set of lines in a plane is not reflexive.

4. The relation 'is a brother of' in the set of all human beings is not reflexive.

**Symmetric Relation** : R is said to be a symmetric relation if $(a,b) \in R$ implies $(b,a) \in R$, where $a,b \in A$.

Arrow diagram of a symmetric relation is

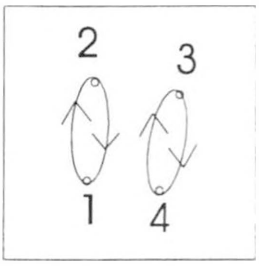



The above definition means that whenever a is related to b under R, b is also related to a.

**Examples :**

1. Let $A = \{ 1,2,3,4 \}$

and $R = \{ (1,2), (2,1), (3,4), (4,3) \}$




**Examples :**

1. Equality on a set A. If $a,b \in A$ such that $a = b$, then $b = a$ holds as well.
2. Perpendicularity on a set of straight lines on a plane. For any two lines $l_1$ and $l_2$ such that $l_1 \perp l_2$, we know that $l_2 \perp l_1$ also holds good.
3. Parallelism on a set of straight lines in a plane l and l' be any two parallel lines. i.e. $l_1 \mid\mid l'$ implies $l' \mid\mid l$ as well.
4. 'a has the same centre as b' on the set of all circles in a plane.

If the circle $C_1$ "has the same centre as circle $C_2$, then circle $C_2$" has the same centre as circle $C_1$ is true.

Non examples :

1. 'Divisibility' relation on a set of numbers is non symmetric.

    a divides b need not imply that 'b divides a'.

2. "Sister of" is not a symmetric relation because a is a' sister of b may not imply b is a sister of a'.

R is symmetric iff $R^{-1} = R$.

$R = \{ (a,b) : a \, R \, b, \; a,b \in A \}$
$R^{-1} = \{ (b,a) : a \, R \, b, \; a,b \in A \}$

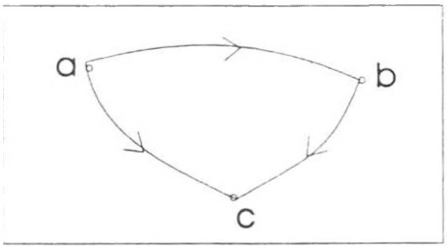
Since R is symmetric $(a,b) \in R \rightarrow (b,a) \in R$, $a,b \in A$. Hence $R = R^{-1}$.

**Transitive Relation :**

A relation R is said to be transitive.

If $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$.

In the arrow diagram for $a,b,c \in A$ whenever a is connected to b and b is connected to c, a is also connected to c for any $a,b,c \in A$.




**Examples of Transitive Relations :**

1. Equality relation
2. 'Greater than' (or less than) on a set of number
3. 'divisor of ' on a set of numbers
4. 'Parallelism on the set of straight lines in a plane
5. Subset of on sets.

Non examples :

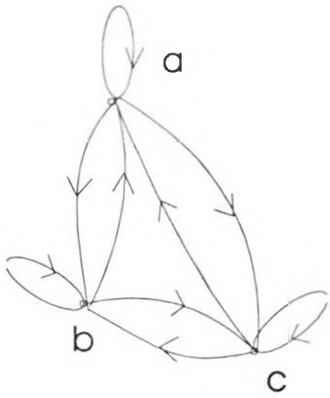
1. 'Perpendicularity' on the set of lines in a plane.
2. "Tangential to" on the set of circles in a plane.

Exercise : Prove that R is transitive iff $R^{-1}$ is transitive.

A relation R is called an equivalence relation on A if R is reflexive, symmetric and transitive. This means

i. For every $a \in A$ $(a,a) \in R$ or aRa.,
ii. If $(a,b) \in R$, then $(b,a) \in R$; $a.b \in A$.
iii. If $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$; $a,b,c \in A$.

The arrow diagram of an equivalence relation is




R = { (a,a), (b,b), (c,c), (a,b), (b,a), (a,c), (c,a), (b,c), (c,b) }

Equivalence Relation

Examples :

1. Equality relation
2. Congruency of triangles
3. Similarity of triangles
4. Parallelism of straight lines
5. The relation "has the same number of digits" on the set of natural numbers.

Non examples :
1. Perpendicularity on the set of lines in a plane
2. Subset relation on sets
3. 'less than' relation on a set of numbers

Worked Example :

Let A = { a,b,c,d,e,}

R = { (a,a), (b,b), (c,c), (d,d), (e,e), (a,b), (b,c), (a,c), (d,e), (e,d) }

Is R an equivalence relation A ?

Solution :

1. R is reflexive because
   aRa, bRb, cRc, dRd and eRe.

2. R is not a symmetric relation
   for, although (a,b), (a,c), (b,c) are elements of R

$(b,a) \notin R$, $(a,c) \notin R$ and $(c,b) \notin R$.

3. R is transitive .
   1. $(a,a), (a,c) \in R$ and $(a,c) \in R$
   2. $(b,b), (b,c) \in R$ and $(b,c) \in R$
   3. $(e,d), (d,d) \in R$ and $(e,d) \in R$.
   4. $(a,a), (a,b) \in R$ and $(a,b) \in R$
   5. $(a,b), (b,c) \in R$ and $(a,c) \in R$
   6. $(e,e) \in R$, $(e,d) \in R$ and $(e,d) \in R$.

$\therefore$ R is not an equivalence relation.

## FUNCTIONS

Let A and B be two non empty sets. Function or mapping from A to B, written as $f : A \rightarrow B$ is a subset of A x B (i.e. a relation from A to B) satisfying the following conditions.

1. For every $a \in A$, we can find $b \in B$ such that $(a,b) \in f$.
2. No two distinct ordered pairs in f have the same first entry (element./coordinates) i.e. if $(a,b) \in f$ and $b_1 \neq b_2$ then $(a, b_2) \notin f$ where $b_1$, $b_2 \in f$.

Thus for every $a \in A$, there is precisely one $b \in B$ called the image of a under f such that $(a,b) \in f$. The image of 'a' under f is denoted by f(a) so that
$(a,b) \in f \Leftrightarrow (a, f(a)) \in f$.

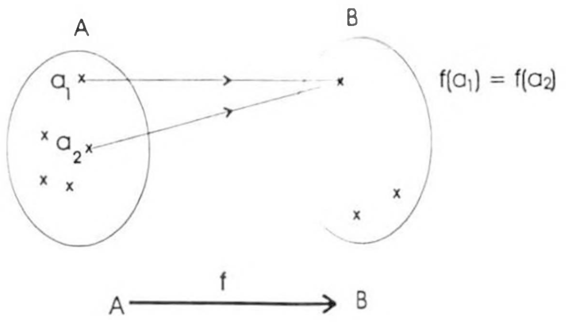
$f = \{ (a, f(a)) : a \in A\}$

f is defined on the set A and A is called the domain of f. The set of all images f(a) for all $a \in A$ is called the range of f. Range of f is denoted by f(A).

$f(A) = \{ f(a) : a \in A \}$

Note :

1. We denote functions by small letters f, g, h, etc.
2. Given a function $f : A \rightarrow B$, every element of the domain A of f should have an image in B. Every element of A is used only once to find its image under f.
3. Every element of B need not appear as an image under f. Hence the range of f may be B itself or a proper subset of B.
   i.e. $f(A) \subseteq B$.
4. Two or more elements of A may have the same image in B under f.
   i.e. $a_1, a_2 \in A$, $a_1 \neq a_2$ may exist such that $f(a_1)$ and $f(a_2)$ may be the same.




5. Every function is a relation, but every relation need not be function.

Examples :

1. The table given below gives the area A of a circle corresponding to the given radius r.

| Radius r in cms | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| Area A in sq.cm | $\pi$ | $4\pi$ | $9\pi$ | $16\pi$ | $25\pi$ |

This information gives rise to the set of ordered pairs.
$f = \{ (1,\pi), (2,4\pi), (3,9\pi), (4,16\pi), (5,25\pi) \}$

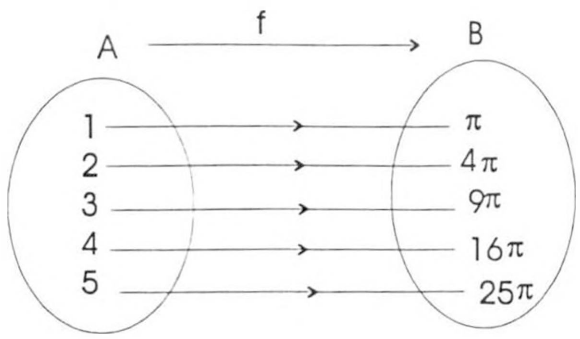
Here f is a function from A to B where domain of $A = \{ 1,2,3,4,5\}$ and range of $f = B = \{ \pi, 4\pi, 9\pi, 16\pi, 25\pi \}$ and we write $f : A \rightarrow B$.

$f(1) = \pi$, $f(2) = 4\pi$ and so on ....$f(5) = 25\pi$.

We explain this as 'area of a circle is a function of its radius'.
For a given radius r, the area A is uniquely determined.

$\therefore$ We write $A = f(r)$.

Arrow diagram of f is



2. The number of bacteria, under certain conditions increase with respect to time as shown in this table.

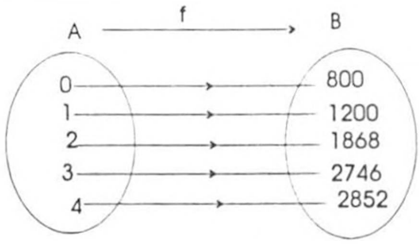
| Time T | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| No of bacteria P | 800 | 1200 | 1868 | 2746 | 2852 |

For every T there is a unique value of P. Hence P the number of bacteria is a function of time T. We write P = f(t).

f = { (0,800), (1,1200), (2,1868), (3,2746), (4,2852) }

Domain of f = {0,1,2,3,4 }
Range of f = { 800, 1200, 1868, 2746, 2852 }
f(0) = 800, f(1) = 1200 and so on; f(4) = 2852

Arrow diagram of f is




Example 3 : Marks scored by the students of A,B,C,D, E,F,G,H in a test out of 10 marks are listed below.

| Students S | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| Marks M | 5 | 8 | 2 | 6 | 5 | 8 | 2 | 7 |

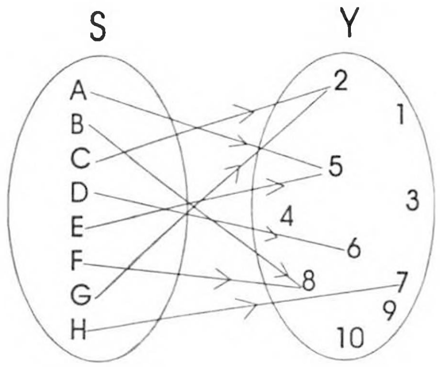
f = { (A,5), (B,8), (C,2), (B,6), (E,5), (F,8), (G,2), (H,7) }
S = the domain of f = {A,B,C,D,E,F,G,H }
M = the range of f (set of all images under f) = { 5,8,2,6,7 }

Here f is a mapping from S to Y.

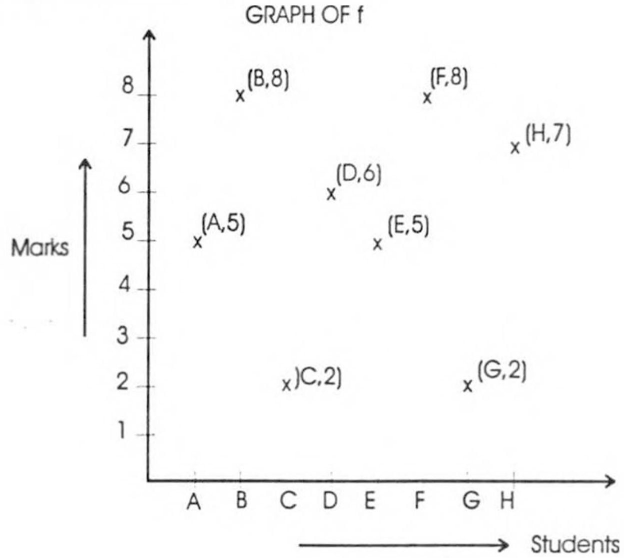
where Y = { 1,2,3,4,5,6,7,8,9,10 } = codomain of f
(assuming that marks are given only in whole numbers).

Note that the range M = f(S) is a proper subset of the codomain Y of the function f.

Arrow diagram of f : S → Y is




$1,3,4,9,10 \in Y$ are not images under f.




Example 4: $f = \{ (1,2), (2,3), (3,4), (4,5) \}$

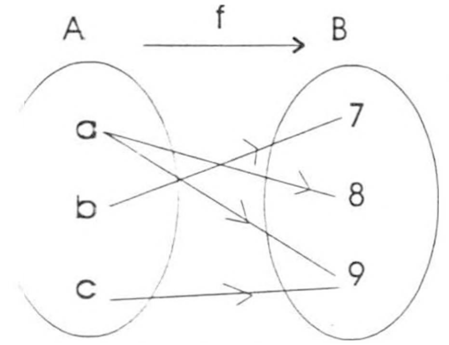
$f(1) = 2$, $f(2) = 3$, $f(3) = 4$, $f(4) = 5$
Domain of $f = A = \{1,2,3,4\}$
Range of $f = B = \{2,3,4,5\}$
Here f is defined by the rule/formula $f(a) = a+1$, $a \in R$.

$\therefore\ f = \{ (a, a + 1) : a \in A \}$

We can also describe f as $f : A \rightarrow B$ defined by $f(a) = a + 1$. $a \in A$.

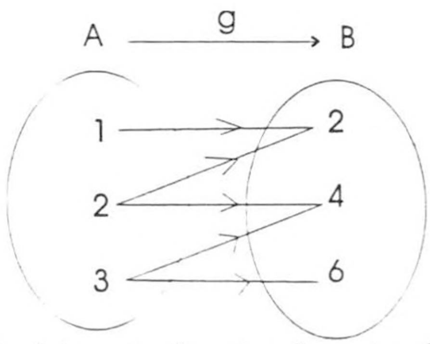
Non-Examples :

1. From the arrow diagram f = { (a, 8), (a, 9), (b,7), (c,9) }.

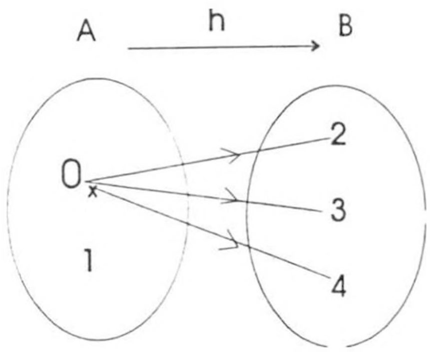



Ordered pairs (a,8) and (a,9) ∈ f. Both have same first entry namely 'a'. This violates the requirement for f to be a function. Hence f is only a relation from A to B and not a function. If two or more arrows proceed from the same element ∈ A in the arrow diagram, then the diagram does not represent a function.

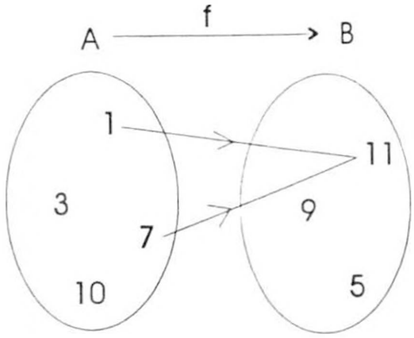
2. g is not a function from A to B, because two arrows are attached to the elements 2 and 3 in the domain A.




3. h is not a function from A to B. Observe three arrows attached to the element O of the domain A.




4. f is not a function since 3 and 10 in the domain of f have no images in the codomain of f.




More Examples :

It is possible to represent/define a function in the form of a rule or a formula.

1. Let f be a function defined on N by $f(x) = 2x$. We write it as $f : N \rightarrow N$ defined by $f(x) = 2x$.

OR

$f : N \rightarrow N : f(x) = 2x.$

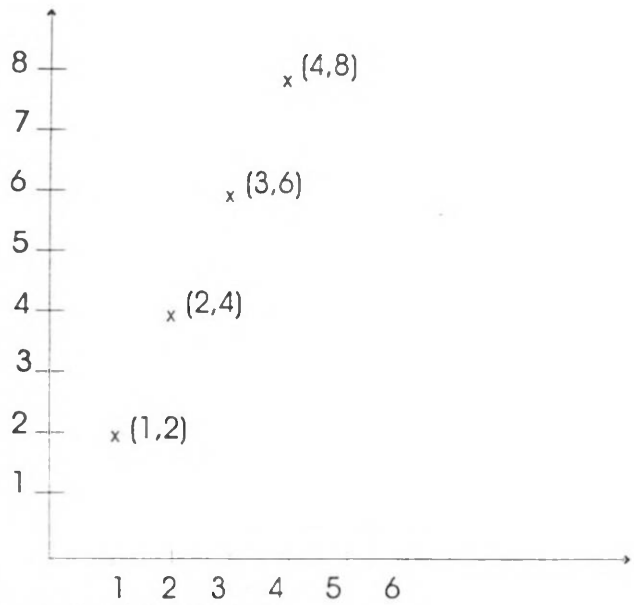
We say f maps or carries every natural number x to a natural number 2x. The mapping is done using the rule $f(x) = 2x$.

Hence we can say a function is a rule which assigns to every element of its domain, a unique element is its codomain.
(Range of f = (even natural numbers) is a proper subset of the codomain N) in the above example.

$f = \{ (x, 2x) : x \in N\}$

Note that the set f is given by rule form because listing all ordered pairs of f is not possible.

Graph of f is an infinite set of points that lie on the line $y = 2x$.

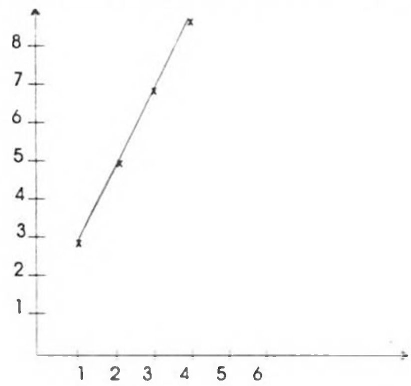



2. Let f : N → N defined by $f(x) = 2x + 1$.
3. f as a set of ordered pairs is given by

$$f = \{ (x.\ 2x + 1) : x \in N \} \text{ or } f = \{ (x,y) : y = 2x + 1, x \in N \}$$

f is an infinite set of ordered pairs. Under the mapping f. 1 → 3 (1 goes to 3 or 1 is carried to 3 or 1 is mapped on to 2), 2 → 4, 3 → 6, etc. which means for every $x \in N$ its image f(x) is given by the rule $f(x) = 2x + 1$.

Graph of f is an infinite set of points that lie on the line $y = 2x + 1$.




3. Let f map each natural number to its square.

f : N → N defined by $f(x) = x^2$.
And $f = \{ (x, x^2) : x \in N \}$
or $f = \{ (x,y) : y = x^2 \text{ and } x \in N \}$

Draw the graph of f.
Here Range of f $\subset$ codomain N of f.

4. Let $f(x) = x^3 + 1$, $x \in A$ where A = { 1,2,3 }, then f = { (1,2), (2,9), (3,28) }
5. $f(x) = 1 - x$, $x \in A$

where A = { ½, 1/3, ¼, 1/5 }
f = { (1/2, ½), (1/3, 2/3), (1/4, ¾), (1/5, 4/5)

Range of f = { ½, 2/3, ¾, 4/5 }

Worked Problems (Exercise on Page 34).

If A = { -1,3, 5} and f : A → R where R is the set of reals, find the range of f where $f(x) = x^2 + 2$.

Range of f is the set of all images under f, namely
{ f(-1), f(3), f(5) } = { 1, 11, 27 }.
f (-1) = 1, f(3) = 11, f(5) = 27.

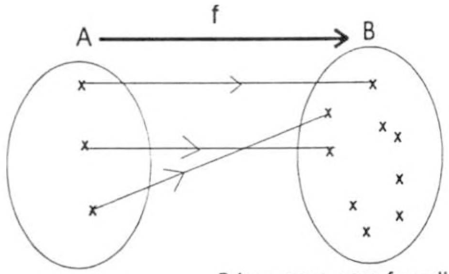
**Types of Functions :**

1. One-one Function :

Let f : A → B be a function. f is called a one-one function if no element of B is an image of more than one element of A, under f. One-one function maps distinct elements of A to distinct elements in B. Hence, for a one-one function f : A → B.

$f(a) = f(b) \Rightarrow a = b$, a. b $\in$ A.

Equivalently, $a \neq b \Rightarrow f(a) \neq f(b)$, a . b $\in$ A. The arrow diagram of the one-one function f : A → B is



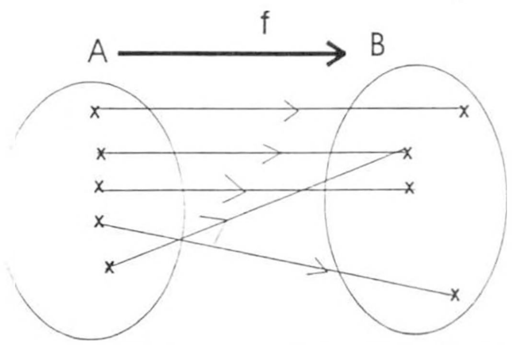
f: A → B is a one-one function

2. Onto function :

Let $f: A \rightarrow B$ be a function from A into B such that every element of B is an image under f. Then the codomain of f and the range of $|f|$ are both equal to B. i.e. $f(A) = B$. (Recall that in general, range of f is not necessarily equal to the codomain).

We call such a function an onto function. $f: A \rightarrow B$ is an onto function, then $\forall\ b \in B$, we can find an $a \in A$ such that $f(a) = b$.

Arrow diagram of an onto function is




f: A ⟶ B is an onto function

Examples of one-one functions :

1. $f: N \rightarrow N$ defined by $f(x) = x$ is a one one function, since $f(x) = f(y) \rightarrow x = y$ $x, y \in N$.
2. $f: N \rightarrow N$ defined by $f(x) = 3x$ is a one-one function.
   Since $f(x) = f(y)$ here means $3x = 3y \rightarrow x = y$, f is one-one.

3. Register numbers of students of a certain class taking the same paper in an examination is a one-one function, since for different students, their register numbers are different.

4. Let $f(x) = x + 1,\ x \in N$,
   then, $f: N \rightarrow N$. f is a one-one function because, $x, y \in N$, $f(x) = x + 1$.
   $f(x) = f(y)$ means $x + 1 = y + 1 \rightarrow x = y$.

5. Non Examples :

1. Heights of students of a certain class is not a one-one function because more than one student may have the same height.
2. Body temperature of a patient taken every hour in a day is not one-one function of the time at which temperature is recorded.

Because same body temperature may be recorded at different times of the day.

Examples of Onto Function :

1. The function $f(x) = x$ defined on N is an onto function.
2. Days of the week matched with the corresponding names of the week days is an onto function.

If A = set of all days of the week and
B = set of all names of the week days
Then, $f : A \rightarrow B$ defined by f (day of the week) = name of the week day.

For example : f ($2^{nd}$ day of the week) = Monday.

Non examples :

1. Let $f(x) = a^2 \ \forall \ a \in N$.

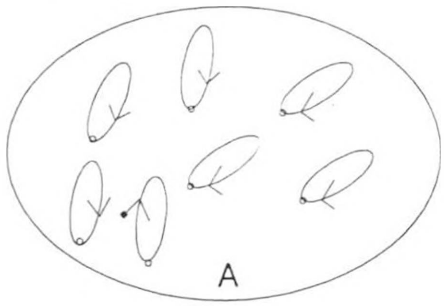
Here $f : N \rightarrow N$ and $f = \{ (a, a^2) : a \in N \}$
f is not an onto function since every natural number need not be a square of a natural number; for ex. $2 \in N$ but 2 is not a square of any natural number. Note that f is one one.

2. $f : z \rightarrow z$ defined by $f(a) = 2a$, $a \in Z$ is not an onto function (Z is the set of integers). Since there are elements like 5, 7 , -11, -25 etc. in Z which are not images under f.

Two important functions :

1. Let f be a function on A defined by $f(x) = x$ for all $x \in A$.

Here every element in A is an image of itself under f. We call such a function f an identity functionon A.
Identity function is denoted by I.
$\therefore$ $I : A \rightarrow A$ such that $I(x) = x$
$I = \{ (x,x) : x \in A \}$

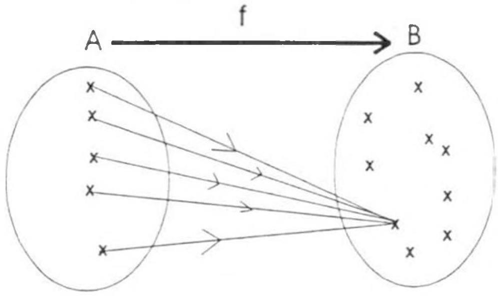



$I : A \longrightarrow A$

As we have already seen, Identity function is a one-one function as well as onto function.

2. f : A → B be afunction such that $f(x) = k$ for every $x \in A$. $K \in B$. Here all elements of A the same image namely, $K \in B$.

Such a function is called a constant function.




Constant function f is not an one-one function. f need not be an onto function unless the codomain is a singleton { K } .

More Examples :

1. S be the set of all even integers,
   then, f : z → S defined by $f(a) = 2a$ is one-one as well as onto.
2. The identity map I is one one and onto.
3. $f(x) = x^2 - x + 1 \ \forall x \in N$ is a 1-1 map. But not onto (why ?)
4. Let A = { 1,2,3 }, B = { 4,5,6 }

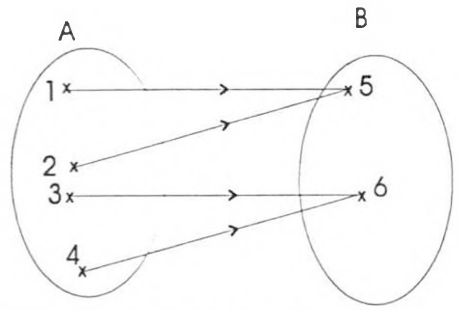
Write all one-one functions from A to B.

Solutions :

i. { (1,4), (2,5), (3,6) }
ii. { (1,4), (2,6), (3,5) }
iii. { (1,5), (2,4), (3,6) }
iv. { (1,5), (2,6), (3,4) }
v. { (1,6), (2,4), (3,5) }
vi. { (1,6), (2,5), (3,4) }

5. Let A = { 1,2,3,4 }, B = { 5,6,7 }

Let f : A → B be as given in the arrow diagram.




f = { (1,5), (2,5), (3,6), (4,6) }

Presence of 5 in two ordered pairs (two arrows to 5) and presence of 6 in two ordered pairs show that f is not one-one.

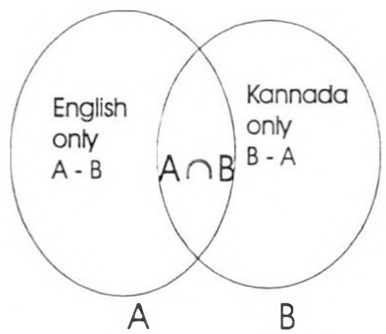
**SOLVED PROBLEMS :**

Exercise on Page 18 (TextBook)

1. Number of Champak Garlands = n(A) = 110.
2. No. of jasmine garlands = n(B) = 50.
   No. of garlands of both kinds of flowers = n (A ∩ B ) = 30.

N (A U B) = n(A) + n(B) - n(A ∩ B).
= 110 + 50 - 30
= 130 ∴ Total number of garlands is 130.

3. No. of students in the class be 100 = n (A U B).
No. of students contributing to NDF = n(A) = 60.
No. of students contributing to NSS = n(B) = 70.
No. of students contributing to both the schemes = n (A ∩ B) = ?
n(A ∩ B) = n(A) + n(B) - n(AUB).




= 60 + 70 - 100.
= 30

Therefore, 30 out of 100 (or 30%) students contribute to both the schemes.

5. No. of people speaking English and Kannada = $n(A \cap B) = 130$.
No. of people who speak English only = $n(A - B) = 160$.
No. of people who speak Kannada only = $n(B - A) = ?$
$n(A - B) + n(A \cap B) + n(B - A) = 420$.
$\therefore$ $n(B - A) = 600 - \ - n(A - B) - n(A \cap B)$.
$\therefore$ $= 420 - 160 - 130$.
$\therefore$ $= 130$
$\therefore$ 130 people speak only English.

1. $A = \{ 0,3 \}$, $B = \{ 2,3,4 \}$, $C = \{ 1,4,5 \}$
i. $A \times B \cap C = ?$
$B \cap C = \{ 4 \}$ and $A \times B \cap C = \{ (0,4), (3,4) \}$
ii. $B' \times A = ?$
$B' = U - B = \{ 0,1,2,3,4,5 \} - \{ 2,3,4 \}$
$B' = \{ 0,1,5 \}$
$B' \times A = \{ (0,1), (0,3), (1,0), (1,3), (5,0), (5,3) \}$
Answer given in the textbook is incorrect.

iii. $(A - B) \times (B \cap C) = \{ (0,4) \}$.

$B \cap C = \{ 4 \}$
$A - B = \{ 0 \}$

Answer is given as $\{ 0,4 \}$ which is not correct. It is to be written as $\{ (0,4) \}$.

4. Solve for x and y if $(x-1, y+2) = (y-2, 2x+1)$. Since the given ordered pairs are equal, their corresponding elements are equal.
i.e. $x - 1 = y - 2 \rightarrow x - y = -1$ (1)
and $y + 2 = 2x + 1 \rightarrow 2x - y = 1$ (2)
Solving the simultaneous equations 1 and 2
We get $x = 2$, $y = 3$.
Note: a) Read question No. 6 on page as follows.
Is the relation "is parallel to" defined on the set of all lines in a plane, an equivalence relation ? (Assume that a line is parallel to itself).
b) Read question No.7 on page as follows. Is the relation "is perpendicular to" defined on the set of lines in a plane an equivalence relation.

To say that "the set of parallel lines" or a "set of perpendicular lines" is an equivalent relation" is incorrect.

# Exercise

1. Fill in the blanks :
i. If (x,y) = (2,5), then x = ------; y = ----------.
ii. If (p,5) = (5,q), then p = ------; q = --------.
iii. If (x + y, 1) = (5, x - y) , then x = ------; y = --------.
iv. If (y, x - 2) = (x,0) then x = -------; y = ----------

2. Let P = {Anitha, Sunil}; Q = {Geeta }
find i)P x Q, ii) Q x P
3. Let A = {2,3}, B = {1,4,5 }, find i) A x B, ii) B x A (using tree diagram)
4. Find A x B and B x A from question 6 using arrow diagrams.
5. Let A = {1,2,3,4}, B = {5,6,7 } Illustrate graphically i) A x B, ii) B x A.
Is A x B = B x A ?
6. How many elements are there in
i. {1 } x { 2 } -----------
ii. {1,2} x {3,4} -----------
iii. {1,2} x {3} -----------
iv. {1,2,3} x {4,5} ------------
Hence, if n(A) = p; n(B) = q then n(A x B) = ---------

7. For A = {r,s,t} and B = {u,v,w}, is A x B = B x A ?
8. Find {1,2,3 } x { }.

The cross product set contains ordered pairs as elements. Since there are no elements in the second set, it is not possible to get any ordered pair. Hence, the cross product of any set A and the empty set will be empty. Thus,
$A \times \phi = \phi$.

9. Let A = {1,2,3,4} ; B = {2,4} ; C = {1,3}
Find i) B U C, ii) A x B, iii) A x C, iv) A x (B U C), v) (A x B) U (A x C).

10. From the sets given in Question 9, find
i. A - B, ii) A x C iii) B x C
ii. (A - B) x C, v) (A x C) - (B x C)

11. If A = {1,2}; B = {4,5}; C = {1,2,3}; D = {3,4,5 }, find i) A x B, ii) C x D

Is it true that if i) $A \subset C$ and $B \subset D$ then ii) $A \times B \subset C \times D$ ?

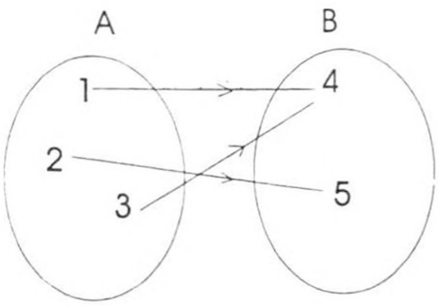
12. Let A = { 1,2,3,4,5 }; B = { 1,2,3,4,5 }. Illustrate graphically

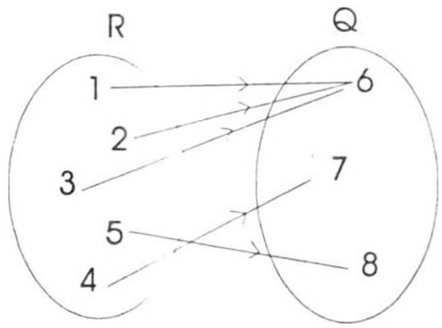
i. $A \times B$
ii. $R = \{ (x,y) / (x,y) \in A \times B ;\ x = y \}$
iii. $G = \{ (x,y) / (x,y) \in A \times B ;\ x < y \}$
iv. $S = \{ (x,y) / (x,y) \in A \times B;\ x > y \}$

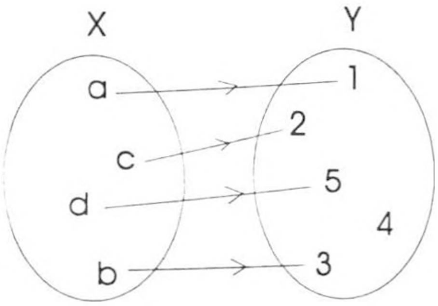
13. Fill in the blanks :
i. A relation is a set of ---------.
ii. In the ordered pair, (x,y), x is called the -------- and y is called the ---------.
iii. If $R \subseteq A \times B$, then R is called ------- from -------- to ---------.
iv. If R is a relation from set A to B, then R is a subset of --------
v. If $(x,y) \in R$, then --------- $\in R^{-1}$.

14. If A = { 1,2,3,4}; B = {5,6,7} which of the following are relations from A to B ?
i. { (1,5), (3,7)}
ii. { (7,3), (5,1) }
iii. { (1,5), (2,7), (4,6) }
iv. { (7,2), (6,4), (1,7) }
v. { (6,1), (1,7), (8,8) }

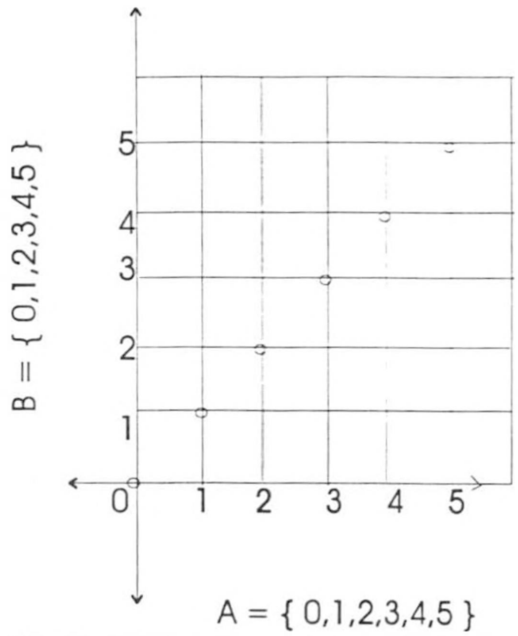
15. Below are given arrow diagrams representing relations. Represent each of the relations in the roster form :

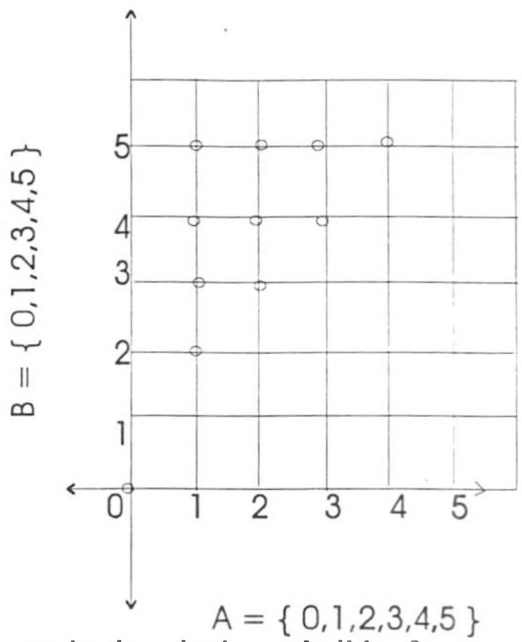


16. Below are given graphs of relations. Write them in the roster form :




17. Below are given relations in the roster form; write them in the set builder form :

i. { (1,1), (2,2), (3,3) ,.... }
ii. { (1,2), (2,3), (3,4) ,... }
iii. { (1,1), (2,4), (3,9), (4,16) ,... }
iv. { (1,2), (2,4), (3,6), ... }
v. { (1,5), (2,10), (3,15), ... }

18. Below are given relations in the set builder form. Represent them as (a) arrow diagram, b) graphs c) roster form.

i. { (x,y) / x = 2y} where $x \in$ {2,3,4,5}; y = { 1,2,3,4 }
ii. { (x,y)/y = x + 3 } where x, y are natural numbers less than 6.
iii. { (x,y) / $y = x^2$ where x,y $\in$N and less than 25.

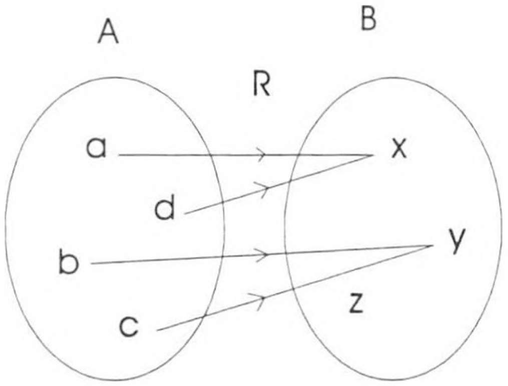
19. Given A = { 3,6,9,12,15} and B = { 4,8,12,16, 20 } write down the relations from A to B each consisting of 4 ordered pairs defined by x + y < 25.
20. Let A = {1,2,3} and B = {1,2,3,4} write the following relations in the roster form :
i. r is a relation satisfying the rule x = y.
ii. s is a relation satisfying the rule x < y
iii. t is a relation satisfying the rule x > y
iv. p is a relation satisfying the rule y = x + 1
v. l is a relation satisfying the rule x = y + 2

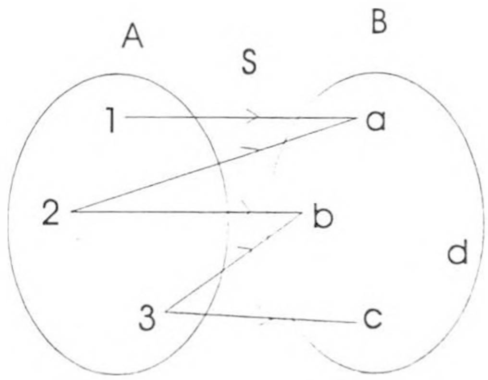
21. If R = {(x,y) / (x,y) $\in$ A x B ; x + y = 10 } where A = { 1,2,3,4} and B = {3,4,5,6}. Write R in the roster form.

22. Write the domain and the range of each of the following relations :

i. $r = \{ (-3,1), (-1,1), (1,0), (3,0) \}$
ii. $s = \{ (1,1), (2,2), (3,3), (4,4), (5,5) \}$
iii. $t = \{ (1,2), (2,3), (3,4), (4,5), (5,6) \}$
iv. $k = \{ (a,x), (b,y), (c,z), (d,p) \}$
v. $m = \{ (0,0), (1,1), (2,2), \ldots \}$

23. Write the domain and the range of the following relations :



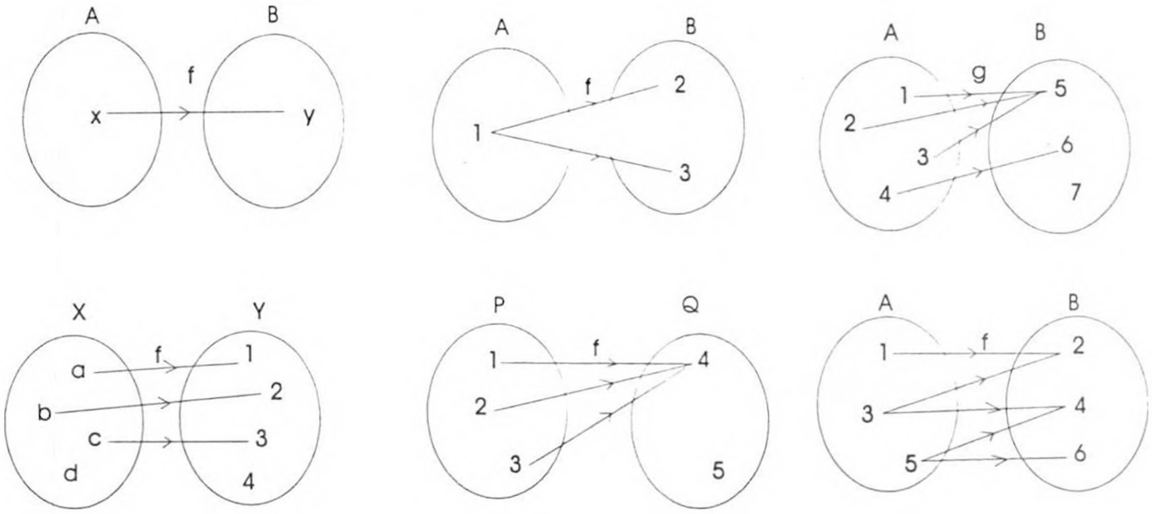

24. Write the domain and the range of the relations given below :

i. $r = \{ (x,y)/x$ is a multiple of 3 and $y$ is a multiple of 5 $\}$
ii. $s = \{ (x,y) / x,y \in N, a = 2b$ and $a,b < 8 \}$
iii. $t = \{ (a,b)/a,b \in N, a = 2b$ and $a,b < 8 \}$
iv. $p = \{(a,b) / a \in N$ and $b = 3 \}$
v. $q = \{ (a,b) / a = b;$ a, b natural numbers less than 5 $\}$

25. Find the inverse relation of the following :

i. $\{ (1,2), (2,3), (3,4) \}$
ii. $\{ (4,5), (5,4), (1,2), (2,1) \}$
iii. $\{ (1,5), (2,10), (3,15), (4,20) \}$
iv. $\{ (x,y)/ (x,y) \in N \times N ;\ x + y = 10 \}$
v. $\{ (x,y) / (x,y) \in N \times N ;\ y = x \}$

26. State whether the following relations are functions are not :




27. Find whether the following relations are functions or not :

i. $A = \{ (1,2), (2,3), (3,4) \}$
ii. $B= \{ (1,2), (1,4), (2,3) \}$
iii. $C = \{ (5,0), (6,0), (7,0) \}$
iv. $D = \{ (a,x), (b,y), (c,z) \}$
v. $E = \{ (1,1), (2,2), (3,3) \}$

28. Which of the following relations are functions ?

i. f = { (x,y) / x = 2;';;  y = 1,2,3 }
ii. $g = \{ (x,y) / x \in N,\ y \in N ;\ x = y \}$
iii. $h = \{ (x,y) / x \in N, y \in N, y = x^2 \}$
iv. $l = \{ (x,y) / x \in N, y \in N, y = 2x + 1 \}$
v. $k = \{ (x,y)/y = \sqrt{x} ;\ x \in N \}$

29. Write the following in roster form :

i. $f(x) = x;\ x \in \{1,2,3,4\}$
ii. $f(x) = 3x + 1;\ x \in \{0,1,2\}$
iii. $f(x) = x^2;\ x \in \{ (0,1,2,3,4,\ldots25)\}$
iv. $f(x) = 2x^2 - 1;\ x \in \{0,1,-1,-2,2 \}$
v. $f(x) = 1 - x;\ x \in \{1/2, 1/3, 1/4\}$
vi. $f(x) = 1/x, x \in \{1,2,3,4 \}$
vii. $f(x) = 1/x; x \in \{1, 1/2, 1/3, 1/4 \}$
viii. $f(x) = x + 1/x;\ x \in \{1,2,3,4 \}$

# PRACTICAL GEOMETRY

Practical geometry is very important in that it brings to fore the importance and application of geometry (and so mathematics in general) in daily life. It also exemplifies the role of deductive reasoning in solving practical problems. So, practical geometry forms a very important part of any mathematics curriculum in school.

Here we solve some construction problems relating to triangles and quadrilaterals which are not given in the text book.

**Problem 1** : Construct a parallelogram when the two diagonals and the angle between them is given.

**Solution :**
We have to construct the parallelogram ABCD, given the diagonals AC, BD and $\angle BOC = \alpha$.

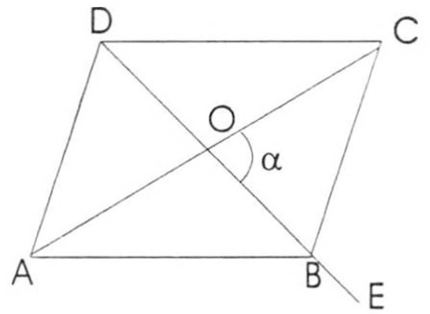



Fig.1

To construct the parallelogram ABCD, we use the property that a quadrilateral is a parallelogram if and only if its diagonals bisect each other.

**Construction :**

Draw AC. Bisect AC at O such that AO = OC. Draw $\angle EOC = \alpha$ such that OE = given length BD, $\angle EOC = \alpha$. Bisect OE at B such that OB = BE. Extend BO towards O so that OD = BO. Join AB, BC, CD and DA. Now ABCD is the required parallelogram.

**Proof**: In triangles OAB and OCD,

OA = OC, OB = OD and $\angle AOB = \angle COD$.

So, $\Delta$s OAB and OCD are congruent by SAS postulate.
So, $\angle CAB = \angle ACD$
So, AB || CD.

Similarly, it can be shown that AD | | BC.
So, ABCD is a parallelogram.

**Problem 2** Construct a parallelogram ABCD where the side AB = a, diagonal AC = b and $\angle BAD = \alpha$.

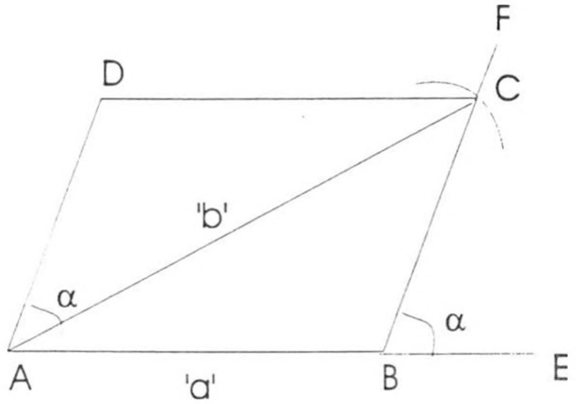



Fig.2

**Construction** : Draw AB = a. Extend AB to E to construct $\angle FBE = \alpha$. Use compass to find a point on BF so that AC = b. Draw by using compass lines CD equal to AB and AD equals to BC so that AD and CD meet at D. Fig. ABCD is the required parallelogram.

**Proof** In fig. ABCD, AB = CD, and BC = AD. So $\Delta$s ABC and ACD are congruent by SSS postulate. So ABCD is a parallelogram satisfying AB = a, AC = b and $\angle DAB = \angle CBE = \alpha$.

**Problem 3** Construct a parallelogram ABCD with the side AB = a and diagonals AC = e and BD = f.

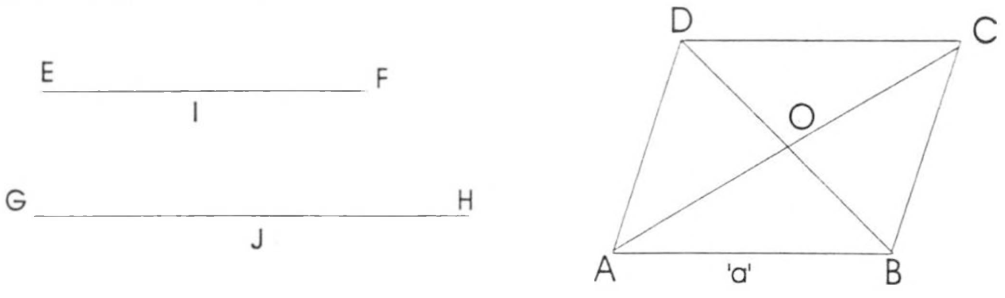



Fig.3

**Construction** Draw two line segments EF and GH equal to given lengths e and f respectively. Bisect EF and GH at I and J respectively. Now draw $\Delta$ABO with AB = a, AO = EI and BO = GJ. Extend AO and CO towards O so that OD = BO and AO = OC. Join BC, CD and DA to form the required parallelogram ABCD.

**Proof** Now by construction $\Delta$s AOB and COD are congruent due to SSS postulate; and as in problem 3, ABCD is a parallelogram.

Here $AC = 2AO = 2.EI = 2 . e/2 = e$
and $BD = 2BO = 2.GJ = 2 . f/2 = f$

**Problem 4 Construct a rhombus, given one side and one of its diagonals.**

Here we have to construct a rhombus ABCD, given AB = a, and diagonal AC = e.

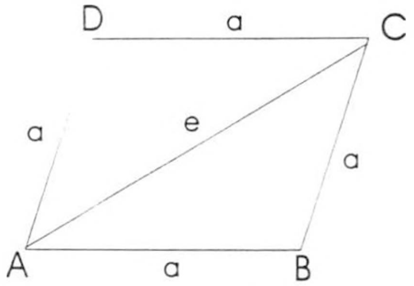



Fig. 4

**Construction** Draw AB = a. From A and B draw arcs with radii e and a respectively to intersect at C. Then from A and C draw arcs each of radius a to intersect at D. Join BC, CD and DA to complete the rhombus ABCD.

**Proof** By construction, AB = BC= CD = DA = a and the diagonal AC = e. So ABCD is the required rhombus.

**Problem 5** Construct a rhombus, given one side and one angle.

Here we have to construct a rhombus ABCD with AB = a and $\angle DAB = \alpha$.

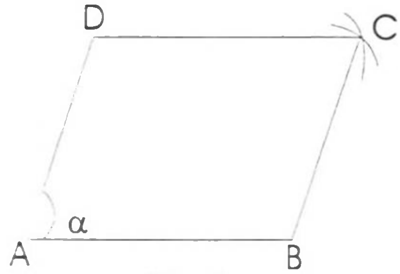



Fig. 5

**Construction** Draw $\angle DAB = \alpha$ with AB and AD = a. Draw arcs each of radius a from B and D to intersect at C. Join BC and DC to complete the rhombus ABCD.
**Proof** From the construction it is clear that AB = BC = CD = DA and $\angle DAB = \alpha$. So, ABCD is the required rhombus.

**Problem 6 Construct a right angled triangle ABC with $\angle B = 90^\circ$, $\angle A = \alpha$ and in-radius = r.**

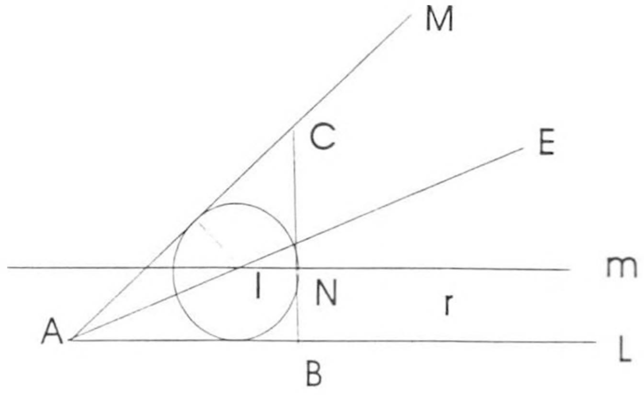



Fig.6

**Construction** Draw a straight line AL so that $\angle LAM = \alpha$. Draw the straight line m $\parallel$ AL so that the distance between m and AL is r. Draw AE, the bisector of the angle $\angle LAM$ to intersect the line m at I. Draw a circle of radius r with centre at I. This circle touches AL and AM and intersects the line m at N. Draw a straight line through N $\perp$ to m intersecting AL and AM at B and C respectively. $\Delta ABC$ is the required triangle.

**Proof** As AE is the bisector of $\angle BAC$, I, the in-centre of $\Delta ABC$ lies on AE. Since the in-radius = r, I is on the line m which is $\parallel$ AB and at a distance r from AB. So, I, the point of intersection of lines AE and m is the incentre. As IN $\perp$ BC and IN = r, IN being a radius of the in-circle, N lies on the side BC of $\Delta ABC$. BC $\perp$ m, m $\parallel$ AB imply BC $\perp$ AB. So $\angle ABC = 90^\circ$. Hence $\Delta ABC$ is the required triangle.

**Problem 7 Construct a square, given the sum of its side and a diagonal.**

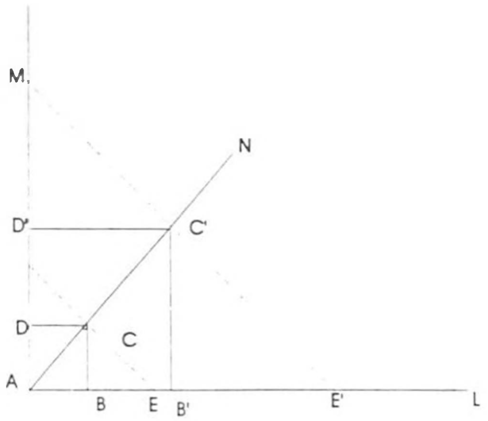



Fig.7

Let the given sum be d.

**Construction** Construct a square ABCD. Extend AB and AD to L and M. Draw BE = AC and AE' = d, where E and E' are on the line AL. Extend AC to N and join CE. Draw E′ C′ || CE so that E′C′ intersect AN at C′. Now complete the square AB′C′D′ which is the required square.

**Hints for proof** Use similarity concept or the theorem on ratio of the intercepts of the transversals intersecting three or more parallel lines to arrive at

$$\frac{AE'}{AE} = \frac{AC'}{AC} = \frac{AB'}{AB}$$

Now here for the square ABCD, AB, AC, AE are respectively the side, a diagonal and the sum of a side and a diagonal. So for the square AB′ C′D′, AB′, AC′, AE′ are respectively the side, a diagonal, and the sum d of a side and a diagonal. So AB′C′D′ is the required square.

**Additional Exercises for Students**

1. Construct a parallelogram ABCD with AC = 10 cm, BD = 6 cm and the angle between them being 30 °.

2. Construct a parallelogram ABCD, where the side AB = 7 cm, the diagonal AC = 10 cm, and ∠BAD = 45 °.

3. Construct a parallelogram ABCD with the side AB = 5 cm and diagonals AC = 7 cm, BD = 9 cm.

4. Construct a rhombus with one side 5 cm and one of its angles being 30°.

5. Construct a rhombus, given its side 5 cm long and one diagonal 8 cm long.

6. Construct a rhombus, given one side 6 cm long and one of its angle 60°.

7. Construct a right angled triangle ABC with ∠A = 75° and in-radius = 4 cm.

8. Construct a square, given the sum of its side and a diagonal as 9 cm.

9. Construct a square if the length of the difference of a diagonal and a side is given. Give the method of construction.
   (Hint: Proceed on the line of Problem 8 solved in this chapter).

10. Construct a square if the length of the difference of a diagonal and a side is 2 cm.

11. Construct an isosceles triangle, if its one acute angle and circum-radius are given.

12. Construct an isosceles right triangle, given its circum-radius.

13. Construct an equilateral triangle, if its in-centre is given.

14. Construct a right triangle, if its one side and in-centre are given.

15. Construct an isosceles trapezium given its bases and a diagonal.

16. Construct an isosceles triangle, if its base and the altitude to one of the congruent sides are given.

17. Is it possible to construct a parallelogram with one side 10 cm and diagonals 6 cms and 8 cms ? Why ?

**Teaching Methodology**

The teaching of practical geometry (i.e. construction of figures satisfying certain geometrical conditions) has a twofold purpose at the school level: 1. Teaching the students how to apply the theorems of geometry in practical situations, 2. Training the students in drawing correct and accurate figures satisfying given conditions. Whereas the first objective involves cognitive exercises, the second objective involves skill of drawing figures using the geometrical instrument box. It is clear that not all students in an average mathematics class will be ready for the first objective - only the better ones of the lot will be able to understand the proof of the method of construction or solve a general constructional problem by evolving a method and a proof for the same because of the abstract nature of the first objective. But the skill mentioned in the second objective is a must for all mathematics students. These considerations lead us to the following strategy - whereas a mathematics teacher may attempt to achieve the first objective only for the good students, he must attempt that all students are able to draw neat, precise and correct figures. This is because mathematics is a core subject in the high school curriculum.

Keeping the above in view, the following teaching strategies may be arrived at for teaching practical geometry in classes IX and X. To start with, a construction problem (e.g. to find the method of construction) may be posed for the class as a challenge or an assignment. In all probability only a few students (the good ones) will be able to solve it or make some headway towards solution. The teacher need not be depressed or feel bad about it. He should identify the students attempting or answering the problem as exceptional or better students of the class. He may treat them differently by giving them more theoretical and challenging problems in geometrical construction. By this, the good students will find more interest in mathematics and develop a challenging spirit towards solving problems.

In the classroom itself, the teacher may give the method of construction for a problem (after preliminary discussion with good students). All the students including the good ones may be directed to draw the figure neatly, correctly and precisely. They should be instructed about the essentials of skill of drawing geometrical figures. 1. The figures should be neat and convincing, 2. Straight lines and curves (e.g. circles) drawn should be as thin as possible and also very neat and clear. For this purpose, they should use pencils with sharp and pointed ends. 3. Students may be trained in the safe upkeep of the instrument box.

**Books consulted**

1. Moise Edwin E and Downs Floyd L. Jr. - Geometry (Addison Wesley, 1964).

# SURFACE AND VOLUME OF SOLIDS

The advancement of mathematics arose out of necessity. For example, the annual inundation of the Nile River Valley, forced the Egyptians to develop some system of measurement of the land. The word 'geometry' means 'measurement of earth'. The need for mensuration formulas was necessary as the taxes in Egypt was paid on the basis of land area.

The Babylonians were famous for their skill in the construction of engineering structures. Hence, the Babylonians needed mensuration and geometry for this purpose.

The Indians, at the Vedic times needed geometry to measure distances, areas and volumes to construct the altars (Yajna Bhumikas) of particular sizes and shapes using special types of bricks for religious purposes. They developed mensuration and had written the formulas in the scriptures called Baudhayana "Sulbha Sutras", which has been composed around 500-300 BC. Those who are interested more in the history of mensuration (geometry) may see the following books :

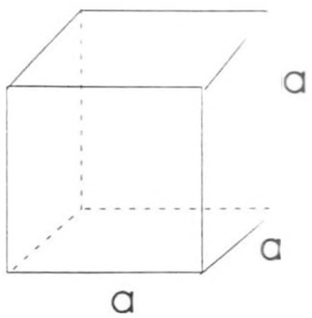
1. Eves and Newson - An introduction to the foundations and Fundamental Concepts of Mathematics. Holt, Rinehart and Winston, London.
2. Batta B.B. and Singh A N (1962) - History of Hindu Mathematics, Asia Publishing House, New Delhi.
3. Scott J.F. - A history of Mathematics, Taylor and Francis, London.
4. Balachandra Rao S - Indian Mathematics and Astronomy, Jnanadeep Publication - Bangalore.

## SOLIDS

In our daily life, we come across different kinds of solids. Some have familiar and regular shapes and others have irregular shapes. We have already learnt how to find the surface area and volume of cube and cuboid. We briefly mention the formula just for completeness of this material.

**Cube** : length = breadth = height = a

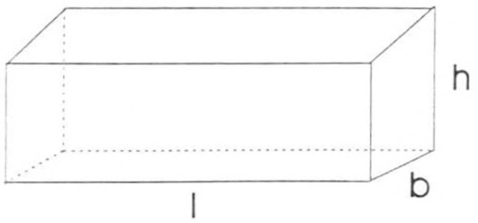
Lateral surface area = $4a^2$
Total surface area = $6a^2$
Volume of the cube = $a^3$.



**Cuboid :**

Length = l, breadth = b, height = h.

Lateral surface area = perimeter of the base x height = 2 ( l + b) h
Total surface area = 2 (l + b) h + 2 lb = 2 (lb + lh + bh)
Volume of the cuboid = length x breadth x height = lbh.




**Polyhedra :**

A polyhedra is a solid whose surfaces (faces) are polygons. The cube is also a polyhedra because each face of the cube is a square (polygon). The cuboid is also a polyhedra because its faces are rectangles (polygons).

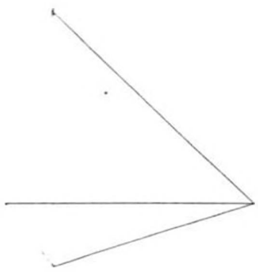
A polyhedra in which all the faces (polygons) are congruent and all the angles at the vertices are equal is called a regular polyhedra. The cube is a regular polyhedra, while the cuboid is not a regular polyhedra.

Although the Greeks had studied the polyhedra even during the early days, it was Descarte (1640) and Euler (1752) who studied all the properties of the polyhedra systematically. There is a famous result

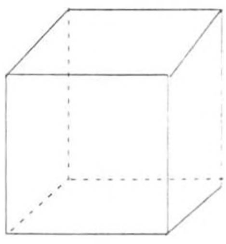
$$V + F = E + 2$$

Vertices + Faces = Edges + 2 known as Euler's formula, which is satisfied by all the polyhedra. Those who wish to read more about polyhedra may see the book : "What is Mathematics" by Courant and Robbins, Oxford University Press.

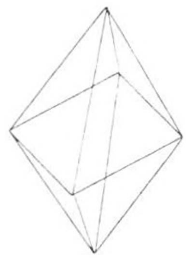
We give the figures of the five regular polyhedra for the information of the readers.



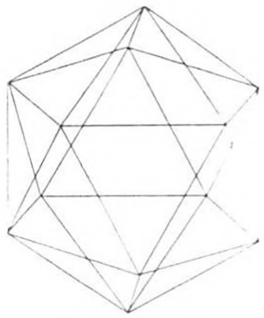
**Tetrahedron(4)**



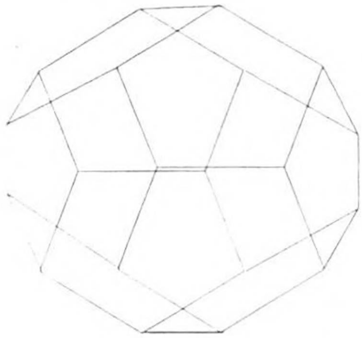
**Hexahedron(6)**



**Octahedron(8)**

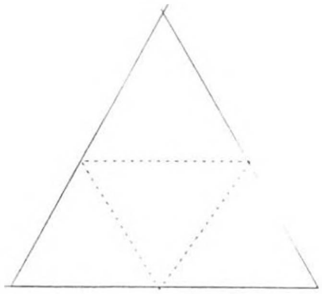


**Icosahedron(20)**

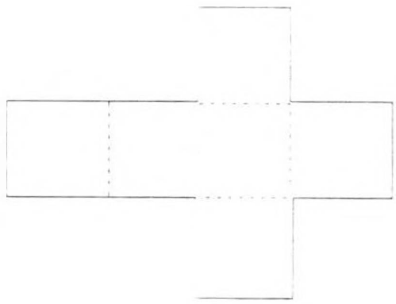


**Dodecahedron(12)**

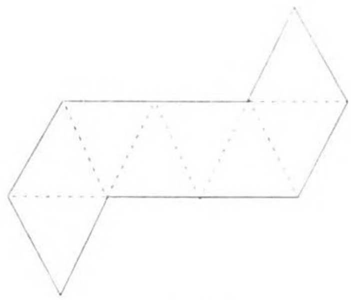
The teacher can ask the students to cut the paper in the following shapes and fold them at the dotted lines to get above five regular polyhedra.



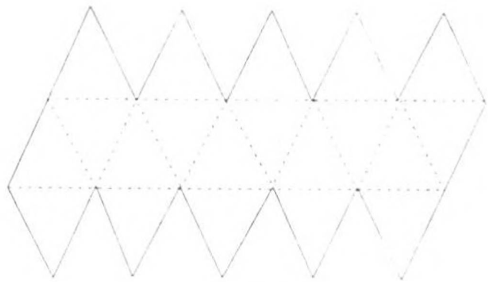
(4)



(6)

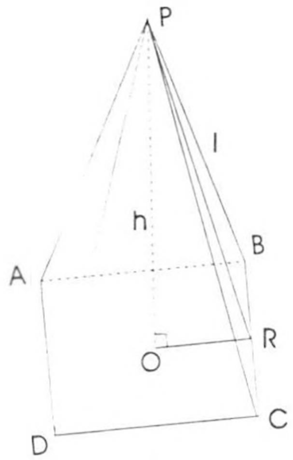


(8)



(20)

Relation between the height and slant height : From the point P, if we drop a perpendicular PO on the base of the pyramid, then PO is the height h of the pyramid. (See Fig. (v).



(v)

L = slant height = PR.
h = height of the pyramid = PO
OR = a/2 if a is the length of the base of the square pyramid.

Then using Pythagorean theorem, we get

$PR^2 = PO^2 + OR^2$

$l^2 = h^2 + (a/2)^2$ (1)

From the above relation (1), if any two of the quantities l, h and a are known, the third one can be calculated. (Please note that PB is not the slant height. It is called lateral edge).

**Example 1 :**

A square pyramid is 12 cm high. The length of its base edge is 10 cm. What is its height ?

$l^2 = h^2 + (a/2)^2$

$= (12)^2 + (10/2)^2$
$= 144 + 25$
$= 169$
Slant height = l = 13 cm

**Example 2 :** The perimeter of the base of a square pyramid is 72 cm. Its slant height is 41 cm. What is its height ?

Here, $4a = 72 \quad \therefore \quad a = 18$

$h^2 = l^2 - (a/2)^2$

$= (41)^2 - (9)^2$

height = h = 40 cms.

**Example 3 :** If the slant height of a square pyramid is l cms and the base edge is a cms, find the length of its lateral edge.

See Fig. (v) PB = lateral edge = e cm
PR = slant height = l cm
BR = ½ (base edge) = a/2 cms

From the right angle triangle PBR,

$PB^2 = PR^2 + PB^2$

$\therefore e^2 = l^2 + (a/2)^2$

$$e = \sqrt{l^2 + (a/2)^2} \qquad \ldots\ldots \quad (2)$$

From expression No (2), we can find the value of e if l and a are known.

**Example 4 :** The lateral edge of a square pyramid is 20 cm and its base edge 24 cm. Find the slant height of the pyramid.

$$l = \sqrt{[e^2 - (a/2)^2]}$$

$$= \sqrt{[(20)^2 - (12)^2]}$$

= 16 cm

l = slant height = 16 cms.

The lateral surface area = sum of the areas of all the lateral surfaces.
Total surface area = sum of the areas of all the lateral surfaces + Area of the base.

The examples of lateral surface area and total surface area are worked out in the textbook and therefore we shall not discuss them here.

**Volume of Prism :**

Volume of the Cuboid is known to the students. They know that
V = base x height
The teacher can cut an object which is in the shape of cuboid, (soap cake or wooden piece etc,) through the diagonal and show that two right prisms will be needed to form a cuboid. Therefore, volume of the right prism is half of the volume of the cuboid.

$\therefore$ V = The volume of the right prism
= ½ (Twice the base area of the triangle) x height
= Base area of the Triangle x height
= Bh.

**Volume of a Pyramid :**

Take six pyramids of equal square bases and equal heights. Arrange the six pyramids in such a way that we get a cuboid whose base is equal to the base of the pyramid and height of the cuboid is equal to double the height of the pyramid.

It should be noted that the height of the cuboid will be double the height of the pyramid.

$\therefore$ Volume of each pyramid = 1/6 Volume of the cuboid

= 1/6 (Base area x 2 height)
= 1/3 (Base area x height)

V = 1/3 Bh where B = the base area and h = height of the pyramid.

The above formula can also be verified by doing an experiment. Take two card board boxes one in the shape of a square prism (cuboid) and the other in the shape of a square pyramid.

They should have the same height and the same base area. Fill the box that has the shape of a pyramid with sand and empty it into the other box which is in the shape of cuboid. Repeat this experiment until the large box is just full. We see that the larger box will be full when we do this exercise three times.

$\therefore$ Volume of the pyramid = 1/3 volume of the cuboid
= 1/3 base x height

Note that the formula

V = 1/3 B h is true for pyramids with other bases also.

**Example 1 :** Length of the base edge of a square pyramid is 16 cm. Its height is 15 cms. Find its volume.

V = 1/3 B h
= 1/3 (16 x 16) x 15
= 1280 $cm^3$

**Example 2 :** Base perimeter of a square pyramid is 96 cm. Its slant height is 20 cm. Find the volume.

Note that to find the volume of the pyramid, we require the base area and height.

Given $4a = 96$. $\therefore$ $a = 24$
Base area = $24 \times 24$ $cm^2$

$h = \sqrt{[l^2 - (a/2)^2]}$, where l is the slant height.
$= \sqrt{[20^2 - 12^2]}$
$= \sqrt{256}$
$=16$

$\therefore$ $V = 1/3 \times 24 \times 24 \times 16$
$= 3072$ $cm^3$.

**Example 3 :**

The lateral surface area of a square pyramid is 2940 $cm^2$ and the length of its base edge is 42 cm. Find the volume.

Given $a = 42$.

$\therefore$ Base area = $42 \times 42$.

Lateral surface area = 4 x Area of each triangle = $4 \times ½ \times 42 \times l$

$2940 = 4 \times ½ \times 42 \times l$, l is the slant height.

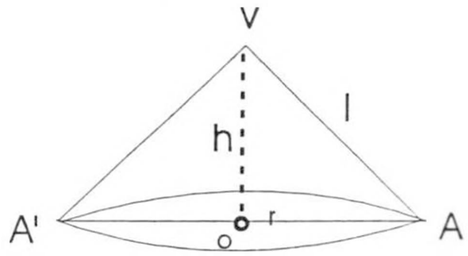
$\therefore$ $l = 2940/84$
$= 35$.

$\therefore$ $h = \sqrt{[35^2 - 21^2]}$, h is the height of the pyramid.
$= 28$

$\therefore$ Volume of the pyramid = $1/3 \times 42 \times 42 \times 28$
$= 16464$ $cm^3$

**CONE**

The base of a triangular pyramid is a triangle, the base of a square pyramid is a square; the base of a hexagonal pyramid is a hexagon. If the base of a pyramid is a circle, then we call such a pyramid as a circular pyramid or a cone. A pile of sand, certain caps, tents and the top of some towers are examples of cones.



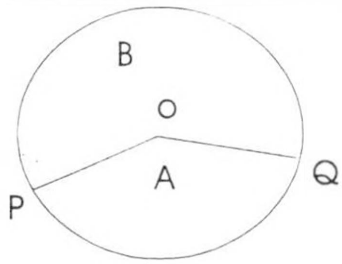

In this figure, V is the vertex of the cone, h is the height, l is the slant height and r = OA is the radius of the circular base of the cone.
By Pythagorus Theorem,

$l^2 = h^2 + r^2$

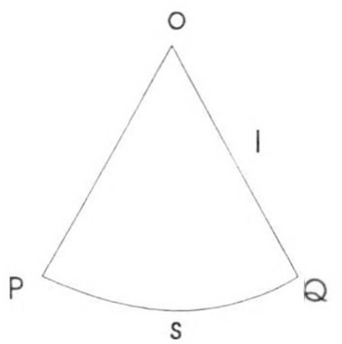
Hence if any two of l, h and r are known, the third one can be calculated.

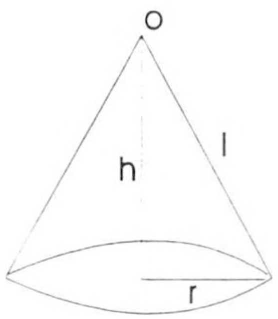
**Curved Surface Area of the Cone :**

The teacher can cut a circle into two sectors.




POQ (smaller) and POQ (bigger sector) as shown in the figure. By folding these sectors, one can show that the surface area of the cone is equal to the area of the corresponding sector of the circle.

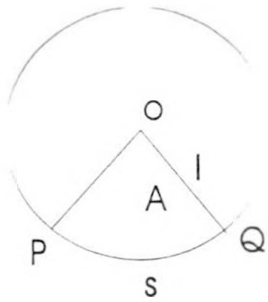



If s is the length of the arc PQ, then we known that

Length of the arc/circumference of circle = area of sector/ area of circle

$$\frac{s}{2\pi l} = \frac{A}{\pi l^2}$$




$$\therefore \frac{s}{2\pi l} \times \pi l^2 = A$$

$= Sl/2$

In this case $s = 2\pi r$

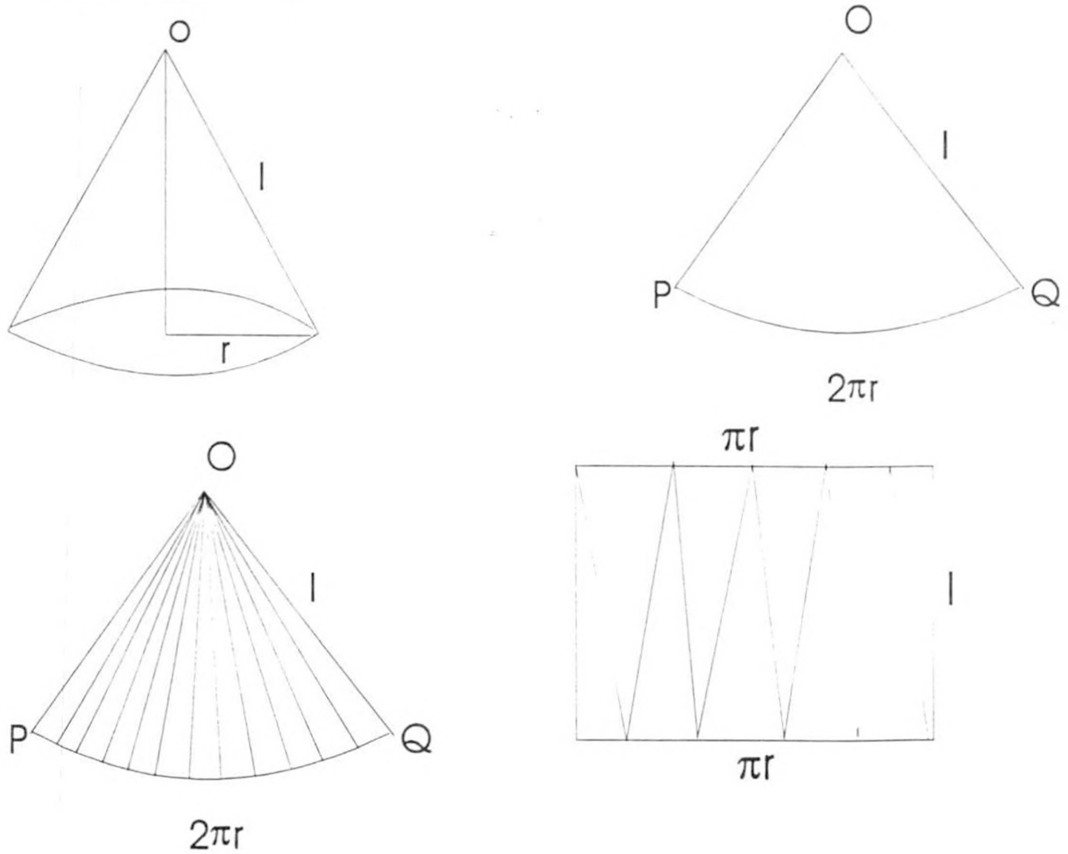
$$\therefore A = \frac{2\pi r l}{2}$$

$= \pi rl$

$\therefore$ Surface Area of the Cone = $\pi rl$

where l is the slant height of the cone and r is the radius of the circular base.

**Alternate Method :** The curved surface area of the cone = $\pi rl$ can also be derived by the following method.




Take a cone of radius r and slant height l. Cut the cone open. It will form a sector OPQ of the circle of radius l and arc length $2\pi r$ as shown.

Cut the sector OPQ into smaller sectors as shown in the figure and arrange them to form a rectangle of length $\pi r$ and breadth l.

$\therefore$ Curved surface area of the cone of radius r, slant height l

= Area of the sector of the circle of radius l and arc length $2\pi r$

= Area of the rectangle of length $\pi r$ and breadth l.

= $\pi rl$

**Volume of the Cone :**

We have seen that the volume of the pyramid = 1/3 x base area x height

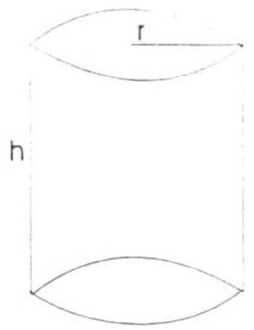
$\therefore$ Volume of the cone = 1/3 x Base area x height

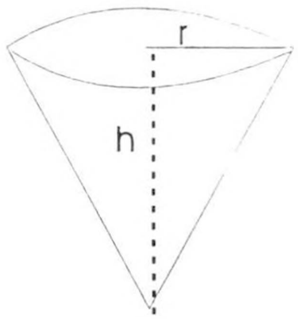
$\therefore$ = $1/3 \times \pi r^2 \times h$

= $1/3\, \pi r^2 h$

where r is the radius of the circular base, and h is the height of the cone.

It is possible to verify the above result experimentally as in the case of a square pyramid. Using cardboard, improvise a cylindrical vessel and a conical vessel, having the same height and the same base radius as shown.




Fill the conical vessel with sand and empty it into the cylindrical vessel. How many times have you to do the above process to fill the cylindrical vessel ? Three times, This verifies that

Volume of the cone = 1/3 volume of the cylinder

= 1/3 x Base Area x height

= $1/3\, \pi\, r^2\, h$

**Example 1 :** The perimeter of the base of a cone is $42\,\pi$ cm. If its height is 28 cm, find its volume and total surface area of the cone.

Given h = 28, $2\pi r = 42\pi$

$\therefore$ $2r = 42$
$\therefore$ $r = 21$ cms

Volume = $1/3\ \pi\ r^2$ x h
= $1/3$ x $\pi$ x 21 x 21 x 28
= 12924.24 $cm^3$

To find the surface area, we require l, the lateral height.
Use the formula :

$l^2 = h^2 + r^2$
$= 28^2 + 21^2$

$\therefore$ $l = 35$ cms.

Total Surface Area
= Surface Area of the curved surface + area of the base
$= \pi rl + \pi r^2$

= ($\pi$ x 21 x 35) + $\pi$ x 21 x 21
= 1176 $\pi$
= 3692.64 $cm^2$.

**Example 2 :**

The volume of the cone is 2592 $\pi$ $cm^3$ and its height is 24 cm. What is the total surface area ?

Given : h = 24, V = 2592 $\pi$
$V = 1/3\ \pi\ r^2 h = 2592\ \pi$

$\therefore$ $1/3\ \pi\ r^2$ x 24 = 2592 $\pi$
$\therefore$ $r^2 = 324$
$\therefore$ $r = 18$ cms.

l = lateral height = $\sqrt{h^2 + r^2}$
$= \sqrt{24^2 + 18^2}$
= 30 cms.

Total surface area :
$= \pi\ r^2 + \pi\ rl$
$= \pi r\ (r + l) = \pi$ x 18 (30 + 18) = 2712.96 $cm^2$

**Example 3 :** A heap of wheat is in the form of a cone of diameter 9m and height 3.5m. Find its volume. How much canvass cloth is required to just cover the heap ( use $\pi$ = 3.14).

Given : r = 4.5 m, h = 3.5 m
Volume : $1/3\ \pi r^2 \times h$
$= 1/3 \times 3.14 \times (4.5)^2 \times (3.5)$
$= 74.18\ m^3$

Convass required = surface area of the cone

$= \pi\, rl$ $\qquad l = \sqrt{h^2 + r^2}$

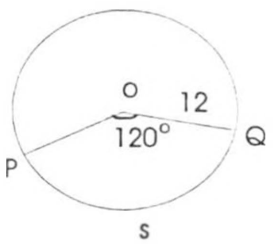
$= 3.14 \times 4.5 \times \sqrt{32.5}$ $\qquad = \sqrt{3.5^2 + 4.5^2}$

$= 80.54\ m^2$ $\qquad = \sqrt{32.5}$

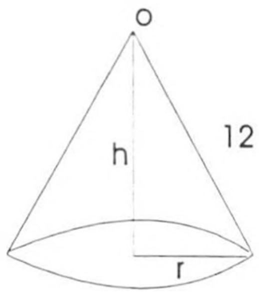
**Example 4 :** A sector of a circle of radius 12 cm has the angle 120°. It is rolled up so that two bounding radii are joined together to form a cone. Find the volume of the cone (Take $\pi$ = 22/7).

$$\frac{Arc\ lengths}{\pi .12} = \frac{120^o}{180^o}$$

$\therefore$ s = $2/3 . \pi . 12$
$= 8. \pi$




When rolled, to form the cone, s = the circumferernce of the base (circle).
$\therefore\ 2\pi r = s = 8\pi$
$\therefore\ r = 4$ cm




$l^2 = h^2 + r^2$
$\therefore\ h^2 = l^2 - r^2$
$h = \sqrt{l^2 - r^2}$
$= \sqrt{12^2 - 4^2}$
$= 8\sqrt{2}$ cms.

$\therefore$ Volume of the cone = 1/3 x Base Area x h

$$= \frac{1}{3} \times \frac{22}{7} \times 4 \times 4 \times 8\sqrt{2}$$
$= 189.5\ cm^3$

**SPHERE**

We are familiar with the objects which are spherical in shape. The Tennis ball and the fully blown volley ball are two examples of spheres.

A sphere is the set of all points in space which are at an equal distance from a fixed point. The fixed point is called the centre of the sphere. The fixed distance is called the radius of the sphere.

If we consider a sphere together with all points inside, then it will be called a solid sphere.

Assume that a plane intersects a sphere. Then the intersection will be a circle. Such a circle is called a cross section of the sphere. Any cross section containing the centre of the sphere is called a great circle of the sphere and its radii will be equal to the radius of the sphere.

If the sphere is cut into two equal halves, each half is called hemisphere. The following formulas may be stated without proof.

A sphere of radius r has
Volume = $4/3\ \pi\ r^3$ Surface Area = $4\pi r^2$

There is no easy proof for the above result. The students will come across the proof when they study the integration at the pre-university level. There are other physical methods by which we verify the result. One of them is given in the textbook.

**Example 1 :** The diameter of a spherical shell is 6 cm. Find its volume and surface area ( $\pi = 3.14$).

Given : $d = 6$, $\therefore r = 3$ cms.

$V = 4/3\ \pi\ r^3$

$= 4/3 \times \pi \times 3 \times 3 \times 3$
$= 113.04\ cm^3$
$A = 4\ \pi\ r^2$
$= 4 \times \pi \times 3 \times 3$
$= 36\ \pi$
$= 113.04\ cm^2$

**Example 2 :** The surface area of the sphere is $900\ \pi\ cm^2$. Find its volume.

$A = 900\ \pi\ = 4\ \pi\ r^2$
$= 4 \text{ x } \pi \text{ x } r^2$
$\therefore\ r^2 = 225$
$r = 15$ cm

$V = 4/3\ \pi\ r^3$
$= 4/3 \text{ x } \pi \text{ x } 15 \text{ x } 15 \text{ x } 15$
$= 14130\ cm^3$

**Hemisphere :**

A plane through the centre of the sphere divides the sphere into two equal parts. Each is called a hemisphere. If r is the radius of the hemisphere, then,

$V = 2/3\ \pi\ r^3$

Curved surface area (CSA) $= 2\pi r^2$

Total surface area (TSA) $= 2\pi r^2 +\ \pi r^2$
$= 3\pi r^2$.

**Example 1:** The inner diameter of a hemispherical bowl is 30 cm. Find its capacity in litres.

Given : d = 30
$\therefore$ r = 15
$V = 2/3\ \pi\ r^3$
$= 2/3 \text{ x } 3.14 \text{ x } 15 \text{ x } 15 \text{ x } 15$
$= 7065\ cm^3$
Capacity = 7.065 litres.

**Example 2 :** Total surface area of a hemisphere is $432\ \pi\ cm^2$. Find its volume.
Given : TSA = $432\ \pi$
$432\pi = 3\pi r^2$
$\therefore\ r^2 = 144$
r = 12
$V = 2/3\ \pi\ r^3$
$= 2/3 \text{ x } 3.14 \text{ x } 12 \text{ x } 12 \text{ x } 12$
$= 3617.28\ cm^3$.

# STATISTICS

## Meaning of the word Statistics

The word "Statistics" comes from the Latin word 'Status' meaning a political state. The origin of Statistics is due to administrative requirements of a state. The administration of the states required the collection of information relating to population and material wealth of the country for purposes of war and finance. For comparison and analysis, the information is expressed in terms of numerical figures so that these numerical figures could be used like a measuring stick to know about the states. Therefore, some people interpret the word 'Statistics' as the combination of the words 'Status' (which it means a state) and 'Stick' (a measuring rod).

Presently the word 'Statistics' is used in two senses. It is sometimes used in the plural and sometimes in the singular. When it is used in plural, it refers to numerical statements of facts. For example, unemployment statistics, statistics of industrial accidents in India, etc. When the word statistics is used in singular, it refers to the subject of statistics. In this module, we consider the word statistics in the singular sense. The word 'Statistics' has been variously explained by different authors. Some of the explanations are listed below.

1. The science of counting
2. The science of averages
3. The science of measurements of social organism, regarded as a whole in all manifestations.

   \- A L Bowley
4. The science of estimates and probabilities - Boddington
5. The collection, presentation, analysis and interpretation of numerical data - Croxton, Cowden and Klein
6. Statistics is the science which deals with the methods of collecting, classifying, presenting, comparing and interpreting numerical data collected to throw some light on any sphere of enquiry. - Seligman.
7. By Statistics we mean aggregate of facts affected to a marked extent by multiplicity of causes numerically expressed, enumerated or estimated according to reasonable standards of accuracy, collected in a systematic manner for a pre-determined purpose and placed in relation to each other. - Horace Secrist
8. Statistics are the classified facts representing the conditions of people in a state, especially those facts which can be stated in numbers or in any tabular or classified arrangement . - Webster

Of all these explanations, the explanation given by Seligman is considered to be more appropriate. Accordingly, the following explanation is accepted.

**Statistics is the subject which deals with the methods of collecting, classifying, presenting, comparing, analysing and interpreting the numerical information regarding any sphere of activity.**

## Historical Background

The modern development of statistics began in the $16^{th}$ century, when Governments of various East European countries became interested in collecting information about their citizens. By the $17^{th}$ century, surveys were conducted which very closely resemble our modern census. In Germany, the systematic collection of official statistics originated towards the end of the $18^{th}$ century in order to have an idea of the relative strength of different German states and information regarding population and industrial, agricultural output was collected. In England, Statistics were the outcome of Napoleonic wars. The wars necessitated the systematic collection of numerical data to enable the government to assess the revenue and expenditure with greater precision and then to levy new taxes in order to meet the cost of war. John Graunt of London (1620-1674), known as the 'Father of Vital Statistics', was the first man to study the statistics of births and deaths. Casper Newman, Sir William Petty (1623-1687), James Dodson, Price led to the idea of 'life insurance' statistics. The French mathematician Pascal (1623-1662), laid the foundation of the theory of probability which is the backbone of the modern theory of statistics. The most notable persons who contributed to the theory of probability were James Bernoulli (1654-1705), De-Moivre (1667-1754), Laplace (1749-1827), Gauss (1777-1855), Euler, Lagrange, Bayes, A.Markoff, Khintchin, Kolmogoroff. Francis Galton (1822-1921) pioneered the use of statistical methods in the field of Biometry. Karl Pearson (1857-1936), the founder of the statistical laboratory in England (1911), was the pioneer in correlational analysis. In 1908, the discovery of student 't' distribution by W.S. Gosset had brought a new turn in statistical inference. Sir Roland A. Fisher (1890-1962) is known as the 'Father of Statistics' for his contributions of statistical methods in the fields of Genetics, Biometry, Education and Agriculture. The main statistical concepts developed by him are Point Estimation, 'Fiducial Inference', 'Exact Sampling Distributions', 'Design of Experiments' and 'Analysis of Variance'.

In the modern days, Statistics has become indispensable tool in all disciplines.

## Importance of Statistics

As Bowley stated, 'A knowledge of statistics is like a knowledge of a foreign language or of algebra; it may prove of use at any time under any circumstances'.

In the modern world, statistical methods are of universal applicability. As a matter of fact, there are millions of people all over the world who have not heard a word about statistics and yet who make a profuse use of statistical methods in their day-to-day decisions.

*Importance to State* : Statistics are very helpful to a state as they help in administration. The state must have information regarding the number of persons living within its boundaries, total imports and exports, production of goods , labour situation, etc.

*Economics* : Statistical data and statistical methods are very helpful in the proper understanding of the economic problems and the formulation of economic policy. Ex. Construction of cost of living index, index number of whole sale prices, National income estimates, etc.

*Planning*: Economic planning is inconceivable without adequate statistical data. Ex. Employment Statistics, National Income, trade statistics, etc.

*Business and Commerce*: Success of business depends on accurate forecasts of sales. That is, a producer or a dealer should first estimate the demand for his products, analyse the possible effects of factors like changes in tastes, fashions, etc. Ex; Consumer research, business forecasting, sales control, stock, etc.

*Industry* : Quality control of goods manufactured, etc. depends upon the statistical methods.

*Insurance*: The work of life insurance companies depends on the compilation of life tables and computation of expectation of life from time to time. The theory of probability finds its full use in the field of insurance.

*Public Utility Concerns* : Public utility concerns like railways, electricity boards, companies, water works, etc. use statistics extensively. On the basis of estimated traffic, additional trains will be introduced. Water work department should know the list of users and amount of water consumption to fix the water rates.

*Banks* : A stock exchange broker or an investor in securities needs a knowledge of interest rates, the fluctuation of interest marked and other related data to strike a timely bargain.

*Research* : Statistics is an indispensable tool of research. For example, in experiments about crop yields and different types of fertilisers and different types of soils, the growth of animals under diets, etc.

*Social Studies* : A social worker has to rely upon statistics for carrying on his activities in the right direction. Ex: Beggar problem, child marriages, illiteracy, unemployment, scholastic performance, extra curricular activities, public opinion, Listeners' research.

*Medicine*: To test the effects of new medicines, statistical data are very useful.
*Biology*: The theory of heredity depends completely on statistics. A new subject called Biometry has developed out of Biology and Statistics.

*Genetics* : Mendalian hypothesis, Michurian hypothesis can be verified by statistical data.

*Agriculture*: Statistical methods can determine the correlation between rainfall, fertiliser, seeds and chemical content of the soil for maximization of crop production.

*Education*: Facilities for starting new schools, curricula, number of teachers, etc.

*Psychology*: Conducting and analysing the I.Q. tests.

*Meteriology:* Weather forecasts mainly depend upon collection of the past information and the use of statistical estimation techniques.

*Physics* : Radio activity, behaviour of atoms, electrons, etc.

*War*: Training, inspection, purchases, etc.

*Astronomy*: Movement of heavenly bodies are to be observed and tabulated.

*Engineering*: Quality control, queuing problem, etc.

Statistical methods are applicable wherever quantitative studies are to be made. Now-a-days, there is hardly any field where statistical methods are not used. Thus "Statistics" affects everybody and touches life at many points". As H G Wells pointed out, "There is nothing wrong if we say that now-a-days statistical thinking is as necessary for efficient citizenship as the ability to read and write".

**Exercises**

1. Critically examine the various definitions of Statistics.
2. Explain the role of Statistics in education and business.
3. Comment on the statement "Planning on the basis of inadequate and inaccurate statistics is worse than no planning at all".
4. "A knowledge of Statistics is like a knowledge of foreign languages or of algebra; it may prove of use at any time under any circumstances".

**Frequency Table**

The statistical data arranged and classified into a number of groups in an orderly manner on the basis of magnitudes of the values, constitute a frequency distribution, and a table presenting them is known as frequency table. For example, marks scored in mathematics by 25 students are presented in the form of the following frequency table.

| Marks | Number of students |
|---|---|
| 0 - 20 | 5 |
| 20-40 | 3 |
| 40-60 | 12 |
| 60-80 | 5 |

In the above example, 0-20, 20-40, 40-60, 60-80 are known as class-intervals. The number of items (here students) falling within each class is known as frequency. For each class interval, the lower and upper values of the interval is known as lower and upper limits of that class - interval respectively. In the above example, 0 and 20 are the lower and upper limits of 0-20 interval respectively. The difference between these limits is the width of the class interval. The value which lies midway between the lower and the upper limits of a class is called the 'mid value' or the 'class mark' of that class. In other words,

$$\text{Mid Value} = \frac{(Lower\ Class\ \text{limit} + upper\ class\ \text{limit})}{2}$$

Now let us discuss the way of preparing the frequency table regarding any given statistical data. The main aim of presenting the data in a frequency table is to condense the huge data on the basis of certain classification based on the magnitude of the observation. Suppose we have the data regarding the Mathematics marks of 200 students. To present all these individual marks, a lot of space is required and no useful purpose is served. But this data can be presented in a frequency table and that table can be utilised for useful inferences.

The first factor to be considered while preparing a frequency table is the number of groups or the number of class intervals. The number of class-intervals should not ordinarily exceed 20 and should not in general, be less than 5. Once we decide the number of class intervals, the next step is to define the class-widths. For the sake of computation and comparison, all the class-intervals should, whenever possible, be of equal width. How to find the width of the class-interval? Find the difference between the smallest and the largest values in the given data. This difference is known as range. Then the width of the class-interval is given by the nearest convenient value to the following ratio.

$$\frac{Range}{Number\ of\ class\ \text{intervals}}$$

For example, if the range is 67 and the number of class intervals is 10, then the width is

$7\left(\approx \frac{67}{10}\right)$.

This approximate value should be as in the given data (nearer to integer or adjusted to the required decimal). After deciding the class-interval, the lower limits and upper limits of all the class-intervals should be identified. The lower limit of the first class-interval is generally the integral value which is nearer and smaller than the smallest value in the given data. Then the value of the upper limit of first class-interval becomes the sum of lower limit value and the class width (generally an integral value). The lower limit of the second interval will be sometimes equal to the upper limit of the first interval and the upper limit of the second interval will be the sum of lower limit of the second interval and class width. For example, if the class width is 10 and the lower limit of the first interval is 0, then first two intervals will be
0 - 10 and
10 - 20.

In the above example, it is not clearly defined whether the value 10 belongs to the first class-interval or to the second class-interval. Therefore, in such situations, it is necessary to specify clearly whether the lower/upper limits are belonging to those class intervals or not. If the lower limit of one class belongs to that concerned class, then all the lower limits of the other classes also should belong to those respective classes. This is also true in case of upper limits. Such specification of class intervals is known as inclusive method. If the upper limits of all the class intervals are not belonging to the respective classes, then the class-intervals are formed with exclusive method.

Instead of having confusion regarding the inclusive or exclusive method as in the above example, the class-intervals may also be considered as follows:

0 - 9
10 - 19
20 - 29 and so on.

Suppose one of the values is 9.5. Here it is not clearly known whether this 9.5 will belongs to the first class interval viz. 0 - 9 or to the second 10 - 19. This type of problem arises only when the values in the given data take any value on the real line (when the observation is a continuous variable). In such cases, we add half of the difference between the upper limit of one class and the lower limit of the succeeding class to the upper limit of each class. Similarly, we subtract half of the above difference from the lower limit of each class. These resultant values are known as the upper and lower practical limits of those classes. In the above example, the interval 0-9 will become as -0.5 - 9.5, because we are adding (10-9)/2 to 9 (upper limit) and subtracting (10 - 9)/2 from 0 (lower limit). Therefore, -0.5 and 9.5 are the practical limits of the class 0 - 9. Similarly, 9.5 and 19.5 are the practical limits of the interval 10 - 19. Here, we consider all the values up to and including 9.5 to the first class and the values above 9.5 (excluding 9.5) and upto 19.5 (including 19.5) to the second class and so on. These practical limits are also known as boundaries.

In case of continuous data by class-limits, we mean practical limits. Once the class intervals are specified, then the last step in preparing the frequency table is to count the number of values belonging to different class-intervals. The number of values belonging to the different class intervals are known as the frequencies of the respective classes. Counting observations will be done by putting a tally mark corresponding to each observation against the appropriate class interval. Finally, count the total tally marks indicated against each class interval. These are the corresponding frequencies of those classes.

Example: Prepare a suitable frequency table for the following marks of 30 students.

| | | | | | | |
|---|---|---|---|---|---|---|
| 72 | 49 | 63 | 72 | 11 | 83 | 93 |
| 38 | 23 | 19 | 26 | 17 | 44 | 67 |
| 29 | 14 | 39 | 53 | 57 | 62 | 56 |
| 68 | 76 | 43 | 91 | 1 | 15 | 42 |
| 17 | 29 | | | | | |

**Solution :**

**Step 1:** Consider the number of classes, say 5.
**Step 2:** Calculation of Range

In the given example,

Maximum value = 93
Minimum value = 1
Range = 93 - 1 = 92

**Step 3:** Calculation of class-width

$$Class\ \ width = \frac{Range}{No.\ of\ \ classes} = \frac{92}{5} = 18.4 \approx 19$$

**Step 4 :** Determination of lower limits

Since the smallest value is 1, the lower limit of the first class is taken as 0. (Generally, the lower limit of the first class is taken as one unit below the smallest value). Hence, the lower limits of the other classes are 0 +19 = 19, 19 + 19 = 38, 38 + 19 = 57, 57 + 19 = 76 respectively.

**Step 5:** Determination of upper limits

The upper limit of first class is 18 (one unit smaller than the lower limit of the second class). Similarly, the upper limits of the other classes are 18 + 19 = 37, 37 + 19 = 56, 56 + 19 = 75, 75 + 19 = 94 respectively.

Note: The upper limits can also be considered as 19, 38, 57, 76, 95 in the case of exclusive upper limits.

**Step 6 :** Indication of classes

The different classes are

| **Class** |
|---|
| **0 - 18** |
| **19 - 37** |
| **38 - 56** |
| **57 - 75** |
| **76 - 94** |

**Step 7 :** Putting the 'tally marks'.

The tally marks corresponding to the given values are as indicated below:

| Class | Tally marks |
|---|---|
| 0 - 18 | |
| 19 - 37 | |
| 38 - 56 | |
| 57 - 75 | |
| 76 - 94 | |

**Step 8:** Obtaining the frequencies.

The frequencies corresponding to the tally marks are

| Class | Tally Mark | Frequency |
|---|---|---|
| 0 - 18 | | 6 |
| 19 - 37 | | 5 |
| 38 - 56 | | 8 |
| 57 - 75 | | 7 |
| 76 - 94 | | 4 |

Step 9 : Indication of the final frequency table.

| Class | Frequency |
|---|---|
| 0 - 18 | 6 |
| 19 - 37 | 5 |
| 38 - 56 | 8 |
| 57 - 75 | 7 |
| 76 - 94 | 4 |

**Exercises**

1. What is a frequency distribution? Also point out the basic principles to be observed in forming the same.
2. Distinguish between the following
   a) Exclusive and Inclusive intervals
   b) Ordinary and cumulative frequencies

3. The following numbers give the weights (in kg) of 30 students of a class. Prepare a suitable frequency table considering 6 classes.

   42, 53, 56, 39, 42, 41, 66, 38, 45, 46, 31, 43, 52, 51, 51, 36, 38, 49, 48, 51, 59, 62, 54, 51, 40, 35, 32, 37, 41, 50

**Mode**

Mode is one of the measures of central tendency (averages). Mode is the value which occurs more frequently than others. For example, in the following raw data mode is equal to 18, because this value occurs more times than the other values.

12, 17, 18, 22, 18, 15, 17, 16, 27, 18

In ungrouped data like this, mode can be obtained by mere inspection. Even in the case of grouped data without class-intervals, mode can be calculated easily.

If there are two or more values occurring with the same maximum number of times, then all those values are the modes of that data. The distribution having only one mode is known as uni-modal distribution, having two modes is bi-modal distribution and having more than two modes is known as multi-modal distribution.

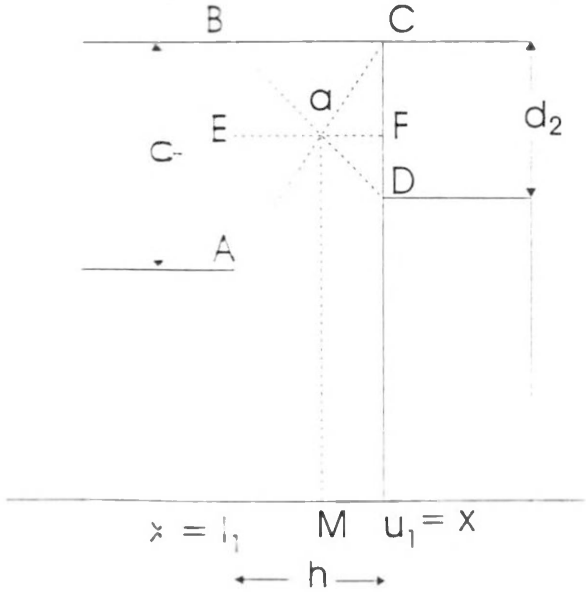
In case of grouped data with class-intervals, mode is given by the following formula.

$$Mode = l + \frac{f_1 - f_2}{2f - f_1 - f_2} h$$

| | | |
|---|---|---|
| Where l | = | the lower limit of the modal class (modal class means the class having the maximum frequency) |
| f | = | frequency of the modal class (or maximum frequency) |
| $f_1$ | = | frequency of the class preceding the modal class |
| $f_2$ | = | frequency of the class succeeding the modal class |
| h | = | width of the modal class |

The above formula can be used only when the width of the modal class is not greater than the width of any other class. This is important, because as otherwise, the frequency of a class can be increased by increasing the width and any class may become a modal class.

The proof for the above mentioned formula for Mode is as given below :




Assume the figure represents three rectangles of the histogram of the frequency distribution, the central rectangle corresponding to the modal class.

Assume also that the class intervals have equal width. We consider the mode as the abscissa M of the point of intersection O of the lines AC and BD.

Let $x = l_1$ and $x = u_1$ represent the lower and upper class boundaries of the modal class, and $d_1$ and $d_2$ represent respectively the excess of the modal class frequency over the class frequencies to the left and right of the modal class.

Now from similar triangles AOB and COD, we have

$$\frac{EO}{AB} = \frac{OF}{CD} \quad OR \quad \frac{M - l_1}{d_1} = \frac{u_1 - M}{d_2}$$

or $(M - l_1)\, d_2 = (d_1 - M)\, d_1$

or $M\,(d_1 + d_2) = u_1 d_1 + l_1 d_2$

or $M = \dfrac{u_1 d_1 - l_1 d_2}{d_1 - d_2}$

But $u_1 = l_1 - h$, where h is the width of the modal class

$$\therefore \quad M = \frac{(l_1 - h)\, d_1 + l_1\, d_2}{d_1 + d_2} = l_1 + \frac{d_1}{d_1 + d_2}\, h$$

We can write $d_1$ as $f-f_1$ and $d_2$ as $f-f_2$ , where f is the maximum frequency, $f_1$ is the frequency of the class preceding the Model class and $f_2$ is the frequency of the class succeeding the model class.

$$\text{So, } M = l_1 + \frac{f - f_1}{2f - f_1 - f_2} h$$

Note: This result has the following interpretation. If a parabola is drawn so as to pass through the true midpoints of the tops of the rectangles in the figure, the abscissa of the maximum of this parabola will be the mode as obtained above. Here the parabola is the part of the frequency curve.

Example: Find the mode for the following distribution.

| Class | Frequency |
|---|---|
| 30-35 | 4 |
| 35-40 | 7 |
| 40-45 | 12 |
| 45-50 | 9 |
| 50-55 | 9 |
| 55-60 | 5 |

To calculate the mode of any frequency data of this kind, the first step is to identify the modal class. Modal class is the class having the maximum frequency.

In the given problem, the maximum frequency is 12 and the class corresponding to this is 40-45. Therefore, the modal class is 40-45.

Now, we can use the formula

$$Mode = l + \frac{f_2 - f_1}{2f - f_1 - f_2} h$$

to calculate the mode.

In the given problem, the modal class is 40-45. Then,

l, the lower limit of the modal class = 40
f, frequency of the modal class = 12
$f_1$, frequency of the preceding class to the modal class = 7
(because the preceding class is 35-40)
$f_2$, frequency of the succeeding class to the modal class = 9
h, width of the modal class = 5

Now by substituting these values in the formula, we get

$$\text{Mode} = 40 + \frac{12 - 7}{2 \times 12 - 7 - 9} 5$$

= 40 + 25/8
= 40 + 3.125
= 43.125

Example: Calculate the mode for the following distribution.

| Class | Frequency |
|---|---|
| 55-60 | 5 |
| 50-55 | 9 |
| 45-50 | 9 |
| 40-45 | 12 |
| 35-40 | 7 |
| 30-35 | 4 |

Here mode can be calculated as in the previous case. Here frequency of the preceding class 1 is 9 and frequency of the succeeding class 2 is 7 as the class intervals are given in the above order. Here the class intervals are given in the descending order and therefore, in the above formula we should interchange $f_1$ and $f_2$ as 7 in the given problem.

$$\therefore \text{ Mode} = l + \frac{f - f_2}{2f - f_2 - f_1} h$$

$$= \quad 40 + \frac{12 - 7}{2 \times 12 - 7 - 9} .5$$

$$= \quad 40 + \frac{5}{8} . 5$$

$$= \quad 40 + \frac{25}{8}$$

= 43.125

Example: Calculate the mode of the following distribution.

| Class | Frequency |
|---|---|
| 1 - 5 | 6 |
| 6 - 10 | 12 |
| 11 - 15 | 13 |
| 16 - 20 | 17 |
| 21 - 25 | 14 |
| 26 - 30 | 9 |

The data here is in the frequency distribution form with class intervals. Therefore, we use the following formula to calculate Mode. :

$$\text{Mode} = l + \frac{f - f_1}{2f - f_1 - f_2} h$$

In this problem, maximum frequency is 17 and therefore the model class is 16-20. Hence here, l = 16, f = 17, $f_1$ = 13, $f_2$ = 14, h = 20 - 16 = 4

$$\therefore \text{ Mode} = 16 + \frac{17 - 13}{2 \times 17 - 13 - 14} 4 = 16 + \frac{4}{7} 4$$

$$= 16 + \frac{16}{7} = 18.2857.$$

Note: In this problem, we consider that the data is discrete (i.e. values are only integral values). When we consider the data as continuous data (i.e. when we assume values can be even between any two integral values) we should consider the class-boundaries or practical limits. In that case, in the above problem, the modal class is 15.5 - 20.5. Therefore. l = 15.5, f = 17, $f_1$ = 13, $f_2$ = 14 and h = 20.5 - 15.5 = 6

$$\text{Hence Mode} = 15.5 + \frac{17 - 13}{2 \times 17 - 13 - 14} . 5$$

$$= 15.5 + \frac{4}{7} 5 = 15.5 + \frac{20}{7}$$
$$= 18.3571$$

Any one of the above methods can be used in such cases.

Example: Calculate the mode of the following distribution.

| Class | Frequency |
|---|---|
| 10 - 20 | 10 |
| 20 - 30 | 11 |
| 30 - 50 | 22 |
| 50 - 60 | 12 |
| 60 - 70 | 11 |
| 70 - 80 | 9 |
| 80 - 90 | 8 |

Here the modal class cannot be 30-50 as it appears. When compared to other classes, the width of the class 30-50 is more and so the maximum frequency may be due to this increased width. In such situations, we regroup the classes so that the class corresponding to the maximum frequency to have the width not more than the width of any other classes.

In the given problem, we combine 10-20 and 20-30 classes into one class as 10-30 and similarly, the two classes 50-60, 60-70 in one class as 50-70 and the other two classes 70-80, 80-90 into one as 70-90.

The new distribution is as follows :

| Class | Frequency |
|---|---|
| 10 - 30 | 21 |
| 30 - 50 | 22 |
| 50 - 70 | 23 |
| 70 - 90 | 17 |

Here, the maximum frequency is 23 and therefore, the modal class is 50-70. Now we can use the formula and obtain the modal value. Since the modal class is 50-70,

$l = 50,\ f = 23,\ f_1 = 22,\ f_2 = 17,\ h = 20$

$$Mode = 50 + \frac{23 - 22}{2 \times 23 - 22 - 17} \times 20$$

$$= 50 + \frac{20}{7}$$
$$= 50 + 2.857$$
$$= 52.857$$

Another method in the above problem is to write the class 30-50 as 2 classes 30-40 and 40-50 with the same frequency 11 each. Sometimes, it is possible that the given data may possess two or more than two modes. Before calculating these modes, we should make sure for the existence of two or more than two modes. This can be seen by the 'method of grouping'.

**Methods of Grouping**

The different steps involved in this method are as follows :

Step 1. The given frequencies are grouped by twos, adding frequencies of classes 1 and 2; 3 and 4 and so on. Suppose there are 7 classes and hence 7 frequencies are given. Now these frequencies are grouped by twos adding the frequencies of classes 1 and 2; 3 and 4; 5 and 6. After the class 6, there is only single class viz. the class 7. Hence, the frequency of this class 7 is left out of consideration. Generally, if the number of classes is odd, the last class is left out of consideration. These sums will be written in a separate column against the mid position of the two concerned frequencies.

Step 2. The frequency of class 1 is left out. The remaining frequencies are grouped by twos, adding the frequencies of classes 2 and 3, 4 and 5 and and so on. While grouping like this, if we are left with a single class in the end as in step 1, we leave it out of consideration. These sums also are written in another column as in step 1.

Step 3. The frequencies are grouped by threes, adding the frequencies of classes 1,2 and 3; 4, 5 and 6; and so on. While grouping like this, if we are left with one class or two classes in the end, they are left out of consideration. These sums will be written in a separate column like the sums in steps 1 and 2. Each sum will be written against the 2nd frequency of the three concerned frequencies for that sum.

Step 4. The frequency of class 1 is left out. The remaining frequencies are grouped by threes, adding the frequencies of classes 2,3 and 4; 5,6 and 7 and so on. Here also, while grouping, if we are left with one class or two classes in the end, they are left out of consideration. These sums also will be written in a separate column as the sums in step 3.

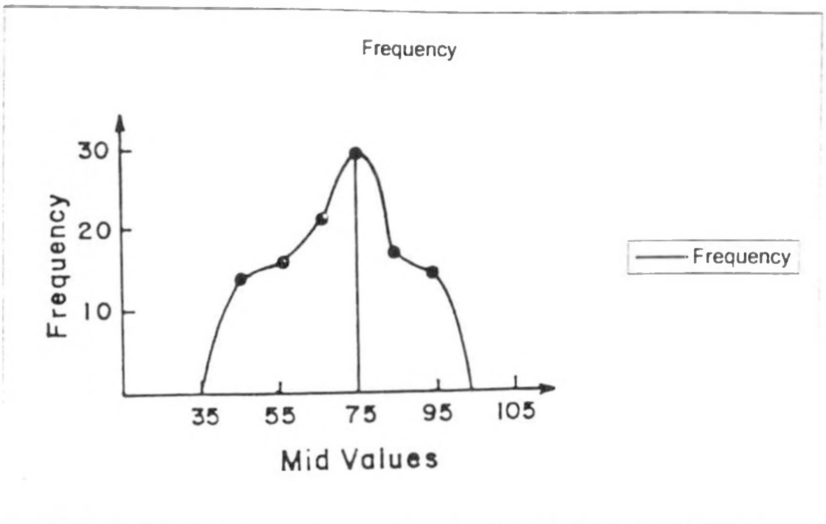
Step 5. The frequencies of classes 1 and 2 are left out. The remaining frequencies are grouped by threes adding the frequencies of classes 3,4 and 5; 6,7 and 8 and so on. These sums will be written in a separate column.

Step 6. Similar to the earlier steps, sums of 4,5......frequencies will be obtained and written in separate columns. This procedure will be continued until we get at least two such sums in a column.

Step 7. The table indicating those sums is known as 'grouping table'. After the preparation of this grouping table, another table, called 'analysis table' is prepared.

Step 8. Identify the maximum values in all frequency columns and the corresponding classes responsible for those maximum frequencies.

Step 9. Count the number of times each class interval is contributing to the maximum frequencies. The class contributing maximum number of times to the maximum frequencies is the modal class. If there are more than one such classes, then all those classes are the modal classes and the distribution is known as multimodal distribution. The modes corresponding to each modal class will be calculated by using the formula.




The frequency curve of the given data is given above.

The value of x corresponding to maximum point is 75 and therefore, mode of the given distribution is 75 approximately.

**Empirical formula to calculate Mode**

Sometimes, the mode can be obtained in terms of the values of arithmetic mean and median by using the following relation,

Mode = 3 (Median) - 2 (Arithmetic Mean)

This relationship is only observed relationship. There is no mathematical proof for this. The value obtained by this formula is only an approximate value.

It is not possible to obtain mode in the situations where the modal class coincides with an open end class or the preceding or succeeding class and the modal class are not having the same width to decide the maximum frequency correctly.

**Exercises**

1. The following table shows the distribution of 100 families according to their expenditure per week. Number of families corresponding to expenditure group Rs.(10-20) and Rs.(30-40) are missing from the table. The median and mode are given to be Rs.25 and Rs.24.74. Calculate the missing frequencies and the arithmetic mean of the data.

| | | | | | |
|---|---|---|---|---|---|
| Expenditure | 0-10 | 10-20 | 20-30 | 30-40 | 40-50 |
| No of families | 14 | - | 80 | - | 15 |

2. From the table given below, find the mode.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| Weekly earnings | 70-75 | 75-80 | 80-85 | 85-90 | 90-95 | 95-100 | 100-105 |
| No of employees | 4 | 8 | 11 | 23 | 20 | 17 | 6 |

**MEASURES OF DISPERSION**

When we compare two similar types of data the measure of central tendency may be the same in both cases, but still we find some differences in the observation of those. For example, one set of observations is 3,5,7,11 and 14; the other set is 8,8,8,8 and 8. For these sets, the arithmetic means are the same but there is a difference in their nature. In the first set, the observations are varying from one another and in the second set all the observations are the same. Therefore, this type of difference is to be studied. This type of difference is known as the 'dispersion' or 'variation'. Here, dispersion means 'scatteredness'. There are different measures of dispersion which are discussed below.

**Quartile Deviation**

Certain values can divide the given set of values into equal number of groups and these values which divide the given observations to form equal parts are known as 'quartiles'. Three values can divide the given set of observations into 4 equal parts. These three values are known as 'quartiles'. As already known, median will divide the observations into two equal parts. Also, we know the method of calculating median in case of any given data. Similarly, the three quartiles (denoted by $Q_1$, $Q_2$, $Q_3$) can be calculated; here $Q_2$ is the same as median.

The first quartile, $Q_1$ is the value which divides the given observed values into two parts such that one-fourth of the observations to have the values less than $Q_1$ and the remaining three-fourths observations to have values greater than $Q_1$.

The third quartile, $Q_3$ is the value which divides the given observations into two parts such that three-fourths of the observations to have the values smaller than $Q_3$ and the remaining one-fourth of the observations to have the values greater than $Q_3$ and the remaining one-fourth of the observations to have the values greater than $Q_3$.

Quartile deviation is half of the difference between the third and first quartiles.

Symbolically, this is equal to $\frac{Q_3 - Q_1}{2}$.

It is also known as semi-interquartile range, because $Q_3 - Q_1$ is the range of the quartiles.

To calculate the quartile deviation, it is necessary to calculate $Q_3$ and $Q_1$.

$Q_1$, $Q_3$ can be calculated after arranging the data in ascending (or descending) order of magnitude. In this ascending order, any value greater than N/4 th value and less than (N/4 + 1) th value is $Q_1$ and any value greater than 3N/4 th value and less than (3N/4 + 1) the value is $Q_3$, where N is total number of values.

Once, we get $Q_1$ and $Q_3$ values, we calculate the quartile deviation as

$$\frac{(Q_3 - Q_1)}{2}$$

Example : If the data are
12, 15, 16. 18, 19, 29, 47, 55, 62
$Q_1 = 16$, $Q_2=19$, $Q_3 = 47$ by definition.

Example: Calculate the quartile deviation for the following data.
12, 18, 15, 16, 29, 47, 19, 55, 62

The first step in calculation is to arrange the observations in ascending order of magnitude. The order of the given observations is

12, 15, 16, 18, 19, 29, 47, 55, 62, 83

Here N = 10 and therefore $Q_1$ is the values greater than (10/4)th value and less than $\left(\frac{10}{4} + 1\right)$ th value.

$Q_1 = \frac{15 + 16}{2} = 15.5$ (generally central value of these (N/4) th and (N/4 + 1) th value.

$$Q_3 = \frac{47 + 55}{2} = 51$$

Quartile Deviation = $\dfrac{51 - 15.5}{2} = \dfrac{35.5}{2} = 17.75$.

In the case of a frequency distribution, with class-intervals, $Q_1$ and $Q_3$ will be calculated by using the following formulae like median formula.

$$Q_1 = l_1 + [(N/4) - C_1)/f_1)\, h_1$$

where $l_1$ = lower limit of the first quartile class
(here first quartile class is the class having the (N/4) th value)
N = total frequency
$f_1$ = Frequency of the first quartile class
$C_1$ = Cumulative frequency of the class preceding the first quartile class
$h_1$ = width of the first quartile class.

Similarly,

$$Q_3 = l_3 + [\,(\,3N/4) - C_3\ /\ f_3\,)\,]\, h_3$$

where $l_3$ = lower limit of the third quartile class
(here, third quartile class is the class which contains the 3N/4the value)
N = total frequency
$f_3$ = frequency of the third quartile class
$C_3$ = Cumulative frequency of the class preceding the third quartile class
$h_3$ = width of the third quartile class

Example : Calculate the quartile deviation for the following distribution.

| Class | Frequency |
|---|---|
| 30 - 40 | 8 |
| 40 - 50 | 6 |
| 50 - 60 | 12 |
| 60 - 70 | 9 |
| 70 - 80 | 10 |
| 80 - 90 | 14 |
| 90 - 100 | 7 |

To calculate the first and third quartiles, the classes to be identified where those quartiles lie. For this, the cumulative frequencies are to be obtained first.

The cumulative frequencies of the different classes are as shown below :

| | Class | Frequency | Cumulative Frequency |
|---|---|---|---|
| | 30 - 40 | 8 | 8 |
| | 40 - 50 | 6 | 14 |
| | | | ←N/4 (16.5) |
| $Q_1$ class | $\boxed{50 - 60}$ | 12 | 26 |
| | 60 - 70 | 9 | 35 |
| | 70 - 80 | 10 | 45 |
| | | | ←3N/4(49.5) |
| $Q_3$ class | $\boxed{80 - 90}$ | 14 | 59 |
| | 90 - 100 | 7 | 66 |

Here N = 66, N/4 = 66/4 = 16.5
3N/4 = 3 x 66/4 = 49.5

The class in which 16.5 th value lies is 50 - 60 and the class in which 49.5 th value lies is 80 - 90. Therefore, the first quartile class is 50 - 60 and the third quartile class is 80 - 90 as indicated in the table.

Hence, $l_1 = 50$, $f_1 = 12$, $c_1 = 14$, $h_1 = 10$

$l_3 = 80$, $f_3 = 14$, $c_3 = 45$, $h_3 = 10$

$Q_1 = l_1 + (N/4 - c_1) / f_1) h_1$

$= 50 + ( (66/4 - 14) / 12 ) \times 10$
$= 50 + ((16.5 - 14) /12) \times 10 = 50 + (2.5/12) \times 10$
$= 50 + 25/12 = 50 + 2.083 = 52.083$

$Q_3 = l_3 + (( 3N/4 - c_3) / f_3) h_3 = 80 + (( 3 \times 66/4 - 45) /14) \times 10$
$= 80 + (( 49.5 - 45) /14 ) \times 10 = 80 + (4.5 / 14) \times 10$
$= 80 + 45/14 = 80 + 3.214 = 83.214$

$$\text{Quartile Deviation} = \frac{Q_3 - Q_1}{2} = \frac{83.214 - 52.083}{2} = 15.5655.$$

It is obvious that the quartile deviation may be same even some observations are changed, as the measure is based on the first and third quartiles.

**Exercises**

1. Goals scored by two teams A and B in a football season were as follows :

| No. of goals scored in a match | No of Matches | |
|---|---|---|
| 0 | A | B |
| 1 | 27 | 17 |
| 2 | 9 | 9 |
| 3 | 5 | 5 |
| 4 | 4 | 3 |

Calculate the Quartile Deviation and comment on the consistency of the teams.

## List of Participants of the 2-day Preliminary Workshop

## August 13 - 14, 1997

1. B S Joshi
   K B Boards Composite Pre-university College
   Malamaddi
   Dharwar Dist.

2. D R Rayan Gouder
   Shree Ramakrishna Ashrama High School
   Julkote
   Gadag Taluk

3. Venkatesh Rao
   Tagore Memorial Boys High School
   Raichur

4. G P Bhat
   Govt Pre Univesity College
   Murkwad, Haliyal Taluk
   Uttara Kannada

5. B T Narayana
   Sri Swarnamba High School
   Honudike
   Tumkur Taluk

6. H K Raghavan
   Government High School
   Chekkur, Gubbi Taluk

## List of Participants of the

## Tryout Programme for the Development of Enrichment Materials in Mathematics for Class IX of Karnataka

**24 - 28 November 1997**

1. B P Uthappa
   Graduate Assistant Teacher
   Govt P U College
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2. H N Mohan Kumar
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3. K Thammanagoud
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5. Shivagowda C Biradar
   I/c Headmaster
   G H School
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   Govt P U College for Girls
   Bagalkot

7. Bharamu Tippanna Mallannavar
   Assistant Teacher
   Pandit Neharu P U College
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8. C R Bharathi
Assistant Mistress
Govt High School
Chitradurga

9. T Venkatashiva Reddy
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10. B L Muddegowda
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13. B V Rangaswamy
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Hongere, Hassan Dist

14. J Venkatesh
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15. B N Yoganna
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18. Medavev I Parashatti
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19. Solabanna Rudrappa Wali
Assistant Master
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Arjunag, Bijapur Dist.

20. M Ganganna
Assistant Master
Govt Boys Junior College
Chitradurga

21. Jagannatha Pampanna Bennur
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