CONTENT ENRICHMENT TRAINING PROGRAMME IN MATHEMATICS FOR JUNIOR COLLEGE LECTURERS OF APSWREI SOCIETY, HYDERABAD

(22.6.1998 to 3.7.1998)

Regional Institute of Education, Mysore 570 006
[National Council of Educational Research and Training, New Delhi]

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RIE Faculty

Dr D Basavayya (Coordinator)
Dr V Shankaram
Dr N M Rao
Dr B S Upadhyaya

Regional Institute of Education, Mysore 570 006
[National Council of Educational Research and Training, New Delhi]

Content Enrichment Programme in Mathematics for Junior College Lecturers of APSWREI, Hyderabad

(From 22.6.98 to 3.7.98)

Need for the Programme :

In Andhra Pradesh, most of the residential schools under Social Welfare Department were upgraded to teach Intermediate Course and the performance in Mathematics in these schools is not encouraging. Therefore, Social Welfare Department felt the need to train their Lecturers to acquire required competencies to teach effectively the intermediate subjects. Commissioner of Social Welfare Department of Govt of A P has expressed this need in the State Coordination Committee (SCC) meeting and requested RIEM to take the steps in this regard. Therefore, to fulfil this need of Social Welfare Department, RIE, Mysore has planned to organize an enrichment programme for the lecturers to teach Mathematics at intermediate level.

About the Training

1. To identify the lecturers' difficulties in teaching mathematics at intermediate level.
2. To provide opportunities in solving the difficult problems.
3. To enrich the participants in the content .
4. To expose the Lecturers to choose and identify the suitable teaching strategies to teach the different concepts of Mathematics.
5. To expose the Lecturers to choose and implement different motivational strategies while teaching to their students.
6. To select and use different resources (like teaching aids, video films, Maths laboratory, etc) available to teach mathematics more effectively.

The strategies used in the Programme :

1. Lecture-cum-discussion mode was used in dealing with the different difficult areas in Mathematics.
2. Pedagogical issues like teaching strategies, evaluation, Teaching Aids and Mathematics club activities were discussed for the benefit of the participants.
3. Seminars related to Content and Pedagogy of Mathematics by the participants have been arranged.
4. Films related to topics in Mathematics at +2 level were screened.
5. Demonstrations were arranged to explain the effective role of computers in teaching Mathematics at +2 level.
6. Enrichment lectures were arranged for deeper understanding of some of the concepts.
7. Participants were exposed to study the various teaching aids available in the Maths Lab of RIE Mysore.

8. Opportunities were given to go through various books/literature pertaining to intermediate Mathematics.
9. The role of continuous evaluation in teaching of Mathematics was discussed to a great extent.
10. Participatory approach was followed throughout the programme.
11. Lecture synopses and Graded Problems Lists were distributed to highlight the important issues related to teaching of Mathematics.

Seminar Topics :

1. Sequences and Series
2. Indian Mathematicians
3. Techniques and Simple Methods of Solving Integration Problems
4. History of Mathematics
5. Coaxial System of Circles
6. Projects in Mathematics
7. Graph Theory and Mathematical Modelling
8. Application of Vector
9. General Equations of Second Degree – Geometrical Interpretations
10. Applications of Calculus
11. Computers in Teaching of Mathematics
12. Mathematics Club Activities
13. Curve Tracing
14. Coordinate System
15. Translation and Rotation
16. Motivational Strategies in Mathematics Learning
17. Mathematics Induction
18. Development and uses of Logarithmic and exponential Functions

Issues/Topics covered during the Programme :

1. Permutation :
   - Meaning and explanation
   - Problem solving

2. Combinations :
   - Meaning and explanation
   - Derivation of formula
   - Problem Solving

3. Binomial theorem :
   - Binomial coefficients
   - Properties of Binomial Coefficients
   - Problem Solving

4. Probability :
   - Concept of Probability
   - Introducing Probability
   - Various Definitions of Probability
   - Problem Solving based on Mathematical Probability
   - Addition and Multiplication Theorems of Probability
   - Conditional Probability – Concept, Formula, Problems
   - Bayese Theorem – Proof, Problems based on this theorem
   - Geometric Probability – Problem Solving
   - Random Variable – Concept, Definition, types, Distribution and Properties, Distribution Function Properties, etc.
   - Binomial Distribution – Concept, Derivation of Probability Function, Properties, Problems
   - Poisson Distribution – Concept, Situations, Derivation of P D as a limiting case of B D, Properties, Problem solving.
   - Applications of Probability

5. Analytical Geometry
   a) Coordinates – Systems of Coordinates
   b) Straight line – Various equations
   c) Pair of straight lines – Angle between the lines
   d) Circles – various equations
      - Orthogonality
      - Radical axis of two triangles
      - Coaxal System of triangles – Limiting Points
      - Conjugate family of triangles

   e) Family of triangles

f) Conics – Parabola/Ellipse/Hyperbola
  - Geometrical meanings and Analytical definitions
  - Various Standard Equations
  - Properties of conics
  - Reduction to Standard forms
  - Tangent and normal

g) Transformation of Coordinate axes
  - Translation
  - Rotation

h) Reduction of a general equation of second degree to the standard forms of equations to conics.

6. Complex Numbers

a) Complex Numbers – Cartesian - Polar forms

b) Modulus / Amplitude of a complex number

c) Geometrical Representation of a Complex Number – Argand diagram
Geometrical Constructions for : $Z_1, \pm Z_2, Z_1 .Z_2, Z_1 / Z_2, (Z_2 \neq 0), Z.$

d) Properties of Complex Numbers – Structure of the system

e) De Moivre's Theorem

f) nth root of a complex number and their geometrical representations

g) The cube roots of unity – properties and the nth roots of unity

7. Vectors :

Concept of vectors, operations with vectors (scalr products) and products of vectors, triple products. All the algebraic properties of vectors (distributive, associative, etc). Linear dependency and independency of vectors, coplanar vectors, division of joins of two points (m : n ratio), parametric equations of lines (vectorial form), equation of plane, skew lines and the shortest distance between them, application of vectors in solving geometric problems.

8. Trigonometry : Trigonometric functions ($\sin\theta$, $\cos\theta$, $\tan\theta$, $\sec\theta$, $\mathrm{cosec}\,\theta$, $\cot\theta$), curves of the above functions, inverse functions of trigonometric functions, their curves.

Exponential functions, hyperbolic functions, their graphs, inverse hyperbolic functions and their graphs.

9. Algebra :

Sets – Subsets, Union and intersection of sets, De Morgan's law, Cartesian products of sets

Relations as a subset of the Cartesian product of two sets. Relation as a set, Different kinds of relations, equivalence relations.

Function as a special type of relations, function defined as a map, 1-1 and onto and bijective maps.

Groups – definition and examples, subgroups, elementary theorms as groups.

Sequences and series. Arithmetic, geometric and Harmonic series

10. Calculus :

Functions – Real valued and complex valued functions, notion of limit of a function, results as limits of functions, one sided and two sided limits.
Continuity of function, ε-δ definition and limit definition examples of continuous functions, Simple properties of continuous functions.
Applications of derivatives – Approximations.

The scope and importance of the Fundamental Theorem of Integral Calculus, Application of Definite Integrals.
How to introduce and teach Integral Calculus to the student, Historical perspectives and pictorial representation of $d(x^2) / dx = 2x$, $d(x^8) / dx = 3dx$ and $d(kx)/dx = k$.

Order and degree of an ordinary differential equation, formation of a differential equation and easy examples of application of differential equation.

11. Evaluation Techniques – Error Analysis
12. Motivational Strategies
13. Computers :
    What is a Computer ?
    Parts of computer.
    Capabilities of computer
    Use of Computers in Mathematics Teaching
    MultiMedia

14. Teaching Aids (Maths Lab) : About 60 teaching aids were discussed. Their applications in the classroom situations were given. All the participants have taken note of them.

15. Projects in Mathematics : How a basic result is arrived at by doing projects in Mathematics and how the validity of the results are found out by Mathematical methods was discussed. Examples of 10-15 projects were prepared.

Materials / Test Items / Graded Problems distributed to the participants.
Film Shows screened :

1. The Binomial Theorem
2. Conics

Reference Books :

1. S L Loney : Analytical Geometry
2. George B Thomas : Analytical Geometry and Calculus
3. Hille : Calculus of Several Variables
4. Manicka Vachakam Pillay : Analytical Geometry
5. Analytical Geometry (Schaum's Outline Series)
6. Shanti Narayana : Vector Algebra
7. Thomas and Finny : Calculus with Analytic Geometry
8. J N Kapoor : Reflections of a Mathematician
9. Shanti Narayana : A Textbook of Matrices
10. S Balachandra Rao : History of Indian Mathematics
11. Courant and Robbins : What is Mathematics ?
12. Lancelot Hogben : Mathematics for the Million
13. Gorakh Prasad : Textbook of Coordinate Geometry
14. Hoffman and Kunze : Linear Algebra
15. Richard Dahlke and Others : A sketch of the History of Probability Theory (Mathematical Education, April - June 1989 pp 218 – 232)
16. Fundan and Kapoor : Fundamentals of Mathematical Statistics
17. T Cacoullos : Exercises in Probability
18. Schaum's Series : Probability
    Vector Algebra
    Integral Calculus
19. NCERT : Mathematics – XI and XII class books (3 volumes each)
20. Sudhir Kumar : Teaching of mathematics
21. Kulbir Singh Sidhu : The Teaching of Mathematics
22. Max A Sobel : Teaching Mathematics – A Source Book of Aids, Activities and Strategies

Time Schedule

The programme has been organized according to the following schedule.

Annexure I

TRAINING PROGRAMME IN MATHEMATICS FOR JUNIOR LECTURERS OF APSWREI SOCIETY

Date	Session I 9.30 – 11.00 am	Session II 11.15 – 12.45 pm	Session III 2.00 – 3.15 pm	Session IV 3.30- 4.45 pm
22.6.98	Registration	Exploratory Session	Algebra – I	Maths Club
23.6.98	Permutations	Algebra – II	Problem Solving	Film Show
24.6.98	Vectors	Combinations	Geometry – I	Seminar – I
25.6.98	Vectors	Algebra – III	Geometry – II	Seminar – II
26.6.98	Trigonometry – I	Algebra – IV	Geometry – III	Seminar – III
27.6.98	Probability	Trigonometry – II	Geometry – IV	Seminar – IV
28.6.98	Problem Solving		Library	
29.6.98	Calculus – I	Error Analysis	Geometry – V	Seminar – V
30.6.98	Calculus – II	Complex Analysis	Integration - I	Seminar – VI
1.7.98	Probability - III	Integration - II	Seminar – VII	Seminar – VIII
2.7.98	Differential Equation	Individual Problems	Test	Computer Lab
3.7.98	Probability	Seminar IX – Error Analysis-	Concluding Session	

Annexure II

List of Participants

1. V Veera Pratap
   APSWR Junior College
   Kalasamadram, Kadri
   Anantapur 515501, A P

2. T V Satyanarayana
   APSWR Junior College
   Kanchiti, Srikakulam Dt
   Andhra Pradesh 532 290

3. Bobbili Ramakrishna
   APSWR Junior College
   Sabbavaram, Visakhapatnam Dist
   Andhra Pradesh 531 035

4. M Rajasekhar
   APSWR Junior College
   Badangi
   Vizianagaram Dt 532 578

5. V Venkatesulu
   APSWR School/College
   C Belagal, Kurnool Dist, AP

6. C Jayaprakash
   APSWR Junior College
   Koilakuntla, Kurnool Dist, AP

7. K Atchaiah
   APSWR Junior College
   B Mattam, Cuddapah Dist, AP

8. K Ranga Swamy
   APSWR Junior College
   Shaikpet, Hyderabad Dist
   Hyderabad 500 008, AP

9. Ch. Pratap Reddy
APSWR (Boys) Junior College
Bhongir, Nalgoda Dist
A P 508 116

10. V K Bhaskar
APSWR School and Junior College (Boys)
Ramakuppam 517 401, Chittoor Dist, AP

11. A Chandrasekhar Reddy
A P S W R Junior College
Chinnatekur, Kurnool Dist, AP

12. Panduranga Rao Duggarazu
APSWR Junior College
Atchampet, Guntur Dist
AP 522 409

13. B Ramesh Babu
APSWR Junior College
Karempudi (P & M)
Guntur Dist 522 614, A P

14. Racherla Bhaskar
APSWR Junior College
Velugonda, Kaluzuvvalapadu PO
Mandal : K K Metla
Prakasam 523 241, AP

15. V Rajani Kumari
APSWR Junior College
Narsingi, Ranga Reddy Dist
Andhra Pradesh

16. K Krutha Murthy
APSWR Junior College (Boys)
Maidipally, Karimnagar 505 453, AP

17. T Sivannarayana
APSWR Junior College
A R Pally PO
Khammam Dt 507 316

18. M N V K Durga Rathnam
APSWR Junior College
Polasanipalli 534425, A P

19. K Usha Rani
APSWR Junior College
Cheepurupalli
Vijayanagaram Dist, A P

20. K Murali Krishna
Dr B R A C S W R Junior College
L N Puram,
Ananparthi P O, East Godavari Dist

21. V V Giri
APSWR Junior College
Tiruvuru, Krishna Dist, A P

22. V B V V S S S H S R Sastry
APSWR Junior College
Arugolanu, Via : Prathipadu
West Godavari Dist
A P 534 152

23. V Lakshmi Ganapathi Rao
APSWR Junior College
Golugonda, Visakapatnam Dist, A P

24. Chakradhara Rao Govada
APSWR Junior College (Boys)
Ameenapet, Eluru - 6
West Godavari Dist, A P

25. S V Ramana
APSWR Junior College
Koperla, Vijayanagaram Dist, A P

26. D Srinivasa Raju
APSWR Junior College
Paloncha, Khammam Dist, A P

27. Y Manikya Rao
APSWR Junior College
Asifabad, Adilabad Dt 504 293, A P

28. K Venkateswara Rao
APSWR Junior College
Duppalavalasa 532 005
Srikakulam Dist, A P

29. D Sailaja
APSWR Junior College
Muthukula, Nellore Dist, A P

30. M Ramanamma
Dr B R A C S W R Junior College
Etcherla, A P

31. K Ravindra Babu
APSWR Junior College
Chillakur (P & M)
Nellore Dist, A P 524 412

32. Maha Lakshmi Yelchuri
APSWR Junior College (Girls)
V P South, Guntur Dist, A P

33. Dr A Madhusudhan
APSWR Junior College
J P Nagar, Kalwakurthur
Mahaboob Nagar Dt 509 324

34. V Jagan Mohana Chary
APSWR Junior College
Uppalwai, Nizamabad Dt

35. Krishnamacharyulu K
Dr B R A C S W R Junior College
Hatnoora, Medak Dist,
A P 502 296

36. C H Hariprasad
APSWR Junior College
Naidupeta, Nellore Dist, A P

37. V Sitaramamurthy
APSWR School/College
Devarapalli 531 030
Visakhapatnam Dist, A P

Annexure III

List of Resource Persons

1. Dr D Basavayya
2. Dr V Shankaram
3. Dr N M Rao
4. Dr B S Upadhyaya
5. Dr N B Badrinarayana

Annexure IV

Observations and Suggestions :

1. Majority of the participants were not interested in the beginning of this programme as they thought this also a routine type of programmes which they have attended earlier.
2. Majority of the participants were below the average standard.
3. Some of the participants were very casual in coming and attending the programme and not of hard working type.
4. Few participants were behaving like a trade union leaders.
5. All participantswere appreciating the exposure to giving seminars.
6. Most of the participants were concerned in producing more than 60% results without taking much trouble.
7. Except one (Mr V V Giri), others were unwilling to write the post-test as they were not confident in their knowledge of content and problem solving.
8. Only motivated participants should be allowed to attend such enrichment programmes.
9. Majority of them were interested in sight-seeing.
10. Participants' (lecturers) attitudes and standards might be the main causes of low achieveent in the Social Welfare Institutions.

D. BASAVAYYA
R.I.E.
MYSORE-6.

PROBABILITY

In our day-to-day life we perform certain activities to verify certain known facts or to observe certain phenomena. Such activities usually we call as experiments. In certain experiments, we can predict results exactly before conducting the experiment and in other it will not be possible. The experiments where the results can be predicted exactly are known as deterministic experiments and the experiments where the prediction is not exact are known as non-deterministic or random or probabilistic experiments. For example, a train is running at a uniform speed of sixty k.m. per hour, then we can predict with hundred percent surety that it will cover one hundred twenty kilometers after two hours, assuming that it never stopped during these hours. Similarly, for a perfect gas, PV = constant (P is pressure, V is volume).

In case of non-deterministic experiments, we cannot make predictions with complete reliability. The results are based on some 'chance element'. For example, if we toss a coin, will it show 'head up' or 'tail up'? Although we cannot predict anything with complete surety, yet if we throw the coin a large number of times, it is very likely that the head will turn up fifty percent of the times and also it is very unlikely that the head turns up in every case.

Consider another example of a trained parachuter who is ready to jump. When he jumps then either his parachute will open or it will not. But experience says that most of the time it opens, though there are occasions on which it does not i.e. the uncertainty associated with the head or tail coming up when we toss a coin.

How will you proceed in answering the following questions ?

1. How should a businessman order for replenishment (filling once again) of his stocks (inventory) so that he has not carried very large stocks, yet the risk of refusing customers is minimized ? (Inventory problem).
2. At what intervals should a car owner replace the car so that the total maintenance expenses are minimized ? (Replacement problem).
3. How many trainees should a large business organisation recruit and train them in certain intervals so that at any time it does not have a large number of trained persons whom it cannot employ and yet the risk of its being without sufficient persons when needed is minimized ?
4. How should the bus service in a city be scheduled so that the queues do not become too long and yet the gains by the bus company are maximized ? (Queing problem).
5. How many booking counters should be provided at a station to serve in the best way the interests of both the railways and the travelling public ? (Queing problem).

6. What should be the strength of a dam (or a bridge) so that its cost is reasonable and yet the risk of its being swept away by the floods is minimized ?

7. How many telephone exchanges should be established in a given city so as to give the best service at a given cost ?

8. Which variety is the best out of given varieties of wheat, on the basis of yields from experimental fields ?

9. What should be the minimum premia charged by an insurance company so that the chance of its running into loss is minimized ?

10. How to decide whether a given batch of items is defective when only a sample of the batch can be examined ?

Answers for all such questions are based upon certain facts and then try to measure the uncertainty associated with some events which may or may not materialise. The theory of probability deals with the problem of measuring the uncertainty associated with various events rather precisely, making it these by possible today, to a certain extent of course, to control phenomena depending upon chance.

The 'measure of uncertainty' is known as probability.

History of Probability Theory

Probability had its birth in the seventeenth century and over the last three hundred years, it has progressed rapidly from its classical heritage of simple mathematical and combinatorial methods to its present rigorous development based on modern functional analysis. The probability had its origin in the usual interest in gambling that pervaded France in the seventeenth century. Eminent mathematicians were led to the quantitative study of games of chance. The Chevalier de Mere, a French nobleman and a notorious gambler, posed a series of problems to B Pascal (1623-1662) like the following :

> Two persons play a game of chance. The person who first gains a certain number of points wins the stake. They stop playing before the game is completed. How is the stake to be divided on the basis of the number of points each has got ?

Though Galileo (1564-1642) had earlier solved a similar problems, this was the beginning of a systematic study of chance and regularity in nature. Pascal's interest was shared by Fermat (1601-1665), and in their correspondence the two mathematicians laid the foundation of the theory of probability. Their results aroused the interest on the Dutch physicist Huyghens (1629-1695) who started working on some difficult problems in games of chance, and published in 1654 the first book on the theory of probability. In this book, he introduced the concept of mathematical expectation which is basic to the modern theory of probability. Following this, Jacob Bernoulli (1654-1705) wrote his famous 'Art Conjectandi' the result of his work of over twenty years. Bernoulli approached this subject from a very general point of view and clearly foresaw the wide applications of the theory. Important contributions were made by Abraham de Moivre (1667-1754) whose book 'The Doctrine of Chance' was published in 1718. Other main contributors were T.Bayse (Inverse Probability), P.S. Laplace (1749-1827) who after extensive research published 'Theoric Analytique des probabilities' in 1812. In addition to these Levy, Mises and R.A.Fisher were the main contributors. It was, however, in the work of Russian mathematicians Tschebyshev (1821-1874), A Markov (1856-1922), Liapounov (Central Limit theorem), A Kintchine (Law of Large Numbers) and A. Kohnogorov that the theory made great strides. Kolmogoroff was the person who axiomised the calculus of probability.

The probability theory itself has developed in many directions, but at present the dominant area is the stochastic processes, which has wide applications in physics, chemistry, biology, engineering, management and the social sciences.

Calculus of Probability

In our day-to-day vocabulary we use words such as 'probably', 'likely', 'fairly good chances', etc. to express the uncertainty as indicated in the following example. Suppose a father of a XII class student wants to know his son's progress in the studies and asks the concerned teacher about his son. Teacher may express to the father about the student's progress in any one of the following sentences.

It is <u>certain</u> that he will get a first class.
He is <u>sure</u> to get a first class.
I <u>believe</u> he will get a first class.
It is <u>quite likely</u> that he will get a first class.
<u>Perhaps</u> he may get a first class.
He <u>may or he may not</u> get a first class.
<u>I believe he will not</u> get a first class.
<u>I am sure he will not</u> get a first class.
I am <u>certain he will not</u> get a first class.

Instead of expressing uncertainty associated with any event with such phrases, it is better and exact if we express uncertainty mathematically. The measure of uncertainty or probability can be measured in three ways and these are known as the three definitions of probability. These methods are

Mathematical or Classical or Priori Probability
Statistical or Empirical Probability and
Axiomatic Probability

Before discussing those methods, we define some of the terms which are useful in the definition of probability.

Experiment : An act of doing something to verify some fact or to obtain some result. (Ex. Throwing a die to observe which number will come up (Die is a six-faced cube).

Trial : Conducting experiment once is known as the trial of that experiment. Ex. Throwing a die once.

Outcomes : The results of an experiment are known as outcomes. Ex. In throwing a die, getting '1' or '2' or '6' are the outcomes.

Events: Any single outcome or set of outcomes in an experiment is known as an event.

Ex: 1. Getting '1' in throwing of a die is an event.

Also getting an even number in throwing a die is also an event.

Ex: 2. Drawing two cards from a well shuffled pack of cards is a trial and getting of a king and a queen is an event.

Exhaustive Events : The total number of possible outcomes in any trial are known as exhaustive events.

Ex: 1. In tossing a coin there are two exhaustive events.

2. In throwing a die, there are six exhaustive cases viz (1,2,3,4,5,6).

Favourable Events (Cases): The number of outcomes which entail the happening of an event are known as the favourable cases (events) of that event.

Ex: In throwing two dice, the number of cases favourable for getting a sum of 5 are (1,4), (2,3), (3,2) and (4,1).

Mutually Exclusive Events : Events are said to be mutually exclusive or incompatible if the happening of any one of them precludes or excludes the happening of all others.

Ex: In tossing a coin, the events head and tail are mutually exclusive (because both cannot occur simultaneously).

Mathematical or Classical or 'a priori' probability

If a trial results in 'n' exhaustive, mutually exclusive and equally likely cases and 'm' of them are favourable to the happening of an event E, then the probability 'p' of happening of E is given by

p = Favourable number of outcomes / Total No of outcomes = m/n

We write p = P(E).

Ex:1. Probability of getting head in tossing of a coin once is ½ because the number of exhaustive cases are 2 and these are mutually exclusive and equally likely (assuming the coin is made evenly) and of these only 1 case is favourable to our event of getting head.

Ex: 2. The probability of getting a number divisible by 3 in throwing of a fair (evenly made) die is 2/6 because the favourable cases are 3 (viz. 3 and 6) and exhaustive cases are 6.

The probability 'q' that E will not happen is given by

$$q = \frac{n-m}{n} = 1 - \frac{m}{n} = 1 - p$$

Always $0 \le p \le 1$.

If $p = P(E) = 1$, E is called a certain event and if $P(E) = 0$, E is called an impossible event.

In this method, the mathematical ratio of two integers is giving the probability and therefore, this definition is known as mathematical definition. Here we are using the concept of probability in the form of 'equality likely cases' and therefore, this definition is a classical definition. Before using this definition, we should know about the nature of outcomes (viz. Mutually exclusive, exhaustive and equally likely) and therefore, it is also known as 'a priori' probability definition.

The definition of mathematical or classical probability definition breaks down in the following cases: 1. If the various outcomes of the trial are not equally likely. 2. If the exhaustive number of cases in a trial is infinite.

Ex:1. When we talk about the probability of a pass of a candidate, it is not ½ as the two customers 'pass' and 'fail' are not equally likely.
Ex: 2. When we talk about the probability of a selected real number is to be divided by 10, the number of exhaustive cases are infinite.

In such above mentioned circumstances it is not possible to use mathematical probability definition. Therefore, probability is defined in the other way as below :

Statistical or Empirical Probability :

If a trial is repeated a number of times under essentially homogeneous and identical conditions, then the limiting value of the ratio of the number of times an event happens to the number of trials, as the number of trials becomes indefinitely large, is called the probability of happening that event.

Mathematically, we write

$$p = P(E) = \lim_{n \to \infty} \left(\frac{m}{n}\right)$$

Here n is the number of trials and m is the number of times of the occurrence of event E. The above limit should be finite.

Ex: When you throw a die 10000 times and if you get 1600 times the number '1', then the probability of getting '1' is 1600/10000. This ratio is nothing but the relative frequency of '1'.

But this definition is also not applicable always because it is very difficult to maintain the identical conditions throughout the experiment. Therefore, the probability is defined in another way by using certain axioms. This definition is known as 'Axiomatic Probability' definition.

Here we define some of the terms which are useful in the 'Axiomatic Probability' definition.

Sample Space: The set of all possible outcomes of an experiment is known as the sample space of that experiment. Usually we denote it by S. Ex: In tossing a coin, S = { H, T }.

Sample Point : Any element of a sample space is known as a sample point.

Ex: In tossing a coin experiment, H or T is a sample point.

Event: Any subset of a sample space is an event.
Ex: In throwing a die, (1,3,5), (2,4,6) or (5,6) are the events where S = {1,2,3,4,5,6}.

If A and B are any two events then $\bar{A}$, $\bar{B}$, AUB, $A \cap B$ are also events because they are also subsets of S.

The event S (entire sample space) is known as certain event and the event Φ (empty set) is known as impossible event.

Mutually Exclusive Events : Events are said to be mutually exclusive if the corresponding sets are disjoint.
Ex: In throwing of a die experiment, if A = (1,3,5) and B = (2,4,6) then A and B are mutually exclusive because we cannot get both odd number and even number simultaneously. That is, if $A \cap B = \Phi$, then A and B are mutually exclusive events.

Axiomatic Probability :

Let S be a sample space and ξ be the class of events. Also let P be a real valued function defined on ξ. Then P is called a probability function and P(A) is called the probability of the event A if the following axioms hold :

i) For every event A, $0 \leq P(A) \leq 1$.
ii) P(S) = 1.
iii) If A and B are mutually exclusive events, then P(AUB) = P(A) + P(B).
iv) If A_1, A_2, ... is a sequence of mutually exclusive events, then
$P(A_1 U A_2 \ldots) = P(A_1) + P(A_2) + \ldots\ldots$

In the above definition axiom (iv) may seem to be not necessary. But it is necessary to stress that axiom (iii) should be extended to more than two events.

Theorem 1 : If Φ is the empty set, then $P(\Phi) = 0$.

Proof : We know that $S = S \cup \Phi$ and $P(S) = P(S \cup \Phi) = P(S) + P(\Phi)$.
(because S and Φ are disjoint and according to axiom (iii)). But $P(S) = 1$ and therefore,
$1 = 1 + P(\Phi)$.

$\therefore P(\Phi) = 0$.

Theorem 2 : If $\bar{A}$ is the complement of an event A, then

$P(\bar{A}) = 1 - P(A)$.

Proof : $A \cup \bar{A} = S$.

$P(A \cup \bar{A}) = P(A) + P(\bar{A}) = P(S)$ (A and $\bar{A}$ are disjoint).

But $P(S) = 1$, therefore,
$P(A) + P(\bar{A}) = 1$
Or $P(\bar{A}) = 1 - p(A)$.

Theorem 3 : If $A \subseteq B$, then $P(A) \leq P(B)$.
Proof : We know that if $A \subsetneq B$, then
$B = A \cup (B - A)$ (here we may use the notation B/A)
So, $P(B) = P(A) + P(B-A)$
But from axiom i, $P(B-A) \geq 0$
$\therefore P(B) \geq P(A)$.

Theorem 4 : If A and B are any two events, then
$P(A-B) = P(A) - P(A \cap B)$

Proof: We can write, $A = (A \cap B) \cup (A - B)$

But $(A \cap B)$ and $(A - B)$ are disjoint and according to axiom (iii).
$P(A) = P(A \cap B) + P(A - B)$.
Or $P(A - B) = P(A) - P(A \cap B)$.

Theorem 5 : (Addition Theorem)

If A and B are any two events, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Proof: We can write, AUB = BU(A-B). But B and (A-B) are disjoint and therefore, by axiom (iii),
P(AUB) =P(B) + P(A-B).

Also, from theorem 4, P(A-B) = P(A) – P(A $\cap$ B)

Hence, P(AUB) = P(B) + P(A-B)
= P(B) + P(A) – P(A $\cap$ B)

This theorem is known as addition theorem and it can be extended to any number of events as follows :

Theorem 6 : (Addition Theorem in case of n events)

If $A_1, A_2, \dots, A_n$ are any n events, then

$$P(A_1 \cup A_2 \cup \dots A_n) = \sum_{i,j=1}^{n} P(A_i \cap A_j) - \sum_{i,j,k=1;\ i<j<k}^{n} P(A_i \cap A_j \cap A_k) + \dots + (-1)^{n-1} P(A_1 \cap A_2 \cap \dots \cap A_n)$$

Proof: This theorem can be proved by the method of induction. For the events A_1 and A_2 we have from theorem 5,

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$$

$$= \sum_{i=1}^{2} P(A_i) + (-1)^{2-1} P(A_1 \cap A_2)$$

Hence the theorem is true for n = 2.
Now, suppose the theorem is true for n = r, say

Then,

$$P(A_1 \cup A_2 \cup \dots \cup A_r) = \sum_{i=1}^{r} P(A_i) - \sum_{i,j=1,\ i<j}^{r} P(A_i \cap A_j) + \dots + (-1)^{r-1} P(A_1 \cap A_2 \cap \dots \cap A_r)$$

Now,

$$P(A_1 \cup A_2, \dots, \cup A_r \cup A_{r+1}) = P(A_1 \cup A_2 \cup \dots \cup A_r) \cup A_{r+1})$$

$$= P(A_1 \cup A_2 \cup \dots \cup A_r) + P(A_{r+1}) - P((A_1 \cap A_{r+1}) \cup (A_2 \cap A_{r+1}) \dots \cup (A_r \cap A_{r+1})$$

$$= \sum_{i=1}^{r} P(A_i) - \sum_{i,j=1,\ i<j}^{n} P(A_i \cap A_j) + \dots (-1)^{r} P(A_1 \cap A_2 \cap \dots A_{r+1})$$

$$+ P(A_{r+1}) - \sum_{i=1}^{r+1} P(A_i \cap A_{r+1}) + \sum_{i,j=1,\ i<j}^{r+1} P(A_i \cap A_j \cap A_{r+1}) + \dots + (-1)^{r} P(A_1 \cap A_2 \cap \dots \cap A_{r+1})$$

$$= \sum_{i=1}^{r+1} P(A_i) - \sum_{i,j=1,\ i<j}^{r+1} A(A_i \cap A_j) + \dots + (-1)^{r} P(A_1 \cap A_2 \cap \dots A_{r+1})$$

Hence, if the theorem is true for n=r, it is also true for n=r+1. But we have proved that the theorem is true for n = 2. Hence by the method of induction, the theorem is true for all positive integer values of n.

Corollary 1 : If A and B are two mutually exclusive events, then,
$P(A \cup B) = P(A) + P(B)$.

Corollary 2 : If $A_1, A_2, \ldots A_n$ are n mutually exclusive events,

Then $P(A_1 \cup A_2 \ldots \cup A_n) = P(A_1) + P(A_2) + \ldots + P(A_n)$

Conditional Probability :

So far, we have assumed that no information was available about the experiment other than the sample space while calculating the probabilities of events. Sometimes, however, it is known that an event A has happened. How do we use this information in making a statement concerning the outcome of another event B ?

Consider the following examples.

Ex.1: Draw a card from a well-shuffled pack of cards. Define the event A as getting a black card and the event B as getting a spade card. Here P(A) = ½ and P(B) = ¼ . Suppose the drawn card is a black card then what is the probability that card is a spade card? That is, if the event A has happened then what is the probability of B given that A has already happened? This probability symbolically we write as P(B/A). In the given example,

$$P(B/A) = \frac{1}{2} = \frac{P(A \cap B)}{P(A)} = \frac{(1/4)}{(1/2)}$$

Because probability of simultaneous occurrence of A and B is ¼ and probability of A is ½.

Ex.2: Let us toss two fair coins. Then the sample space of the experiment is S = { HH, HT, TH, TT}. Let event A = { both coins show same face } and B = { at least one coin shows H }. Then P(A) = 2/4. If B is known to have happened, this information assures that TT cannot happen, and P {A, conditional on the information that B has happened } =

$$P(A/B) - 1/3 - \frac{1/4}{3/4}$$

$$= \frac{P(A \cap B)}{P(B)}$$

In the above two examples, we were interested to find the probability of one event given the condition that the other event has already happened. Such events based on some conditions are known as conditional events. In the above examples B/A and A/B are the conditional events. The probability of a conditional event is known as conditional probability of that event. We write the conditional probabilities as P(A/B), P(E/F), etc.

Definition of conditional probability : The conditional probability of an event A, given B, is denoted by P(A/B) and is defined by

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

Where A, B and $A \cap B$ are events in a sample space S, and $P(B) \neq 0$.

From the definition of conditional probability we know that

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

Therefore, we can write from the above

$P(A \cap B) = P(B)\, P(A/B)$

Also, we know that $P(A \cap B) = P(B \cap A)$ and

$P(B \cap A) = P(A)\; P(B/A)$

Hence we can write

$P(A \cap B) = P(A)\; P(B/A)$ or $P(B)\; P(A/B)$.

The above result is known as multiplication law of probabilities in case of two events.

Multiplication Theorem of Probabilities : If A and B are any two events of a sample space S, then

$P(A \cap B) = P(A)\; P(B/A)$ or $P(B)\; P(A/B)$.

The above theorem can be extended to any n events as follows :

If $A_1, A_2, \ldots, A_n$ are any n events, then

$P(A_1 \cap A_2 \ldots \cap A_n) = P(A_1)\; P(A_2/A_1)\; P(A_3/A_1 \cap A_2) \ldots\ldots P(A_n / A_1 \cap A_2 \ldots \cap A_{n-1})$.

This theorem can be proved by method of induction or generalization.

Baye's Theorem : If $E_1, E_2, \ldots, E_n$ are mutually exclusive events with $P(E_i) \neq 0$, $(i = 1,2,\ldots n)$ then for any arbitrary event A which is a subset of $\bigcup_{i=1}^{n} E_i$ such that $P(A) > 0$, we have

$$P(E_i / A) = \frac{P(E_i)\, P(A/E_i)}{\sum_{i=1}^{n} P(E_i)\, P(A/E_i)} \quad \text{for all i.}$$

Proof: Since $A \subset \bigcup_{i=1}^{n} E_i$ we have

$$A = A \cap (\bigcup_{i=1}^{n} E_i) = \bigcup_{i=1}^{n} (A \cap E_i) \text{ (by distributive law).}$$

Since $(A \cap E_i) \subset E_i$ (for i = 1,2,...., n) are mutually exclusive events, we have by additional theorem of probability

$$P(A) = P[\bigcup_{i=1}^{n} (A \cap E_i)] = \sum_{i=1}^{n} P(A \cap E_i) = \sum_{i=1}^{n} P(E_i)\, P(A/E_i)$$

(By multiplication theorem in case of two events.)

Also, we have

$P(A \cap E_i) = P(A) \quad P(E_i/A)$ and

$$P(E_i / A) = \frac{P(A \cap E_i)}{P(A)} = \frac{P(E_i)\, P(A/E_i)}{P(A)}$$

Hence, $P(E_i/A) = \dfrac{P(E_i)\, P(A/E_i)}{\sum_{i=1}^{n} P(E_i)\, P(A/E_i)}$

This theorem is very useful in calculating the conditional probabilities in certain situations.

If $P(A \cap B) = P(A)\, P(B)$, then we see that $P(B/A) = P(B)$ and hence we say that the probability of B is not depending upon the happening of A. That is the conditional probability of B is same as the unconditional probability of B. Such events are called independent events.

Two events A and B are independent if and only if

$P(A \cap B) = P(A)\; P(B)$.

Ex: Let two fair coins be tossed and let

A = { head on first coin }, B = { head on the second coin }.

Then P(A) = P {HH, HT } = ½

P(B) = P { HH, TH } = ½ and

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{1/4}{1/2} = 1/2 = P(A)$$

Thus,

P (A ∩ B) = P(A) P(B).

and we know that the probability of getting head on the first coin does not depend upon the probability of getting head on the second coin. Hence A and B are independent. Also we see that the condition P(A ∩ B) = P(A) P(B) is both necessary and sufficient for those events A and B to be independent.

If there are three or more than three events, we will have the situation where every pair of these events are independent or the situation where the events in every set of events are independent. In the first case, we call the events as pairwise independent and in the second case we call as complete or mutual independent events.

Geometric Probability :

Sample space can be countably finite or countably infinite or uncountably finite or uncountably infinite depending upon the situation. If the sample space is countably finite, then it is easy to calculate the probability of any event by using either mathematical probability or axiomatic probability definition. Even if the sample space is countably infinite say S = (e_1, e_2,...) we obtain a probability space assigning to each $e_i \in S$ is a real number p_i, called its probability, such that

$$P_i \geq 0 \text{ and } p_1 + p_2 + \ldots = \sum_{i=1}^{\infty} P_i = 1$$

The probability P(A) of any event A is then the sum of the probabilities of its points.

Consider the sample space S ={ 1,2,... } of the experiment of tossing a coin till a head appears; here n denotes the number of times the coin is tossed. A probability space is obtained by

$$P(1) = 1/2, \quad P(2) = 1/4, \ldots, P(n) = \frac{1}{2^n}, \ldots$$

But the calculation of probability of events regarding an uncountably finite or infinite sample space is not so easy.

Consider a situation of selecting a point at random on a line segment of length 'l'. Here the sample space is uncountably finite and the procedure to find the probability of any event in case of countable sample space is not applicable.

Consider another example. Suppose that two friends have agreed to meet at a certain place between 9 a.m. to 10 a.m. They also agreed that each would wait for a quarter of an hour and, if the other did not arrive, would leave. What is the probability that they meet ?

In the above example also both the sample space and the given event are uncountable and the ordinary procedures of calculation of probability are not applicable. So we need different procedure in such cases.

If the sample space is uncountably finite, we present that sample space by some geometrical measurement, m(S) such as length, area of volume, and in which a point is selected at random. The probability of an event A, i.e. the selected point belongs to A, is then the ratio of m(A) to m(S) is

$$P(A) = \frac{\text{length of } A}{\text{length of } S} \quad \text{or} \quad P(A) = \frac{\text{area of } A}{\text{area of } S} \quad \text{Or} \quad P(A) = \frac{\text{volume of } A}{\text{volume of } S}$$

Such probability is known as 'geometrical probability'.

Solved Problems :

1. A bag contains 5 red, 4 white and 3 blue balls. What is the probability that two balls drawn are red and blue ?

Sol : Total number of balls = 5 + 4 + 3 = 12

The number of ways of drawing two balls out of 12 balls = $12^{c}2 = \frac{12\times 11}{2} = 66$ *ways*

The number of ways of drawing 1 red ball out of 5 red balls = 5 ways.

The number of ways of drawing 1 blue ball out of 3 blue balls = 3 ways.

The number of ways of drawing 1 red ball out of 5 red balls and 1 blue ball out of 3 blue balls = 5 x 3 = 15 ways.

The required probability = 15/66 = 5/22, by using Mathematical probability definition.

2. If the letters of the word 'STATISTICS' are arranged at random to form words, what is the probability that three S's come consecutively ?

Sol: Total no. of letters in the word 'STATISTICS' = 10. Total no. of arrangements of these 10 letters in which 3 are of one kind (viz. S) , 3 are of second kind (viz. T), 2 are of third kind (viz. I), 1 of fourth kind (viz. A) and 1 of fifth kind (viz. C).

$$= \frac{10!}{3!\ 3!\ 2!\ 1!\ 1!}$$

Following are the 8 possible combinations of 3 S's coming consecutively.

i) in the first three places

ii) in the second, third and fourth places

iii) in the eighth, ninth and tenth places

Since in each of the above cases, the total number of arrangements of the remaining 7 letters, viz. TTTIIAC of which 3 are of one kind, 2 of second kind, 1 of third kind and 1 of fourth kind

$$= \frac{7!}{3!\ 2!\ 1!\ 1!}$$

and the required number of favourable cases = $\frac{8 \times 7!}{3!\ 2!\ 1!\ 1!}$

Hence the required probability

$$= \frac{\textit{Favourable Cases}}{\textit{Total No of cases}} = \frac{8 \times 7!}{3!\ 2!\ 1!\ 1!} \Big/ \frac{10!}{3!\ 3!\ 2!\ 1!\ 1!}$$

$$= \frac{8 \times 7! \times 3!}{10!} = \frac{1}{15}$$

3. What is the probability that a leap year selected at random will contain 53 Sundays ?

Sol : In a leap year, there are 366 days of 52 complete weeks and 2 days more. In order that a leap year selected at random should contain 53 Sundays, one of these extra 2 days must be Sunday. But there are 7 different combinations with these two extra 2 days viz. Sunday and Monday, Monday and Tuesday, etc. Out of these 7 possible ways, only in 2 ways we are having an extra Sunday.

∴ Required probability =2/7.

4. Two dice are thrown simultaneously. What is the probability of obtaining a total score of seven?

Sol: Six numbers (1,2,3,4,5,6) are on the six faces of each die. Therefore, there are six possible ways of outcomes on the first die and to each of these ways, there corresponds 6 possible number of outcomes on the second die.

Hence the total number of ways, n = 6 x 6 = 36. Now we will find out of these, how many are favourable to the total score of 7. This may happen only in the following ways (1,6), (6,1), (2,5), (5,2), (3,4) and (4,3) that is, in six ways where first number of each ordered pair denotes the number on the first die and second number denotes the number on the second die.

m = 6.

Hence required probability = $\dfrac{\textit{Favourable No of Cases}}{\textit{Total No of cases}}$

$= \dfrac{m}{n} = \dfrac{6}{36} = \dfrac{1}{6}$

5. Two digits are selected at random from the digits 1 through 9. If the sum is even find the probability, p that both numbers are odd.

Sol: If both numbers are even or if both numbers are odd, then the sum is even. In this problem, there are 4 even numbers (2,4,6,8) and hence there are 4^C2 ways to choose two even numbers. There are 5 odd numbers (1,3,5,7,9) and hence there are 5^C2 ways to choose two odd numbers. Thus there are $4^C2 + 5^C2 = 16$ ways to choose two numbers such that their sum is even. Since 10 of these ways occur when both numbers are odd, the required probability,

$p = \dfrac{10}{16} = \dfrac{5}{8}$

6. Six boys and six girls sit in a row randomly. Find the probability that a) the six girls sit together, b) the boys and girls sit alternately.

Sol: a) Six girls and six boys can sit at random in a row in 12 ways. Consider six girls as one object and the six boys as six different objects. Now these seven objects can be arranged in 7! different ways. But the six girls in the first object can be arranged in 6! ways. Thus the favourable number of cases to the event of sitting all girls together is 7! 6! ways.

Therefore, the required probability = $\dfrac{\textit{Favourable No of Cases}}{\textit{Total No of Cases}} = \dfrac{7!\ \ 6!}{12!} = \dfrac{1}{132}$

b) Since the boys and girls can sit alternately in 6! 6! ways if we begin with a boy and similarly they can sit alternately in 6! 6! ways if we begin with a girl. Thus the total number of ways sitting the boys and girls alternately = 2 6! 6!.

$\therefore$ The required probability = $\frac{\textit{Favourable No of Cases}}{\textit{Total No of Cases}} = \frac{2 \;\; 6! \;\;\; 6!}{12!} = \frac{1}{462}$

7. Out of (2n+1) tickets consecutively numbered, three are drawn at random. Find the chance that the numbers on them are in A.P.

Sol: Suppose that the smallest number among the three drawn is 1. Then the groups of three numbers in A.P. are (1,2,3), (1,3,5), (1,4,7)....., (1, n+1, 2n+1) and they are n in number.

Similarly, if the smallest number is 2, then the possible groups are (2,3,4), (2,4,6),...(2, n+1, 2n) and their number is n-1. If the lowest number is 3, the groups are (3,4,5), (3,5,7),....(3, n+2, 2n+1) and their number is n-1.

Similarly, it can be seen that if the lowest numbers selected are 4,5,6,..., 2n-2, 2n-1, the number of selections respectively are (n-2), (n-2), (n-3), (n-3),..., 2,2,1,1. Thus the favourable ways for the selected three numbers are in A.P.

$= 2\,(1 + 2 + 3 + \ldots + n\text{-}1) + n$

$= \frac{2\,(n-1)\,n}{2} + n = n^2$

Also the total number of ways of selecting three numbers out of (2n+1) numbers

$= \binom{2n+1}{3} = \frac{(2n+1)\,(2n)\,(2n-1)}{1\,.\,2\,.\,3} = \frac{n\,(4n^2-1)}{3}$

Hence the required probability = $\frac{\textit{Favourable No of cases}}{\textit{Total No of cases}} = \frac{n^2}{n\,(4n^2-1)/3} = \frac{3n}{4n^2-1}$

8. If a coin is tossed (m+n) times (m >n), then show that the probability of at least m consecutive heads is $\frac{n+2}{2^{m+1}}$.

Sol: Let us denote by H the appearance of head and by T the appearance of tail and let X denote the appearance of head or tail. Now P(H) = P(T) = 1/2 and P(X) = 1.

Suppose the appearance of m consecutive heads starts from the first throw, we have

(H H H....m times) (X X ...... n times)

The chance of this event = (½ . ½ .... m times) = $\frac{1}{2^m}$

If the sequence of m consecutive heads starts from the second throw, the first must be a tail and we have

T (H H .... m times) (X X ....(n-1) times)

The chance of this event = ½ (½ . ½ .. m times) = $\frac{1}{2^{m+1}}$

If the sequence of m consecutive heads starts from the (r+1)th throw, the first (r-1) throws may be head or tail but rth throw must be tail and we have

(X X,,,, r-1 times) T (H H .... m times) (X X... (m+n-r) times)

The probability of this event = $\frac{1}{2} \frac{1}{2^m} = \frac{1}{2^{m+1}}$

In the above case, r can take any value from 1,2,... n. Since all the above cases are mutually exclusive, the required probability when r takes 0,1,2,... n

$$= \frac{1}{2^m} + \left(\frac{1}{2^{m+1}} + \frac{1}{2^{m+1}} + \ldots n \text{ times} \right)$$

$$= \frac{n+2}{2^{m+1}}$$

Hence the result.

9. What is the probability that in a group of N people, at least two of them will have the same birthday ?

Sol: We first find the probability that no two persons have the same birthday and then subtract from 1 to get the required probability. Suppose there are 365 different birthdays possible in a year (excluding leap year).

Any person might have any of these 365 days of the year as birthday. A second person may likewise have any of these 365 birthdays and so on. Hence the total number of ways of N people to have their birthdays = $(365)^N$.

But the number of possible ways for none of these N birthdays to coincide is =

$365.\ 364 \ldots (365 - \overline{N-1})$

$$= \frac{(365)!}{(365-N)!}$$

The probability that no two birthdays coincide is

$$\frac{(365)!}{(365-N)!} \Big/ (365)^N$$

Hence the required probability (for at least two people to have the same birthday)

$$= 1 - \frac{(365)!}{(365-N)!\ (365)^N}$$

10. A and B are two independent witnesses (i.e. there is no collusion between them) in a case. The probability that A will speak the truth is x and the probability that B will speak the truth is y. A and B agree in a certain statement. Show that the probability that the statement is true is $xy / (1 - x - y + 2xy)$.

Sol: A and B agree in a certain statement means either both of them speak truth or make false statement. But the probability that they both speak truth is xy and both of them make false statement is $(1 - x)(1 - y)$.

Thus the probability of their agreement in a statement

$= xy + (1 - x)(1 - y) = 1 - x - y - 2xy$

Therefore, the conditional probability of their statement is true = $\dfrac{xy}{1 - x - y + 2xy}$

(by using the definition $P(A/B) = \dfrac{P(A \cap B)}{P(B)}$, where A is the event of correct statement and B is the event of common statement).

11. Two friends have agreed to meet at a certain place between nine and ten O' clock. They also agreed that each would wait for a quarter of an hour and, if the other did not arrive, would leave. What is the probability that they meet ?

Sol: Suppose x is the moment one person arrives at the appointed place, and y is the moment the other arrives.

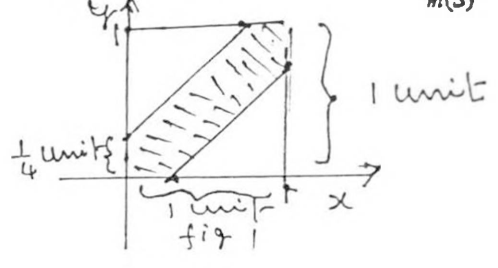
Let us consider a point with coordinates (x,y) on a place as an outcome of the rendezvous.

Every possible outcome is within the area of square having side corresponds to an hour as shown in the figure.

The outcome is favourable (the two meet) for all points (x,y) such that $| x - y | \leq 1/4$. These points are within the shaded part of the square in the above figure 1.

All the outcomes are exclusive and equally possible, and therefore, the probability of the rendezvous equals the ratio of the shaded area to the area of the square. That is, $m(A) = 7/16$ and $m(S) = 1$.

Hence by geometric probability, the required probability = $\dfrac{m(A)}{m(S)} = \dfrac{7/16}{1} = \dfrac{7}{16}$



Exercises :

1. A factor of 60 is chosen at random. What is the probability that it has factors of both 2 and 5 ?

2. The numbers 3,4 and 5 are placed on three cards and then two cards are chosen at random.

   a) The two cards are placed side-by-side with a decimal point in front. What is the probability that the decimal is more than 3/8 ?

   b) One card is placed over the other to form a fraction. What is the probability that the fraction is less than 1.5 ?

   c) If there are 4 cards with numbers 3,4,5 and 6, then what are the probabilities of the above two cases ?

3. A vertex of a paper isosceles triangle is chosen at random and folded to the midpoint of the opposite side. What is the probability that a trapezoid is formed ?

4. A vertex of a paper square is folded onto another vertex chosen at random. What is the probability that a triangle is formed ?

5. Three randomly chosen vertices of a regular hexagon cut from paper are folded to the centre of the hexagon. What is the probability that an equilateral triangle is formed?

6. A piece of string is cut at random into two pieces. What is the probability that the short piece is less than half the length of the long piece ?

7. A paper square is cut at random into rectangles. What is the probability that larger perimeter is more than 1 ½ times the smaller ?

8. The numbers 2, 3 and 4 are substituted at random for a,b,c in the equation $ax + b = c$.

9. Each coefficient in the equation $ax^2 + bx + c = 0$ is determined by throwing an ordinary die. Find the probability that the equation will have real roots.

10. The numbers 1, 2 and 3 are substituted at random for a,b and c in the quadratic equation $ax^2 + bx + c = 0$.

    a) What is the probability that $ax^2 + bx + c = 0$ can be factored?

    b) What is the probability that $ax^2 + bx + c = 0$ has real roots ?

11. Two faces of a cube are chosen at random. What is the probability that they are in parallel planes ?

12. Three edges of a cube are chosen at random. What is the probability that each edge is perpendicular to the other two ?

13. A point P is chosen at random in the interior of square ABCD. What is the probability that triangle ABP is acute ?

14. Find the probability of the event of the sine of a randomly chosen angle is greater than 0.5.

15. Suppose you ask individuals for their random choices of letters of the alphabet. How many people would you need to ask so that the probability of at least one duplication becomes better than 1 in 2 ?

16. Six boys and six girls sit in a row randomly. Find the probability that i) the six girls sit together, ii) the boys and girls sit alternately ?

17. If the letters of the word 'MATHEMATICS' are arranged at random, what is the probability that there will be exactly 3 letters between H and C ?

18. The sum of two non-negative quantities is equal to 2n. Find the probability that their product is not less than ¾ times their greatest product.

a) What is the probability that the solution is negative ?

b) If c is not 4, what is the probability that the solution is negative ?

19. If A and B are independent events then show that $\bar{A}$ and $\bar{B}$ are also independent events.

20. Cards are dealt one by one from well-shuffled pack of cards until an ace appears,. Find the probability of the event that exactly n cards are dealt before the first ace appears.

21. If four squares are chosen at random on a chess-board, find the chance that they should be in a diagonal line.

22. Prove that if $P(A/B) < P(A)$ then $P(B/A) < P(B)$?

23. If n people are seated at a round table, what is the chance that the two named individuals will be next to each other ?

24. A and B are two very weak students of Mathematics and their chances of solving a problem correctly are 1/8 and 1/12 respectively. If the probability of their making common mistake is 1/1001 and they obtain the same answer, find the chance that their answer is correct.

25. A bag contains an unknown number of blue and red balls. If two balls are drawn at random, the probability of drawing two red balls is five times the probability of drawing two blue balls. Furthermore, the probability of drawing one ball of each colour is six times the probability of drawing two blue balls. How many red and blue balls are there in the bag ?

26. A thief has a bunch of n keys, exactly one can open a lock. If the thief tries to open the lock by trying the keys at random, what is the probability that he requires exactly k attempts, if he rejects the keys already tried ? Find the probability of the same event when he does not reject the keys already tried.

27. A problem in Mathematics is given to three students and their chances of solving it are ½, 1/3 and ¼. What is the probability that the problem will be solved ?

28. A bag A contains 3 white balls and 2 black balls and other bag B contains 2 white and 4 black balls. A bag and a ball out of it are picked at random. What is the probability that the ball is white ?

29. Cards are drawn one-by-one at random from a well-shuffled pack of 52 cards until 2 aces are obtained for the first time. If N is the number of cards required to be drawn, then show that

$$p(N = n) = \frac{(n-1)\ (52-n)\ (51-n)}{50.59.\ \ 17.\ \ 13}$$

Where $2 \leq n \leq 50$.

30. A,B, C are events such that

P (A) = 0.3, P(B) = 0.4, P(C) = 0.8, P(A ∩ B) = 0.08, P (A ∩ C) = 0.28,

P(A ∩ B ∩ C) = 0.09

If P(AUBUC) ≥ 0.75, then show that P(B ∩ C) lies in the interval (0.23, 0.48).

31. A man takes a step forward with probability 0.4 and backwards with probability 0.6. Find the probability that at the end of eleven steps, he is one step away from the starting point.

32. Huyghens Problem. A and B throw alternately a pair of dice in that order. A wins if he scores 6 points before B gets 7 points, in which case B wins. If A starts the game, what is his probability of winning ?

33. A Doctor goes to work following one of three routes A, B,C. His choice of route is independent of the weather. If it rains, the probabilities of arriving late, following A, B, C are 0.06, 0.15, 0.12 respectively. The corresponding probabilities, if it does not rain, are 0.05, 0.10, 0.15.

a) Given that on a sunny day he arrives late, what is the probability that he took route C ? Assume that, on average, one in every four days is rainy.

b) Given that on a day he arrives late, what is the probability that it is a rainy day.

34. Bonferroni's Inequality. Given n(>1) events $A_1, A_2, \ldots A_n$ show that

$$\sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j) \le P\left(\bigcup_{i=1}^{n} A_i\right) \le \sum_{i=1}^{n} P(A_i)$$

35. Show that for any n events $A_1, A_2, \ldots, A_n$

i) $P\left(\bigcap_{i=1}^{n} A_i\right) \ge 1 - \sum_{i=1}^{n} P(\bar{A}_i)$

ii) $P\left(\bigcap_{i=1}^{n} A_i\right) \ge \sum_{i=1}^{n} P(A_i) - (n-1)$

36. If A and B are mutually exclusive and $P(A \cup B) \neq 0$, then prove that

$$P(A/A \cup B) = \frac{P(A)}{P(A) + P(B)}$$

37. If 2n boys are divided into two equal groups, find the probability that the two tallest boys will be a) in different subgroups, and b) in the same subgroup.

38. A small boy is playing with a set of 10 coloured cubes and 3 empty boxes. If he puts the 10 cubes into the 3 boxes at random, what is the probability that he puts 3 cubes in one box, 3 in another box, and 4 in the third box ?

39. The sample space consists of the integers from 1 to 2n, which are assigned probabilities to their logarithms. A) Find the probabilities, b) Show that the conditional probability of the integer 2, given that an even integer occurs is

$$\frac{\log 2}{n \log 2 + \log n!}.$$

40.a) Each of n boxes contains four white and six black balls, while another box contains five white and five black balls. A box is chosen at random from the (n+1) boxes, and two balls are drawn from it, both being black. The probability that five white and three block balls remain in the chosen box is 1/7. Find n.

40b). A point is selected at random inside a circle. Find the probability p that the point is closer to the centre of the circle than to its circumference.

41. What is the probability that two numbers chosen at random will be prime to each other?

42. In throwing n dice at a time, what is the probability of having each of the points 1,2,3,4,5,6 appears at least once ?

43. A bag contains 50 tickets numbered 1,2,3,..., 50 of which five are drawn at random and arranged in ascending order of magnitude ($x_1 < x_2 < x_3 < x_4 < x_5$), what is the probability that $x_3 = 30$?

44. Of the three independent events, the probability that the first only to happen is ¼, the probability that the second only to happen is 1/8 and the third only to happen is 1/12. Obtain the unconditional probabilities of the three events.

45. What is the least number of persons required if the probability exceeds ½ that two or more of them have the same birthday (year of birth need not match) ?

46. If m things are distributed among 'a' men and 'b' women, then show that the chance that the number of things received by men is

$$\frac{1}{2}\;\frac{(b+a)^m - (b-a)^m}{(b+a)^m}$$

47. A pair of dice is rolled until either 5 or a 7 appears. Find the probability that a 5 occurs first.

48. In a certain standard tests I and II, it has been found that 5% and 10% respectively of 10^{th} grade students earn grade A. Comment on the statement that the probability is $\frac{5}{100}\;\frac{10}{100} = \frac{1}{200}$ that a 10^{th} grade student chosen at random will earn grade A on both tests.

49. A bag contains three coins, one of which is coined with two heads while the other two coins are fair. A coin is chosen at random from the bag and tossed four times in succession. If head turns up each time, what is the probability that this is the two headed coin ?

50. A man stands in a certain position (which we may call the origin) and tosses a fair coin. If a head appears he moves one unit of length to the left. If a tail appears, he moves one unit to the right. After 10 tosses of the coin, what are his possible positions and what are the probabilities ?

51. There are 12 compartments in a train going from Madras to Bangalore. Five friends travel by the train for some reasons could not meet each other at Madras station before getting aboard. What is the probability that the five friends will be in different compartments ?

52. The numbers 1,2,3,4,5 are written on five cards. Three cards are drawn in succession and at random from the deck, the resulting digits are written from left to right. What is the probability that the resulting three digits number will be even ?

53. Suppose n dice are thrown at a time. What is the probability of getting a sum 'S' of points on the dice ?

54. A certain mathematician always carries two match boxes, each time he wants a match-stick he selects a box at random. Inevitably, a moment comes when he finds a box empty . Find the probability that the movement the first box is empty, the second contains exactly r match sticks (assume that each box contain N match-sticks initially).

55. There are 3 cards identical in size. The first card is red both sides, the second one is black both sides and the third one red one side and black other side. The cards are mixed up and placed flat on a table. One is picked at random and its upper (visible) side was red. What is the probability that the other side is black ?

56. N different objects 1,2,...., n are distributed at random in n places marked 1,2,... n. Find the probability that none of the objects occupies the place corresponding to its number.

16. i) $\frac{7! \quad 6!}{12\,!}$ ii) $\frac{2 \quad (6!)^2}{12!}$

17. 7/55

18. ½

20. $\frac{4\ (51-n)\ (50-n)\ (49-n)}{52.51.50.49}$

21. $\frac{91}{158844}$

23. $\frac{2}{n-1}$

24. 13/24

25. Red = 6, Blue = 3

26. 1/n, 1/n $\left(1 - \frac{1}{n}\right)^{k-1}$

27. 3/4

28. 7/15

31. $(0.4)^5\,(0.6)^5$

32. 30/61

33. a) 0.5 b) 41/131

37. a) $\frac{n}{2n-1}$ b) $\frac{n-1}{4n-2}$

38. $\frac{3 \quad 10!}{3! \quad 3! \quad 4! \quad 3^{10}}$

39. a) K log 2i b) (log 2i) (n log 2 + log n!)

40. a) 4 b) 1/4

41. $\pi \left(1 - \frac{1}{r^2}\right) = \frac{6}{\pi^2}$

42. $1 - n \left(\frac{5}{6}\right)^n + \binom{n}{2} \left(\frac{4}{6}\right)^n - \binom{n}{3} \left(\frac{3}{6}\right)^n + \binom{n}{4} \left(n\frac{2}{6}\right)^n - \binom{n}{5} \left(\frac{1}{6}\right)^n$

Answers :

1. ½
2. A) 2/3 b) 5/6 c) 3/4, 3/4
3. 1/3
4. 1/3
5. 1/10
6. 2/3
7. 2/5
8. a) ½ b) 3/4
9. 43/216
10. a) 1/3 b) 1/3
11. 1/5
12. 2/55
13. $1 - \pi/8 = 0.6073$
14. 2/3
15. 7

43. $\dfrac{\binom{29}{2}\binom{20}{2}}{\binom{50}{2}}$

44. ½, 1/3, 1/4

45. 23

47. 2/5

49. 8/9

50.

Distance from origin	-10	-8	-6	-4	-2	0	2	4	6	8	10
Prob	$\left(\frac{1}{2}\right)^{10}$	$\binom{10}{1}\left(\frac{1}{2}\right)^{10}$	$\binom{10}{2}\left(\frac{1}{2}\right)^{10}$	$\binom{10}{3}\left(\frac{1}{2}\right)^{10}$	$\binom{10}{4}\left(\frac{1}{2}\right)^{10}$	$\binom{10}{5}\left(\frac{1}{2}\right)^{10}$	$\binom{10}{6}\left(\frac{1}{2}\right)^{10}$	$\binom{10}{7}\left(\frac{1}{2}\right)^{10}$	$\binom{10}{8}\left(\frac{1}{2}\right)^{10}$	$\binom{10}{9}\left(\frac{1}{2}\right)^{10}$	$\left(\frac{1}{2}\right)^{10}$

51. 55/144

52. 1/5

53. $(-1)^k \binom{n}{k} \binom{s-6k-1}{n-1} | 6^n$

54. $\dfrac{\binom{2n-r}{n}}{2^{2n-r}}$

55. ½

56. $\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} = \ldots + (-1)^n \frac{1}{n!}$.

RANDOM VARIABLES AND PROBABILITY DISTRIBUTIONS

In the earlier pages, the idea of a function, subject to certain postulates, which assigned weights called probabilities, to the points of the sample space, was introduced. We then had a probability function which allowed us to compute probabilities for events. Now we deal with the concept of Random Variable.

Random Variable :

Scientific theories on models are our way of depicting and explaining how observations come about. Such theories are simplified statements containing essential features and make for easier comprehension and communication. In statistics, we use a mathematical approach since we quantify our observations. Random variable is the result of such mathematical approach dealing with the probabilities assigning to different events of a random experiment. The set of possible outcomes for a random experiment can be described with the help of a real-valued variable by assigning a single value of this variable to each outcome. For a two coin tossing experiment, the outcomes are two tails, a tail and a head , a head and a tail, or two heads. The sample space can be represented as (TT, TH, HT, HH). Here we express the outcomes by using the number of heads and so assigning the values (0,1,1,2) respectively to those outcomes. Therefore, the outcomes of this experiment can be denoted by the different values of the real-valued variable viz. 0,1,2.

Any function or association that assigns a unique, real value to each sample point is called a chance or random variable. The assigned values are the values of the random variable.

Random variables are symbolised by capital letters, most often X, and their values by lower case letters. The outcome of a random experiment determines a point i.e., the sample space, called the domain of the random variable, and the function transform each sample point to one of a set of real numbers. This set of real numbers is called the range of the random variable. If the sample space is discrete, then the outcomes will be denoted by certain discrete values. The random variable associated with a discrete sample space is known as discrete random variable. Similarly, the random variable associated with continous sample space is known as continuous random variable.

Probability Function :

The association of probabilities with the various values of a discrete random variable is done by reference to the probabilities in the sample space and through a system of relationships or a function is called a probability set function or, simply, a probability function.

Let the discrete random variable X assume the values $x_1, x_2 \ldots x_n$ Then the system of relations can be written as

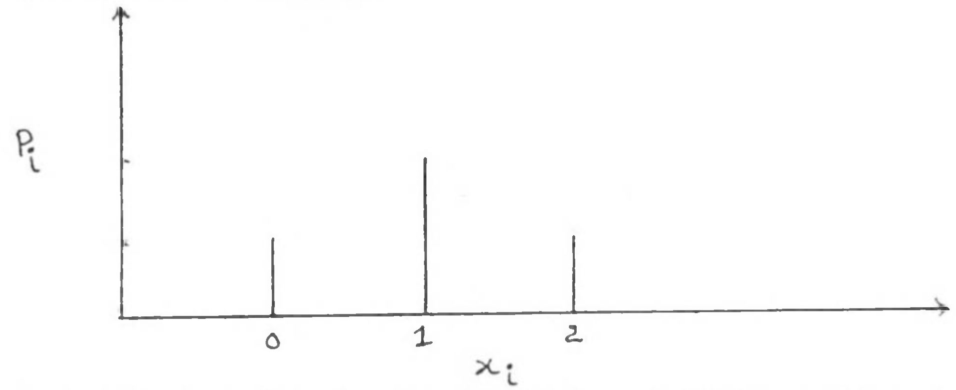
$$P(X = x_i) = p_i$$

This is read as 'the probability that the random variable X takes the value of x_i is p_i'. The set of ordered pairs (x_i, p_i) constitutes a probability function with numerical values to be provided for the x_i and p_i's such that $p_i \geq 0$ for all i and $\sum_i p_i = 1$.

A discrete probability function is a set of ordered pairs of values of a random variable and the corresponding probabilities.

For a two coin experiment, X takes the values 0,1,2 with the probabilities ¼, ½, ¼ respectively.

Sometimes probability function can be represented by a graph or a mathematical function. In case of above example, the X values and the corresponding probabilities can be represented with the help of the following graph.



Suppose X assume the values 1 and 0 with the probabilities p and 1-p respectively. This information can be given with the help of the following function p(x) defined by

$$P(x) = p^x(1-p)^{1-x}, x = 0, 1.$$

This type of function which gives the probabilities of the different values assumed by a random variable is known as probability mass function or simply probability function. Therefore, a function p(x) is said to be a probability function of random variable or a distribution if

i) $p(x) \geq 0$ for all x.

$$\sum_x p(x) = 1$$

where p(x) denotes the probability of the events that the random variable X assumes the value x.

Distribution Function :

The law of probability distribution of a random variable is the rule used to find the probability of the event related to a random variables. For instance, the probability that the variable assumes a certain value or falls in a certain interval. The general form of the distribution law is distribution function, which is the probability that a random variable X assumes a value smaller than a given x i.e. $F(x) = P(X \le x)$.

The distribution function F(x) for any random variable possesses the following properties :

i) $F(-\infty) = 0$

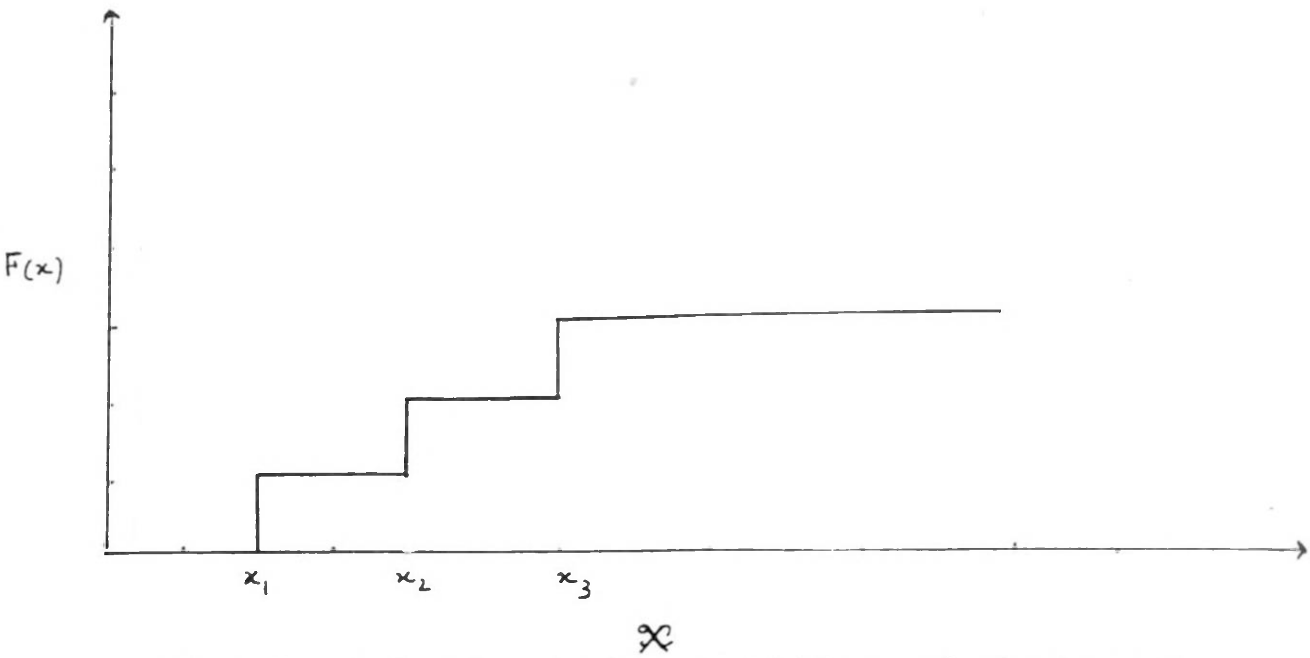
ii) $F(+\infty) = 1$

iii) F(x) does not decrease with an increase in x.

In the case of discrete random variable

$$F(x_k) = \sum_{i=1}^{k} p(x_i)$$

Where $x_1, x_2, \ldots, x_k \ldots$ are the values of the random variable. The graph of F(x) in discrete random variable case is generally as shown below :



It is seen from the above figure that the graph of F(x) is a 'step function' having jump $p(x_i)$ at $x = x_i$ and is constant between each pair of values of x. It can also be proved that

$$F(x_i) - F(x_{i-1}) = p(x_i)$$

Therefore, distribution function can also be used to indicate the distribution of the random variable instead of probability function.

Example :

A student is to match three historical events (Mahatma Gandhi's birth year, India's freedom, and first Word War) with three years (1947, 1914, 1869). If he guesses, with no knowledge of the correct answers, what is the probability distribution of the number of answers he gets correctly ?

Solution : Here the number of correct answers is the random variable, say X. Therefore, X assumes the values 0,1,2,3 because there are three events to match with only three years. Suppose the events are E_1, E_2, E_3 and the corresponding correct years are Y_1, Y_2, Y_3. Student gets the correct answers when he/she matches E_1 to Y_1, E_2 to Y_2 and E_3 to Y_3.

All matchings are wrong only when he/she matches E_1 to Y_2, E_2 to Y_3, E_3 to Y_1 or E_1 to Y_3, E_2 to Y_1, E_3 to Y_2. But the total possible matchings are 6. Therefore, the probability of all matchings to go wrong is 2/6 =1/3. That is, the probability that X to take the value '0' is 1/3.

Similarly X assumes the value '1' with probability 3/6 (= ½) the value '2' with 0 probability and the value '3' with 1/6 probability.

So the probability distribution of the correct answers in the given matching is

No of correct answers (x)	0	1	2	3
Probability	1/3	½	0	1/6

Example : Suppose a number is selected at random from the integers 10 through 30. Let X be the number of its divisors. Construct the probability function of X. What is the probability that there will be 4 or more divisors ?

Solution : X is the number of divisors of randomly selected number from the integers 10 through 30. Therefore, X is a random variable. The possible values that X assumes are :

2, 3,4, 5,6 depending upon the selected number. For example, if the selected number is either 1,2,3,5,7,11,13,17,19 then X takes the value 2. Similarly when the selected number is 4,6,8,10,14,15 X takes 4. Therefore, the different values of X and the number of their appearances we get the following :

X values	1	2	3	4	5	6
No of appearances out of 20	1	g	3	4	1	3

Now the required probability distribution is

x	1	2	3	4	5	6
p(x)	1/20	g/20	3/20	4/20	1/20	3/20

The probability of X to take 4 or more

= P (x = 4 or 5 or 6) = P (x = 4) + P (x = 5) + P (x = 6)

$$= \frac{4}{20} + \frac{1}{20} + \frac{3}{20} = \frac{8}{20} = \frac{2}{5}$$

Mean, Variance, Standard Deviation of the Random Variable.

Let X be a random variable with probability function as follows :

x	x_1	x_2	..	..	..	x_n
p(x)	$p(x_1)$	$p(x_2)$	..	..	..	$p(x_n)$

The mean of X is defined as

$$x_1\ p(x_1) + x_2\ p(x_2) + \ldots + x_n\ (n_n)$$

or

$$\sum_{i=1}^{n} x_i\ p(x_i)$$

This is also known as mean of the distribution and generally denoted by μ.

The variance of X is defined as

$$\sum_{i=1}^{n} x_i^2\ p(x_i) - \left[\sum_{i=1}^{n} x_i\ p(x_i) \right]^2$$

$$\sum_{i=1}^{n} x_i^2 \; p(x_i) - \mu^2$$

where μ is the mean of X.

The variance is generally denoted by σ^2.

The standard deviation is the positive square root of variance and is denoted by σ.

Example : A single 6-sided die is tossed. Find the mean and variance of the number of points on the top face.

Solution : Let X represents the number of points on the top face. The probability function of X is

x	1	2	3	4	5	6
p(x)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

The mean, μ is given by

$$\sum_{i=1}^{n} x_i \; p(x_i) = x_1 \, p(x_1) + x_2 \, p(x_2) .... + x_n \; p(x_n)$$

$$\textit{Here } \mu = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} \cdot 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6}$$

$$= \quad \frac{1}{6}(1 + 2 + 3 + 4 + 5 + 6)$$

$$= \quad \frac{1}{6} \; \frac{6 \times 7}{2} = \frac{7}{2}$$

Variance, σ^2 is given by

$\sum_{i=1}^{n} x_i^2 \; p(x_i) - \mu^2$ where μ is mean.

Here

$$\sum_{i=1}^{n} x_i^2 \; p(x_i) = 1^2 \cdot \frac{1}{6} + 2^2 \frac{1}{6} + 3^2 \frac{1}{6} + 4^2 \cdot \frac{1}{6} + 5^2 \cdot \frac{1}{6} + 6^2 \cdot \frac{1}{6}$$

$$= \frac{1}{6}\left[1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2\right]$$

$$= \frac{1}{6}\ \frac{6\times 7\ \ (2\times 6\times 1)}{6} = \frac{91}{6}$$

Variance $\sigma^2 = \frac{91}{6} - \left(\frac{7}{2}\right)^2 \qquad (\because \mu = \frac{7}{2})$

$$= \frac{91}{6} - \frac{49}{4} = \frac{35}{12}$$

Exercises :

1. One cube with faces numbers 1,2,3,4,5 and 6 is tossed twice, and the recorded outcome consists of the ordered pair of numbers on the hidden faces at the first and second tosses.

   a) Let the random variable X takes on the value 0 if the sum of the numbers in the ordered pair is even and 1 if odd. What is the probability function for this random variable ?

   b) Let the random variable X takes on the value 2 if both numbers in the ordered pair are even, 1 if exactly one is even, and 0 if neither is even. What is the probability distribution of this random variable ?

   c) Let the random variable X be the number of divisors in the sum of the two faces. What is the probability function of X ?

2. Of six balls in a bag, two are known to be black. The balls are drawn one at a time from the bag and observed until both black balls are drawn. If X is the number of trials (draws) required to get the two black balls. Obtain the probability distribution of X.

3. Suppose that the random variable X has possible values 1,2,3,... and $P(x = j) = \frac{1}{2^j}$, j = 1,2,...

   i) compute P(x is even), ii) compute P(x is divisible by 3).

4. The probability mass function of a random variable X is zero except at the points x= 0,1,2,. At these points has the values $p(0) = 3c^3$, $p(1) = 4c - 10c^2$ and $p(2) = 5c - 1$ for some $c > 0$.

i) Determine the value of c.

ii) Compute $P(1 < X \leq 2)$.

iii) Describe the distribution function and draw its graph.

iv) Find the largest x such that $F(x) < \frac{1}{2}$.

5. Let X denote the profits that a man makes in business. He may earn Rs.3000 with probability 0.5, he may lose Rs.5000 with probability 0.3 and he may neither earn nor lose with probability 0.2. Calculate his average profits.

6. A man wins a rupee for head and loses a rupee for tail when a coin is tossed. Suppose that he tosses once and quits if he wins but tries once more if he loses on the first toss. What are his expected winnings ?

7. Three boxes contain respectively 3 red and 2 black balls, 5 red and 6 black balls and 2 red and 4 black balls. One ball is drawn from each box. Find the average number of black balls drawn.

8. If the random variable, X takes the values 1,2,....n respectively with probabilities $\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}$ find the mean and variance of X.

Answers :

1. a)

X	Prob
0	½
1	½

b)

X	Prob
0	¼
1	½
2	¼

DISCRETE DISTRIBUTIONS

In the previous pages, we discussed about 'random variable', 'probability function', etc. Here we discuss some theoretical discrete distributions in which variables are distributed according to some definite law which can be expressed mathematically.

Bernoulli Distribution : Suppose you want to study the probability of different events corresponding to tossing of a single coin experiment. The two possible events are getting a head or getting a tail. Define a random variable x assuming the values 1 and 0 corresponding to these two events viz. Head and tail respectively. If the probability of getting a head in tossing that coin is 'p' then the probability that the random variable to take '1' is p and the probability that the random variable to take '0' is 1-p. Therefore, the distribution of the random variable X becomes

X	Prob
1	p
0	1-p

Any experiment where there are only two possible outcomes viz. Success and failure is called as Bernoulli experiment. A single trial of a **Bernoulli experiment** is known as Bernoulli trial.

Corresponding to any Bernoulli experiment, it is possible to define a random variable X as given above.

A random variable X which takes two values 0 and 1, with probability q(=1-p) and p respectively is called Bernoulli variate and is said to have a Bernoulli distribution.

Binomial Distribution :

Let a Bernoulli experiment be performed repeatedly and let the occurrence of an event in any trial be called a success and its non-occurrence a failure. Consider a series of n independent Bernoulli trials (n being finite), in which the probability 'p' of success in any trial is constant for each trial. Then q = 1-p is the probability of failure inany trial. Let the random variable X be the number of successes in these trials.

The probability of x successes and consequently (n-x) failures in n independent trials, in a specified order (say) SS FF SSS .... FSFF (where S represents success and F failure) is given by compound probability as given below :

P (SSFF, ... FSFF) = P(S) P(S) P(F) P(F) ....P(F) P(S) P(F) P(F)
= p.p.qq....q p q q
= pp ....p qq ....q (x p's and (n-x) q's)
= $p^x q^{n-x}$

c)

X	Prob
2	15/36
3	12/36
4	8/36
6	1/36

2.

X	Prob
2	1/15
3	2/15
4	3/15
5	4/15
6	5/15

3. i) 1/3 ii) 1/7

4. i) 1/3 ii) 2/3 iii) 1

5. 0

6. 0

7. $\frac{266}{165}$

8. Mean = $\frac{(n+1)}{2}$, Variance = $\frac{n^2 - 1}{12}$

But x successes in n trials can occur in $\binom{n}{x}$ ways and the probability for each of these ways is $p^x q^{n-x}$. Hence the probability of x successes in n trials in any order whatsoever is given by the addition of individual probabilities and is given by $\binom{n}{x} p^x q^{n-x}$. The number of successes in n trials will be either 0 or 1 or 2 ... or n in any experiment.

$p(x) = P(X = x) = \binom{n}{x} p^x q^{n-x}$

Is true for all $x = 0,1,2,\ldots n$.

This function $p(x) = \binom{n}{x} p^x q^{n-x}$, $x = 0, 1, \ldots, n$ is called the probability mass function of the Binomial distribution, for the obvious reason that the probabilities of 0,1,2,...n successes, viz. q^n, $\binom{n}{1} q^{n-1} p$, $\binom{n}{2} q^{n-2} p^2, \ldots, p^n$ Are the successive terms of the binomial expansion $(q + p)^n$.

A random variable X is said to follow binomial distribution if its probability mass function is given by

$$P(X = x) = p(x) = \binom{n}{x} p^x q^{n-x}, x = 0,1,2,\ldots n;\ q = 1 - p.$$

The values n and p of this distribution are known as the parameters of the distribution.

Mean and Variance of Binomial Distribution

We know, mean of any discrete distribution

$$= \sum_r r\, p(r)$$

where p(r) is the probability that the random variable X to take the value r. In case of binomial distribution x takes the values $r = 0,1,2,\ldots, n$ and $pr = \binom{n}{r} p^r q^{n-r}$ where n and p are the parameters of the binomial distribution.

$$\therefore \text{Mean} = \sum_{r=0}^{n} r \binom{n}{r} p^r q^{n-r}$$

$$= \sum_{r=0}^{n} r \frac{n!}{r!\,(n-r)!} p^r q^{n-r}$$

$$= np \sum_{r=1}^{n} \frac{(n-1)!}{(r-1)!\,(n-r)!} p^{r-1} q^{n-r}$$

$= n\, p \left[q^{n-1} + (n-1)\, C_1\, q^{n-2} + \ldots + p^{n-1} \right]$

$= n\, p\, (q + p)^{n-1}$

$= np \quad (\because p + q = 1)$

Also we know Variance = $\sum_r r^2\, p(r) - \left[\sum_r r\, p(r) \right]^2$

$= \sum_r r^2\, p(r) - (Mean)^2$

In case of binomial distribution

Variance = $\sum_{r=0}^{n} r^2 \binom{n}{r} p^r\, q^{n-r} - (n\, p)^2$

$= \sum_{r=0}^{n} r^2 \frac{n!}{r!\, (n-r)!} p^r\, q^{n-r} - (np)^2 \quad (\because Mean = np)$

$= \sum_{r=0}^{n} [\, r\, (r-1) + r] \frac{n!}{r!\, (n-r)!} p^r\, q^{n-r} - (n\, p)^2$

$= \sum_{r=0}^{n} r(r-1) \frac{n!}{r!\, (n-r)!} p^r\, q^{n-r} + \sum_{r=0}^{n} r \frac{n!}{r!\, (n-r)!} p^r\, q^{n-r} - (n\, p)^2$

$= \sum_{r=2}^{n} \frac{n!}{(r-2)!\, (n-r)!} p^r\, q^{n-r} + np - (n\, p)^2$

$(\because \sum r \frac{n!}{r!\, (n-r)!} p^r\, q^{n-r} = np \text{ provedabove})$

$= n(n-1)\, p^2 \left[\sum_{r=2}^{n} \frac{(n-2)!}{(r-2)!\, (n-r)!} p^{r-2}\, q^{n-2} \right] + np - (n\, p)^2$

$= n(n-1)\, p^2 \left[q^{n-2} + (n-2)C_1 + (n-2)\, C_2 + \ldots p^{n-2} \right] + np - (np)^2$

$= n(n-1)\, p^2\, (q + p)^{n-2} + np - (np)^2$

$= n(n-1)\, p^2 + np - (np)^2$

$= np\, [(n-1)\, p + 1 - np]$

$= n\, p\, [\, np - p + 1 - np]$

$= np\, [\, 1 - p\,] = n\, p\, q$

So, Mean = np

Variance = npq

Standard Deviation = $\sqrt{Variance} = \sqrt{npq}$

Example : The mean and variance of binomial distribution with parameters n and p are 16 and 8. Find i) P (x = 0), ii) P (x ≥ 2).

Solution : We know mean = np and variance = npq.

∴ np = 16 and npq = 8

Solving for n and p we get n = 32 and p = ½

Now $P(X=0) = \binom{n}{0} p^0 q^{n-0} = q^n$

(Because $p(x=r) = \binom{n}{r} p^r q^{n-r}$

$\therefore P(x=0) = (1-p)^n = \left(1-\frac{1}{2}\right)^{32} = \left(\frac{1}{2}\right)^{32}$

$(\because n=32,\ q=1-p = 1-\frac{1}{2})$

ii) $P(x \geq 2) = 1-P(x<2) = 1-[P(x=0)+P(x=1)] = 1-P(x=0)-P(x=1)$

But P (x = 0) = $\left(\frac{1}{2}\right)^{32}$ (As obtained above)

and $P(x=1) = \binom{n}{1} p^1 q^{n-1} = 32\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{32-1}$

$\therefore P(x \geq 2) = 1-\left(\frac{1}{2}\right)^{32} - 32\left(\frac{1}{2}\right)^{32} = 1-33\left(\frac{1}{2}\right)^{32}$

Example : A perfect cube is thrown a large number of items in sets of 8. The occurrence of a 2 or 4 is called a success. In what proportion of the sets would you expect 3 successes.

Solution : In this problem we have to find the probability of getting 3 successes out of 8 trials. Tossing of a single cube is our trial. The probability of success, p is getting either 2 or 4. The number of cubes in the set is the number of trials. If we define x as the number of successes in 8 trials, then x is distributed as a binomial variate with parameters 8 and p where p is the probability of success.

The probability of getting either 2 or 4 in tossing of a perfect cube = 2/6 = 1/3.

∴ p = 1/3

Hence $P(x=r) = \binom{n}{r} p^r q^{n-r}$

and $P(x=3) = \binom{n}{3} p^3 q^{n-3}$ (∵ x is a binomial variate)

$= \quad \binom{8}{3} \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^{8-3}$ (∵ n = 8, p = 1/3, q = 1-p)

$= \quad 8\times 7 \left(\frac{1}{3}\right)^8 2^5$

$= \quad 56\times 32 \left(\frac{1}{3}\right)^8$

= 0.2731

∴ The proportion of sets in which we expect 3 successes = 27.31 %.

Example :The probability of a man hitting a target is ¼.

i) If he fires 7 times, what is the probability of his hitting the target at least twice ?

ii) How many times must he fire so that the probability of his hitting the target at least once is greater than 2/3 ?

Solutions :

i) Consider 'firing once' as a Bernoulli trial. Firing 7 times is the Binomial experiment with 7 independent Bernoulli trials. If X is the number of hits in 7 trials, then the required probability of hitting the target at least twice = P $(X \geq 2)$.

We know,

$P(X \geq 2) = 1 - p(X < 2)$

$= 1 - P(X = 0) - P(X = 1)$

and $P(X = x) = \binom{n}{x} p^x q^{n-x}$ where n = 7, p = 1/4, and q = 1 – p = 3/4.

$P(X = 0) = (3/4)^7$

$P(X = 1) = \binom{7}{1} \left(\frac{1}{4}\right) \left(\frac{3}{4}\right)^6 = 7\,\frac{3^6}{4^7}.$

The required probability

$= 1 - \left(\frac{3}{4}\right)^7 - 7\,\frac{3^6}{4^7} = \frac{4547}{8192}.$

ii) p = ¼, q = ¾

We want to find n such that $P(X \geq 1) > 2/3$

Or $1 - P(X < 1) > 2/3$

Or $1 - P(X = 0) > 2/3$

Or $1 - q^n > 2/3$ when $q = 3/4$

$\Rightarrow (3/4)^n < 1/3$

$\Rightarrow n = 4.$

POISSON DISTRIBUTION

There are many situations where we must count the number of individuals possessing a certain characteristic yet have difficulty in defining the basic experiment. In turn, it becomes difficult to say what is the probability of the occurrence of a single event. For example i) number of telephone calls received at a particular telephone exchange, ii) emission of radioactive particles, iii) number of printing mistakes in a book. In all these situations, it is easy to count the events, but what are the non events.

In situations like those mentioned above, we customarily resort to specifying a unit size or a time interval in which to observe the events etc. We find then that we are observing events that fluctuate around some mean value that might be defined in terms of some sort of underlying binomial parameters p and n as np, a product never separable into its component parts and simply give the mean value. Therefore, in such situations, we assume that for a short enough unit of time or space, the probability of an event occurring is proportional to the length of time or size of the space. We also assume that for non overlapping units, the results in one unit are of no value in predicting when or where another event will occur (independently). The above assumptions underlie the probability function given by

$$P(X = x) = \frac{e^{-\lambda} \lambda^x}{x!}, \quad x = 0,1,2,3,....$$

where λ is the average number of times an event occurs in a unit interval and is called the parameter of a Poisson distribution. Poisson Distribution as a limiting case of Binomial Distribution.

The above mentioned Poisson distribution can be viewed as a limiting case of the binomial distribution under the following conditions.

i) n, the number of trials in the binomial experiement is infinitely large i.e. $n \to \infty$.
ii) p, the probability of success in each trial is indefinitely small, i.e. $p \to 0$.
iii) $np = \lambda$ is finite so that $p = \frac{\lambda}{n}$, $q = 1 - \frac{\lambda}{n}$.

We know , if X is a binomial variate with parameters n and p then

$$P(X=x) = p(x) = \binom{n}{x} p^x q^{n-x}, \; x = 0, 1, 2, ... n$$

where $n \to \infty$ and $p \to 0$.

Therefore, this probability

$$= \lim_{\substack{n\to\infty \\ p\to 0}} \binom{n}{x} p^x q^{n-x}$$

$$= \lim_{n\to\infty} \binom{n}{x} \left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x} \qquad \left(\because p = \frac{\lambda}{n}, \; q = 1 - \frac{\lambda}{n}\right)$$

$$= \lim_{n\to\infty} \frac{n!}{x!\,(n-x)!} \left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{\lambda^x}{x!} \lim_{n\to\infty} \frac{n\,.\,(n-1)\;\ldots\ldots\;(n-x-1)}{n^x} \frac{\left(1 - \frac{\lambda}{n}\right)^n}{\left(1 - \frac{\lambda}{n}\right)^x}$$

$$= \frac{\lambda^x}{x!} \lim_{n\to\infty} \left[1\,.\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \ldots \left(1 - \frac{x-1}{n}\right)\left(1 - \frac{\lambda}{n}\right)^n\right]$$

$$\lim_{n\to\infty} \frac{1}{\left(1 - \frac{\lambda}{n}\right)^x}$$

$$= \frac{\lambda^x}{x!} \lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^n \qquad \left(\because \lim_{n\to\infty} \frac{1}{\left(1 - \frac{\lambda}{n}\right)^x} = 1\right)$$

$$= \frac{\lambda^x}{x!} \lim_{m\to -\infty} \left(1 + \frac{1}{m}\right)^{-m\lambda}$$

$$= \frac{\lambda^x}{x!} \lim_{m\to -\infty} \left[\left(1 + \frac{1}{m}\right)^m\right]^{-\lambda}$$

$$= \frac{\lambda^x}{x!} e^{-\lambda}$$

$$\therefore P(X = x) = \frac{e^{-\lambda} \lambda^x}{x!}, \quad x = 0,1,2,\ldots\ldots$$

This function is known as the Probability function of the Poisson distribution and λ is the parameter of the distribution.

Mean and Variance of the Poisson Distribution :

$$\text{Mean} = \sum_{x=1}^{\infty} x \; P(x_i)$$

$$= \sum_{x=1}^{\infty} x \; \frac{e^{-\lambda} \lambda^x}{x_i!} \qquad \left(\because P(x) = \frac{e^{-\lambda} \lambda^x}{x!} \text{ In case of Poisson distribution,}\right.$$

$$= \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^{x}}{x!} \qquad (\because \text{ the values of Poisson variate are } 0,1,2,\ldots)$$

$$= \lambda e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^{x-1}}{x!}$$

$$= \lambda e^{-\lambda} \left(1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \ldots\right)$$

$$= \lambda e^{-\lambda} e^{\lambda}$$

$$= \lambda$$

$$\text{Variance} = \sum_{i=1}^{\infty} x_i^2\, p(x_i) - \left[\sum_{i=1}^{\infty} x_i\, p(x_i)\right]^2$$

Continued in last page.

Exercises :

1. A random variable X has a binomial distribution with parameters n = 4 and p = 1/3
   i) Describe the probability mass function and sketch its graph.
   ii) Compute the probabilities $P(1 < X \leq 2)$ and $P(1 \leq X \leq 2)$.

2. In a binomial distribution consisting of 5 independent trials, probabilities of 1 and 2 successes are 0.4096 and 0.2048 respectively. Find the parameter p of the distribution.

3. The probability of a man hitting a target is 1/3.
   i) If he fires 5 times what is the probability of hitting the target at least twice ?
   ii) How many times must he fire so that the probability of hitting the target at least once is more than 90% ?

4. The random variable X has a binomial distribution with n = 4, p = 0.5.
   Find $\{ |X-2| \geq 1 \}$

Answers :

1. 8/27, 56/81
2. 0.2
3. i) 131/243
   ii) 6
4. 5/16

PROBLEM SHEET – 4

1. Given A = (1, -1), B = (0,2) and the O^{le} S : $x^2 + y^2 - 3x + 2y - 1 = 0$
   a) A and B are both inside S
   b) A and B are both outside S
   c) A is inside and B outside S
   d) none of these

2. If S_1 : $x^2 + y^2 + 2x - 4y - 3 = 0$, S_2 : $2x^2 + 2y^2 + 4x - 8y - 3 = 0$
   a) S_1 and S_2 are non-overlapping
   b) S_1 and S_2 are intersecting
   c) S_1 and S_2 are concentric
   d) S_1 and S_2 are touching

3. For what value of k are the circles S_1 and S_2 given by $x^2 + y^2 + 2(k+1)x + 2(k-1)y + k^2 = 0$ of radius $\sqrt{11}$?
   a) 3 and –3
   b) 3
   c) –3
   d) none of these

4. Given the straight line L : $3x + 4y - 5 = 0$ and the circle S : $x^2 + y^2 - 4x - 2y = 0$

   a) L touches S
   b) L intersects S
   c) L passes through the centre of S
   d) L does not meet S

5. The angle between the lines $2x^2 - 7xy + 3y^2 = 0$ is
   a) $\pi/3$
   b) $\pi/6$
   c) $\pi/4$
   d) $\tan^{-1}(3/4)$

6. If the slopes of the lines : $x^2 + 2hxy + 6y^2 = 0$ are as 1 : 2 then h^2 equals
   a) 1/3
   b) 1/6
   c) ¼
   d) none of these

7. Two diameters of a circle are $x + y = 8$, $2x - y = 4$. Its centre is

a) (0,8)
b) (4,4)
c) (8,12)
d) none of these

8. For what value of k, the radius of the circle $x^2 + y^2 + 2x - 4ky - 3 = 0$, is $\sqrt{5}$.

a) 0
b) ½
c) 1
d) 2

9. A circle with centre at (-2,1) touches the x-axis. Its radius is

a) 1
b) 2
c) 5
d) none of these

10. The intercept of the circle $x^2 + y^2 - 4x + 6y = 0$ on the x-axis is

a) 2
b) 3
c) 4
d) 6

11. A circle in the second quadrant touches both axes. Its radius is 2. Then its centre is

a) (2,2)
b) (-2, 2)
c) (-2,-2)
d) (2,-2)

12. A circle through the origin makes +ve intercepts of 4 units and 6 units on the x-axis and y-axis respectively. Its radius is

a) 2
b) 5/2
c) 3
d) none of these

13. A circle of radius A_1, touches a circle of radius A_2 internally. The distance between the centres of the circles is

a) $A_1 - A_2$
b) $A_1 + A_2$
c) $A_2 - A_1$
d) None of these

14. $x^2 + y^2 = a^2$ touches $lx + my + n = 0$. Then

a) $a^2 (l^2 + m^2) = n^2$

b) $a^2 = (l^2 + m^2) n^2$

c) $a^2n^2 = l^2 + m^2$

d) none of these

15. Which circle does not exist ?

a) $x^2 + y^2 - 4x + 2y = 0$

b) $3x^2 + 3y^2 - 6x + 9y - 2 = 0$

c) $2x^2 + 2y^2 + 4x - 2y + 11 = 0$

d) $x^2 + y^2 - 5x + 3y + 8 = 0$

16. The circle on the joining of A (3,-1), B (-2,3) on diameter is

a) $(x-3)(x+2) + (y+1)(y-3) = 0$

b) $(x+3)(x-2) + (y-1)(y+3) = 0$

c) $(x+3)(x+2) + (y-1)(y-3) = 0$

d) none of these

17. Equation of the circle through the center of $x^2 + y^2 - 4x + 1 = 0$ and having the centre at (0, -1) is

a) $x^2 + y^2 + 2y - 4 = 0$

b) $x^2 + y^2 - 2x + 4 = 0$

c) $x^2 + y^2 - 4x + 2y - 2 = 0$

d) none of these

18. The line $(x/a + y/b) = 1$ touches the circle $x^2 + y^2 = 1$ if

a) $a^2 + b^2 = 1$

b) $a^2 + b^2 = a^2b^2$

c) $a + b = 1$

d) none of these

19. The radical axis of the circles $2x^2 + 2y^2 - 4x + 2y - 1 = 0$ and $x^2 + y^2 + 2x - 2y = 0$ is

a) $4x + 3y = 1$

b) $4x - 3y = ½$

c) $4x - 3y + ½ = 0$

d) none of these

20. If $x = 0$ is the radical axis of a circle $x^2 + y^2 - x + y - 1 = 0$ and another circle $x^2 + y^2 - 2x + \lambda y - 1 = 0$, then the value of λ is

a) 0

b) 1

c) –1

d) 2

21. P is a point whose powers w.r.t. two circles S_1 and S_2 are equal and L is the radical axis of the circles. Then P lies on
a) L
b) S_1
c) S_2
d) both S_1 and S_2

22. C is the radical centre of circles S_1, S_2, S_3 and t is the length of the tangent from C to one of the circles. S is the circle with the centre at C and radius t. Then S cuts orthogonally the circles
a) S_1
b) S_1 and S_2
c) S_1, S_2 and S_3
d) none of these

23. The circles $x^2 + y^2 - 8x - 6y + 21 = 0$ and $x^2 + y^2 - 2y + c = 0$ cut orthogonally. The value of c is
a) 21
b) 5
c) −15
d) none of these

24. Every circle $x^2 + y^2 + 2gx + C = 0$ cuts
a) every circle $x^2 + y^2 + 2fy + C = 0$
b) every circle $x^2 + y^2 + 2gx - C = 0$
c) every circle $x^2 + y^2 + 2fy + C = 0$
d) every circle $x^2 + y^2 + 2fy - C = 0$

25. Limiting points of a family of coaxial circles are
a) circles at infinity
b) point circles
c) circles in the limit
d) none of these

26. The eccentricity of a conic is $\sqrt{(2/3)}$. The conic is
a) a circle
b) an ellipse
c) a hyperbola
d) a parabola

27. The distance of the focus from the vertex of the parabola is 10 units. What is the distance of the directrix from the vertex ?
a) 10
b) 5
c) 20
d) 0

28. Which of the following is true ? The directrix of a parabola
a) intersects the parabola at two points
b) is parallel to the axis of the parabola
c) touches the parabola
d) does not meet the parabola

29. The vertex and focus of the parabola $y^2 - 4y + 4x + 8 = 0$ are respectively
a) (-1, 2), (-2,-2)
b) (1,-2), (-2,2)
c) (-1,2), (-2,2)
d) none of these

30. The distance between the vertex and an end of the latus rectum of $x^2 - 4x + 4y - 8 = 0$ is
a) 1
b) 2
c) $\sqrt{5}$
d) none of these

31. The equation of the parabola whose vertex is (0,1) and focus (0,0) is
a) $x^2 - 2x + 4y + 1 = 0$
b) $x^2 + 4y = 4$
c) $y^2 = -4x$
d) $y^2 + 4x - 2y + 1 = 0$

32. The equation of the parabola with vertex at (1,0) and directrix y-axis is
a) $x^2 = 4y - 4$
b) $y^2 = 4x - 4$
c) $y^2 = 4x + 4$
d) $x^2 = 4 - 4y$

33. The equation of the parabola with focus at (1,0) , directrix $x + 3 = 0$ is
a) $y^2 = 8(x+1)$
b) $y^2 = 8(x - 1)$
c) $(x - 1)^2 = 8y$
d) none of these

34. The length of the latus rectum of the parabola $4x^2 - 8x = 3y$ is
a) 4
b) 3
c) ¾
d) 4/3

35. In a parabola which of the following is true ?
a) the focus is equidistant from the vertex and the ends of latus rectum
b) the focus is equidistant from the directrix and the ends of latus rectum
c) the vertex is equidistant from the focus and the ends of latus rectum
d) the directrix is equidistant from the vertex and the ends of latus rectum

36. The straight line $x + y = 1$ touches the parabola.
a) $y^2 = 4x$
b) $y^2 = -4x$
c) $x^2 = 4y$
d) $x^2 = -4y$

37. When will $lx + my = 1$ is a tangent to $y^2 = 4X$?
a) $l + m^2 = 0$
b) $l = m^2$
c) $l^2 = m$
d) none of these

38. Two perpendicular tangents to a parabola always intersect on
a) the tangent at its vertex
b) the axis of parabola
c) the directrix of the parabola
d) the latus rectum of the parabola

39. The equation of the tangent to $y^2 - 4y + 2x = 2$ at (1,0) is
a) $x - 2y = 1$
b) $x + 2y = 1$
c) $2x + y + 1 = 0$
d) $2x + y = 1$

40. The equation to the tangent parallel to $x + y = 1$ to the parabola $y^2 = 8x$ is
a) $x - y - 2 = 0$
b) $x + y + 2 = 0$
c) $x - y + 2 = 0$
d) none of these

41. The equation of the normal at (2, -1) to the parabola $y^2 = -4x$ is
a) $(y + 1) = 2(x - 2)$
b) $x + 2y = 0$
c) $x = 2y$
d) $x - 2y - 4 = 0$

42. Which point will lie on the parabola $y^2 + x = 0$ for all values of t
a) (t^2, t)
b) (t, t^2)
c) $(-t, t^2)$
d) $(-t^2, t)$

43. For which parabola the axis is parallel to the x-axis ?

a) $y = 4x^2 + 1$
b) $y + x^2 = 1$
c) $y^2 - 2y = 2x + 1$
d) $x^2 + 4x = 4y$

44. The equation of the focal chord making an angle $\pi/4$ with ox for the parabola $y^2 + 8x = 0$ is

a) $x - y + 2 = 0$
b) $x + y - 2 = 0$
c) $x + 2y = 0$
d) none of these

45. In which case, the tangent is never parallel to the x-axis

a) $x^2 - x + y + 1 = 0$
b) $x^2 + y = 0$
c) $y^2 - 2x + 2y + 2 = 0$
d) none of these

46. The lengths of the major and minor axes of the ellipse $4x^2 + 9y^2 = 36$ are

a) 4 and 9
b) 18 and 16
c) 6 and 4
d) 3 and 2

47. The eccentricity of an ellipse whose major axis is of length 2 is ½. The minor axis length is

a) 1
b) ½
c) 3/2
d) $\sqrt{3}$

48. The distance between the foci of the ellipse $4x^2 + y^2 = 4$ is

a) $2\sqrt{3}$
b) 3/2
c) $\sqrt{3}$
d) none of these

49. The length of the latus rectum of an ellipse whose eccentricity is $1/\sqrt{2}$ and major radius 2 is

a) 2
b) 3/2
c) 3
d) none of these

50. The center of the ellipse $x^2 + 2x + 4y^2 - 8y + 1 = 0$
a) (0,0)
b) (-1,2)
c) (-1,-2)
d) (-1,1)

51. The eccentricity of an ellipse is $\sqrt{(2/3)}$ and length of its minor diameter is $2\sqrt{3}$. The sum of focal distance of any point on the ellipse is
a) $2\sqrt{3}$
b) 6
c) $\sqrt{3}$
d) $\sqrt{6}$

52. The centre of an ellipse whose major and minor axes are 6 and 4 is (0, -1). The equation of the ellipse is
a) $6x^2 + 4(y + 1)^2 = 1$
b) $6(x + 1)^2 + 4y^2 = 24$
c) $4x^2 + 9 (y + 1)^2 = 36$
d) $9x^2 + 4(y + 1)^2 = 36$

53. For which ellipse in the following, the major axis is parallel to the x-axis

a) $3x^2 + 4y^2 = 1$

b) $3(x +1)^2 + 4 (y - 1)^2 = 12$

c) $4(x+1)^2.+ 3(y-1)^2 = 12$

d) none of these

54. The ends of the minor axis of the ellipse : $x^2 - 6x + 4y^2 - 8y + 9 = 0$ are

a) (5,1), (1,1)

b) (3,2), (3,0)

c) (0,1), (0, -1)

d) none of these

55. The parametric equations of the ellipse $9x^2 + 16y^2 = 1$ are

a) $x = 3 \text{ Cos } \theta, y = 4 \sin \theta$

b) $x = 4 \text{ Cos } \theta, y = 3 \text{ Sin } \theta$

c) $x = \frac{1}{4} \cos \theta, y = 1/3 \sin \theta$

d) $x = 1/3 \cos \theta, y = \frac{1}{4} \sin \theta$

56. The eccentricity of the ellipse $9x^2 + 25 y^2 = 1$ is

a) 4/5

b) 3/5

c) ¾

d) none of these

57. Equations of the directrices of the ellipses $(x-1)^2 / 25 + (y + 1)^2 / 16 = 1$ are

a) $x = 25/3$; $x = -25/3$

b) $x = 28/3$; $x = -22/3$

c) $y = -25/3$; $y = 25/3$

d) none of these

58. The straight line $y = 2x + c$ touches the ellipse $4x^2 + 9y^2 = 36$ when

a) $c = 4\sqrt{10}$

b) $c = 2\sqrt{10}$

c) $c = 5$

d) $c = 25$

59. If $y = mx + c$ touches the ellipse $x^2 / a^2 + y^2 / b^2 = 1$, the point of contact is

a) $(ma^2/c, b^2/c)$

b) $(mb^2/c, a^2/c)$

c) $(-ma^2/c, b^2/c)$

d) $(a^2/c, -mb^2/c)$

60. If x Cos α + y Sin α = p touches the ellipse $x^2/a^2 + y^2/b^2 = 1$, the perpendicular distance of the tangent from the centre of the ellipse is

a) $\sqrt{(a^2 \cos^2 \alpha + b^2 \sin^2 \alpha)}$

b) $\sqrt{(a^2 \text{Sin}^2\alpha + b^2 \text{Cos}^2 \alpha)}$

c) $\sqrt{(a^2 \text{Sec}^2 \alpha + b^2)}$

d) $\sqrt{(a^2 + b^2 \text{Cot}^2 \alpha)}$

61. The equation to the tangent at $\theta = \pi/4$ on $x^2 / 8 + y^2 / 2 = 1$ is

a) $x + 2y = 4$

b) $2x + y = 4$

c) $x + 2y = 2$

d) $2x - y = 3$

62. The y-intercept of the normal to the ellipse $x^2/9 + y^2/4 = 1$ at $(3/\sqrt{2}, \sqrt{2})$ is

a) $1/\sqrt{2}$

b) $1/2\sqrt{2}$

c) $2\sqrt{2}$

d) none of these

63. Which point lies inside the ellipse $4x^2 + 9y^2 = 36$?

a) (1, -1)

b) (1,3)

c) (4,1)

d) (3,2)

64. The area of the ellipse whose major axis is 10 units and eccentricity is 4/5 is

a) 15π

b) 20π

c) 12π

d) 18π

65. The length of any diameter of the ellipse $x^2 + 4y^2 = 4$ lies between

a) 1 and 3

b) 2 and 4

c) 3 and 5

d) 4 and 6

66. The eccentricity of the hyperbola $(x^2/9) - (y^2/16) = 1$ is

a) 5/2

b) 5/3

c) 5/4

d) none of these

67. The centre of the hyperbola $x^2 - 4y^2 - 6x + 8y = 11$ is

a) (0,0)

b) (6, -2)

c) (3,1)

d) (2,3)

68. The lengths of the transverse and conjugate axes of the hyperbola

$4x^2 - 16x - y^2 + 2y = 1$ are respectively

a) 2 and 1

b) 4 and 8

c) 8 and 4

d) 4 and 16

69. The hyperbola $(x^2/16) - (y^2 / 9) = 1$ exists then

a) $3 < x < 4$

b) $|x| \geq 3$

c) $|x| < 4$

d) $|x| \geq 4$

70. The conjugate hyperbola of $(x^2/4) - (y^2/3) = 1$ is

a) $(x^2/3) - (y^2/4) = 1$

b) $(x^2/4) - (y^2/3) + 1 = 0$

c) $(x^2/3) - (y^2/4) + 1 = 0$

d) none of these

71. The parametric equations of the hyperbola with centre at (0,0). Transverse and conjugate axes of lengths 10 and 6 on x and y – axes respectively are

a) $x = 10 \cos \theta, y = 6 \text{ Sin } \theta$

b) $x = 5 \text{ Cos } \theta , y = 3 \text{ Sin } \theta$

c) $x = 5 \text{ Sec } \theta, y = 3 \tan \theta$

d) $x = 5 \tan \theta, y = 3 \text{ Sec } \theta$

72. The angle between the asymptotes of a rectangular hyperbola is

a) $\pi/4$

b) $\pi/2$

c) $\pi/3$

d) $2\pi/3$

73. The asymptotes of $(x^2/16) - (y^2 / 25) = 1$ are

a) $5x = \pm 4y$

b) $4x = \pm 5y$

c) $(x/5) \pm (y/4) = 0$

d) none of these

74. The eccentricity of a rectangular hyperbola is

a) 2

b) $\sqrt{2}$

c) 1

d) 0

75. The foci of the hyperbola $(x^2/16) - (y^2/9) = 1$ are.

a) $(\pm 5, 0)$

b) $(0, \pm 4)$

c) $(\pm 3, 0)$

d) none of these

76. The straight line $y = mx + c$ touches the hyperbola $3x^2 - 9y^2 = 9$ when $c^2=$

a) $3m^2 - 9$

b) $3m^2 - 1$

c) $m^2 - 3$

d) $m^2 + 1$

77. The tangent parallel to the line $x + 2y = 1$ to the hyperbola $(x^2/16) - (y^2 / 1) = 1$ is

a) $x + 2y = 2\sqrt{3}$

b) $x + 2y = \sqrt{2}$

c) $x + 2y = \sqrt{3}$

d) $2x + y = 2\sqrt{3}$

78. The equation to the normal at 'θ' to the hyperbola $(x^2/a^2) - (y^2/b^2) = 1$ is

a) $ax \cos\theta + by \sin\theta = a^2 + b^2$

b) $ax \cos\theta + by \cot\theta = a^2 + b^2$

c) $ax \sec\theta + by \tan\theta = 1$

d) none of these

79. The eccentricity of the conjugate hyperbola of $(x^2/9) - (y^2/16) = 1$ is

a) 5/3

b) 5/4

c) 4/3

d) 16/9

80. A tangent at a point on one branch of a hyperbola

a) intersects the other branch at two points

b) touches the other branch

c) intersects the other branch at one point

d) does not meet the other branch

PROBLEM SHEET – 1

1. Suppose

$$f(x) = \sqrt{1 + \sqrt{x}}\,,\quad g(x) = \frac{1}{\sqrt{x}}\,,\quad h(x) = \frac{4}{x}$$

Find the function fogohof. Find the value of this function at 225.

2. Let $f(x) = 2x^2 + 3$, $g(x) = \sqrt{x-1}$ Where are f+g, f-g, fg are defined ?

3. Where is fog defined if $f(x) = \sqrt{x+3}$ $g(y) = 1 + y^2$?

4. Determine any points where the given function f(x) is undefined, and if possible extend the definition of the function to these points so that the graph is continuous.

   i) $f(x) = (x-1)^2 / (x^2 - 1)$

   ii) $f(x) = (x^2 - 6x + 9) / [(x^2-9)(x-3)]$

   iii) $f(x) = x^2 / (\sqrt{x^2+1} - 1)$

5. Find

   i) Lim $(x^n - 1) / x - 1$, n a positive integer

   ii) $Lim \dfrac{x^2 - 7x + 10}{x - 2}$

6. Give an example of functions f and g such that f(x) + g(x) approaches to a limit as $x \to 0$ even though f(x) and g(x) separately do not approach limits as $x \to 0$.

7. Prove or dispprove : If lim f(x) and lim g(x) do not exist then, does not exist as $x \to 0$.

8. a) Graph the following functions. Then answer the following questions about them.

b) At what points C in the domain of f does lim f(x) exist ?

c) At what points does only the left hand limit exist ?

d) At what point does only the right hand limit exist ?

i) $f(x) = \sqrt{1 - x^2}$ *if* $0 \leq x \leq 1$
$= 1$ if $-1 \leq x < 1$
$= 2$ if $x = 2$

ii) $f(x) = x$ if $-1 \leq x < 0$ or $0 < x \leq 1$
$= 1$ if $x = 0$
$= 0$ if $x < -1$ or $x > 1$

9. For what values of C does the function $f(x) = [x]$ approaches a limit as $x \to c$ ($[x]$ = the greatest integer not greater than $x \varepsilon R$).

10. For what values of C does

$$f(x) = \frac{x}{|x|}$$

approaches a limit as $x \to c$.

11. Find

i) $lim \dfrac{Sin\ 2x}{2x^2 + x}$

ii) $\lim \dfrac{Sin\ 5x}{Sin\ 3x}$

iii) $\lim\limits_{x \to \infty} \dfrac{2x^2 - x + 3}{3x^2 + 5}$

iv) $\lim\limits_{x \to \infty} \left(2 + \dfrac{Sin\ x}{x}\right)$

12. Define h(2) in a way that extends $h(x) = (x^2 + 3x - 10) / (x - 2)$ to be continuous at $x = 2$.

13. Graph the function

$$f(x)= \begin{cases} x & 0 \le x \le 1 \\ 2-x & 1 < x \le 2 \end{cases}$$

b) Is f continuous at x = 1 ?

c) Is f differentiable at x = 1 ?

14. What value should be assigned to make the function

$$f(x) = \begin{cases} x^2 - 1 & x < 3 \\ 2ax, & x \ge 3 \end{cases}$$

continuous at 3 ?

15. Show that the function

$$f(x) = x \sin 1/x \qquad x \neq 0$$
$$= 0 \qquad x = 0$$

is continuous at x = 0.
(Hint : $|x \sin 1/x| \le |x|$).

16. Let f be a continuous function, and suppose that f(c) is positive : Show that there is some interval about C say $c - \delta < x < c + \delta$, throughout which f(x) remains positive.

17. Give an example of a function that is defined on $0 \le x \le 1$, continuous in the open interval $0 < x < 1$ and discontinuous at x = 0.

18. Let f(x) = 9x – 5. Find a $\delta > 0$ so that $|f(x) - 4| < 1/10$ for all x such that $|x - 1| < \delta$. Is f(x) continuous at x = 1 ?

19. Let $f(x) = 2x^2 + 3$. Find an 'a' > 0 so that $|f(x) - f(0)| < ½$ for $|x| < a$.

20. Prove that $f(x) = (x^2 - 1)/3$ is continuous. Is f(x) continuous at all other points wherever it is defined ? Justify your answer.

21. Prove that the function $f(x) = x^2 + 1$, for $x < -1$, $-2x$ for $x \ge -1$ is continuous everywhere. Draw the graph of the function to see that it has no cuts.

22. Prove that the function $K(u) = u^3 - 1$ for $u \le 1$, 2u for $u > 1$ is not continuous at 1. Is the function continuous at all other points ? Justify.

23. Prove that the function
F(x) = 0 if x is rational
= 1 if x is irrational
at every point in R (the reals)

24. Find the greatest common divisor of the following polynomials over the field of rational numbers.

a) $x^3 - 6x^2 + x + 4$ and $x^5 - 6x + 1$
b) $x^2 + 1$ and $x^6 + x^3 + x + 1$

25. Prove that $1 + x + x^2 + \dots + x^{p-1}$ where p is a prime number is a prime polynomial over the rationals

26. Give two non trivial application of prime polynomials over the rationals.

27. The polynomial $p(x) = x^5 - 7x^3 + 2x^2 + 12x - 8$ has a double root at -2 and a single root at 2. Find the other roots.

28. If $\alpha_1, \alpha_2, \alpha_3$ are the roots of the polynomial $x^3 + 7x^2 - 8x + 3$, find the cubic polynomial whose roots are
a) $\alpha_1^2, \alpha_2^2, \alpha_3^2$
b) $1/\alpha_1, 1/\alpha_2, 1/\alpha_3$
c) $\alpha_1^3, \alpha_2^3, \alpha_3^3$

PROBLEM SHEET - 2

a) Prove or disprove the following :

1. Empty set is a subset of any set. 2. Function is a set 3. Every real valued continuous function is an equivalence relation on R. 4. Given any set A, there is always a family F of subsets of A such that the intersection of all the subsets in F is nonempty whenever the member of F pairwise intersect. 5. A constant function is on-to if and only if its codomain is sigleton. 6. A collection X is a set if and only if X = { x/x satisfies a mathematical statement P(x) }, 7. A number is irrational if it is not rational, 8. If A and B are finite sets such that the number of elements in A-B is zero, then the number of elements in AUB is the number of elements in A plus the number of elements in B. 9. There is a 1-1 correspondence between the set of integers and set of rational numbers. 10. The set of integers is infinite . 11. The number of sand particles on a sea shore is infinite . 12. If a and b are any integers, then the product of L.C.M. and g.c.d. of a and b is the product of a and b. 13. If a,b,c are three integers such that a divides bc, then a divides c or b. 14. 19 divides $4n^2 + 4$ for some integers , 15. 4 divides $n^2 + 2$ for some integer n. 16. If x and y are odd, $x^2 + y^2$ is divisible by 4. 17. There are integers x and y such that x + y + 100 and the g.c.d. of x and y is 3. 18. If b and c are relatively prime, and if r divides b, then r and c are relatively prime. 19. If a^2 divides b^3, then a divides b. 20. $6x = 8 \pmod 3$ is solvable 21) Equivalence relations help in counting principles.

b) If S is a finite set, then prove : If f maps S onto S, then f is 1-1. Does the same hold true if S is infinite.

c) Given two sets A and B, prove that $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$:

d) Draw and justify impplication diagram (Venn diagram) of the following :
i) Relations; ii) Function; iii) 1-1 functions; iv) Onto functions;
v) Equivalence relations; vi) constant functions; vii) polynomial functions;
viii) Continuous functions

e) Give an example of real valued function $f : R \rightarrow R$ in each of the following cases :

i) f is 1-1 and on to	ii) f is 1-1 but not onto
iii) f is not 1-1 but onto	iv) f is neither 1-1 nor onto

f) Let S be the set of points in the plane. Two points a and b are related if they are equidistant from the origin. Is such a relation an equivalence relation ? If so find the equivalence classes.

g) Define an equivalence relation on R such that the equivalence class is R itself.

h) For the given set and relation below determine which define equivalence relation.
- i) S is the set of all people in the world today, a R b if a and b have an ancestor in common.
- ii) S is the set of all people in the world today, a R b if a lives within 100 miles of b.
- iii) S is the set of integers, a R b if both a > b and b < a.

i) Prove that an integer is divisible by 3 if and only if the sum of the digits is divisible by 3.

j) Prove that the number of primes is infinite.

k) Evaluate (ab, P^4) and $(a + b, p^4)$ given that $(a, P^2) = P$ and $(b, p^3) = p^2$ where p is a prime and (x,y) denotes gcd of x and y.

l) If x and y are relatively prime to 3, prove that $x^2 + y^2$ cannot be a perfect square.

m) List all integers x in the range $1 \leq x \leq 100$ that satisfy $x \equiv 7 \pmod{17}$.

n) What are the last two digits in the ordinary decimal representation of 3^{400} ?

o) If n is composite and $n > 4$, prove that $(n-1)\,! \equiv 0 \pmod{n}$. Hence or otherwise prove that $(n-1)\,! + 1$ is not a power of n.

p) Find a formula for the number of positive divisors of n.

q) Give a counter example to disprove the following :

   i) if $(a,b) = (a,c)$ then $[a,b] = [a,c]$ ([x,y] LCM of x,y).

   ii) if $p / (a^2 + b^2)$, $p / (b^2 + c^2) \Rightarrow p / (a^2 + c^2)$

r) Find the least positive remainder when 3^{26} is divided by 7.

s) Prove that any number which is a square must have one of the following for its units digit : 0,1,4,5,6,9.

t) Prove that if n is odd, then $n^2 - 1$ is divisible by 8.

u) Prove that the g.c.d. of a and (a + 2) is 1 or 2 for every integer a.

v) Prove that the g.c.d. of a, (a + k) divides k for all integers, a, k not both zero.

w) Let a/b and c/d be fractions in lowest forms. Prove that if their sum is an integer, then $|b| = |d|$.

x) Give a necessary and sufficient condition (with justification) for a function f to have the inverse.

PROBLEM SHEET – 3

1. Find the complex number with modulus four times that of 4 + 3I and amplitude less than that of 4 + 3 i by $\pi/4$.
2. If $| z_r | = 2$ and amp $z_r = \pi/6 + r\ p/2$ for r = 1,2,3,4 show that the points $1 + z_1$, $1 + z_2$, $1 + z_3$, $1 + z_4$ form a square.
3. $Z = x + iy$ is such that the amplitude of the fraction

$$\frac{z-1}{z+1}$$

Is always equal to p/4. Show that $x^2 + y^2 - 2y = 1$.

4. Let (r, θ) denote the complex number whose modulus is r and amplitude is θ. If $a = (1, \alpha)$, $b = (1, \beta)$, $c = (1, \gamma\gamma)$ and $a + b + c = 0$, prove that $1/a + 1/b + 1/c = 0$.

5. If

$$x + iy = \frac{3}{2 + Cos\ \theta\ i\ Sin\ \theta}$$

Prove that $(x - 1)(x - 3) + y^2 = 0$.

6. If $l^2 + m^2 + n^2 = l$ and $(m + in) = (1 + i) z$ show that

$$\frac{1 + iz}{1 - iz} = \frac{1 + im}{1 + n}$$

7. If z_1, z_2, z_3 are the vertices of an equilateral triangle drawn in the Argand diagram, prove that $z^2_1 + z^2_2 + z^2_3 = z_2z_3 + z_3z_1 + z_1z_2$.

8. If z_1 and z_2 are two complex numbers, prove that $| z_1 + z_2|^2 = 2 |z_1|^2 + 2 |z_2|$. Prove also that $|z_1 + z_2|^2 = | z_1 |^2 + | z_2 |^2$ if and only if z_1/z_2 is purely imaginery.

9. Simplify :

$$\frac{(Cos\ \pi/15 + i\ Sin\ \pi/15)^{10} + (Cos\ \pi/15 - i\ Sin\ \pi/15)^{10}}{(Cos\ \pi/3 + i\ Sin\ \pi/3)^6}$$

10. If n is a positive integer, prove that $(Sin\ \theta + i\ Cos\ \theta)^n = Cos\ n(\pi/2 - \theta) + i\ Sin\ n(\pi/2 - \theta)$.

11. Prove that $(a + ib)^{m/n} + (a - ib)^{m/n} = 2 (a^2 + b^2)^{m/2n} Cos (m/n \tan^{-1} b/a)$.

12. Prove that $(1 + i)^n + (1 - i)^n = 2^{n/2+1}$ Cos n $\pi/4$ where n is a positive integer.

13. Prove that

$$(1+i\sqrt{3}\ )^8 + (1 - i\ \sqrt{3}\ )^8 = -2^8$$

14. If x_r = Cos $(\pi/2^r)$ + i Sin $(\pi/2^r)$, prove that x_1. X_2. X_3...to ∞ = -1.

15. If α and β are the roots of $x^2 - 2x + 4 = 0$, prove that

$$\alpha^n + \beta^n = 2^{n+1}\ Cos\ \frac{n\pi}{3}$$

16. If2 Cos θ = x + 1/x and 2 Cos θ = y + 1/y, prove that $x^m y^n + 1/ x^m y^n$ = 2 Cos (m θ + n θ) and $x^m y^n$ - $1/ x^m y^n$ = 2i Sin (mθ + nθ).

17. If 2 Cos 2θ = x + 1/x, prove that $x^6 + 1/x^6$ = 2 Cos 12 θ and $X^6 - 1/x^6$ = 2 i Sin 12 θ.

18. If x = C is α, y = Ċ is β, z = C is γ and if x + y + z = 0, prove that 1/x + 1/y + 1/z = 0.

19. If Cos α + Cos β + Cos γ = Sin α + Sin β + Sin γ = 0, prove that

i) Σ Sin^2 α = Σ Cos^2 = 3/2

iii) Σ Sin^2 α = Σ Cos^2 α = 0.

20. If Cos α + 2 Cos β + 3 Cos γ = 0 = Sin α + 2 Sin β + 3 Sin γ prove that Cos 3α + 8 Cos 3 β + 27 Cos 3γ = 18 Cos (α + β + γ) Sin 3α + 8 Sin 3 β + 27 Sin 3r = 18 sin (α + β + γ).

21. Find all the values of :

i) $(1 + i)^{2/3}$

ii) $(1 - i\ \sqrt{3})^{1/3}$

iii) $1^{1/3}$

iv) $(-1)^{1/3-}$

v) $16^{1/4}$

vi) i^4

vii) $(1 + i)^{1/3}$

viii) $(1 - i)^{1/4}$

ix) $(\sqrt{3} + i)^{1/3}$

x) $27^{1/3}$

22. Solve the equations :

i) $x^4 - x^3 + x^2 - x + 1 = 0$

ii) $x^7 + x^4 + i(x^3 + 1) = 0$

iii) $x^7 + x^4 + x^3 + 1 = 0$

iv) $x^9 - x^5 + x^4 - 1 = 0$

23. Solve the equation $x^{12} - 1 = 0$ and find which of its roots satisfy $X^4 + x^2 - 1 = 0$.

24. Using De Moivre's theorem, prove that $\text{Cos } 3\theta = 4 \text{ Cos}^3 \theta - 3 \text{ Cos } \theta$ and $\text{Sin } 3\theta = 3 \text{ Sin } \theta - 4 \text{ Sin}^3 \theta$.

25. If w is an imaginery cube root of 1, prove that

1. $(1 + w - w^2)^3 - (1 - w + w^2)^3 = 0$
2. $(1 - w + w^2)^5 + (1 + w - w^2)^5 = 32$
3. $(2 - w)(2 - w^2)(2 - w^{10})(2 - w^{11}) = 49$
4. $(a - b)(a - wb)(a - w^2 b) = a^3 - b^3$.

26. Find the continued product of the four values of

$$\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right)^{3/4}$$