REGIONAL INSTITUTE OF EDCUATION, MYSORE-570 006
[National Council Educational Research Training]

A Training Programme for Resource Persons on Teaching of Selected Hardspots in Mathematics at Secondary Level
13.10.2003 to 17.10.2003

Dr.N.M.Rao
Academic Coordinator

REPORT

Title of the PAC Programme :

A training programme for Resource Persons on Teaching of selected Hardspots in Mathematics at secondary level. 13.10.2003 to 17.10.2003

The above programme was conducted at RIE, Mysore from 13 to 17, October 2003 in which twenty six resource persons from Andhra Pradesh, Kerala and Pondicherry participated. Dr N M Rao was the coordinator. All the faculty members of the Mathematics Section and the Principal, gave the intensive training programme for the resource persons for 5 days. They were trained to identify the hardspots in Mathematics, to develop different methods to teach them and to organize the project based teaching of these hardspots for classes IX and X. The topics covered during the programme include Computing, Flow Charts, Number Systems, Algebra, Probability, Linear Programming, Logarithms, Commercial Mathematics, Mensuration, Graphs, etc.

In the end of the programme, these resource persons were capable of solving most of their difficult problems in Mathematics themselves and could develop some projects and teaching aids independently on some concepts in the school mathematics. They, in turn, will train the other Mathematics teachers in their States.

The list of participants is enclosed separately.

(N M Rao)
Coordinator

REGIONAL INSTITUTE OF EDUCATION, MYSORE 570 006

Five-day Training Programme for Resource Persons on Teaching of Selected Hardspots in Mathematics at Secondary Level

TIME-TABLE

Day/ Date	9.30 to 11.00	11.00 - 11.15	11.15 – 12.45	12.45 – 2.00	2.00 – 3.30	3.30 – 3.45	3.45 – 5.15
Monday 13.10.2003	Inauguration (NMR / DB / BSPR / BSU / BCB)		Identification of Hardspots initiation and group work.		Identification of Hardspots and Group Work (NMR / DB / BSPR / BSU / BCB)		Reporting and selecting the hardspots.
Tuesday 14.10.2003	NMR		BCB		CB		BSU
Wednesday 15.10.2003	GR	TEA BREAK	BSPR	LUNCH BREAK	DB	TEA BREAK	NMR
Thursday 16.10.2003	NMR		BSPR		BSU		DB
Friday 17.10.2003	BCB		BSU		Reporting Session		Valedictory

GR – Dr G Ravindra, Principal;
DB – Dr D Basavayya; Professor in Mathematics;
BSU – Dr B S Upadhyaya, Reader in Mathematics;
NMR – Dr N M Rao, Professor in Mathematics and Coordinator;
BSPR – Dr B S P Raju, Reader in Mathematics;
BCB – Sri B C Basti, Senior Lecturer in Mathematics

Areas to be discussed by different faculty members will be intimated after the selection of hardspots.

8.10.03

COORDINATOR

PROJECT BASED LEARNING IN MATHEMATICS

Dr N M Rao

Primary Level

Concrete and semi-concrete materials are effective means to develop mathematical concepts and skills at the primary level. Their use helps children to understand the underlying structures and principles of mathematical concepts and skills. Learning becomes more meaningful and permanent in their mind by the use of concrete materials.

Secondary Level

When the child comes to secondary level, the nature of mathematics changes slowly. Mathematics is a subject, most of which are abstract. The change from concrete objects at primary level to an abstract subject at the secondary level, cannot be made too abrupt. Any change has to be gradual and continuous. Therefore, wide range of materials available in and outside the school environment has to be used by the teachers in the teaching process. But, in this process, the rigor of mathematics as an abstract subject should not be lost. Therefore, it is suggested to develop project based teaching learning of mathematics at least for some topics which are hardspots. This method can also be adopted to show the applications of mathematics in the day today life.

Steps :

The following steps may be adopted in the Project Based Learning.

1. Title of the Project : Name of the Teaching Aid.
2. Objective : Purpose of doing this experiment / what the students learn out of this experiment.
3. Description : How do you conduct this experiment / The student has to arrive at some conjecture by doing the experiment repeatedly. This result will be only a proposition till it is proved.

4. Conjecture : The student arrives at some result or the proposition by doing the experiment repeatedly and by doing some guess work also. This proposition need not be true always.

5. Proof /Disproof : (The above conjecture has to be either proved or disproved by the mathematical methods. If it is proved, then the conjecture will be called 'Theorem' or 'Result'. If the conjecture is disproved by giving a counter example, then the conjecture will be considered 'false'.

6. Open Questions : The above procedure helps in inventing a result. But it is always better to give some more open questions to the students so that they can continue their investigations.

   Here we give some of the projects conducted during the programme.

PROJECTS IN MATHEMATICS

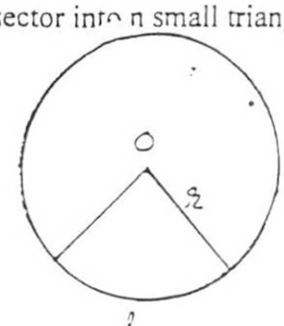
Prof N M Rao

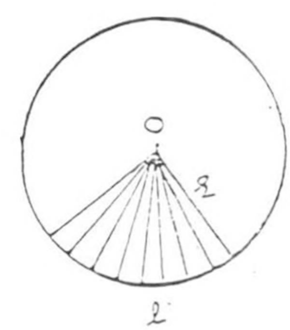
1. Area of the Circle

Objective : To find the area of the circle by using the area of small sectors.

Description :

Take a circle of radius r. Consider a sector of the circle of arc length l and divide the sector into n small triangles as shown in the figure.





The area of each triangle = $\frac{1}{2}rb$ where b = $\frac{l}{n}$.

The total area of the sector of arc length l = n (½ rb)
= ½ r (nb)
= ½ rl
since l = nb

In the same way, the area of the circle = ½ rc, where c is the circumference of the circle.

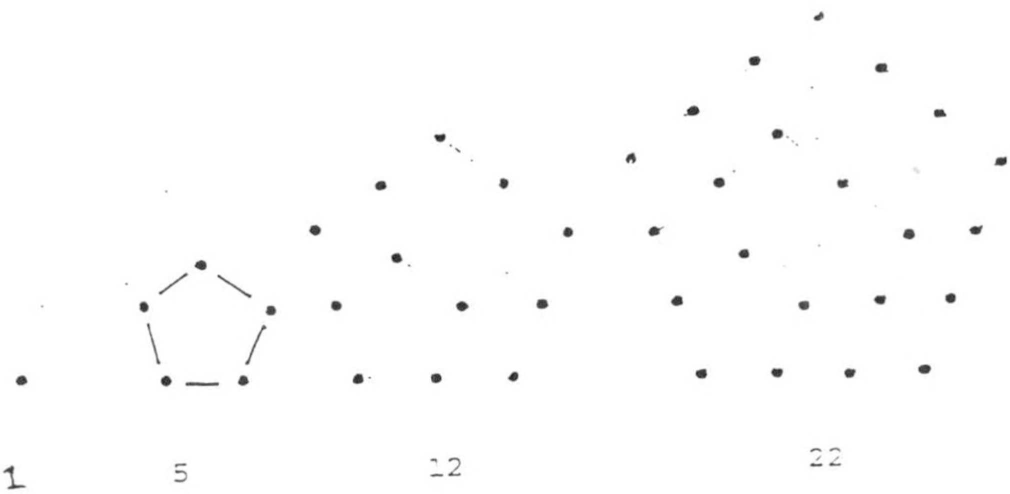
Area of the circle = ½ rc
= ½ r (2 π r)
= πr^2

2. Pentagonal Numbers

Objective : To enable the students to acquire the knowledge of pentagonal numbers.

Description:

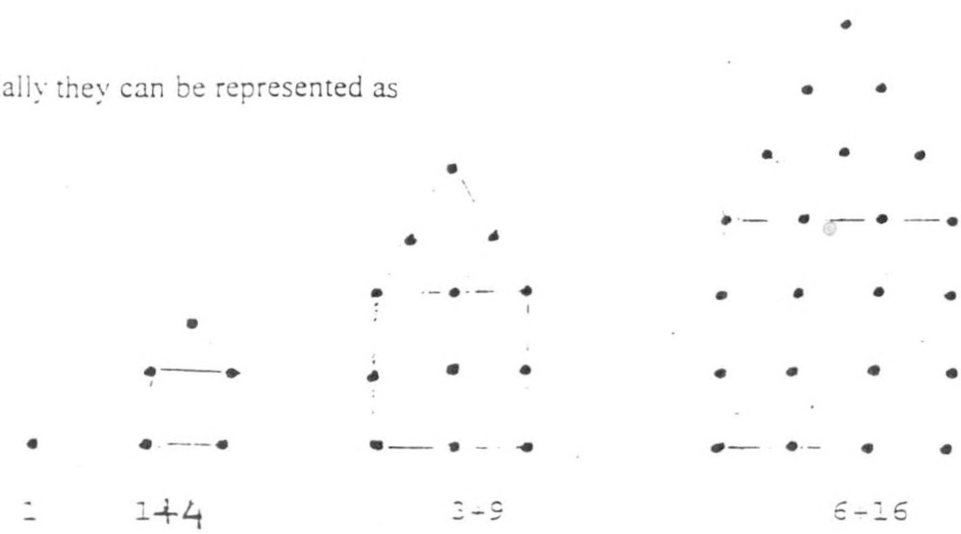
Numbers can be represented in certain patterns. One of the patterns is by representing the dots. The below shown are the pattern of pentagonal numbers.



1,5,12,22,..... are called pentagonal numbers. These pentagonal numbers are obtained by adding triangular numbers and square numbers. The pattern thus formed with these numbers are

Triangular Numbers	+	Square Numbers	=	Pentagonal Numbers
	+	1	=	1
1	+	4	=	5
3	+	9	=	12
6	+	16	=	22
10	+	25	=	?

Thus pictorially they can be represented as



The bindis can be pasted on chart paper and the patterns of the pentagonal numbers can be enjoyed by the students.

1. The students can be asked to guess the next pentagonal numberand verify it afterwards by adding the corresponding triangular and square numbers.
2. The students can also be asked to find a formula to represent the triangular, square and pentagonal numbers

3. Tetrahedral Numbers

Objective : To enable th students to acquire the knowledge of the development of fifth tetrahedral number through Pythagorean, triangular numbers.

Procedure :

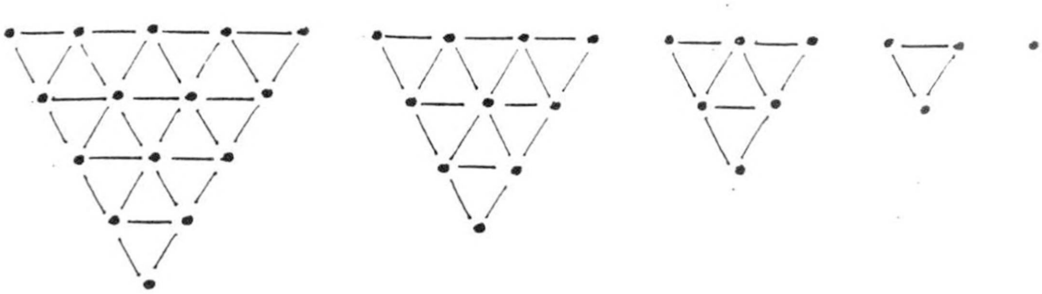
The first six Pythagorean numbers are 1,3,6, 10, 15 and 21. They are represented as follows:

T1	T2	T3	T4	T5	T6
=1	(1+2)	(1+2+3)	(1+2+3+4)	(1+2+3+4+5)	(1+2+3+4+5+6)
=1	=3	=6	=10	=15	=21

Look at the pattern down below and the series in the fourth line :

1	1	1	1	1	1	1	1	
1	2	3	4	5	6	7	8	The Natural Numbers
1	3	6	10	15	21	28	36	The Triangular Numbers
1	4	10	20	35	56	84	120	The Tetrahedral Numbers

The tetrahedral number is built up from Pythagorean, triangular numbers as follows :



15 + 10 + 6 + 3 + 1

1; (1+3)=4; (1+3+6)=10; (1+3+6+10)=20; (1+3+6+10+15)=35

Taking clue from the above table, a model of the tetrahedral numbers is formed by keeping the patterns one upon the other as follows :

1. Keep one ball on the top step.
2. Below that, keep a step having three balls.
3. Next step contains 6 balls.
4. Next lower step contains 10 balls.
5. The fifth step contains 15 balls.

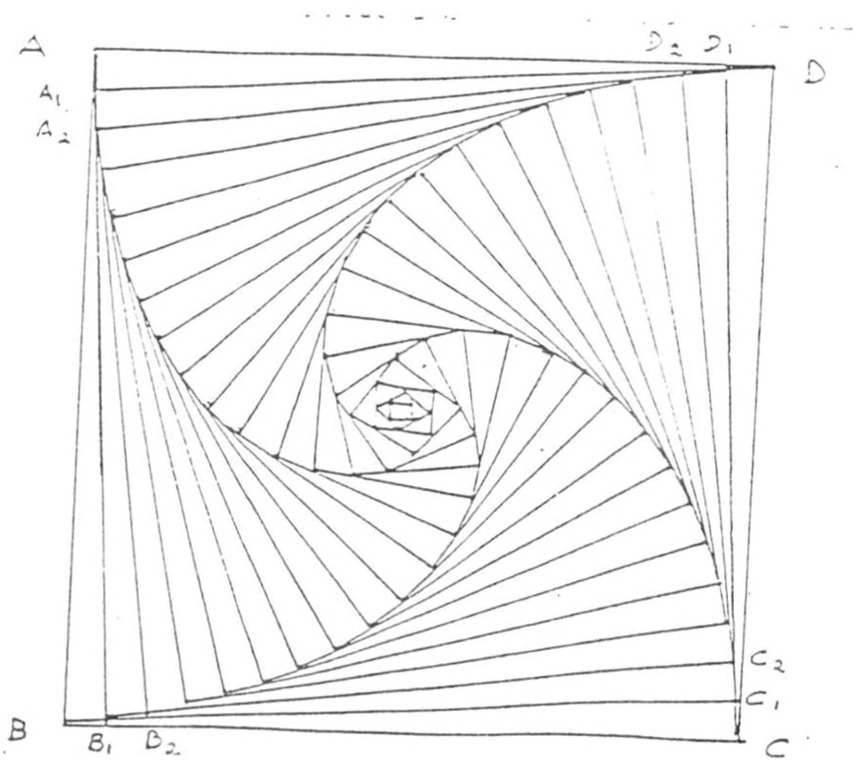
Now the complete model contains 15 + 10 + 6 + 3 + 1 = 35 balls – A tetrahedral number is built up from triangular numbers. Similarly any tetrahedral number can be built up as the sum of triangular numbers.

The students can be asked to prepare a vertical model of the above. They can also be asked to guess a formula to find tetrahedral numbers.

4. Path of Pursuits

Objective: To find the paths of four ants placed at the corners of the square, each one moving in the direction of the ant in front of it. (This path is called the path pursuits).

Take a piece of stiff card board and mark a square ABCD of side 10 cm. Mark the point A_1 on AB at ½ cm distance from A. Similarly mark B_1, C_1 and D_1 at ½ cm from B, C and D respectively. Now mark A_2 at a distance of ½ cm from A_1, on the line A_1B_1, B_2 at ½ cm from B_1 on the line B_1C_1 and so on. Continue in this way until the center of the square is reached. These envelopes are known as curves of pursuit. Since they are the paths which four ants originally placed at the corners of the square, would follow if they were always to walk in the direction of the ant in front of them.

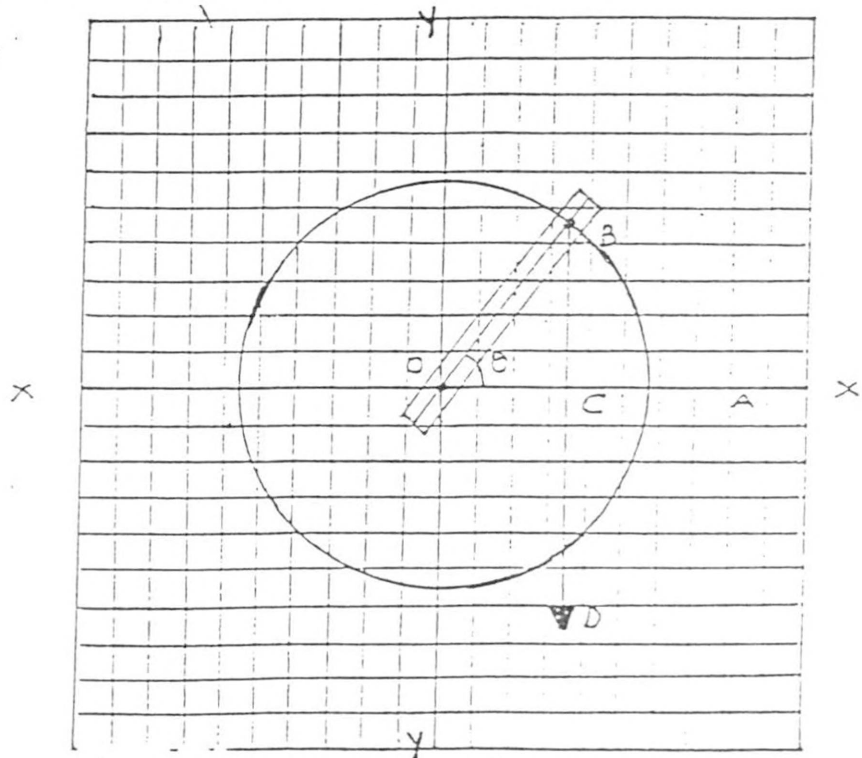


1. Can you stitch the path of pursuits on a black coloured cloth using white thread ?
2. Where is the point at which all four ants meet each other in the end ?
3. Read the chapter on envelopes and evolutes (geometry) to understand the significance of this path.

5. Building Trignometrical Tables

Objective: A simple device can be constructed by the students that will enable them to make their own table of trigonometric ratios for the sine and cosine.

Procedure :



1. On a graph paper, draw a circle with a radius of 10 cm.
2. Cut thin strip of cardboard atleast 12 cm long.
3. Draw a line down the center of the strip.
4. Attach one end of the strip to the center of the circle.
5. At the other end of the strip, 10 cm from the point where it is attached to the circle, make a small hole and attach a piece of thread.
6. At the opposite end of the string, attach a weight to serve as a plumb line.

The strip OB can be rotated around the point O so that OB makes different angles θ with x-axis. The hanging plummet BD cuts the x-axis at the point C. Count the number of spaces of length of the cord BC. Since hypotenuse is fixed at 10 cm, we can easily determine sine ratio. Sin θ = BC/10. As we change the angle by moving the cardboard strip, we can observe the change in the value of sin θ. Similarly the value of cos θ can also be read by counting the number of spaces of horizontal axis OA. Cos θ = OC/10.

From this we can get the value of tan θ, cot θ, sec θ and cosec θ. There may be some error in counting the lengths of BC and OC. Therefore, students are asked to compare these values of sin θ, cos θ, etc. with the standard values given in the trigonometric tables.

6. Solids of Revolution

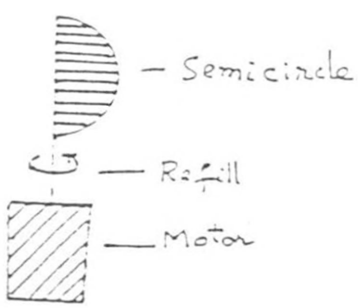
Objective: To show that various geometrical figures when revolved around a particular axis give various solids.

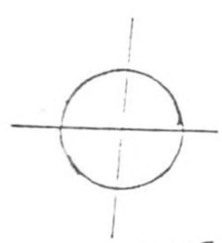
How to use this aid

The teaching aid consists of a motor and various objects of following shapes :

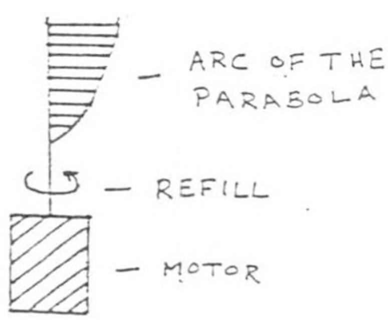
a) circular
b) parabolic
c) triangular or angular
d) square or rectangular

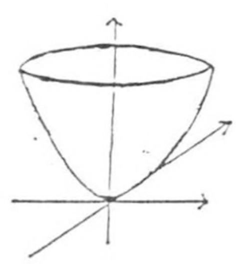
The objects are fixed to a pen refill. that should be attached to the motor which rotates about its axis. We get the following solids of revolution.





SPHERE





PARABOLOID

7. Path of the Moving Chord Inside a Circle

Objective : To illustrate that, the path of the moving chord of constant length inside a circle is a circle and to find out the radius of this inner circle.

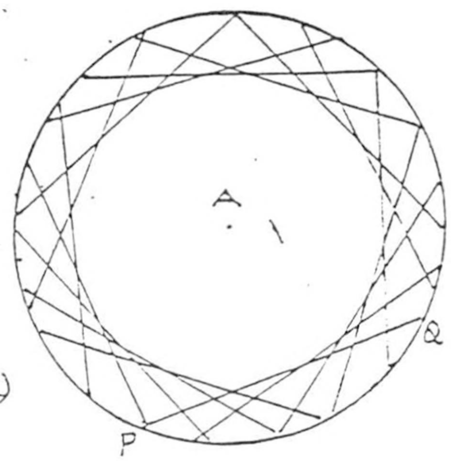


FIG (a)

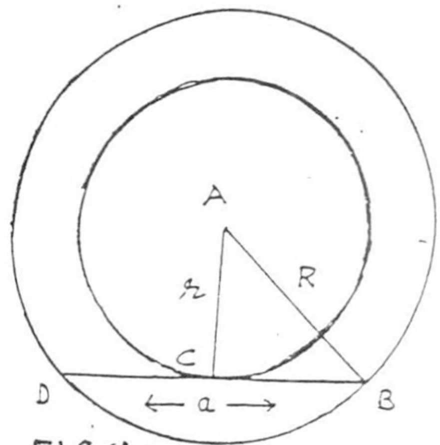


FIG (b)

PQ is a chord of constant length which moves inside the circle of a radius R, centred at the point A. What is the path of PQ ? The students can move the stick PQ inside the circle and convince themselves that the path of the moving chord PQ of constant length inside a circle in a circle. They can repeat the experiment and verify the above fact. It is also clear that the center of the new circle is also A. What is the radius of this inner circle?

To find the radius of the inner circle see Fig. (2).
In which BD = a (length of the chord)
AB = R (radius of the outer circle)
AC = r (radius of the inner circle)

By Pythagoras theorem,

$$AB^2 = AC^2 + BC^2$$

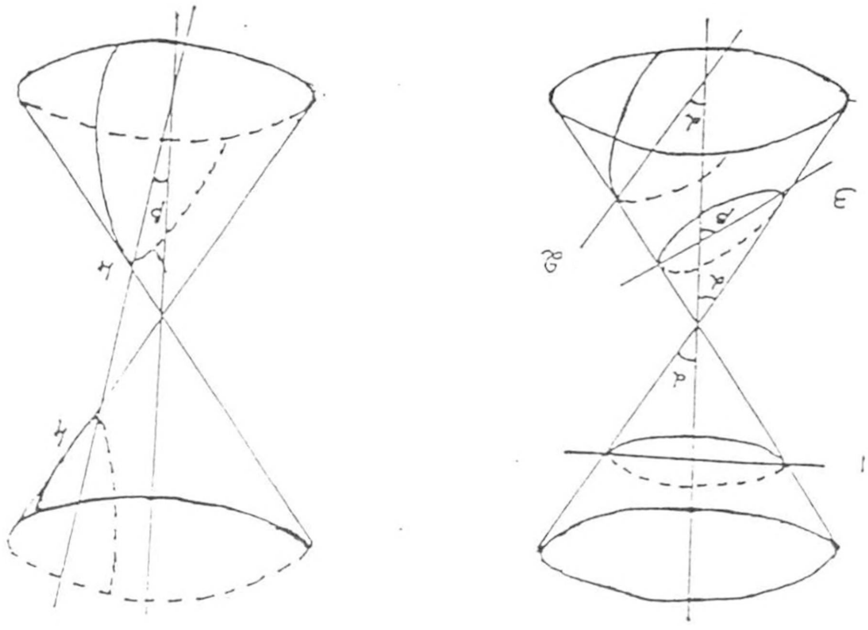
$$R^2 = r^2 - \left(\frac{a}{2}\right)^2$$

$$r = \sqrt{R^2 - \frac{a^2}{4}}$$

1. What happens if the length of the chord PQ is equal to the diameter of the bigger circle ?
2. What happens if the length of the chord PQ is equal to the radius of the bigger chord?

8. Conic Sections

Objective: To show that when a right circular cone is cut in four specific ways we get conic sections namely (1) circle, (2) parabola, (3) ellipse and (4) hyperbola.



How to use

- Hold the model and chart side by side, disjoint the right circular cone at the place marked '1' and see that the edge of the surface is a circle i.e. we get a circle by cutting the right circular cone perpendicular to its axis by a plane.
- Similarly disjoint the cone at the place marked '2' and see that the edge of the surface is a parabola. i.e. when we cut the cone parallel to one of its side we obtain parabola.
- Disjoint the cone at the place marked '3' and see that the edge of the surface is an ellipse. i.e. when we cut the cone at an inclined angle we get ellipse.

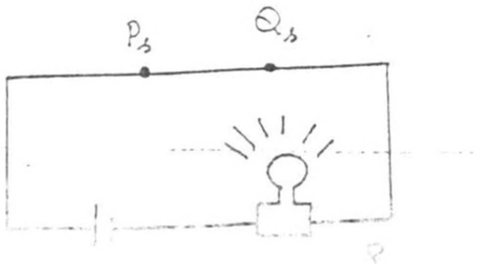
9. Logic Box

Objective: To enable the students to understand the conjunction (^) and Disjunction (v) of two statements and draw their truth tables.

P ^ Q = P and Q (Conjunction)
P v Q = P or Q (Disjunction)

How to use the Teaching Aid :

1. Connect the battery to the circuit. The circuit is now ready to operate.
2. The Ps and Qs switches, are connected in the series circuit. The circuit is given by



(i) When the switches Ps and Qs are both switched on the light is on (T ^ T = T).

(ii) When either of the switches are off the light is off (T ^ F = F).

(iii) When both the switches are off, the light is off (F ^ F = F).

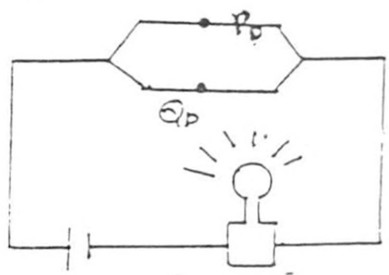
The truth table for the given "And" circuit is :

Ps	Qs	Ps ^ Qs
T	T	T
T	F	F
F	T	F
F	F	F

This is called the *conjunction.*

3. Now see the disjunction (v) circuit.

Pp and Qp are connected in parallel circuit. The circuit is shown as



(i) When both Pp and Qp are switched on, the light is on (T v T = T).

(ii) When either Pp or Qp are switched on, the light is on (T v F = T).

(iii) When both Pp or Qp are switched off, the light is off (F v F = F).

The truth table is given by

Pp	Qp	Pp v Qp
T	T	T
T	F	T
F	T	T
T	F	F

The 'OR' circuit is off only when both Pp and Qp are off. This is called the disjunction of P, Q (Read as P or Q).

Verify whether the following statements are true or false :

1. (Conjunction) : Either $2 + 3 = 6$ and $4 + 5 = 9$.
2. (Disjunction) : Either $2 + 3 = 6$ or $4 + 5 = 9$

Justify your answer using the logic box.

10. Magic Square

Problem

Prepare a magic square by putting the given numbers between 1 and 20 in the holes of given 3 × 3 box such that sum of columns, rows and diagonals should be 21.

x_1	x_2	x_3
y_1	y_2	y_3
z_1	z_2	z_3

Solution

The least sum from 3 × 3 magic square will be 15, a multiple of 3. Let the sum be "a". To find the numbers in the magic square first subtract 15 from "a" divide by 3 and add 1.

$$\frac{a - 15}{3} + 1 = C \qquad \ldots\ldots (1)$$

C will occupy the position of z_2. The number at x_3 will be $c + 1$. Similarly $y_2 = x_3 + 3$, $z_1 = y_2 + 3\ (x_3 + 6)$. From these four numbers we get $z_3 = a - (z_1 + z_2)$, $x_2 = a - (z_2 + y_2)$, $z_1 = a - (x_2\ x_3)$, $y_3 = a - (x_3 + z_3 0$.

Here the given sum is 21. From (1) $z_2 = \frac{21 - 15}{3} + 1 = 3$

$x_3 = 3 + 1 = 4$, $y_2 = 4 + 3 = 7$, $z_1 = 7 + 3 = 10$ or $4 + 6 = 10$,
$z_3 = 21 - (10 + 3) = 8$, $x_2 = 21 - (3 + 7) = 11$, $y_1 = 21 - 16 = 5$
$x_1 = 21 - (11 + 4) = 6$, $y_3 = 21 - (4 + 8) = 9$.
Therefore, a magic square of sum 21 is as follows:

6	11	4
5	7	9
10	3	8

This method can be applied for any 3 × 3 magic square.

1. Students are advised to try to form a different magic square in which the sum is 21.
2. Form a magic square of sum 15.

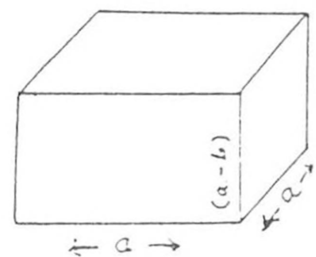
11. Model of $a^3 - b^3$

Objective : This model is to illustrate that

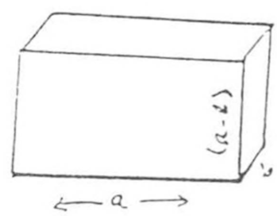
$$a^3 - b^3 = (a - b)\,a^2 + (a - b)\,ab + (a - b)\,b^2$$
$$= (a - b)\,(a^2 + ab + b^2)$$

Model : There are three wooden blocks of the following dimensions as shown :

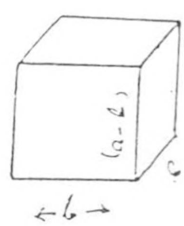
(i) $(a - b) \times a \times a$



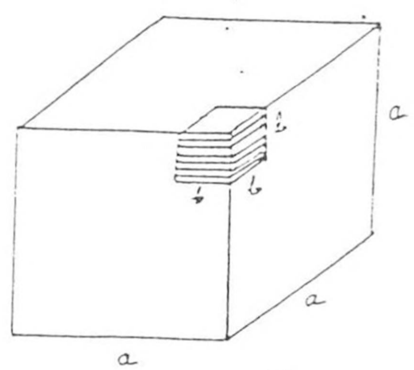
(ii) $(a - b) \times a \times b$



(iii) $(a - b) \times b \times b$



The three wooden blocks can be arranged in such a way that the complete assembly looks like $a^3 - b^3$. i.e. a small cube of volume b^3 has been removed from cube of volume a^3 units.



The students are requested to assemble the wooden blocks and convince themselves about the result :

$$
\begin{aligned}
& (a-b)\times a\times a+(a-b)\times a\times b+(a-b)\times b\times b\\
=\ & (a-b)\,a^2+(c-b)ab+(a-b)b^2\\
=\ & (a-b)\,(a^2+ab+b^2)\\
=\ & a^3+a^2b+ab^2-a^2b-ab^2-b^3\\
=\ & a^3-b^3
\end{aligned}
$$

12. Envelopes

Objective: To enable the students to understand the locus of a point and envelope of a set of lines.

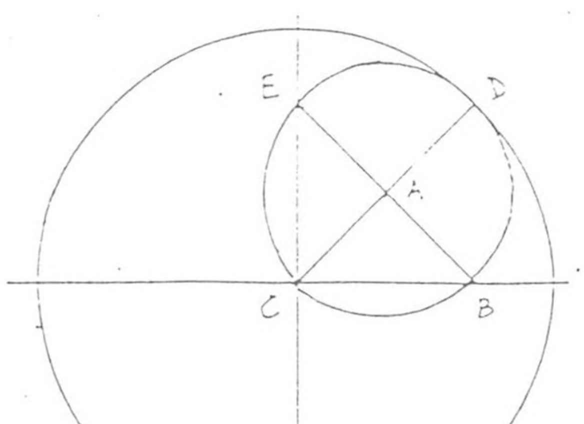
Analysis : A set of points obeying a rule is called locus and a set of lines obeying a rule is called an envelope.

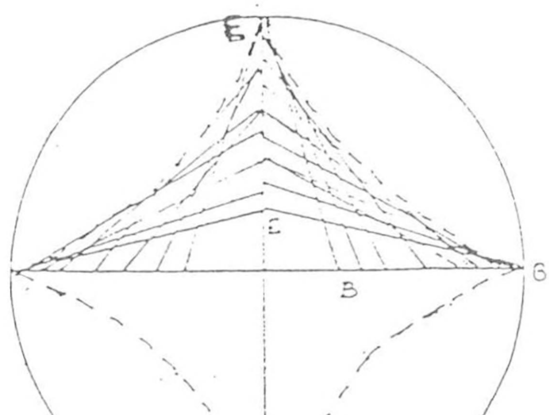
Experiment :

Cut a hole whose radius is the diameter of a one rupee coin, in a piece of cardboard. Roll the coin, without slipping, round the hole. What is the locus of

(a) the center of the coin ?

(b) a point on the circumference ?





Answer :

(i) If B is any point on the coin, then the locus of B is the diameter of the hole.

(ii) If BE is the diameter of the coin, the locus of E is the perpendicular diameter of the hole.

(iii) The locus of A, the center of the coin, is a circle.

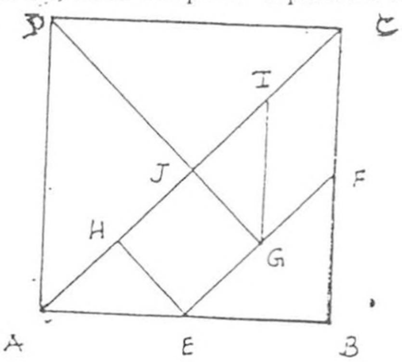
(iv) The envelope of BE is an astroid.

13. Tangrans

Objective : To form the geometrical shapes of squares, rectangles, hexagon, trapezium, etc. from the given pieces and to improve the mental ability of students.

Construction:

1. Take a square cardboard ABCD of side length 20 cms.
2. Draw the diagonal segment AC as shown in the figure.
3. The points E and F are mid points of AB and BC respectively. Draw EF.
4. G is the midpoint of EF. Draw GD.
5. Construct the line segment EH perpendicular on AC from the point E.
6. Draw a line segment GI, from the point G parallel to BC to cut the line AC at the point I.

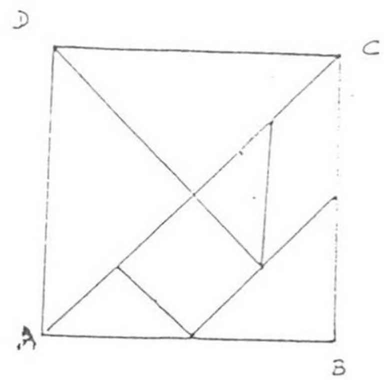


By cutting along the lines as shown in the figure, we get tan gram pieces.

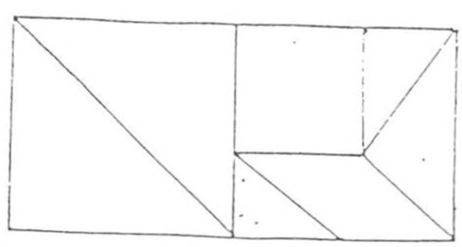
How to use

Take out all the seven tan gram pieces. Ask the learner to arrange the given pieces.

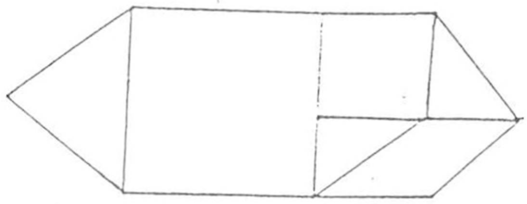
(i) to form a square



(ii) to form a rectangle

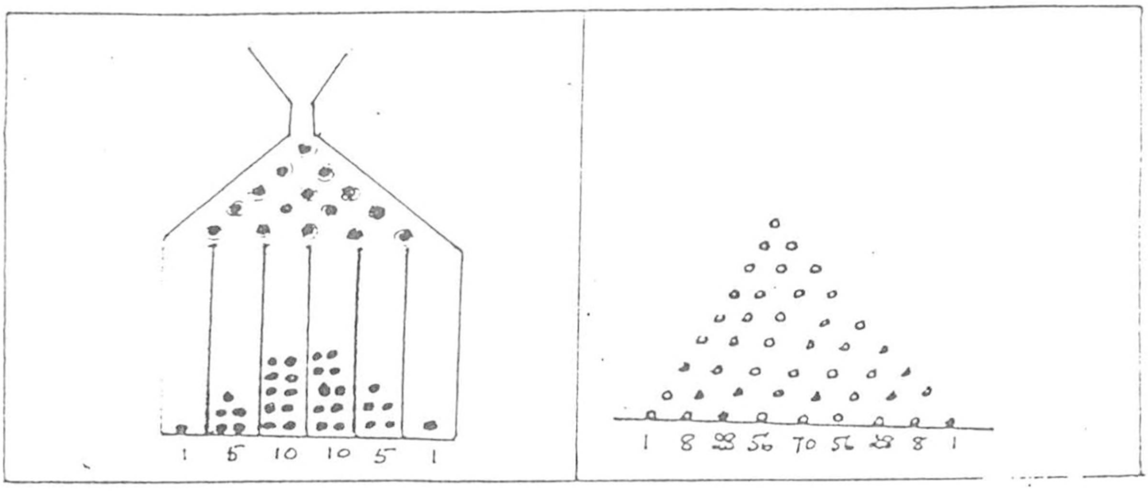


(iii) to form a hexagon



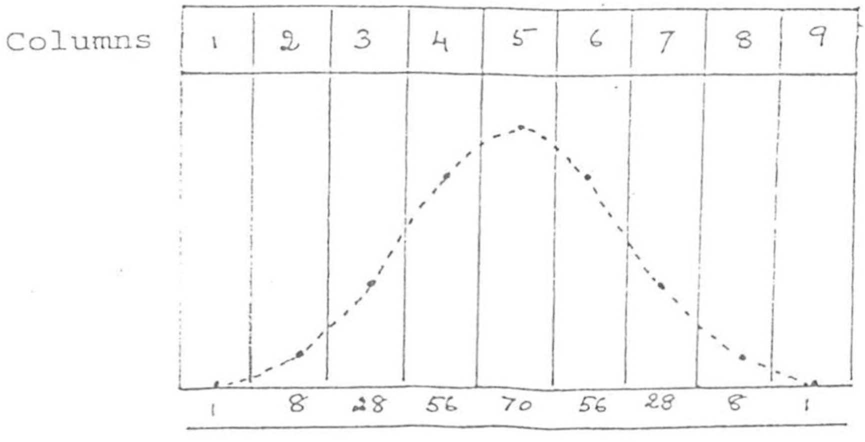
14. Probability Curve Experiment

Objective : This is a wooden model to show that the marbles flowing through a series of nails in the form of Pascal's triangle, will settle down in the shape of a Normal Probability Curve.



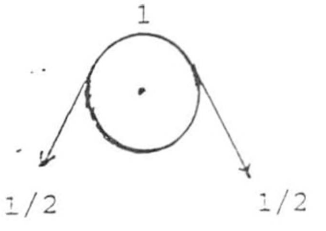
How to use it

The nails are fixed on a wooden board according to the Pascal's Triangle as shown in the figure. Above the nails, a metal box is fitted to pour the marbles uniformly. As we pour the marbles in the metal box, they come and settle in the columns in the form of normal probability curve as shown.



Principle

When we pour the marbles at the top, at the first nail, half of the marbles will flow by the left side of the nail and half the marbles will flow by the right side of the nail as shown :



The marbles coming to left nail in the second row will have two equal possibilities to go to the 3rd row, ¼ to the left and ¼ to the right. Similarly the marbles coming to the right nail in the second row will have two equal chances to go to the 3rd row, ¼ to the left of it and ¼ to the right of it. Hence in the second row, the marbles flow will be as follows : ¼ in the left, ¼ + ¼ = ½ in the center and ¼ in the right. Similarly in the third row, the marbles flow as follows : 1/8, 3/8, 3/8 and 1/8. If we continue in this way, in the eight row marbles flow as follows:

$$\frac{1}{256}, \frac{8}{256}, \frac{28}{256}, \frac{56}{256}, \frac{70}{256}, \frac{56}{256}, \frac{28}{256}, \frac{8}{256}, \frac{1}{256}$$

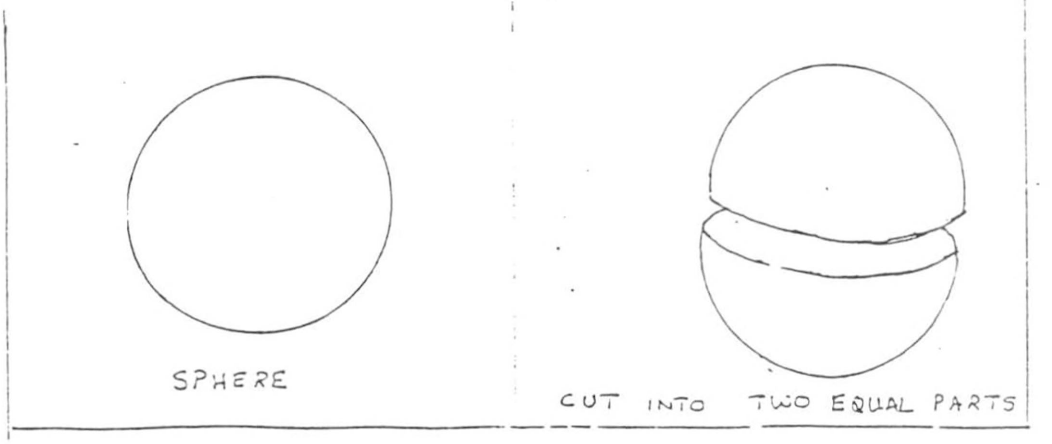
In other words, if we pour 256 marbles from the top, then in a normal case, the number of marbles setting in each column will be as shown below.

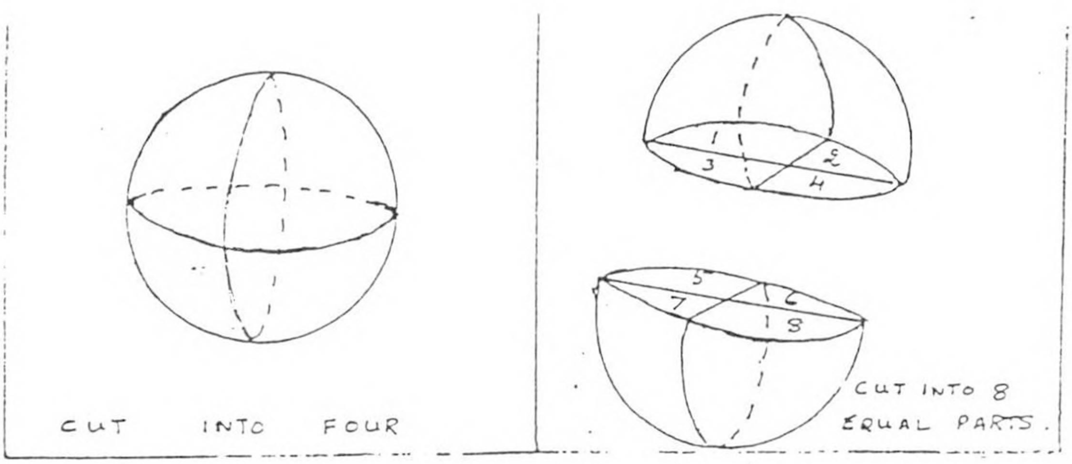
15. Relation between the volume of sphere and volume of cube, constructed from the sphere

Objective: To see the relation between the volume of the original sphere and the volume of the interior of the simple cube constructed from the sphere.

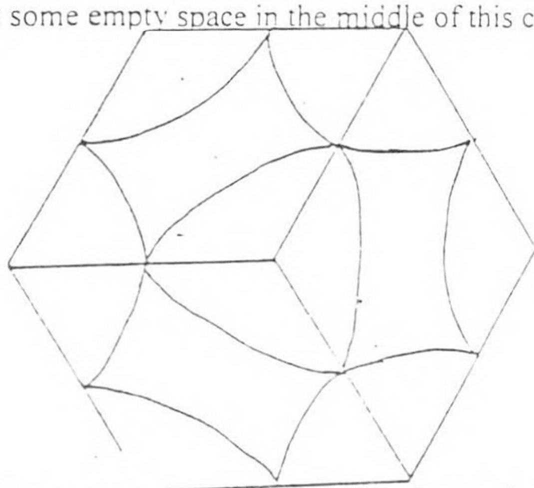
How to use:

Take a sphere of radius 'R' and cut the sphere into eight equal parts as shown below:





Join them from the reverse direction to form an object similar to a cuboid as shown below. Note that there will be some empty space in the middle of this cuboid.



The comparison is between the volume of the sphere and volume of the interior (empty space) of the constructed simple cube.

Volume of sphere = $\frac{4}{3}\pi R^3$

Volume of cube = $(2R)^3 = 8R^3$

Volume of the interior empty space of = $8R^3 - \frac{4}{3}\pi R^3$.

$$\% \text{ of empty space in the cube} = \frac{8R^3 - (4/3)\pi R^3}{(4/3)\pi R^3} \times 100$$

$$= \left(\frac{6-\pi}{\pi}\right) \times 100$$

Note that this expression is independent of R.

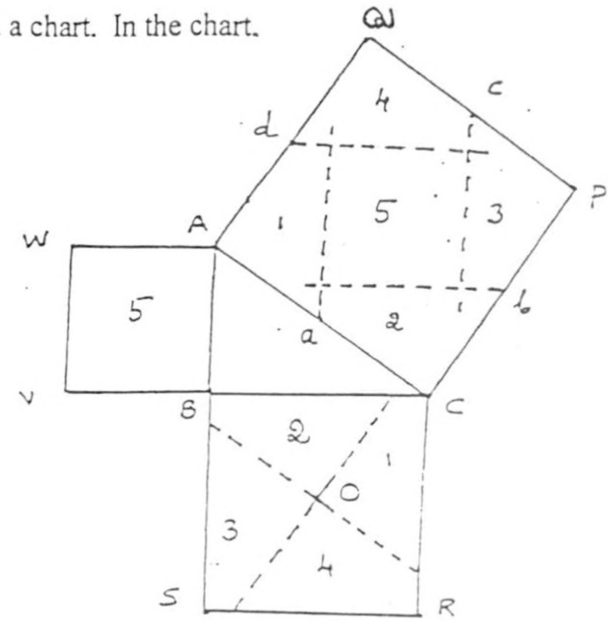
The percent of empty space remains the same, even if the diameter of the sphere changes.

16. Pythagoras Theorem (Perigal's Dissection Method)

Objectives: To show that in a right angled triangle ABC, $AC^2 = AB^2 + BC^2$ where AC is the hypotenuse, by the Perigal's Dissection Method.

Procedure :

There is a wooden model and a chart. In the chart.



ABC is the given right angled triangle. BCRS is the square on the side BC. O is the point of intersection of the diagonals BR and CS. Draw a line parallel to AC through O. Also draw a line perpendicular to AC through O. They divide the square BCRS to four parts 1.2.3.4 as shown in the figure.

a. b. c. d are mid points of AC. CP, PQ and QA respectively. Draw lines parallel to the line AB through a and c. Draw lines perpendicular to the line AB through b and d. These four lines divide the square ACPQ into five parts 1,2,3,4 and 5 as shown.

There are five plastic cut pieces which are congruent to the shapes 1,2,3,4 and 5.

Place these plastic pieces numbered 1,2.3 and 4 on the square on BC and piece numbered 5 on the square on AB as shown in the figure.

Now place the same five pieces on the square on the hypotenuse AC. The five places exactly fit in the square on the hypotenuse (the areas are equal).

The above method justifies that

$AC^2 = AB^2 + BC^2$

Remember that it is not a proof.

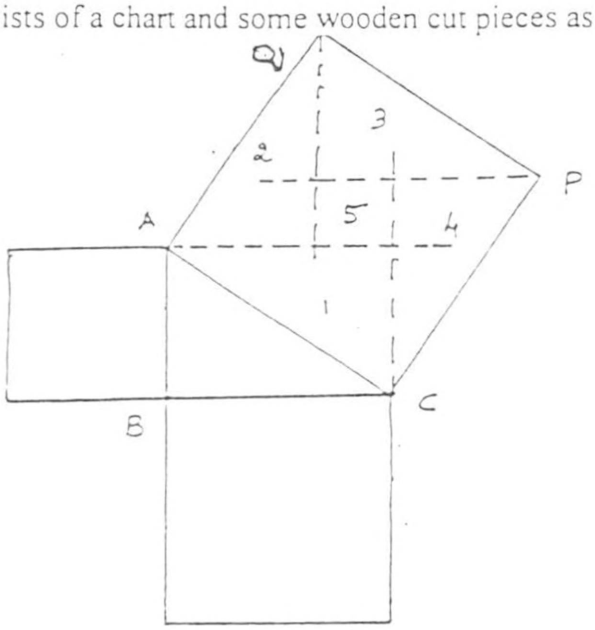
The teachers and the students are welcome to give a mathematical proof for the Perigal's method.

17. Pythagoras theorem (Bhaskaracharya's Dissection Method)

Objectives: To show that in a right angled triangle ABC, $AC^2 = AB^2 + BC^2$ where AC is the hypotenuse, by Bhaskaracharya's Dissection Method.

Procedure

This teaching aid consists of a chart and some wooden cut pieces as shown in the following figure.



ABC is a right angled triangle. ACPQ is the square on the side AC. Draw lines parallel to AB from the vertices Q and C. Also draw lines parallel to BC from the vertices P and A, and hence divide the square ACPQ into four triangles congruent to the triangle ABC and a square in the center whose side length is (BC – AB) as shown in the figure.

Now,

Area of the square ACPQ
$= 4\,(1/2 \times AB \times BC) + (BC - AB)^2$
$= 4\,(1/2 \times AB \times BC) + BC^2 + AB^2 - 2BC\,.\,AB$
$= 2\,AB.\,BC + BC^2 + AB^2 - 2AB\,.\,BC$
$= BC^2 + AB^2$

$\therefore AB^2 + BC^2 = AC^2$

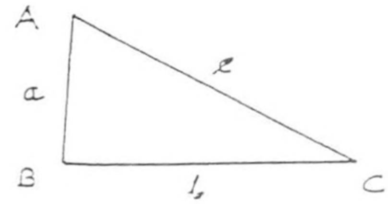
By keeping the wooden pieces in the appropriate places, the students can convince themselves that the result is true.

Now try to give a complete mathematical proof for Bhaskaracharya's method.

18. Pythagoras Theorem (Chau Pei's Dissection Method)

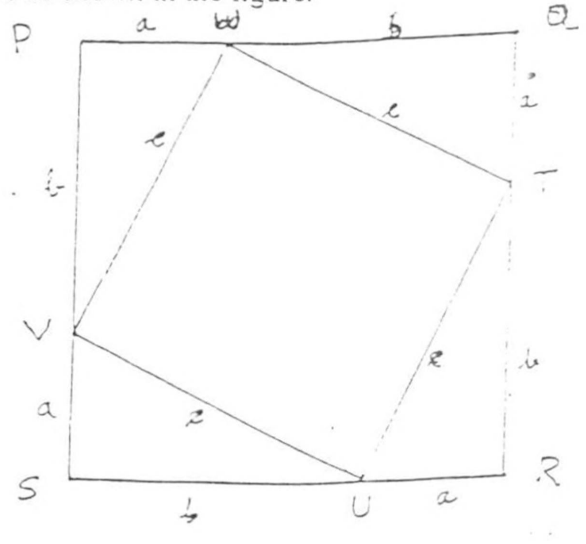
Objectives: To show that in a right angled triangle ABC, $AC^2 = AB^2 + BC^2$ where AC is the hypotenuse, by using the expansion of the expression $(a + b)^2$:

This teaching aid consists of a chart and some wooden cut pieces.



In the right angled triangle ABC, the lengths of the sides are a, b respectively while the length of the hypotenuse is c.

Take a plastic square piece PQRS of side length a + b as shown in the figure.



Then TUVW is a square whose side length is c.

Area of PQRS = $(a + b)^2$

$= a^2 + b^2 + 2ab$ (1)

Area of PQRS = Area of the square TUVW + 4 (Area of the triangle PVW)

$= c^2 + 4\ (1/2 \times a \times b)$

$= c^2 + 2ab$ (2)

From (1) and (2)

$$a^2 + b^2 + 2ab = c^2 + 2ab$$
$$a^2 + b^2 = c^2$$
$$AB^2 + BC^2 = AC^2$$

By keeping the plastic pieces in the appropriate places, the students can convince themselves that the result is true.

The students can also be asked to prove mathematically that the four triangles are congruent to each other. Probably this method was adopted by the Chinese Mathematician Chou Pei (AD 40). Please see the book 'History of Mathematics' by Smith.

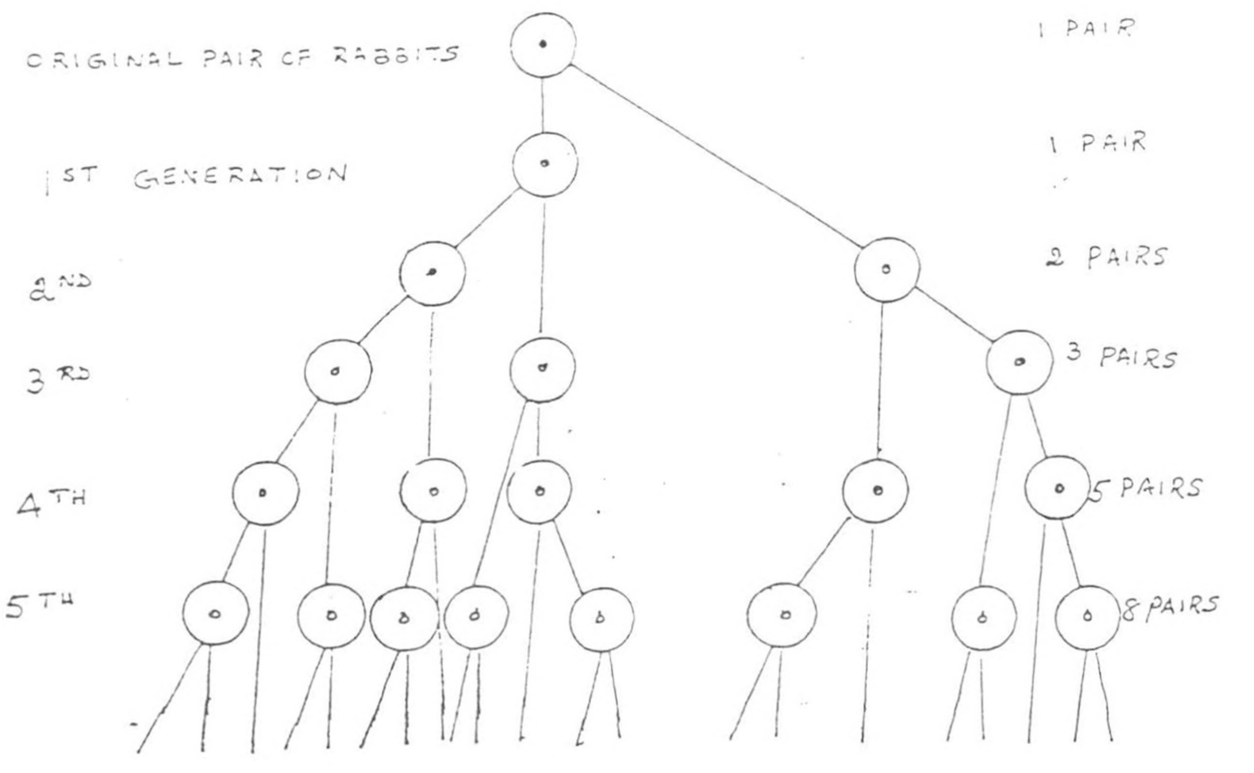
19. Fibonacci Sequence

Objective: This is a model, to show the physical meaning of the 'FIBONACCI SEQUENCE'.

The Fibonacci sequence is 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89,.....

The Problem :

The problem can be stated as follows. A man brought a pair of rabbits and bred them. The pair produced one pair of offspring after one month and a second pair of offspring after the second month. Then they stopped breeding. Each new pair also produced tow more pairs in the same way and then stopped breeding. How many new pairs of rabbits did he get each month ?



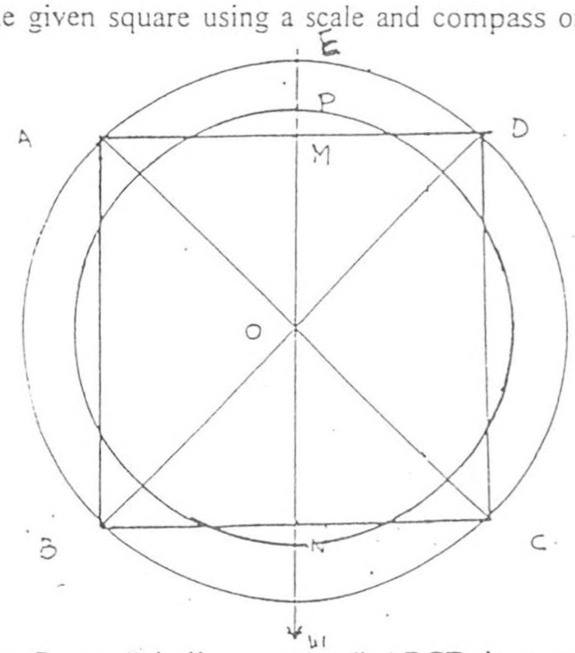
Let us write down in a line, the number of pairs in each generation of rabbits.

1. First we write the number 1 for the single pair we started with (1 new pair).
2. Next we write the number 1 for the pair they produced after a month (1 new pair).
3. The next month, both pairs produced. So the next number is 2 (2 new pairs).
4. Now the original pair stopped producing. The first generation (1 pair) produced 1 pair. The second generation (2 pairs) produced 2 pairs. So the next number we write is 1 + 2 or 3. (Total 3 new pairs).
5. Now the first generation stopped producing. The second generation (2 pairs) produced 2 pairs. The third generation (3 pairs) produced 3 pairs. So, the next number we write is 2 + 3 or 5. (Total 5 new pairs).
6. Each month, only the last two generations produced. So, we can get the next number by adding the last two numbers in the line.
7. The numbers we get in this way are called Fibonacci numbers.

Reference : Land – Language of Mathematics

20. Circling a Square

Objective: This chart can be used to explain "How to construct a circle whose area is equal to the area of the given square using a scale and compass only". (approximately equal).



How to use it

1. In the above figure, "circling a square" ABCD is a square which is to be transformed into a circle so that their areas are equal.
2. AC and BD are the diagonals of the square intersecting at O.

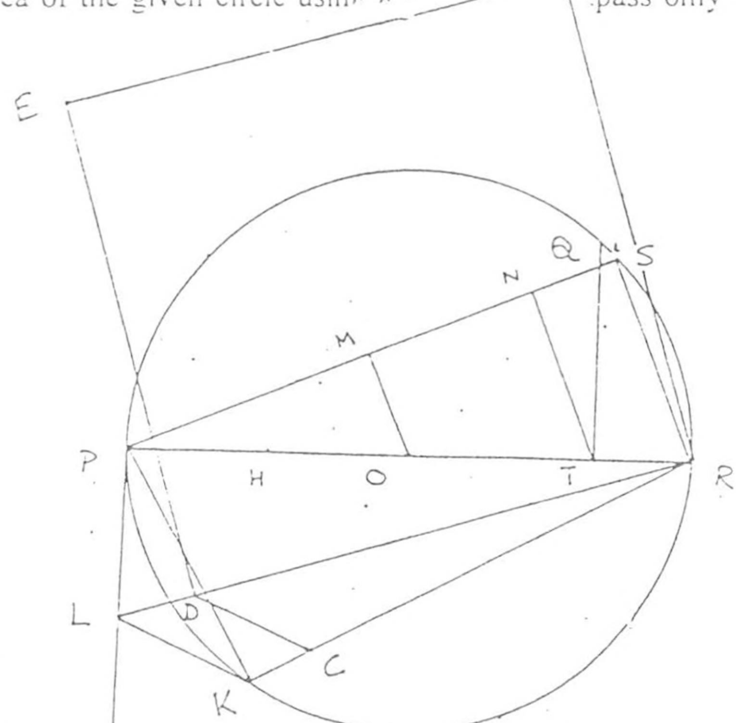
3. EW is a line passing through M, O and N, where M and N are the midpoints of AD and BC respectively.
4. With 'O' as center and OA as radius, a circle is drawn such that it intersects EW at E.
5. EM is divided such that EP = 2PM.
6. With 'O' as center and OP as radius another circle is drawn. The area of this circle is approximately equal in the area of the square ABCD.

Reference : Indian Mathematics and Astronomy by S. Balachandra Rao.

Note : The above problem, "Constructing a circle whose area is equal to the area of the given square" had remained unsolved for centuries in the history of Mathematics. The above method of construction is given by the ancient "Indian Mathematicians" in "Sulva Sutra".

21. Squaring a circle

Objective: This chart can be used to explain "How to construct a square whose area is equal to the area of the given circle using a [illegible] pass only". (Approximately equal).



How to use it :

1. In the figure, "Squaring a circle", PQRS is a circle which is to be transformed into a square, so that their areas are equal. O is the center of the circle and PR is the diameter of the circle.
2. PO is bisected at H and OR is trisected at T nearer R.
3. TQ is drawn such that TQ $\perp$ PR and a chord RS is placed such that RS = TQ.
4. 'P' and 'S' are joined and OM and TN are drawn parallel to RS.
5. A chord is drawn such as PK = PM and a tangent PL is drawn to the circle at P such that PL = MN. RL, RK and KL are drawn.

6. A point 'C' is marked on RK such that RC = RH and CD is drawn such that CD is parallel to KL, meeting RL at D. Now a square is constructed on RD. Area of this square is equal to the area of the circle PQR approximately.

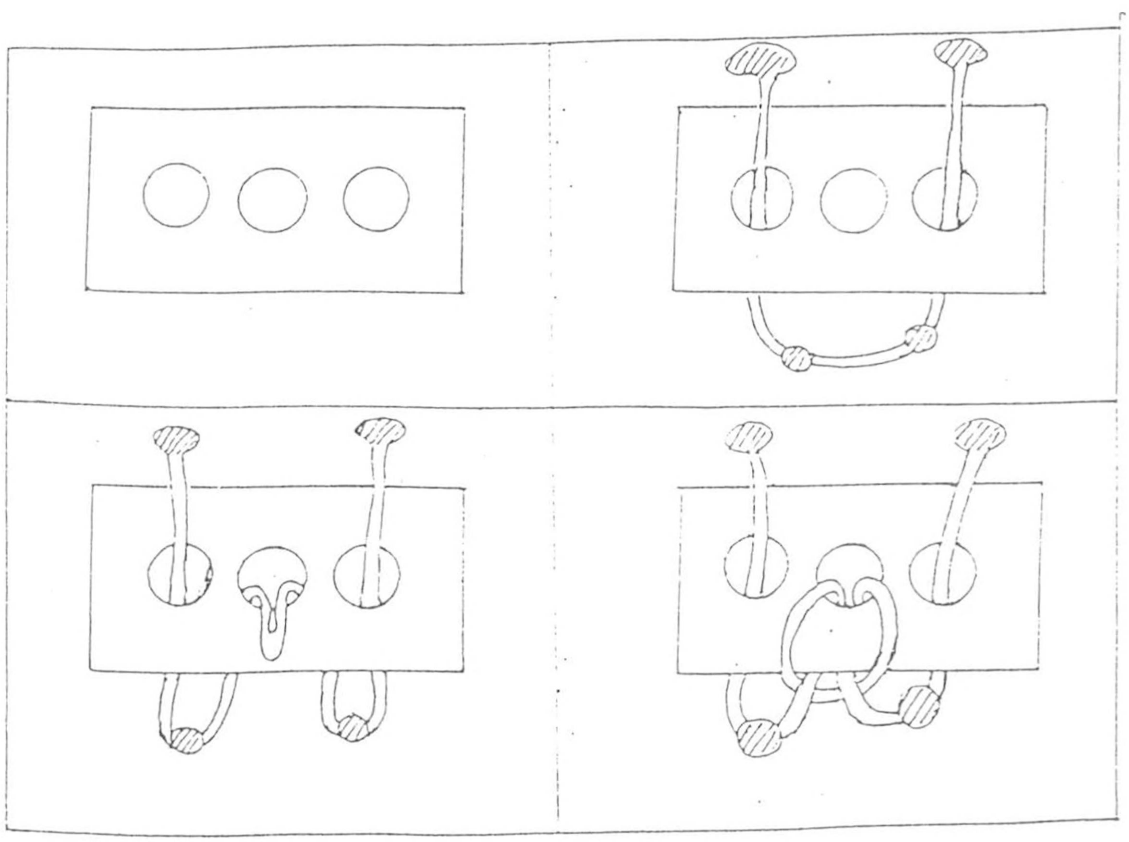
Reference: Indian Mathematics and Astronomy by S. Balachandra Rao.

Note : The above problem "Squaring a circle", i.e. to construct a square whose area is equal to the area of the given square using a scale and compass, had remained unsolved for centuries in the history of Mathematics. But ancient Indian Mathematicians solved the above problem in "Sulva Sutras". The above method of construction is given by "Srinivasa Ramanujan".

22. Buttons and Beads Puzzle

Objective: To improve the mental ability of students
Needed : Cardboard, string, two buttons and two beeds

How to prepare it :



Insert the string through the two beads and insert one end of the string, through hole A and attach a button larger than the hole. In the same direction, thread the other end of the string through hole C and attach a button as in figure 2.

How to use it

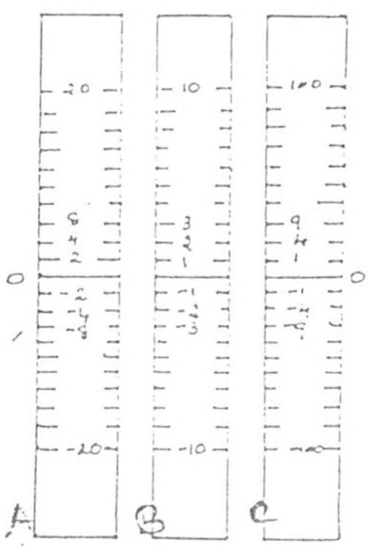
The string is looped through hole B, as in figure 3. Now to loop it back under itself as in figure 4, the loop is first threaded up in hole A and cover the button and then likewise in hole C. Now the puzzle is ready for someone to try to undo the loop and get the beads together.

23. Quadratic Equation Solver

Objective : To enable the user to solve quadratic equations.

How to use it :

The aid consists of three scales namely A, B, C of which the scale B can be moved.



Suppose we have to solve the quadratic equation $x^2 + 6x + 8 = 0$.

Step 1 : Move sliding scale B so that 0 (zero) on it coincides with 6 (six) (coefficient of x) on scale A.

Step 2: Note corresponding reading on scale C which is 9 (nine).

Step 3: Subtract 8 (eight) (constant term or equation) from 9 (nine). The result is 1 (one).

Step 4: On scale C search the position of number 1 (one). There will be 2 (two) positions on scale C where you find 1 (one).

Step 5 : Note the corresponding two readings on scale B. They are – 2 and –4. Hence, -2 and –4 are the solutions of the equation $x^2 + 6x + 8 = 0$.

Reference: Teaching of Mathematics by S K Aggarwal.

24. Four Colour Theorem

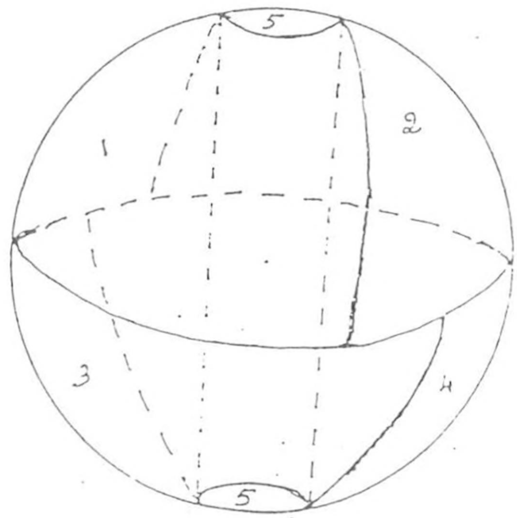
Objective : This is a model to show that the four colour theorem fails to hold in a 3-dimension object.

Analysis :

There is a celebrated theorem called 'four colour theorem' which states that four colours are sufficient to colour any map in the plane in such a way that the neighbouring states do not get the same colour.

Model

A cylindrical hole is constructed in the center of a sphere (football). A horizontal line is drawn around the sphere to make it into two semi-spheres. The horizontal line is connected to the two poles in four different places as shown in the figure. Now the sphere has five regions each having a common boundary with all the remaining four regions.



Therefore, this model requires five colours.

1. Does it disprove the four colour theorem ? (If not, why?).
2. Can you produce a map in the plane, which actually requires four colours ?

25. Euler's Formula V + F = E + 2

Objective: To show that the Euler's formula 'Vertices + Faces = Edges + 2' is satisfied by al the convex polyhedra.

Teaching Aid:

This teaching aid consists of a vertical stand in which all the five regular polyhedra (tetrahedron, hexahedron, octahedron, dodecahedron and icosahedron) made from thermacol and fixed. There are some other convex polyhedra also.

Procedure

The children will have to count the number of vertices, faces and edge of each one of these objects and make a table to find the relation between them.

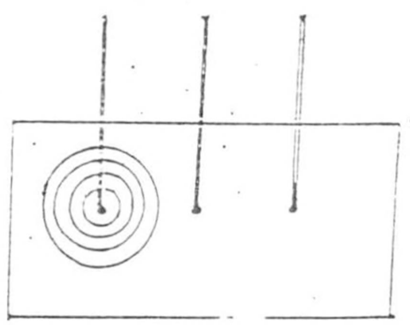
	Name	No. of Faces	No. of Edges	No. of Vertices	V + F	E + 2
1.	Tetrahedron	4	6	4	4 + 4	6 − 2
2.	Hexahedron	6	12	8	8 − 6	12 − 2
3.	Octahedron	8	-	-	-	-
4.	Dodecahedron	12	-	-	-	-
5.	Icosahedron	20	30	12	32	32

Can you produce a polyhedra which does not satisfy the Euler's formula ?

26. Tower of Hanoi

Objective : It is a puzzle called Tower of Hanoi for high school students to develop the inductive reasoning.

Puzzle :



Three vertical rods are fixed on a metallic plate. On one end of the rods, five discs of different sizes have been inserted, the largest disc being at the bottom, in the decreasing order of size.

You will have to put all the discs on any other rod, replacing one at a time and not placing a larger disc on a smaller disc. How many trials are needed to replace all the five discs ?

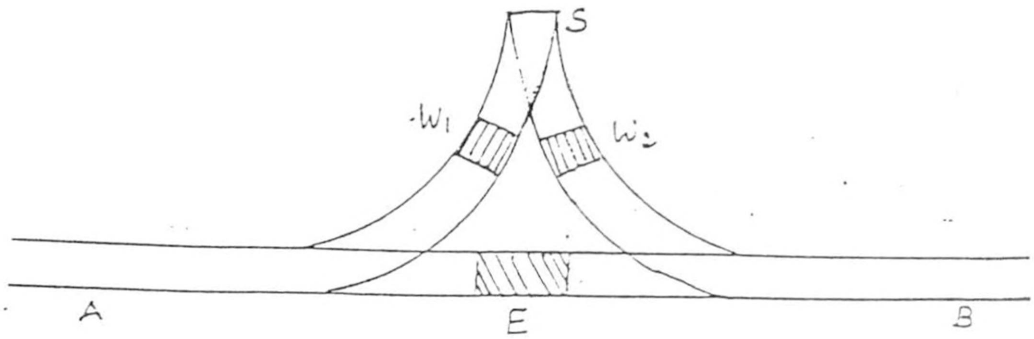
How to do

Children can try by taking two discs first. They will see that the number of trials needed are3. They can repeat this experiment by increasing the number of discs.

At last they can see that if the number of discs are n, then the number of trials required to replace them is $2^n - 1$.

27. Interchanging the Railway Wagons

Problem :



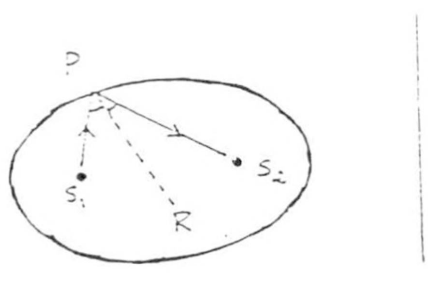
There is a railway line along AB and a slanting line S is connected to AB as shown in the figure. The length of the shunting place S will be sufficient for the wagons W_1 and W_2, but will not be sufficient for the engine E to move. Using the engine E, interchange the positions of the wagons W_1 and W_2.

The children can find the answer to this questions, themselves, by moving wagons W_1 and W_2 through the engine E to different directions. Repeated trials will help to improve their thinking and reasoning powers.

28. Elliptic Carrom Board

Objective: To enable the students to experience a geometrical property of ellipse.

Carrom Board :



An elliptic carom board is prepared, in which the two foci S_1 and S_2 are marked. Keep one carom coin each at S_1 and S_2. The coin at S_1 is pushed to hit any side of the wall of the board. After hitting the wall, the coin S_1 will hit the coin at S_2 and throw it away.

Reason

The perpendicular PR to the wall of the ellipse at any point P divides the angle S_1 PS_2 equally. Hence the angle of incidence S_1PR and the angle of reflexion RPS_2 are equal.

1. What happens if the point P is on the line S_1S_2 ?
2. What happens if the point P is on the perpendicular bisector of the line S_1S_2 ?

NUMBER SYSTEM

B. C. Basti

The following hardspots identified by teachers were discussed and also some classroom notes were given to teachers for their reference.

1. Proof to show that (-a) × (-b) = ab for any two real numbers a and b.
2. Use of fundamental theorem of arithmetic for explaining/proving the following facts.
   i) 1 is not a prime number.
   ii) The number of primes is infinite.
   iii) Irrationality of $\sqrt{2}$.
   iv) The construction of an uncountable set using natural numbers alone.
   v) N × N is an uncountable set.

3. Bertrand Roussel's paradox on sets leading to an axiom of axiomatic set theory.
4. Principle of mathematical induction and simple problems based on it.
5. Proof of Dense Property of irrationals in the real number system, by using dense property of rationals in the real number system.
6. Partition of a set and its connection with an equivalence relation.
7. Equivalence relation on a set and its relation with partition.
8. Functions and different types of functions.
9. Proof to show that power set of a set of n elements contains 2^n elements.
10. Proof to show that $2^n > n$ for any natural number n, without using principle of mathematical induction.

4. BUSINESS MATHEMATICS

Dr B S P Raju

Sinking Fund

Sinking fund is a kind of reserve by which a provision is made to

a) reduce a liability i.e. redemption of debentures or repayment of loan,
b) replace depreciating assets,
c) renew a lease,
d) replace wasting assets i.e. mines.

Let the amount of debt be A; E be the installment amount to credit to sinking fund and 'r' be the interest rate per annum in decimal form that accrues to sinking fund.

Let us consider the case for 3 years.

At the end of the first year, the amount in S.F. (Sinking Fund) is Rs.E.

At the end of II year, this becomes E (1 + r) rupees. (by compound interest formula).

At the beginning of III year, he adds another E rupees so the amount in S.F. is $E(1 + r) + E$.

At the end of III year, this becomes

$$\{ E (1 + r) + E\} \{ 1 + r \}$$

At the beginning of the IV year again he adds E Rupees.

Hence S.F. = $[\{ E (1 + r) + E \} \{ 1 + r \}] + E$.

But this is equal to A.

i.e. $E (1 + r)^2 + E(1 + r) + E = A$.

$E \{ 1 + (1 + r) + (1 + r)^2 = A$.

$$\therefore \ E = \frac{A}{1 + (1+r) + (1+r)^2}$$

But $1 + (1 + r) + (1 + r)^2$

$$= 1 \left\{ \frac{(1+r)^3 - 1}{1 + r - 1} \right\}$$

$$= 1 \left\{\frac{(1+r)^3 - 1}{r}\right\}$$

$$\therefore \; E = \frac{Ar}{(1+r)^3 - 1}$$

In general for n years,

$$E = \frac{A}{1 + (1+r) + (1+r)^2 + (1+r)^3 + ... + (1+r^n)}$$

$$= \frac{A}{1\left\{\frac{(1+r)^n - 1}{1+r-1}\right\}} = \frac{A}{\left(\frac{(1+r)^n - 1}{r}\right)}$$

$$= \frac{A\,r}{(1+r)^n - 1}$$

Problem :

A mortgage of Rs.10,000/- is due in 5 years. It calls for interest payments of 8% payable annually to the creditor. What is the annual payment ? The debtor decides to make equal payments at the end of each year for 5 years into a sinking fund investment that earns 4% compounded annually, to accumulate Rs.10,000/- in 5 years. What is the annual payment to the sinking fund, construct a sinking fund schedule.

Interest $\frac{10,000 \times 1 \times 8}{100} = 800/-$ payable annually.

Installment for payment to sinking fund = $E = \frac{A\,r}{(1+r)^n - 1}$

$$= \frac{10000 \times 0.04}{(1+0.04)^5 - 1} = \frac{400}{0.2166528}$$

$$= 1846.272$$

Period	Interest at 4%	Payment to Sinking Fund	Increase in S.F. Col. 2 + 3	Amount in S.F.	Book Value of Debt
0	--	--	--	--	10,000
1	0	1846.272	1846.272	1846.272	8,153.728
2	73.85088	1846.272	1846.272 + 73.85088 = 1920.1228	1846.272 + 1920.1228 = 3766.3948	6233.606
3	$3766.3948 \times \frac{4}{100} =$ 15065579	1846.272	1846.272 + 150.65 = 1996.9277	5763.3225	4236.678
4	230.5329 \	1846.272	2076.8049	7840.1274	2159.873
5	313.60509	1846.272	2159.877	10000	0000

1. In order to purchase new carpeting and furniture, the Healys decided to deposit Rs.50/- in a S.B. account at the end of each month for 2 years. How much will they have available at that time, if the interest rate is 5% compounded monthly.

Problems on Partnership

X starts a business on 1st January 1987 with Rs.5000/-.
Y joins on 1st May 1987 with Rs.10,000/-.
On 1st July, Z comes in as a partner with Rs.15,000/-.
And on the same date, X contributes Rs.5000/- and Y contributes Rs.10,000/- as further capital.
The profits for the year ended 31st December 1987 amounted to Rs.16,000/-. The partners agree to share the profits in proportion of their capitals. Find their profits.

X : 5000 for 12 months — $5000 \times 12 = 60,000$
5000 for 6 months — $5000 \times 6 = 30,000$
Total: 90,000

Y : 10,000 for 8 months — $10000 \times 8 = 80,000$
10,000 for 6 months — $10000 \times 6 = 60,000$
Total: 1,40,000

Z : 15,000 for 6 months — $15000 \times 6 = 90,000$

Their profits should be 90 : 140 : 90 i.e. 9 : 14 : 9

$\therefore$ X's profit is $16,000 \times \frac{9}{32} = 4,500/-$

Y's profit is $16,000 \times \frac{14}{32} = 7,000/-$

Z's profit is $16{,}000 \times \frac{9}{32} = 4{,}500/-$

Admission of a Partner

1. Change in the profit sharing ratio.

Ex: If A, B and C are partners sharing in the ratio 6 : 5 : 3 and later they admit D for $\frac{1}{8}$ share. What is the new and sacrificing ratio ?

Solution : Old ratio is 6 : 5 : 3

D's ratio is $\frac{1}{8}$ (given).

A's, B's and C's combined share in the new firm $= 1 - \frac{1}{8} = \frac{7}{8}$.

A will get $\frac{6}{14}$ th of the remaining $\frac{6}{14} \times \frac{7}{8} = \frac{6}{16}$.

B will get $\frac{5}{14}$ of the remaining $\frac{5}{14} \times \frac{7}{8} = \frac{5}{16}$.

C will get $\frac{3}{14}$ of the remaining $\frac{3}{14} \times \frac{7}{8} = \frac{3}{16}$.

New profit sharing ratio $\frac{6}{16} : \frac{5}{16} : \frac{3}{16} : \frac{2}{16}$ i.e. 6 : 5 : 3 : 2.

Sacrificing ratio of A is $\frac{6}{14} - \frac{6}{16} = \frac{48 - 42}{112} = \frac{6}{112}$.

Sacrificing ratio of B is $\frac{5}{14} - \frac{5}{16} = \frac{5}{112}$

Sacrificing ratio of C is $\frac{3}{14} - \frac{3}{16} = \frac{24 - 21}{112} = \frac{3}{112}$.

$\therefore$ Sacrificing ratio $= \frac{6}{112} : \frac{5}{112} : \frac{3}{112} = 6 : 5 : 3$.

Goodwill

Goodwill is the attracting force, which attracts the customers towards products of the firm. It is the value of customer's confidence in the business. It is an intangible and invisible asset.

Goodwill = Actual profit earned – Normal profit.
Goodwill = Certain number of times the average profit.

Ex : A and B are equal partners in a firm. Their capitals show credit balances of Rs.18000/- and Rs.12000/- respectively. A new partner C is admitted with $1/5^{th}$ share in the profits. He brings Rs.14000/- for his capital. Find the value of goodwill of the firm at the time of C's admission.

Solution : For $1/5^{th}$ of share C contributes Rs.14000/- (given).
Full capital of the new firm = 14000 × 5 = 70,000/-.
But combined total capital of the three partners = 18000 + 12000 + 14000 = 44000.
∴ Total value of firm's goodwill = 70000 – 44000 = 26000.

Adjustment of Capital

Ex : A, B and C have been sharing their profit and loss in the ratio of 6: 5 : 3. They admit D to a $1/8^{th}$ share. D brings Rs.16000/- for his share of capital. All the partners decide to make the balance of their capital accounts in the profit sharing ratio, calculate their capital.

Solution : Combined share of A, B and C in the new firm = $1 - \frac{1}{8} = \frac{7}{8}$.

A's new share $\frac{6}{14}$ of $\frac{7}{8} = \frac{6}{14} \times \frac{7}{8} = \frac{6}{16}$.

B's new share $= \frac{5}{14} \times \frac{7}{8} = \frac{5}{16}$.

C's new share $= \frac{3}{14} \times \frac{7}{8} = \frac{3}{16}$.

∴ New profit sharing ratio among A, B, C and D is $\frac{6}{16} : \frac{5}{16} : \frac{3}{16} : \frac{2}{16}$ = 6:5:3:2.

For $1/8^{th}$ share, the new partner D brings Rs.16,000.

∴ Total capital of new firm will be 8 × 16,000 = 1,28,000/-.

∴ A's capital in new firm = $1,28,000 \times \frac{6}{16}$ = 48,000/-.

B's capital in new firm = $1,28,000 \times \frac{5}{16}$ = 40,000/-.

C's capital in new firm = $1,28,000 \times \frac{3}{16}$ = 24,000/-.

D's capital in new firm = 1,28,000 × $\frac{2}{16}$ = 16,000/-.

On the Retirement or Death of a Partner

Ex : If A, B, C and D are partners sharing in the ratio of 6 : 5 : 3 : 2. D retires from the firm. Calculate the new ratio after D's retirement.

Combined share of A, B and C (after excluding D).

$$= 1 - \frac{2}{16} = \frac{14}{16}$$

A's share out of $\frac{14}{16}$ is $\frac{6}{16}$.

$\therefore$ A's share out of 1 is $\dfrac{\frac{6}{16}}{\frac{14}{16}} = \frac{6}{16} \times \frac{16}{14} = \frac{6}{14}$

$|||^{ly}$ B's share is $\frac{5}{14}$.

C's share is $\frac{3}{14}$.

$\therefore$ New ratio is $\frac{6}{14} : \frac{5}{14} : \frac{3}{14}$.

Gain in ratio :

A's gain = New share – old share

$$= \frac{6}{14} - \frac{6}{16} = \frac{6}{112}.$$

$|||^{ly}$ B's gain = $\frac{5}{112}$.

C's gain = $\frac{3}{112}$.

NUMBER SYSTEM

B C Basti

The following hardspots identified by teachers were discussed and also some classroom notes were given to teachers for their reference.

1. Proof to show that (-a) × (-b) = ab for any two real numbers a and b.
2. Use of fundamental theorem of arithmetic for explaining/proving the following facts.
   i) 1 is not a prime number.
   ii) The number of primes is infinite.
   iii) Irrationality of $\sqrt{2}$
   iv) The construction of an uncountable set using natural numbers alone.
   v) N × N is an uncountable set.

3. Bertrand Roussel's paradox on sets leading to an axiom of axiomatic set theory.
4. Principle of mathematical induction and simple problems based on it.
5. Proof of Dense Property of irrationals in the real number system, by using dense property of rationals in the real number system.
6. Partition of a set and its connection with an equivalence relation.
7. Equivalence relation on a set and its relation with partition.
8. Functions and different types of functions.
9. Proof to show that power set of a set of n elements contains 2^n elements.
10. Proof to show that $2^n > n$ for any natural number n, without using principle of mathematical induction.

ALGEBRA

B S Upadhyaya

Following concepts/areas identified as difficult in High School Algebra were discussed :

1. Defining a function as a set of ordered pairs.
2. Geometric Proof of $(a+b+c)^2 \neq a^2 + b^2 + c^2$.
3. The proof that 'on division of f(x) by ax+b, the remainder is $f\left(-\frac{b}{a}\right)$'.
4. Reflexive, symmetric and transitive relations.
5. Concept of empty set and examples.
6. Method of finding the inverses of bijective functions.
7. Is it necessary to have two equations to solve for two unknowns?
8. Proof of irrationality of $\sqrt{2}$.
9. Proof that the sum of a rational number and an irrational number is irrational.
10. Simplification of expressions like $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$.
11. Factorization of quadratic polynomials.
12. Logic – specially implications and their negations.

1. Defining a function as a set of ordered pairs :

Intuitively we define a function from a set A to a set B as a rule which associates to every element of the set A, a unique element in set B. To make the definition more rigorous, we use ordered pair of elements to define a function. We consider the Cartesian product $A \times B$ of the set of all ordered pairs (a,b) where $a \in A$ and $b \in B$. Any subset $R \subset A \times B$ we call as a relation from A to B. We call a relation R to be a function if every $a \in A$ appears in some $(a,b) \in R$ and further if $(a,b) \in R$ and $(a,c) \in R$ then $b = c$.

i.e. corresponding to every $a \in A$, there exists a unique $b \in B$ such that $(a,b) \in R$. So this defines a rule which assigns to every $a \in A$, a unique element b in B, making our intuitive definition meaningful. Thus,

Definition : A function R from A to B is a subset R of $A \times B$ such that

i) For every $a \in A$, $\exists$ $(a,b) \in R$

ii) $(a,b), (a,c) \in R \Rightarrow b = c$.

If $(a,b) \in R$ we call b to be the image of the element a under the function R and write $b = R(a)$.

For example, If $A = \{1,2,3,4\}$, $B = \{1,2,3,4,5,\ldots 16\}$ then the function which associates to every element $a \in A$, its square in B, i.e.

$f(a) = a^2$ for every $a \in A$

then f is defined as the set of ordered pairs is given by

$R = \{ 1,1), (2,4), (3,9), (4,16) \} \subset A \times B$

Note that the second entry is the square of the first entry.

Conversely, if $A = \{1,2,3,4\}$, $B = \{1,2,3,4,5,6\}$ and $R = \{(1,2), (2,3), (3,4), (4,5) \}$. Then it is defined by the rule $f(x) = x + 1$ for every x in A.

2. Geometric Proof for $(a+b+c^2) \neq a^2 + b^2 + c^2$

Teachers were complaining that students always make the mistake that $(a+b+c)^2 = a^2 + b^2 + c^2$ and that it is difficult to convince them. But then same must be true about $(a+b)^2 = a^2 + b^2$. The difficulty is probably because of the abstractness of notations and the insufficiency of illustrative examples given by the teachers. Best way probably is to give geometrical proof for the fact that $(a+b)^2 \neq a^2 + b^2$ and that $(a+b+c)^2 \neq a^2 + b^2 + c^2$. $(a+b)^2$ is the area of square on the side of length a+b whereas $a^2 + b^2$ is the sum of the areas of the squares on the sides of lengths a and b. Definitely the two are not equal as can be seen.

Similarly, $(a+b+c)^2$ is the area of square on the side of length a+b+c whereas $a^2 + b^2\ \ c^2$ is the sum of areas of the squares as the sides of lengths a,b and c. Definitely the two are not equal as can be seen.

3. How to conclude that on division of f(x) by ax+b, the remainder is $f\left(-\frac{b}{a}\right)$?

On division of f(x) by ax+b, let the quotient be g(x) and remainder be r. Then,

$$f(x) = g(x) . (ax + b) + r$$

$$= a\ q(x) \left(x + \frac{b}{a}\right) + r$$

$$= q_1(x) . (x - c) + r \text{ where } q_1(x) = a\ g(x) \text{ and } c = \frac{-b}{a}.$$

$\therefore$ r is the remainder of f(x) on division by x – c.

$$\therefore \quad f(c) = q_1(c)\ (c - c) + r$$

$$= q_1(c) \times 0 + r$$

$$= r$$

$$\therefore r = f(c) = f\left(-\frac{b}{a}\right).$$

4. Reflexive, Symmetric and Transitive Relations

A relation from a set A to a set B is a subset R of A $\times$ B.

If (a,b) $\in$ R, then we write aRb and say that a is related to b. For example, if A = {1,2,3} and B = {a,b,c,d,e} then R = { (1,b), (1,c), (2,d), (2,b), (2,e) } is a relation from A to B. Then 1Rb, 1Rc, 2Rd, 2Rb, 2Rc. But note that 1 is not related to a. We write this as 1 R a. Also 3 is not related to any element of B.

Let us take one more example. Let A = {-2, -1, 1, 2,3,} and B = {1,2,3,4,5,9}. Let R = {(-2,4), (-1, 1), (1,1), (2,4), (3,9)}. Then R is a relation from A to B. Further (a,b) $\in$ R iff b is the square of a i.e. aRb if $b = a^2$.

In case B = A, i.e. if R is a subset of A $\times$ A we say that R is a relation on the set A. For example, let A = {1,2,3,4}. Then the subset
R = { (2,1), (3,1), (3,2), (4,1), (4,2), (4,3) } is a relation on A. Here R is in fact defined by the rule that the first entry is bigger than the second entry.

Note that in the above example (1,1) $\notin$ R and (4,4) $\notin$ R.

If A = {1,2,3,4} and R = {(1,1), (2,4)} then (1,1) $\in$ R but (2,2) $\notin$ R, (3,3)$\notin$R and (4,4)$\notin$R. However if A = {1,2,3,4} and R= {(1,1), (2,2), (2,1), (3,3), (3,2), (3,1), (4,4), (4,3), (4,2),(4,1) } i.e. (a,b) $\in$ R iff a $\geq$ b, then (a,a) $\in$ R for every a $\in$ A. Such a relation is called a reflexive relation.

A relation R on a set A is said to be reflexive if (a,a) $\in$ R for every a $\in$ A.
If A = {1,2,3,4,5}, R = {(1,2), (2,3), (1,3), (3,4), (1,4), (2,4) }
Then,

(1,2), (2,3) $\in$ R as also (1,3) $\in$ R

(1,3), (3,4) $\in$ R as also (1,4) $\in$ R

(2,3), (3,4) $\in$ R as also (2,4) $\in$ R

i.e. wherever (a,b), (b,c) $\in$ R so does (a,c). Such relations are called transitive.

A relation R is said to be **transitive** if (a,b) $\in$ R, (b,c) $\in$ R $\Rightarrow$ (a,c) $\in$ R.
If A = { 1,2,3,4,5} and K = {(1,2), (2,1), (3,4), (4,3), (2,5), (5,2) }, then
(1,2) $\in$ R, (2,1) $\in$ R
(3,4) $\in$ R, (4,3) $\in$ R
(2,5) $\in$ R, (5,2) $\in$ R

i.e. whenever $(a,b) \in R$ so is (b,a). Such relations are said to be symmetric relations.

A relation R is said to be symmetric if

$(a,b) \in R \Rightarrow (b,a) \in R$

A relation which is reflexive, symmetric and transitive is called an equivalent relation.

Ex. A = {(1,2,3)}, R = { (1,1), (2,2), (3,3), (2,3), (3,2) }

5. Concept of Empty Set and Examples

Concept of empty set should be introduced through examples. Consider the following

1. set of Women Presidents of India
2. set of integers which are greater than 7 and less than 8.
3. set of integers whose squares are negative.
4. set of integers which are bigger than themselves

None of the above can have any element. Such sets which have no element are called empty sets. Number of elements in an empty is zero.

6. Method of finding inverses of given bijective functions

The problem as pointed out by the teachers was when a bijective function is given by a linear function or quotient of liner functions how to find its inverse. We will illustrate the method by taking examples. Let

$f(x) = 2x + 3$

be the given bijective function. To find its inverse put $y = 2x + 3$.

$\therefore \quad 2x = y - 3$

$\therefore \quad x = \frac{y-3}{2}$

Take $g(x) = \frac{x-3}{2}$. Then g(x) is the inverse function of f(x). We can actually verify this :

$$(f \circ g)(x) = f(g(x)) = f\left(\frac{x-3}{2}\right) = 2\left(\frac{x-3}{2}\right) + 3 = 2x - 3 + 3 = x$$

$$(g \circ f)(x) = g(f(x)) = g(2x+3) = \frac{2x+3-3}{2} = \frac{2x}{2} = x$$

Thus g(x) is the inverse of f(x).

Let us take another example, now of a rational function.

Let $f(x) = \frac{3x+1}{x+2}$ Then f(x) is defined for all $x \neq -2$.

Now to find its inverse put

$$y = \frac{3x+1}{x+2}$$

Then $y(x + 2) = 3x + 1$

$\therefore xy + 2y = 3x + 1$

$\therefore xy - 3x = 1 - 2y$

$\therefore x(y - 3) = 1 - 2y$

$\therefore x = \frac{1-2y}{y-3}$

Let $g(x) = \frac{1-2x}{x-3}$

Then g(x) is defined for all $x \neq 3$ and is the inverse of f(x). We can verify this as follows :

$$(f \circ g)(x) = f(g(x)) = f\left(\frac{1-2x}{x-3}\right) = \frac{3\left(\frac{1-2x}{x-3}\right)+1}{\frac{1-2x}{x-3}+2}$$

$$= \frac{3(1-2x)+x-3}{1-2x+2(x-3)} = \frac{3-6x+x-3}{1-2x+2x-6} = \frac{-5x}{-5} = x$$

and

$$(g \circ f)(x) = g(f(x)) = g\left(\frac{3x+1}{x+2}\right) = \frac{1-2\left(\frac{3x+1}{x+2}\right)}{\frac{3x+1}{x+2}-3}$$

$$= \frac{x-2-2(3x+1)}{3x+1-3(x+2)} = \frac{x+2-6x-2}{3x+1-3x-6} = \frac{-5x}{-5} = x$$

Thus g(x) is the inverse of f(x).

7. **Is it necessary to have two equations to solve for two unknowns :**

No, it is sufficient just to have one equation to solve for two unknowns. The difficulty experienced here probably is due to the fact that in Mathematics a problem has a unique solution and how is it that a single equation in 2 unknowns has more than one solution. Note that here also the set of solutions is unique. When we say solve ax+by = c, we mean, what are all the values of x and y for which the relation ax + by = c holds. For example, when we say solve x+y = 4, we have to find all values of x and y, whose sum is 4. The relation x+y = 4 can also be written as y = 4-x. Whatever value we give for x we get a value for y and this set of values for x and y satisfies x+y = 4. So, x = k, y = 4 – k is a solution for any real number k.

8. **Proof for the irrationality of $\sqrt{2}$.**

Mathematics is Deductive and one of its Hallmarks is consistency. It is using consistency that we prove many of the results in Mathematics. We also call this method as reductio absurdum or as method of contradiction. In this method we assume the negation of the statement to be proved and using deduction arrive at a statement which negates one of the Axioms or theorems or our own assumption. This inconsistency is not allowed by Mathematical Logic and hence our assumption that negation of the statement to be proved is true is wrong. Hence the statement to be proved is true.

So we assume that $\sqrt{2}$ is not irrational. Hence $\sqrt{2}$ is rational. Hence $\sqrt{2} = \frac{p}{q}$ where p and q are integers and by canceling common factors if any we can assume that gcd (p,q) = 1. Now squaring we get $2 = \frac{p^2}{q^2}$. Hence

$p^2 = 2q^2$. Since $2q^2$ is even, p^2 is even. Hence p is even (as p is odd implies p^2 is odd). Hence p = 2m for some integer m.

$\therefore p^2 = 4m^2$,

$\therefore 2q^2 = 4m^2$

$\therefore q^2 = 2m^2$

$\therefore q^2$ is even and hence q is even.

Hence 2 is a common factor of p and q which contradicts gcd (p,q) = 1. This proves that $\sqrt{2}$ is irrational.

9. **Proof that the sum of a rational number and an irrational number is irrational.**

Let a be a rational number and b be irrational. If possible, let a+b be a rational number denoted by C. Now since a and c are rational, $a = \frac{p}{q}$, $c = \frac{r}{s}$ where p,q,r, s are integers $q \neq 0$ and $s \neq 0$. Now,

$$a+b=c \Rightarrow \frac{p}{q} + b = \frac{r}{s} \Rightarrow b = \frac{r}{s} - \frac{p}{q}$$

$$= \frac{rq - sp}{sq}$$

Now since $s \neq 0$, $q \neq 0$, $sq \neq 0$.

$\therefore \frac{rq - sp}{sq}$ is a rational number i.e. b is a rational, a contradiction to b being irrational. This contradiction proves that a+b cannot be rational.

$\therefore$ a+b is irrational.

10. **Simplification of expressions of the form** $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$

$$\frac{1}{a} + \frac{1}{b} = \frac{b+a}{ab} = \frac{a+b}{ab} = \frac{e}{f} \text{ where } e = a+b,\ f = ab.$$

$$\therefore \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \left(\frac{1}{a} + \frac{1}{b}\right) + \frac{1}{c} = \frac{e}{f} + \frac{1}{c} = \frac{e.c + 1.f}{fc}$$

$$= \frac{(a+b).c + a.b}{(ab).c} = \frac{ac + bc + ab}{abc}$$

$$= \frac{ab + bc + ca}{abc}$$

Apart from the above, Logic was also discussed in detail as the teachers were finding it difficult especially implications and the negation of implications.

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5-DAY TRAINING PROGRAMME FOR RESOURCE PERSONS ON THE TEACHING OF SELECTED HARDSPOTS IN MATHEMATICS AT SECONDARY LEVEL

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