TRAINING MODULE
FOR
MAJOR PEDAGOGIC ISSUES IN TEACHING SCIENCE AND MATHEMATICS AT HIGHER SECONDARY LEVEL (FOR KERALA STATE)

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FOREWORD

There is growing population of students in school education who find disinterest in learning science and mathematics . The phenomenon is global. Its effect is too visible at the national level. We have become specialized in scoring high marks but fail to carry beyond that point. This perhaps may be the reason in mushrooming of coaching centers across the country. Is there a cure for this malady ?. If at all, it should begin at the school level. The teaching of science and mathematics must generate sense of wonder and aesthetics to understand the intricacies and beauty of the world and its functioning both in micro and macro scale. While describing scientific endeavor Henri Poincare writes "The scientist does not study nature because it is delightful, he studies it because he delights in it, and he delights in it because it is beautiful". Surely it is possible to engage the students in a way that heightens their delight and their appreciation of the simple principles and facts of the world around them. Obviously the issue is pedagogical — of how to teach but not of what to teach. When the child is very young it is easy to bring out the sense of wonder by bringing the child into contact with nature and the nature herself will teach the child in the best way possible. The pedagogy of the nature is inimitable. But as the child grows and the child's intellectual abilities mature, the teacher has to work hard to teach the principles that govern the functioning of the world which science and mathematics seek to find and describe in the simplest terms possible.

Often students lament that they fail to relate the science to the real world as it is taught in their classroom. They fail to understand its application while dealing with the real world. The real issue is to make the students recognize that doing Science is a creative process and mathematics is rich in elegance and simplicity. For instance many mathematics teachers teach routinely geometry without realizing that it is one of the pedagogically meaningful ways of introducing formalized logical reasoning. Regional Institute of Education, Mysore among its multifarious activities concerning school education is also engaged in teacher education and development of pedagogic materials of school subjects. The present package on pedagogic issues in teaching science and mathematics, developed by Dr G R Prakash Reader in chemisty along with team of experts, provides exemplar teaching strategies on selected topics of science and mathematics. It is hoped that this package serves the purpose for which it is prepared.

PRINCIPAL

Preface

Most of the learning theories and teaching methodologies are applied to the Secondary School level. Very rarely we see teachers at higher secondary level willing to apply different methodologies at higher secondary level. This is quite obvious and understandable in view of the fact that the students have the required cognitive capabilities to abstract thinking and logical reasoning. On the other hand, teachers have a heavy load of content to be taught and hence cannot afford to try out methods which may require more time.

However, when the request came from Kerala State, to train the higher secondary teachers on pedagogic issues at higher secondary level, it was taken up in the right earnest. If one goes through available literature, one would see bits of information here and there and definitely not in a compiled form. So this work was all the more challenging and unique in nature. While suggesting the strategies here, we have ensured that we are realistic in our approach. The activities suggested are exemplary ones and not exhaustive. Similar attempts have to be made by the practicing teachers on topics so suited. Here is an attempt to provide variety of methods within the nature and scope of the subject whether it is science or mathematics. Since this is a training module, some demonstration lessons may also be tried out on the suggested lines. The authors and the Coordinator will be highly indebted for readers' suggestions and views.

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SCIENCE

PHYSICS

Introduction

The aim of teaching physics is to develop the following objectives.

1. To develop concepts and theories in physics.
2. To understand different pedagogical approaches in learning physics.
3. Use of activities to promote scientific attitude.
4. To appreciate the role of experimental work in physics.
5. To design simple and innovative experiments to learn physics.
6. To appreciate the process of physics such as acquisitive skills, organization skills, manipulative skills, and communication skills.
7. To sustain interest in physics by making it lively, exciting and intelligible.
8. To develop a critical mind.
9. To appreciate the importance of measurements and transform data to ideas and concepts.

At present the lecture method is predominant in classrooms and importance is not given to pedagogical approaches. In the practical sessions, standard experiments are done mechanically by using detailed laboratory manuals. These approaches do not help to achieve the objectives of teaching physics.

So all over the world, work is being done to find suitable approaches to teach physics in the classroom and in the laboratory. In this booklet, an attempt is made to introduce those approaches.

In what follows different approaches that are suitable to teach physics at the higher secondary stage are illustrated.

The strategies used and the corresponding topics developed are listed below.

A. Theory

1. Guided discovery method – Uniform motion by using air bubble experiment.
2. Innovative method – conservation of energy by using a simple pendulum.
3. Deductive method – Derivation of Equations of motion and derivation of gas laws by using the generalized equation.
4. Inductive method.

5. Laboratory method – Data analysis.
6. Investigatory method : Application of principles of mechanics in sports activities.
7. Simulation method – Radioactivity using dice game.
8. Problem solving strategy.
9. Demonstration method – Fluid flow.
10. Project Method – Damping of oscillations

B. Laboratory Work

Guided open-ended approach is described for doing activities related to propagation of light in different media.

It is hoped that these novel methods to teach physics will help students to sustain and increase their interest in physics.

The Role of Experiments in Science Teaching

Science has not only become an integral part of social and political life of today but also its methods such as observation, recording, making models, experimentation, reasoning, drawing conclusions, etc. have become a part of all disciplines.

How science is taught is of special importance. Students pose many questions. Why is it so? How do we know? What does it mean? etc. They wonder, speculate, examine, predict, experiment and interpret. These experiences are facilitated if science teaching is accompanied by experiments.

Objectives of Doing Experiments :

Some important objectives are :

1. Making theoretical knowledge more real.
2. To develop acquisitive skills such as observing, searching, gathering data, investigating, drawing conclusions.
3. To develop organisational skills such as recording, classifying, comparing, analyzing and organizing a project.
4. To develop manipulative skills such as using instruments, repairing, performing experiments.

5. To develop communicative skills such as explanation, discussion, preparing reports.
6. To develop creative skills like inventing, new approaches, planning ahead, etc.
7. To sustain interest in physics.

How are experiments done at present ?

Different types of experiments such as demonstration experiments, standard experiments, projects are presently being done at the higher secondary stage to achieve these objectives. The extent to which these different types of experiments help to achieve the different objectives in an open question and no detailed work in this connection seems to have been done.

Demonstration experiments are generally intended to sustain the student interest in Physics and to make theoretical knowledge more real. As such many of the objectives listed above are not achieved by this type of experiments.

The standard experiments are by and large the type of experiments that are usually done in the school laboratories at present. Experiments like verification of Hook's Law, finding the focal length of a lens, finding the velocity of sound are some of them. In these experiments what the students should find out, what procedures they should follow and the expected answers are well known. As such, they do not help much in developing a curious or a creative mind. Besides, there is ample scope for the students to copy one another's work.

Project work as is commonly understood involves investigation of a specific problem or fabrication of a gadget. It is a mini research or development work. In this, the procedure is generally well defined and is the same for all students. At times, it involves use of certain apparatus which are beyond the reach of average schools. Help from the teacher or from senior pupils is often required and is being availed. Besides, because of the limited resources in the schools, it is generally not possible for the students to do more than one project.

Need for Experiments based on Inquiry and Discovery Approach

Science is an ongoing enterprise. Old ideas are continuously refined and replaced. Formerly light was supposed to be made of particles which propagate in straight lines. Then it was said that light is a form of radiation which travel in waves.

Now it is said that light is a form of radiation and its energy is transferred in packets. In science new and exciting questions crop up and solutions are found. Scientific ideas remain open for further exploration and experimentation.

The facts of today may be changed tomorrow. The facts and ideas about the nature of the interior of the earth may stand changed tomorrow. The ideas and theories about the planets and the universe are being continuously changing. So it is obvious that stress has to be laid on the methods of obtaining the facts, on the knowledge and the skill required to develop and apply the methods. Well conceived ideas and established facts are not of primary importance in science.

There should be discussion of the activities. There should be reflection and cognition. Questions like, what does this mean? What could have happened? Should be posed and discussed. Students should try to generalize from what they have learnt. Highly limited experience like the one they get by doing standard experiments block their development.

In recent years, inquiry methods for teaching science are being emphasized. Besides helping students to understand the scientific concepts and to cultivate their scientific skills such a method will help them to develop the true scientific spirit. When a student observes a phenomenon which he does not understand he does not throw up his hands in despair, or ask someone to explain it. He learns to gather data, experiment, formulate and test a hypothesis. He analyses the intricate events into factors which can be examined in relation to each other. He learns to study the effect of one variable upon the other. He learns to control and predict events. He learns to go beyond the concrete situations to abstraction and generalization.

The above discussion suggests that we should encourage experiments based on discovery and inquiry approach. One type of experiment which highlights this is called open-ended experiment.

Open-ended Experiments

The salient features of open-ended experiments are :

1. The area of investigation is identified.
2. Students do not know the answer in the beginning but they should have a clear idea about the problem;
3. Students are acquainted with the relevant apparatus and materials;

4. Students follow the procedures they think best;
5. Students make their own observations and record them in the way they think best;
6. Students will interpret, explain and generalise the results in their own way;
7. Students should be able to suggest additional problems related to the investigation.

In doing this type of experiments, students will have the same kind of experience scientists have when they explore the unknown. They learn various processes and realize that those processes are only components of inquiry. They also come across some of the frustrations when things do not work as they are supposed to and find out ways and means of overcoming these obstacles.

Experiences and playing with apparatus become education only if the children reflect on them, relate them to similar experiences and generalize them. Reflection and generalization make the experience open-ended. These experiences lead into the future. They make the children more curious and help in their future studies.

While performing open-ended experiments, children will set up their own experiments and follow their own procedures. They will then store the results using words, symbols and graphs, in a method they think best. On the basis of their previous knowledge they will carefully study the results. They will then form their own patterns, observe relationships, generalize and draw conclusions. They will also discuss with their classmates who might have followed different procedures.

This approach is different from the laboratory method in which certain concepts are theoretically developed and some pre-planned experiments pertaining to them are performed. The children are not supposed to memorise facts and theories. They will be exploring the unknown. They have their own experimental procedures. They will present the result in a way they think appropriate. They may come out with novel and fresh explanations. Children will derive great joy in following this enquiry and discovery approach. They will take it as a challenge and perform their experiments with all their drive and talent. This approach broadens children's knowledge and provides opportunities for originality.

The role of the teacher in the open-ended experiments is an important one. The teachers certainly know in most cases the details of what the children are supposed to do though these things are unknown to the students. He should assess the children's background and accordingly help them to organize a particular investigation. He should see that students do not go astray by doing unrewarding activities. He should provide the setting on which the students discover something and get the thrill of the discovery.

A student sheet on the basis of open-ended approach is given below. Some guidance is provided to them by listing the possible activities and providing evaluatory questions which indicate as to what they are expected to know after the experiment.

Student Sheet

Time : 6 hours

A. Area : Light

B. Topic : Propagation of light in different media.

C. Pre-requisite knowledge : Laws of refraction, critical angle, parallex.

D. Apparatus and materials : Glass slab, prism, rectangular glass tank, hollow glass prism, convex lens, plane mirror, concave mirror, pins, traveling microscope, protractor, slits, liquid, light source, wooden board, spherometer.

E. Activities : Study the passage of light through (i) solid and (2) liquid and estimate its refracture index.

Hint: Some of the possible methods to obtain refractive index are :

1. By finding the minimum deviation angle.
2. By finding apparent depth.
3. By critical angle method.
4. By using hollow prism.
5. By using a concave mirror.
6. By adjusting the volume of the liquid, etc.

Use at least one method for each medium to calculate its refractive index and describe the rest by discussing with other students.

F. Evaluatory questions.

1. Is refractive index of a substance dependent on the colour of light ?
2. Is the refractive index of a medium dependent on the shape of the substance?
3. Is it possible to find the refractive index of glass using critical angle method by keeping the object in air? Explain.
4. Why do the parallel rays of light converge to a point after refraction through a convex lens?
5. On what factors does the lateral shift of the light ray while propagating through a glass slab depend ?
6. List the factors which influence the maximum deviation angle in the prism?

G. List further activities in this topic.

From this we may draw a wrong conclusion that T is directly proportional to 'l'. However, if the graph is extrapolated, one finds that the line cuts the time axis at a finite point, corresponding to pendulum of zero length. This is absurd. Hence, the interpretation, l – T graph is a straight line, is wrong. On the other hand l- T^2 graph is as shown in Fig. 2.

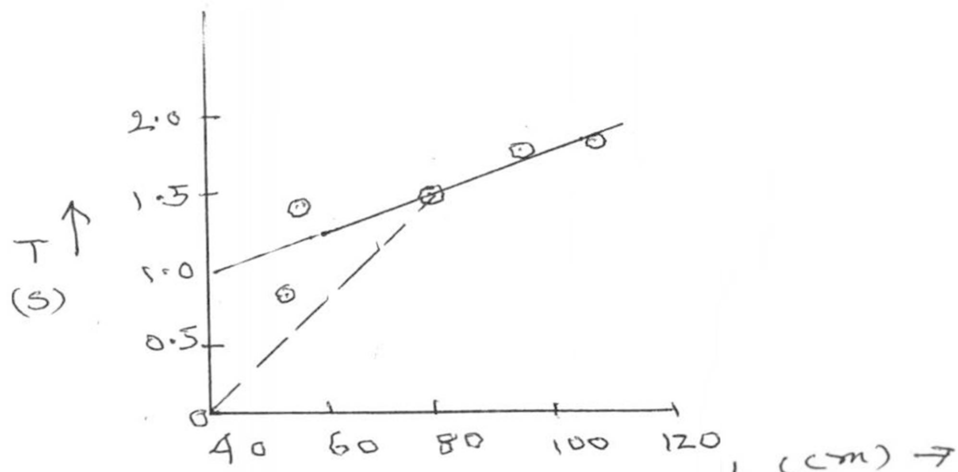
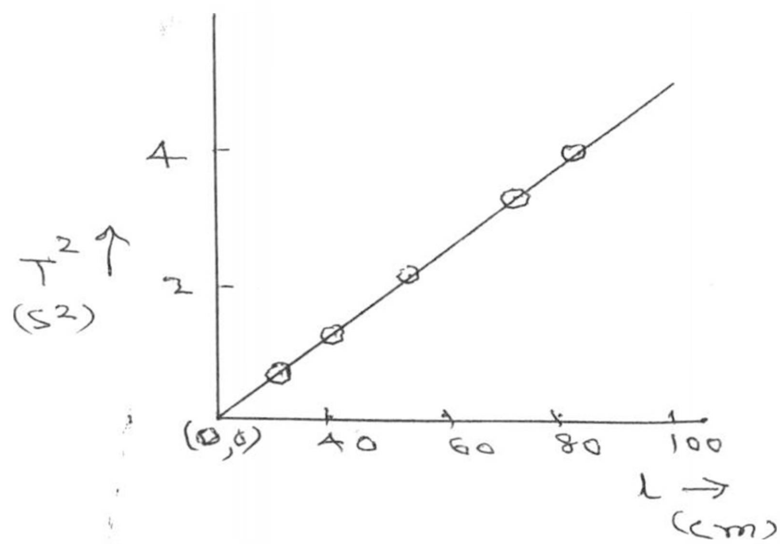


Fig (1)



Laboratory Method of Teaching – Data Analysis

In this method, the teacher need not take students to the laboratory but can use the data collected earlier in a laboratory session. The data collected from an experiment can be analysed by arithmetic and graphical methods. The latter method is very convenient because (i) the comprehension of the relation is easy and quick, and (ii) a graph drawn properly makes the uncertainties to affect on either side – thus 'ironing' out the errors. The required value thus computed is a better one.

In data analysis, it is either a line or a diagram showing how one quantity depends on or changes with another. The following are the important features of graphical method :

a) The nature of the graph

b) The intercept in the graph

c) The area under the curve and

d) The slope of the graph

1. Plotting a graph : Choice of quantities to be plotted on x and y-axes

We need not follow strictly the connectivity – independent variable on x-axis and dependent variable on y-axis. Choose them in a way that we get useful quantities in less number of steps. For example, if we plot the displacement on y-axis and time on x-axis the slope gives the mean speed. If the quantities are interchanged, the reciprocal of the slope gives the mean speed. In this case, an additional step is involved which could be avoided.

2. Choice of the origin

When studying the nature of the relationship between two quantities the origin has to be (0,0). For example, in the simple pendulum experiment, suppose we plot the graph between l and T it appears to be a straight line (see Fig.1) if (0,0) is not chosen.

3. Drawing Inference

From the graph state the nature of the graph and the relationship. For example,

i) A straight line passing through the origin : There is a direct proportionality between the elements of the graph i.e. $y \propto x$.

Experimentally if the line does not pass through the origin, contrary to the theoretical expectations, then discrepancy has to be accounted for.

ii) When two measured quantities indicate inverse relationship, then one quantity has to be plotted against the reciprocal of the other. For instance, in Boyle's law P vs 1/V graph is a straight line, whereas P – V graph is a rectangular hyperbola.

iii) If the graph is linear but has an intercept, then measure the intercept and state what does that quantity represent. For example, in the temperature vs. resistance graph, the intercept on resistance axis gives the resistance of the wire at $0^{o}C$ (Fig. 3).

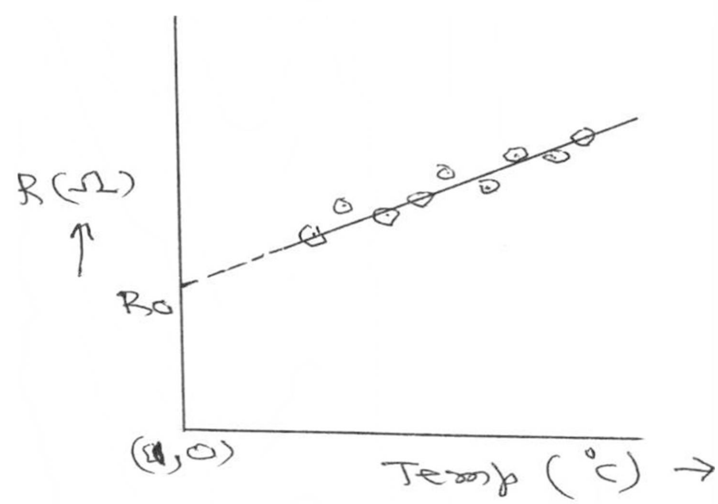


Fig 3.

4. Determining the slope of the graph

While determining the slope, experimental points are not to be selected for drawing the triangle. Instead go along the graph and identify two points on the graph as far apart as possible (Fig. 4).

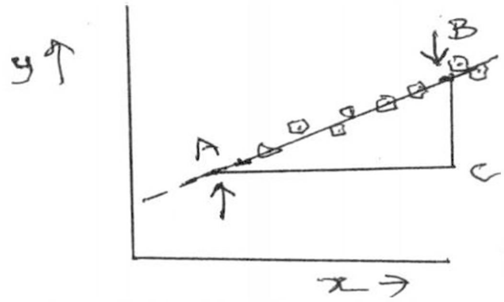


Fig 4.

The slope has to be calculated as the ratio of the values, which the lines BC and AC represent and **not** of their geometrical lengths.

5. Drawing a tangent to a curve at a given point

Consider the temperature time graph in an experiment (Newton's law of cooling). We wish to find the rate of cooling at a given temperature t_r. (Fig. 5)

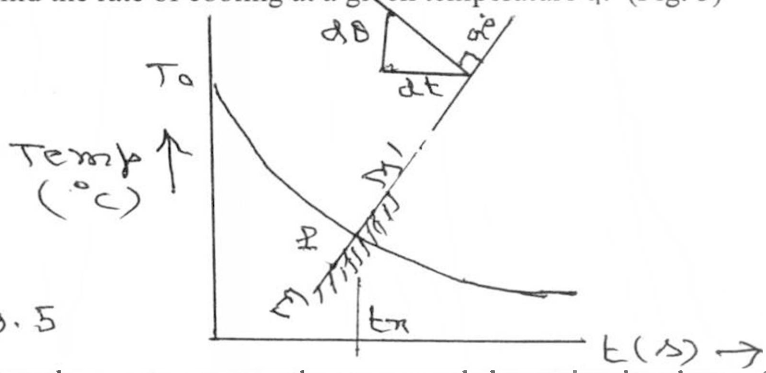


Fig. 5

One method is to draw a tangent to the curve and determine its slope. This gives the instantaneous rate of cooling. For finding the slope, place a plane mirror strip MM′ across the curve corresponding to the given point P. Rotate the mirror about this point till the portion of the curve in front of the mirror and its image through the mirror appears continuous. Trace the mirror surface line MM′. Determine the slope. As another example, instantaneous velocity at a given instant can be measured by this method, on displacement – time graph.

1. Guided Discovery Method – An example

An air bubble trapped in a column of water enclosed in a burette, can be used to study the uniform motion. How Guided Discovery approach method can be used is illustrated below :

A burette is filled with water into a **small** air bubble trapped inside. Then both the ends are sealed. This arrangement can now be used by students to analyse various concepts in motion in one dimension. Equal distances (say 10 cm) are marked on the burette. Observe the motion of the bubble as it traverses from one end to the other, when the burette is kept at a **small angle** (about 2^{o}) to the horizontal (Ref. Fig.1)

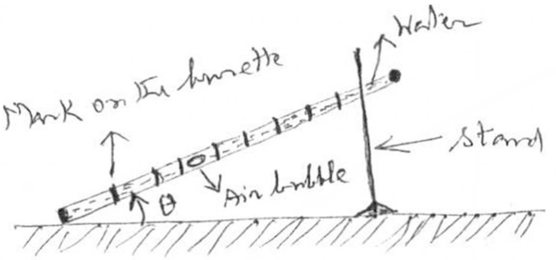


Fig. 1

Tilt the burette so that the bubble can be brought to the lower end. Repeat the process described earlier. What can you say about the motion of the bubble ?

Note the time taken for the bubble to travel a distance of say, 20 cm. Repeat the observation five times and take the average of the best of **three** readings. Record the time taken for the bubble to travel distances of 30 cm, 40 cm, 50cm etc. Plot a graph between distance and time.

What is the nature of the graph?

Find the slope of the graph. What does it represent?

What is your conclusion from the slope ?

Make θ more (say 10^{o}) and repeat your observation. Again plot distance vs. time on the same graph sheet. What do you observe as the angle of inclination increases? Why should the bubble be **small** in this activity ?

Why does the bubble move from the lower end to the higher end? Does it violate the principle of gravity ?

2. Problem Solving Strategy

Problem solving strategy when employed during classroom transaction enables the learners to think critically and apply the knowledge acquired in new situations. Illustrated below are two examples which will be useful in enabling students to understand certain concepts in equations of motion and the laws of motion.

Example 1 :

Consider the following equations of motion when acceleration is constant :

$$v = u + at \quad (1)$$

$$S = ut + \frac{1}{2} at^2 \quad (2)$$

Let u = 0. then, $v = at \quad (3)$

$$S = \frac{1}{2} at^2 \quad (4)$$

Now (4) can be written as,

$$S = \frac{1}{2}\ v/t\ \ t^2$$

$$= \frac{1}{2}\, vt$$

$$= \frac{1}{2}\, S$$ (symbols have the usual meaning)

$\therefore$ 1 = 2 !!!

What went wrong with this conclusion ?

Example 2 : A monkey and a bunch of banana of equal weight are at the two ends of a rope that passes over a frictionless pulley.

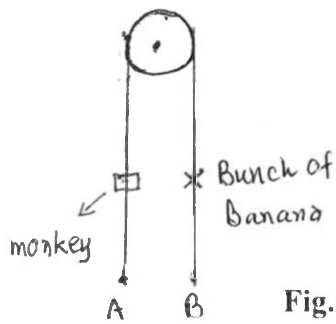


Fig.2

Initially both are balanced and so no motion is observed. Suddenly, the monkey accelerates upwards on the rope. Explain what does happen to the bunch of banana? Which principle did you apply to come to this conclusion?

3. Simulation Method – An example : Radioactive decay process

Simulation method helps a teacher to illustrate some concepts, which otherwise cannot be developed through experiments during classroom teaching. In the example described below, we illustrate how the concepts of radioactivity and half-life can be easily understood by the leaner when the simulation technique is used.

Let us play a game of dice. For illustrative purposes, we have used 36 (thirty six) dices in our activity. The number of dices represent the number of radioactive nuclei in the beginning of the radioactive process. We shuffle the dices in a box and throw them on a table.

Let us pick up those with their faces '6' up.

Is this a random event?

Can any other number apart from '6' (say 1,3 etc.) be picked up ?

The number of dices that are picked up with their face '6' are equivalent to particles emitted by radioactive decay during a given time interval.

What do the remaining dices represent ?

The number of remaining dices is equivalent to remaining nuclei. They are counted and recorded. The above procedure is repeated with the remaining dices until a few are left with. The results of our activity for three games are recorded in Table 1.

Table 1

No. of Throws	No. of remaining dices without face '6'			Total	Average
	Game 1	Game 2	Game 3		
1	32	30	31	93	31
2	28	26	27	81	27
3	25	19	24	68	23
4	21	15	22	58	19
5	15	13	19	47	16
6	11	11	15	37	12
7	10	9	14	33	11
8	7	6	13	26	9
9	6	6	11	23	8
10	5	5	9	19	6

Can you draw a graph using the data shown in Table 1 ?

A graph can be drawn between the number of throws and the average number of remaining dices without face '6'.

What is the nature of the graph ?

From the graph find how many throws are required for the dices to reduce to one half of its initial number. Choose different points on the graph. Does this quantity remain constant ?

If we term the number of throws (equivalent to time interval) where the number of dices reduce to half, as 'half-life' how can you explain half-life for radioactive decay ?

If 100 dices were used instead of 36 in our activity, will the result be affected significantly ?

Damped Oscillation – Project Method

In this project, some experiments using simple pendulum are performed to develop the concept of damped oscillatory motion.

In the beginning, the teacher may perform experiments using simple pendulum and a) show that periodic time (T) is independent of size of the bob, b) show that Periodic time (T) is independent of mass of the bob, c) calculate acceleration due to gravity (g), d) demonstrate damping due to air by doing the following activity.

Take a simple pendulum with a small metal bob. Set up another simple pendulum with same length but having a hollow bob of large radius but same mass.

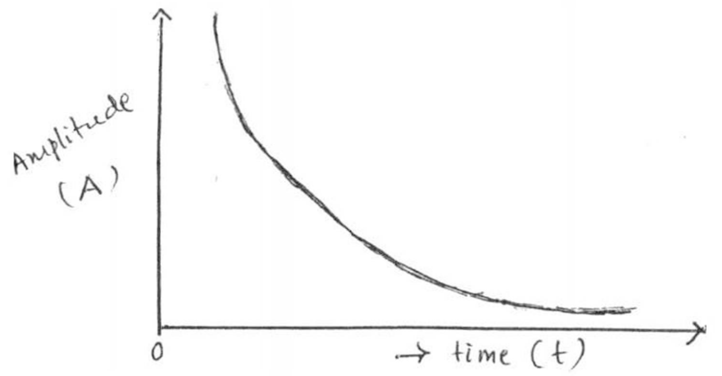
Give same initial displacement to both these pendulums and allow them to oscillate. Both of them will have same initial energy. Amplitude decreases due to air resistance. Also it is observed that the pendulum with larger bob decreases its amplitude more rapidly. This suggests that due to its larger area, the second pendulum has to overcome more resisting force due to air.

e) Demonstrate that damping is different in different media. It can be easily seen that a simple pendulum loses energy more rapidly if it is allowed to oscillate in a large trough containing water or any other liquid. Next calculate the half life time of the damped oscillatory motion.

Students usually get the concept of half-life time while studying radio-activity. But the teacher can easily introduce the idea of half-life time by doing the following experiment.

A simple pendulum of length more than a metre is made to oscillate just above the table-top. A measuring tape or a metre scale is placed on the table horizontally below the bob such that the amplitude of oscillation can be noted on the scale.

Release the pendulum with an initial amplitude of 20 cm. The pendulum is allowed to oscillate. Note the time when the amplitude decreases to 18 cm, 16 cm, 14 cm, etc. Plot a time versus amplitude graph.



The graph will be exponential. It can be described by a formula $A = A_o e^{-kt}$ where A_o is the initial amplitude and K is a constant called damping constant.

Take two amplitudes A_1 and $A_2 = A_{1/2}$. From the graph, note the time t_1 and t_2. The interval $(t_2 - t_1)$ which is time required for amplitude to become half its original value is called half-life time. Take another set of amplitudes A'_1 and $A'_2 = A'_1 / 2$.

Note the half-life time $t'_2 - t'_1$.

Find values for half-time for different set of t_1 and t_2.

Note that the value of all the half-life time will be nearly same.

So under normal conditions, the half-life time of oscillation is independent of the amplitude of the pendulum. We already know that periodic time is a constant.

Students may try to answer the following questions :

4. Will the damping force change if the bobs are at same diameter but different mass?
5. Will the damping force change if the bobs are of same size and material but of different shape?
6. Will the half-life time of the oscillation change if the length of the pendulum is changed ?

Estimation of the value for acceleration due to gravity (Experimental Method)

Fill a burette with water and clamp it vertically. Adjust it such that its nozzle is about one metre above the top of the table.

Slowly open the cock such that water comes out in drops. By trial and error, adjust the cock such that the interval between formation of two drops is equal to the time required for the drop to fall in the beaker kept on the table.

Note the time required for 25 drops to fall and calculate the interval T between the formation of two successive drops.

Knowing the height (h) and time interval T, acceleration due to gravity can be estimated using the formula $h = \frac{1}{2} gT^2$.

This experiment can be repeated for different heights and for different liquids and observe that g is a constant.

Flow of liquids and Reynold's number (Demonstration Method)

A glass tube of length about 10 cm and internal diameter of about 4 mm is attached using a rubber tube to an arrangement which provides water at constant pressure. This arrangement is obtained by maintaining constant level of water in a vessel connected to a water-tap. The water after flowing through the tube is collected in a bucket.

The tube is kept horizontally on the table. Obtain a slow flow of water. Using a doctor's syringe inject potassium permanganate to the rubber tube such that the coloured solution is seen flowing in the glass tube along with the water column. Students can see a red column moving in the tube. The red conduit does not mix up with the adjacent water conduit.

As the rate of flow of water is increased stream line flow changes into turbulent flow and the entire water column in the glass tube appears red.

The velocity of water just before the flow becomes turbulent is called Critical Velocity.

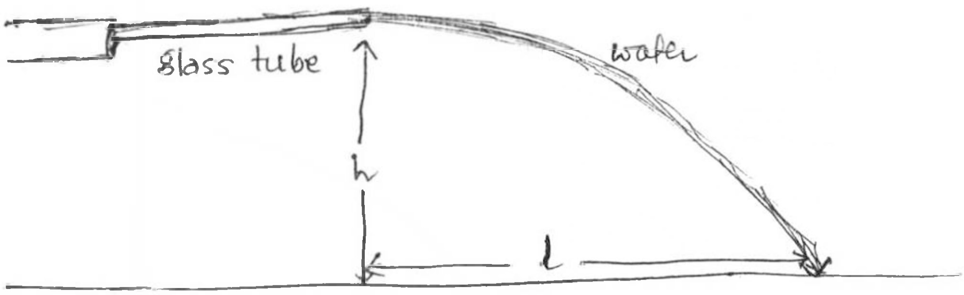
The above set up can be used to calculate Reynold's number (R).

$R = ud\,\rho / \eta$

where u is the critical velocity of water, d is the internal diameter of the glass tube and ρ is the density of water and η is the viscosity of water.

The above experimental set up can be used to calculate u. The glass tube is kept horizontally at a height h from the level where the water coming out of the tube falls. l is the horizontal distance between the end of the tube and the point where the water falls.

Measure h and l.



Since the vertical component of the velocity at the end of the tube is zero, the time of travel (t) is given by $t^2 = 2h / g$ and its horizontal component $u = l / t$.

By measuring d by any suitable method, the value of the Reynold's number R can be estimated.

Teacher can make use of deductive method in the following situation.

Situation : Uniformly accelerated motion.

Accepted General Rules:

1. Rate of change of velocity is acceleration.
2. Distance traveled by a particle is the product of average velocity and time.

Deductive method helps to deduce the following kinematic equations by uniformly accelerated motion.

i) $v_t = v_o + at$

ii) $x - x_o = V_ot + ½ at^2$

iii) $2a (k - k_o) = v_t^2 - v_o^2$

Development of lesson by using the deductive method in classroom :

i. At the time of observation t = 0, let the particle has the initial velocity v_o. Let the particle be uniformly accelerated. What if the velocity of a particle after time 't' seconds?

v_t is the final velocity.

What is the change in velocity ?

Velocity of $v_t - v_o$ particle changes from v_o to v_t. What is the change in time?

$t - 0 = t$

What is the uniform acceleration of a particle during its motion ?

$$a = \frac{v_t - v_o}{t}$$

Rearrangement of the variables give

$$v_t = v_o + at$$

This is one of the equations of motion.

ii) If v_o and v_t are the initial and final velocities, what is the average velocity of a particle?

$$\bar{v} = \frac{v_o + v_t}{2}$$

We know that $s = \bar{v}\,t$.

After substitution for $\bar{v}$ in $s = \bar{v}\,t$, we get $s = \left(\frac{v_o + v_t}{2}\right) t$.

Already $V_t = v_o + at$ is deduced , we know that $s = x - x$.

$$\Rightarrow k - x = \left(\frac{v_o + v_o + a}{2}\right) t.$$

$\therefore\ x - x_o = v_o t + \frac{1}{2} at^t$ another equation of motion.

iii) From the accepted rule $s = \bar{v}\,t$ we may deduce another equation of matrix.

We know that $\bar{v} = \frac{v_o + v_t}{2}$, $s = x - x_o$ and $t = \left(\frac{v_t - v_o}{a}\right)$

Substituting these values in $s = \bar{v}\,t$, we get $2a\,(x - x_o) = v_t^2 - v_o^2$. This is another equation of motion.

Further Examples :

1. Application of Gauss theorem in determining intensities of charges conductors of different shapes.

Another example for Deductive Method :

Generalised concept in kinetic theory of gases :

$$p = \frac{1}{3}\rho\,\bar{v}^2$$

Where p = pressure

ρ = density of the gas

$\bar{v}^2$ = mean square velocity of gas molecules.

Specific concepts such as Boyle's law, Charle's law, Perfect gas equation, Avagadro's law, Dalton's law of partial pressure and Graham's law of diffusion may be deducted from generalized concept.

$$p = \frac{1}{3} \rho \, \bar{v}^2$$

i) We know that $p = \frac{M}{V}$ and $\bar{v}^2 \propto T$

$$\therefore \; p = \frac{1}{3} \rho \, \bar{v}^2 \Rightarrow PV \propto T$$

when 'T' is constant $P \propto \frac{1}{V}$ (Boyle's law). If the temperature of the given mass of gas is constant, its pressure is inversely proportional to volume.

ii) $P = \frac{1}{3} \rho \, \bar{v}^2$

$$P = \frac{1}{3} \frac{M}{V} \bar{v}_2 = \frac{1}{3} \frac{M}{V} T$$

Rearrangement gives $V = \left(\frac{1}{3} \frac{M}{P} \right) T \Rightarrow V \propto T$ where $\frac{1}{3} \frac{M}{P}$ is constructed. (Charle's law).

If pressure of given mass of gas is constant, its volume is directly proportional to the absolute temperature.

iii) $P = \frac{1}{3} \rho \, \bar{v}^2 \Rightarrow PV \propto T$

$\Rightarrow$ PV = RT where R is the universal gas constant (perfect gas equation).

iv) $P = \frac{1}{3} \rho \, \bar{v}^2$

For equal volumes of two gases having same pressure and temperature may be having different number of molecules n_1 and n_2 and masses m_1 and m_2 of each molecule. Then $PV = \frac{1}{3} m_1 n_1 \bar{v}_1^1 = \frac{1}{3} m_2 n_2 \bar{v}_t^t$

We know that $\frac{1}{2} m_1 v_1^1 = \frac{1}{2} m_2 v_2^t$ ..... (1) for different gases at constant temperature.

From (1) and (2) we get $n_1 = n_2$ (Avagadro's law). For equal volumes of all the gases under similar conditions of temperature and pressure contain equal number of molecules

$$P = \frac{1}{3} \rho \, \bar{v}^2$$

If $P_1, P_2, P_3, \ldots$ are pressures exerted by individual bases, $n_1, n_2, n_3, \ldots$ are the number of molecules in the mixture of various gases in a given volume 'v' and assuming that no chemical affinity for each other than

$$P_1 = \frac{1}{3} \frac{m_1 \, n_1}{V} \bar{v}_1^1, \quad P_2 = \frac{1}{3} \frac{m_2 \, n_2}{V} \bar{v}_2^2, \quad P_3 = \frac{1}{3} \frac{m_3 \, n_3}{V} \bar{v}_3^3, \text{ etc.}$$

$$\therefore \; P_1 + P_2 + P_3 + \ldots = \frac{1}{3} \frac{m_1 \, n_1}{V} \bar{v}_1^1 + \frac{1}{3} \frac{m_2 \, n_2}{V} \bar{v}_2^2 + \frac{1}{3} \frac{m_3 \, n_3}{V} \bar{v}_3^3 + \ldots \quad (1)$$

But temperature is same for all the gases in a mixture.

$$\text{Therefore, } \frac{1}{2} m_1 \, \bar{v}_1^t = \frac{1}{2} m_2 \, \bar{v}_2^t = \frac{1}{2} m_3 \, \bar{v}_3^t = \ldots\ldots \quad (2)$$

From (1) and (2)

$$P_1 + P_2 + P_3 + .. = \frac{1}{3V} [n_1 + n_2 + n_3 + ..] \; m \bar{v}^t$$

$$= \frac{1}{3} \frac{mn}{v} \bar{v}_t$$

$\Rightarrow$ $P = P_1 + P_2 + P_3 + \ldots$ (Dalton's law of partial pressure).

To the pressure exerted by a mixture of gases which do not interact with each other, is equal to sum of the partial pressures which would exert, if it alone occupied the same volume at the given temperature.

vi) $\quad P = \frac{1}{3} \rho \, \bar{v}^2$

If two gases A and B having densities ρ_1 and ρ_2 at pressure P, diffuse each other than

$\bar{v}_1^2 = \frac{3P}{\rho_1}$ and $\bar{v}_2^2 = \frac{3P}{\rho_2} \Rightarrow \frac{v_{1\,rms}}{v_{2\,rms}} = \sqrt{\frac{P_2}{\rho_1}}$. If r_1 and r_2 are the rates of diffusion then,

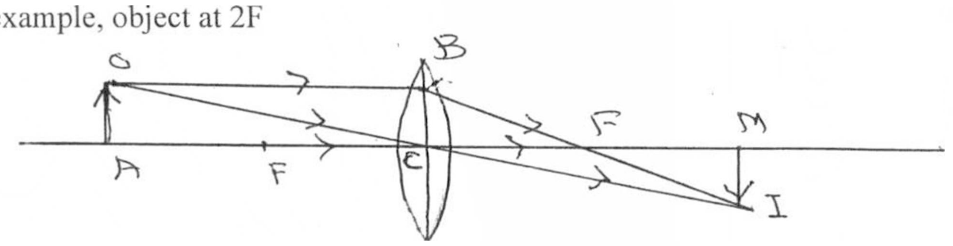
$$\frac{r_1}{r_2} = \sqrt{\frac{P_2}{\rho_1}}$$ (Graham's law of diffusion).

The rates of diffusion of two gases are inversely proportional to the square roots of their densities at the given pressure and temperature.

Inductive method may be used while determining the power for lenses in contact

In the case of a single double convex lens the teacher may show the ray diagram by using rules to locate the position of the object.

For example, object at 2F



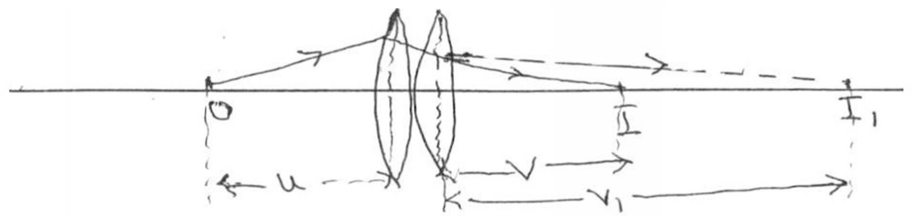
Using the knowledge of high school geometry, one can easily prove the similarity of triangles COA and CIM and FCB and ~~FAD~~ FMI. By considering the correspondence ratios and proper ....law of distances can be established.

i.e. $\frac{1}{v} - \frac{1}{u} = \frac{1}{f}$

Power of a lens if the reciprocal of focal length.

$$\therefore \ P = \frac{1}{f} = \frac{1}{v} - \frac{1}{u}$$

Now if we use two lenses of focal length f_1 and f_2 and they are in contact, then,



$$\frac{1}{f_1} = \frac{1}{v_1} - \frac{1}{u} \text{ and } \frac{1}{f_t} = \frac{1}{v} - \frac{1}{v_1}$$

$$\therefore \ \frac{1}{f_1} + \frac{1}{f_2} = \frac{1}{v} - \frac{1}{u} \Rightarrow \frac{1}{f}$$

$P_1 + P_2 = P$

$\therefore$ For three lenses in contact $P = P_1 + P_2 + P_3$, for 'n' lenses $P = \sum_{i=1}^{n} P_i$.

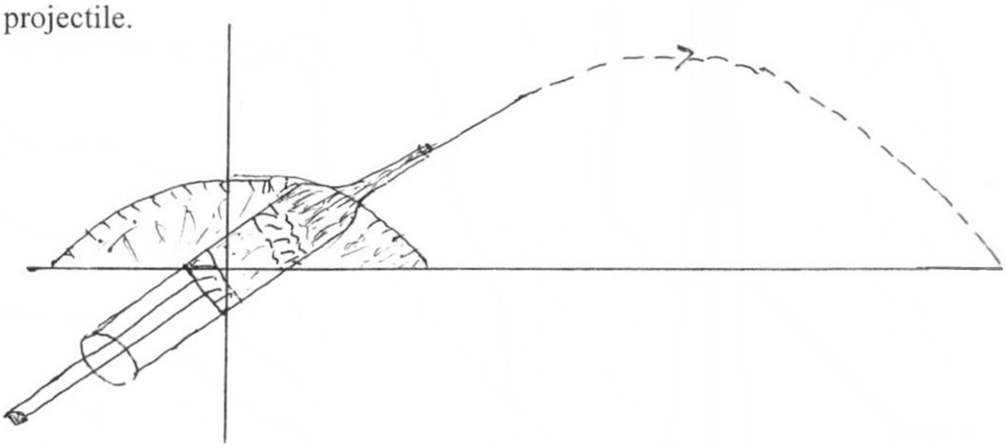
This method of teaching may be used in series and parallel connections of resistances, capacitors, cells, etc.

Applications of Principles of Mechanics in Sports Activities

Investigatory method is a kind of project method. This method may be used by a teacher in many occasions while teaching physics. One example may be projectile motion.

Activity in the Classroom

Disposal syringe may be pivoted to a protractor. Metre scale may be fixed horizontally to the protractor. Adjustments must be made in such a way that syringe is freely moveable on the protractor. One blotting paper may be placed on the table and another blotting paper may be fixed on the vertical board. A few drops of ink is added to water. Coloured water is taken in the syringe. Syringe is adjusted to various angles say 20°, 25°, 30°, 35°, 180°, 45°, 50°, 55°, 60°, etc. with respect to the horizontal distances on the x-axis may be noted down. This gives the concept of 'Range' of projectile.



For every launch angle 'θ' vertical board may be moved in such a way to get the trace of the projectile. Maximum height can be measured and time of flight can be easily calculated by using standard formula.

Knowledge of maximum height, range and time of flight are very helpful for an athlete in the sports field. This may be investigated as applications of projectile motion in sports activities such as shot-put, discus-throw, Javelline throw, high jump, long jump, etc.

Several groups may be made in the classroom. Each group may take one sports activity and investigate the solution for the problems such as – method of holding, method of throw, angle of throw, etc. Physics teacher may take the help of physical education teacher in such activities.

Example 1 : Javelline throw is a good example of projectile motion. If arrangements are made to measure the angle of throw by using modern photographic techniques, many questions related to projectile motion may be asked. Training athletes become easy.

Example 2 : When the golf ball or baseball is in flight, modern photographic techniques may be used to analyse the projectile motion. Here appropriate heights and horizontal distances may be measured for different hits, angle of flight, time of flight, etc. may be calculated.

CHEMISTRY

To study the relative strength of chemical species as oxidizing agents and hence to construct electrochemical cells of different electromotive forces- Investigatory approach.

It is known that elements differ from one another in their ability to function as oxidizing agents. A measure of this ability is their tendency to gain electrons. Since redox reaction involve transfer of electrons from one species to another, we can regard such reactions as combination of two half reactions-one in which a species is under going oxidation and second, in which a species in undergoing reduction. In part-1 of this investigation, the electrons generated by oxidation of one species transferred through a metal wire to the other, which undergo reduction. The electrons that are being transferred through the metal wire may be able to perform work-they can light a bulb, they can drive the needle of a galvanometer and so on. The device is thus able perform electrical work.

Since these essentially chemical processes are responsible for any electrical work that the device can perform, the device is called as electrochemical cell. Just as the cell reaction is combination of two half reactions, the cell is a combination of two half cells. In each half cell a suitable electrolyte solution is in contact with a metal. Electrons enter and leave the solutions through the metals in contact with them. Electrical contact between the solutions in the two half cell is maintained usually through a salt bridge which consists of a concentrated solution of KCl or ammonium nitrate in a U-tube.

Since electrons can be transferred from one point to another only if there is a potential difference between the prints, we can infer that the electrodes in the two half cells are at different potentials. The difference in these 'electrode potentials' is called electromotive force, emf of the cell. The potential difference between the electrodes that is the emf of the cell can however be measured either with a high resistance volt meter or potentiometer .In this investigation, you will measure the emf of several cells.

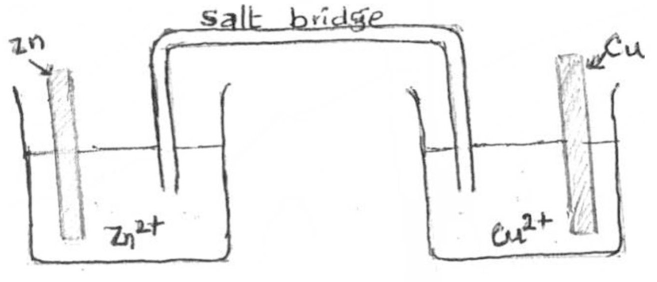
In part-2 of this investigation, you will determine the emf of cells in which the calomel electrode is one of the electrons. Since one of the electrodes is common, this method could serve the purpose of comparing the potentials of several electrodes with reference to calomel.

- *Digital potentiometers at an affordable price are supplied by many instrument manufacturers. Normally a digital potentiometer is supplied along with reference calomel electrode*

Procedure Part-1

1) Dip clean strips of metals in their respective salt solutions (approx 0.1M) taken in separate beakers (copper in copper sulphate solution, magnesium in magnesium sulphate solution, lead in lead acetate solution, silver in silver nitrate solutions, tin in stannous chloride solution etc).
2) Connect the Cu/Cu^{2+} half cell to the Zn / Zn^{2+} half cell using a salt bridge as indicated in the diagram.

   This is a typical arrangement of an electrochemical cell.



3) Connect the copper and zinc strips to separate terminals on the millivoltmeter or potentiometer using crocodile clips and connecting wire. If the voltage indicated is positive note the polarities of the two metal strips. Note the emf in table 1. If the emf indicated is negative, change the polarities.
4) Combine the Cu/Cu^{2+} half cell with each of the other half cell in turn and note the emf and polarities of the metal strips in each case. Record the data inn table 1.
5) Repeat steps 2 to 4 with another pair of half cells. Work with all possible combination of cells of step 1. Record all the emf data table 1.

Procedure-Part-2

1. Take one of the half cells constructed in part 1 and dip the calomel electrode in it. The construction of the calomel electrode supplied to in such that you can avoid the salt bridge in combining it with the half cell.

2. Connect the calomel to one terminal and metal strip to the other terminal of the millvoltmeter a d note the emf . Record in table 1.
3. Repeat 1 and 2 with the other half cells of part 1.

Treatment of data:

1. Write balanced total reactions for the cases where redox reactions were observed. An example is given below.

$$Zn(s) + Cu^{2+}_{(aq)} \parallel Zn^{2+}_{(aq)} + Cu_{(s)}$$

2. In the electrochemical cells assembled by you in part 1 of this experiment, an oxidation half reaction takes place at the electrode in one half cell and a reduction half reaction at the electrode in the other cell. The former electrode is called the anode and the latter, the cathode. The electrodes can be readily identified. The one which is connected to the negative terminal on the millivoltmeter is the anode and the other is the cathode.

In each of the half cells used by you, identify the two electrodes. Combine the two half reactions to write the cell reaction. You have learnt the method of

3. You have learnt the method of representation of the electrodes. The cells, which are combinations of two such electrodes can be represented as in the following example.

$$Zn_{(s)} \mid Zn^{2+}_{(aq)} \parallel Cu^{2+}_{(aq)} \mid Cu_{(s)}$$

Note that the anodes or the negative electrode is on the left and the positive electrode is on the right. The double line indicates that the salt bridge has been used.

4. In part 2 of this experiment you have used calomel electrode as a reference electrode as a reference electrode. The calomel electrode can be represented as follows

$$Pt \mid Hg_{(l)} \mid Hg_2\,Cl_{2(3)},\, KCl_{(sat.\,aq)}$$

5. In the case of each half cell identify the combination in which
a) reduction was the spontaneous process in the half cell.
b) oxidation was the spontaneous process in the half cell.

Discussion

1. In this electrochemical cells constructed in part 1,

   a) Which is the direction of electron flow in the external circuit?
   b) What is the direction of flow of negative ions in the solution?
   c) Why is it appropriate to designate the electrode at which oxidation occurs as negative electrode?

2. In a cell it is found that the Calomel electrode is the cathode. What is the spontaneous process in a cell?

   a. The cell is represented with negative electrode on the left.
   b. The cell emf is $E = E_R - E_L$, where E_R and E_L are reduction potentials of electrodes on the right and left side respectively. If this is positive, the cell representation is correct and the cell process is spontaneous. If not, the position of the electrodes are reversed in representation.
   c. The reactions at both electrodes are written as reductions.
   d. The cell reaction is R_R-R_L , where R_R and R_L are the reduction process at the two electrodes.

Use the conventions to answer the following questions:

1. If it is assumed that the reduction potential for the saturated calomel electrode + 0.241 at 25 c., Calculate, from the measured emfs of cells in part –2, the reduction potentials of the other electrodes in combination within the calomel electrode in the various cells.

2. Using the values of the potentials calculated above, predict the emfs of the cells in part-1. Enter the predicted values in a column adjacent to the experimentally measures value in table 1.

pH of very dilute solution of strong acids -An Enquiry Approach

Many properties of aqueous solutions depend on the concentration of $H^{\oplus}$ ions of the solutions. The $H^{\oplus}$ concentration in dilute solutions is expressed in terms of pH [where $pH = -\log\{[H^+]/\text{mol L}^{-1}\}$]. Thus 10^{-4} M HCl solution will have pH = 4. In pure water at 298 K, the concentration of hydrogen is 10^{-7} M, hence its pH = 7. If an aqueous is acidic, its pH is less than 7 and if it is basic its pH is greater than 7. Pure water is neutral and its pH is 7. The pH 7 is associated with neutrality.

What should be the pH of 10^{-7} M HCl ? Will it be neutral ?

Before we answer this question, let us examine the dissociation of water.

Teacher : Water behaves as a weak acid (or weak base). Write chemical equation representing dissociation equilibrium of water.

Student : $H_2O \rightleftharpoons H^{\oplus} + OH^{-}$

Teacher : What will be he equilibrium expression for the above equation ?

Student : $K = \dfrac{[H^+][OH^-]}{[H_2O]}$

Teacher : What was the concentration of water before dissociation ?

Students : 55.56 mole per liter.

Teacher : Since the dissociation of water is so feeble, the concentration of water at equilibrium is not much altered assuming $[H_2O]$ is constant.

Student : $K[H_2O] = [H^+][OH^-]$

Teacher : $K[H_2O]$ is K_w. K_W is called ionic product of water. At 298 K it is equal to 10^{-14}. Therefore, $[H^+][OH^-] = 10^{-14}$. Water dissociates into the same amount of $[H^+]$ and $[OH^-]$. If we represent the concentration of $H^{(+)}$ as x M then, what should be the concentration of $[OH^-]$? And what should be the pH ?

Student : The concentration of $[OH^-] = x$ M.
$[H^+][OH] = x^2 = 10^{-14}$
$\therefore\ x = 10^{-7}$
$pH = -\log 10^{-7} = 7$

Teacher : Now let us revert back to our original problem – What will be the pH of 10^{-7} M HCl ?
When we add 10^{-7} M HCl into water, the $[H^+]$ from HCl would be 10^{-7}M. When we add this $H^{\oplus}$ from HCl what will happen to the dissociation of water at equilibrium?

Student : The addition of an ion to an equilibrium, having the same ion makes the equilibrium reaction in a direction to consume that ion – common ion effect or suppress the dissociation of H_2O.

Teacher : If the amount to which water dissociate be x^1 in presence of 10^{-7} HCl, then what would be the $[H^+]$ at equilibrium from it?

Student : At equilibrium, $[H^+][OH^-] = 10^{-14} = (X^1 + 10^{-7})(x^1) = 10^{-14}$

Teacher : Calculate the X^1 concentration.

Student : $[X^1 + 10^{-7}]\ x^1 = 10^{-14} \Rightarrow x^2 + 10^{-7} X^1 - 10^{-14}$.

Solving for X^1 using quadratic equation,

$$X^1 = -10^{-7} \pm \frac{\sqrt{10^{14} + 4 \times 10^{-14}}}{2}$$

$= 0.618 \times 10^{-7}$ M

Teacher : You can clearly see the common ion effect in action. Water which was dissociating to 10^{-7} M $H^{\oplus}$ ions, has now experiencing common effect that has finally yielded 0.618×10^{-7} M $H^{\oplus}$ ions . Calculate the pH of 10^{-7} M . HCl ?

Student : $[H^+] = 10^{-7} + 0.618 \times 10^{-7}$
$= 1.618 \times 10^{-7}$ M
pH = - log $(1.618 \times 10^{-7}) = 6.7910$
$\therefore$ pH of 10^{-7} HCl is 6.7910 at 298K.

Teacher : Concentrations higher than 10^{-6} M, the $H^{\oplus}$ from water will be even less than 0.618×10^{-7}M and would be so small in comparison to the $[H^+]$ from HCl, that we can ignore this contribution . Write the expression for K.

Verify the pH of 10^{-7} M HCl using a pH meter and compare with the theoretical value.

Problem Solving

Problem solving is one of the methods often used in teaching of science. In fact, it has been one of the goals of secondary and higher secondary education, to develop problem solving skills in children, so that they can later use it in their life situations. Gagne considers problem solving as reaching the cognitive category of the highest order. Ausubel regards problem solving as a form of discovery learning. Many more psychologists also have given their own formats for problem solving. However, all of them agree on the fact that problem solving is a higher order thinking which is nothing but sheer information processing connecting problems and solutions. Ashmore, Frazer and Casey define problem solving as the result of the application of knowledge and procedures to a problem situation and propose four stages:

a) define the problem,

b) selection of appropriate information

c) combining the separate pieces of information and

d) evaluation of the solution

A problem is a combination of several principles, while problem itself is a combination of several concepts. Problems vary in their level of complexities.

Inductive Effect : Problem Solving approach

Teacher presents ionization constants of some carboxylic acids in aqueous solution at 25°C.

Acids	Ka × 10^5	Acids	Ka × 10^5
HCOOH	17.5	$Cl_2CH.COOH$	5530.0
CH_3COOH	1.8	$Cl_2C - COOH$	23200.0
$Cl\text{-}CH_2COOH$	136.0	$F_3C - COOH$	60,000.0
$Br\text{-}CH_2COOH$	125.0	$CH_3CH_2CH_2 - COOH$	1.5
$I\text{-}CH_2COOH$	67.0	$CH_3CH_2CH - COOH$ Cl	130

$F\text{-}CH_2COOH$	260.0	$CH_3\text{-}CH\text{-}CH_2-COOH$ $\vert$ Cl	89
CH_3CH_2COOH	1.35	$CH_2-CH_2-CH_2COOH$ $\vert$ Cl	2.96

Teacher draws the attention of the students to the Ka values of acids. Asks the following questions. Which is the strongest acid among those listed in the table ?

Teacher analyses the student responses and concludes that trifluoroacetic acid ($F_3C.COOH$) is the strongest (the higher the Ka value, strong the acid should be).

Teacher : You know that acid is a substance that donates proton. Which hydrogen in $H-\underset{\underset{O}{\|}}{C}-OH$ is donated as proton? Why ?

Students : Hydrogen bonded to – OH of $-\underset{\underset{O}{\|}}{C}-OH$ is donated as proton because that $\underset{\underset{O}{\|}}{C}$ is more polar due to higher electronegativity of oxygen.

Teacher : Among acetic acid and formic acid, which is stronger ?

Students : $H-COOH$ is stronger than CH_3COOH (students refer the table).

Teacher : Why is HCOOH stronger than CH_3COOH ?
The students cannot straight away answer the question at this stage of the lesson. Prompting questions are asked as follows :

Teacher : Compare the structures of acetic acid and formic acid, what difference do you find in their structures ?

Students : The only structural difference between the two acids is that the – COOH group is attached to hydrogen in formic acid but to a methyl group in acetic acid.

Teacher : How does this simple structural change render the removal of a proton more difficult in acetic acid ?

Students : This change somehow, must have pushed the electron pair in the – OH bond o the carboxyl group in acetic acid, closer to hydrogen than in the case of formic acid.

Teacher : What role does the CH_3 group play with respect to the shared pair of electron in the – OH bond of the carboxyl group ?

Students : The CH_3 group tend to push the electron pair away from itself and towards the hydrogen of the – O – H bond and hence the – CH_3 group is electron repelling.

Teacher : The CH_3 group is not directly linked to the – OH bond. How then can it bring about this effect ?

Students : This effect of repelling electrons must have been transmitted through the intervening bonds.

Teacher : If that is so, then which is the bond affected first ?

Students : The c – c bond between methyl carbon and carbonyl carbon.

Teacher : How will you represent this shift (include formal charges while representing) ?

Students : Represent the bond polarization as given below.

$$\begin{array}{ccc} \delta+ & & \delta- \\ CH_3 & \longrightarrow & C \\ & & \| \\ & & O \end{array}$$

Teacher : How does this bond polarization affect bond polarization in adjacent bonds?

Students : The carbonyl carbon in turn pushes the electron pair towards the oxygen of the O – H group attached to it. The oxygen then pushes the electron pair towards the hydrogen.

Teacher : How does this kind of bond polarization affect the removal of proton from the – COOH group of acetic acid ?

Students : This increased electron density near hydrogen makes the release of a proton more difficult than in the case of formic acid where such shifts do not occur.

One can generalize and say that an electron repelling group though indirectly linked to a carboxyl group reduces the strength of the carboxylic acid.

Teacher : Which is expected to be more acidic, CH_3OH or H_2O ? Explain.

Students : H_2O is more acidic than CH_3OH. (Offers explanation).

Teacher : What would happen if an electron attracting group is present somewhere along the chain ?

Students : The electron density near the hydrogen of the O – H bond would be reduced and a proton would be more easily released.

Teacher : Then what will happen to the acid strength ?

Students : The acid would become stronger.

Teacher : Compare the ionization constants of chloroacetic acid and acetic acid (Ref. Table).

Students : Chloroacetic acid is much stronger than acetic acid.

Teacher : How do you explain this change ?

Students : The C- Cl bond in chloroacetic acid is more polar than H – C bond in acetic acid and the electron pair will be closer to Cl, that is farther from carbon. Chlorine is, therefore, electron attracting. It exerts its pull for electron all along the chain upto the – O – H bond of the carboxyl group. The electron pair in the – OH bond should, therefore, lie closer to oxygen than the O – H bond in acetic acid. The proton can, therefore, be more easily released.

The polarization in one bond is thus able to exert its influence all along the chain and affect the polarization in the adjacent bonds. This type of effect is called **Inductive Effect.**

Explain why the order of acid strengths of halo acids is as follows :

$F.CH_2 - COOH > Cl\ CH_2COOH > BrCH_2COOH > ICH_2COOH$

Teacher : By convention, groups which are more powerful electron attractors than hydrogen are said to exhibit a negative inductive effect (- I effect). The groups which are poorer electron attractors than hydrogen are said to display a positive inductive effect (+I effect). The following table summarises the inductive effect of some groups.

Inductive Effect of some groups
Table 2

- I groups	+ I groups
- F	- O^-
- Cl	$-CH_3$
- Br	- $CH(CH_3)_2$
- I	- $C(CH_3)_3$
- OR	- COO^-
- COOH	
- C ≡ N	
- NO_2	

Arrange the substituents – Sme_2, - $SeMe_2$, - O Me_2 in order of increasing magnitude of their –I effects.

Teacher : Compare the acid strengths of 2 substituted, 3 substituted, 4 substituted – chloro butanoic acids (Table 1). How do their acid strengths vary ?

Students : As the chlorogroup is moved away from the – COOH group, its effect rapidly dwindles. 3-chlorobutanoic acid is only six times as strong as butanoic acid and 4-chlorobutanoic acid is only twice strong as butanoic acid.

Teacher : It is typical of inductive effect that it decreases with distance and is less important when acting through more than four carbon atoms.

Compare the first ionization constant of oxalic acid (Ka = 5400 10^{-5}) with that of formic acid (Ka = 17.7 10^{-5}). How do you account for this change ?

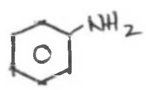
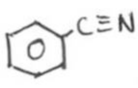
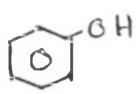
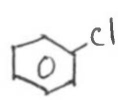
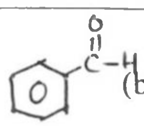
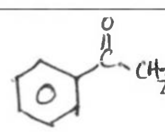
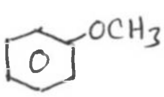
Which of the following represents a stronger acid ?

$^{(-)}OOC.\ CH_2CH_2 - COOH$ and $^{-}OOC - CH_2 - COOH$. Explain.

Aromatic Electrophilic substitution – The effect of substituents on orientation – a problem solving approach.

Teacher presents the table containing some aromatic compounds and major products formed on electrophilic substitution reaction.

Table

Aromatic compounds	Major product	Aromatic compounds	Major product
 (aniline)	o & p	 (benzene nitrile)	m
 (Phenol)	o & p	 (chlorobenzene)	o and p
 (benzaldehyde)	m	 Acetophenone	m
 (anisole)	o & p		

Teacher : Examine the atom which is directly attached to the benzene nucleus, (call this atom as **key atom**) in the compounds listed in the table and identify those compounds in which the key atom contains at least one lone pair of electrons.

Students : Nitrogen in aniline, oxygen in phenol and anisole and chlorine in chlorobenzene contain atleast one lone pair of electrons at the key atom.

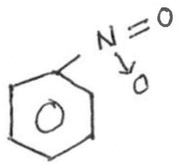
Teacher : Examine the table and identify the major product (s) formed for those compounds in which the key atoms contain atleast one lone pair of electron.

Students : In all these cases, the major product of electrophilic and substitution is ortho and para.

Teacher : Examine the following aromatic compounds and decide which among them can form ortho and para product.

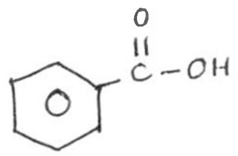
i)

nitrobenzene



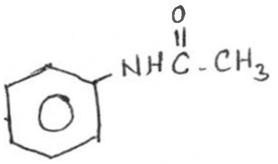
ii)

benzoic acid



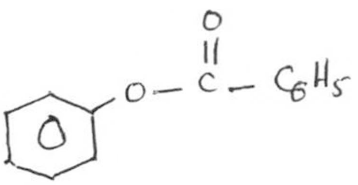
iii)

acetanilide



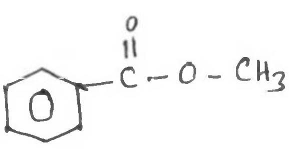
iv)

phenylbenzoate



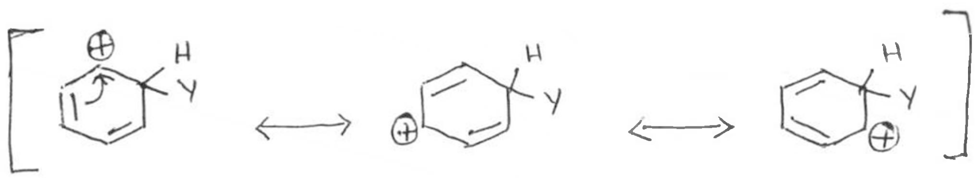
v)

methyl benzoate



Students : Identify the acetanilide and phenyl benzoate can form ortho and para product.

Teacher : We shall examine why the compounds containing lone pair at the key atom form ortho and para products in major amounts ? Aromatic compounds undergo electrophilic substitution by two step mechanism. In the first step the electrophile reacts with benzene nucleus to form an arenium ion intermediate. The arenium ion intermediate is the resonance hybrid of some contributing structures. Formation of arenium ion is the rate determining step. In the fast second step, the arenium ion gives off a proton to form substituted product.

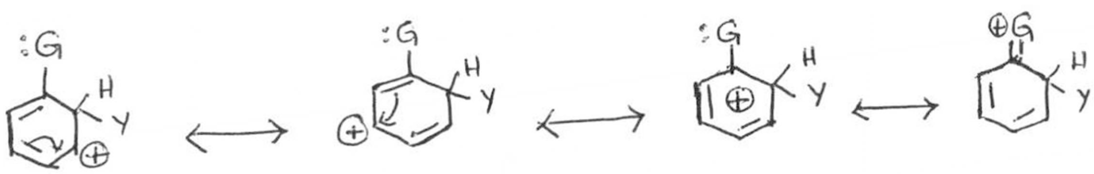


Teacher : Consider the aromatic compounds containing one substituent. What are the different positions at which an electrophile can enter ?

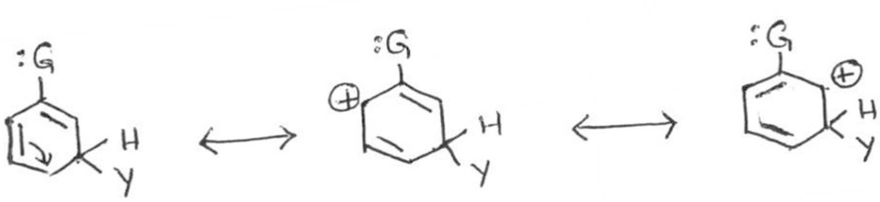
Students : The incoming electrophile can enter at ortho, meta or para with respect to the substituent present in the benzene nucleus.

Teacher : Write the possible arenium ion intermediates that can form a monosubstituted aromatic compound with the key atom containing at least one lone pair of electron.

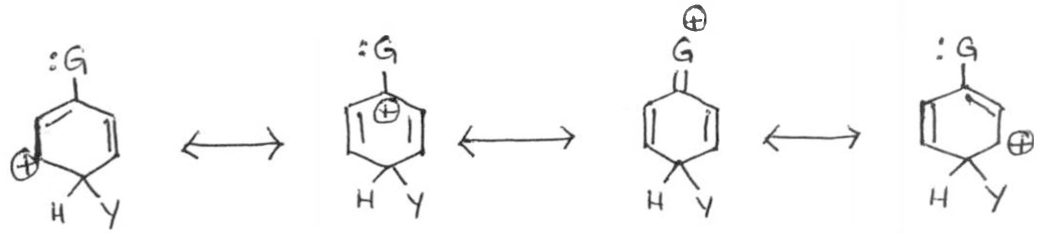
Students : i) For otho attack



ii) For meta attack



iii) For para attack



Teacher	:	Compare the relative stabilities of the 3 arenium ions (ortho, meta and para) keeping in mind that more distribution of charges more stable the species. Which arenium is most stable ?
Students	:	Arenium ion intermediates formed by ortho and para attack are stabilized by four contributing structures whereas in the case of meta, there are only three contributing structures. Hence the arenium ion intermediate formed by the attack of electrophile at ortho and para positions are more stable than the corresponding meta.
Teacher	:	Which product can be formed in larger amount? Why ?
Students	:	Major product will be ortho and para since the activation energy (due to higher stabilities of arenium ion intermediate) for this process is lower compared to the meta.

Rate of reaction, Average rate, order of reaction, rate constant – Data based discussion

Introduction :

Through science teaching if we want to inculcate scientific attitude, it becomes necessary to make keen observations in experiments, collect data and use the data for a meaningful discussion and widening our knowledge. Many a times simple measurements can give a fund of information, if only the teacher can prompt the student think through questions. In this direction, all practical classes should necessarily have a post-lab discussion. It is also desirable that there should be some relevant aspects of theory dealt in practical classes, so that they complement each other.

In view of the above, let us take example of acid catalyzed hydrolysis of ester. The experimental procedure of it is given below.

Procedure :

1. Thermostat 100 cm^3, of 0.5 M HCl and the ester in the separate bottles.
2. Rise and fill the burette with 0.2 M NaOH solution.
3. Take some lumps of ice in each of conical flask.
4. Pipette out 5 cm^3 of the ester in the bottle containing the acid. Shake well and immediately withdraw 5 cm^3 of the reaction mixture using another pipette. Drop the solution onto the ice crystals in one of the conical flask and start the stop watch. When the pipette is half empty add 2 drops of phenolphthalein to the solution and titrate against 0.2 M NaOH till first appearance of pink colour. Note the volume of NaOH as V_0 in Table – 2.
5. Wait for about 2 minutes and withdraw another 5 cm^3 aliquot of the reaction mixture and drop it into ice in another conical flask. Note the time when the pipette is half empty. Titrate as before NaOH. Record volume and time in Table 2.
6. Repeat step 5 after 5′, 10′, 20′, 30′, 50′, 65′, 80′, 100′ and 120′. In each case note the exact time when the pipette is half empty when the solution is being dropped on ice. The conical flasks should be washed between readings and fresh crystals of ice taken in them. Record the volume (V_t) and time in Table 2.

Titrate 5 cc of solution marked infinity against NaOH. Note the volume as infinity (V_∞). Record in the Table 1.

Table 1

Time in Mi.	V. Alkali in mL	$(V_t - V_o)$	$(V_\infty - V_t)$	$-\log (V_\infty - V_t)$
0	5.0	--	--	--
2	6.0	1.0	36.4	1.5611
5	6.4	1.4	36.0	1.5563
10	7.8	2.8	34.6	1.5391
20	13.8	8.8	28.6	1.4564
30	18.8	13.8	23.6	1.3729
40	22.9	17.9	19.5	1.2901
50	26.3	21.3	16.1	1.2068
60	29.1	24.1	13.3	1.124
70	31.5	26.5	10.9	1.0374
80	33.5	28.5	9.0	0.9542
90	34.4	29.4	8.0	0.9031
100	36.2	31.2	6.2	0.7924
110	36.8	31.8	5.6	0.7482
120	38.0	33.0	4.4	0.6435
....				
....				
∞				

Teacher : Based on experimental data, answer the following questions. Write the acid catalysed hydrolysis of ethyl acetate.

Student :

$$CH_3 - COO\ C_2H_5 + H_2O \xrightarrow{\text{Cat. dil HCl}} CH_3 - COOH + C_2H_5 - OH$$

Teacher : What will be the V_{NaOH} (V_t) consumed at time of minute corresponds to?

Student : Volume of NaOH (V_t) at time t min. is time t minutes is proportional to total acid present in the mixture at t minutes.

Teacher : What will NaOH (V_o) at time 0 correspond to ?

Student : The titre V_0 at time 0 is proportional to acid present initially before hydrolysis.

Teacher : Now subtract V_t quantity from V_0 and values are tabulated in Table 1. What will $(V_t - V_0)$ of NaOH corresponds to ?

Student : The volume of NaOH $(V_t - V_0)$ corresponds to acetic acid produced at time t minutes.

Teacher : What will be NaOH litre at time ∞ corresponds to ?

Student : NaOH titre (V_∞) corresponds to total amount acid present after the completion of hydrolysis.

Teacher : Now subtract NaOH titre (V_∞) from NaOH titre (V_t) at time t min. Tabulate the readings in Table 1. What volume of NaOH $(V_\infty - V_t)$ correspond to ?

Student : The volume of NaOH $(V_\infty - V_t)$ is proportional to volume of ester left out at time t minutes.

Teacher : Take – log $(V_\infty - V_t)$ of calculated values of volume of NaOH for various time intervals and tabulate them in Table 1. Now plot a graph $(V_\infty - V_t)$ against the time. Predict how does concentration of ester vary with time.

Student : The concentration of ester decreases with time exponentially.

Teacher : Now drop the ordinate to graph at 5′, 10′, 20′, 30′, 40′ and 50′. Read the concentration of ester compounds to this point. Find the change in concentration between each pair of successive points and calculate average rate in these time intervals.

Student :

Time	5	10	20	30	40	50
Concn.	36.0	34.4	28.4	23.6	19.2	15.6

$$\text{Average rate} = \frac{dc}{dt} = \frac{C_{10} - C_{30}}{30 - 10} = \frac{34.4 - 23.6}{20}$$

$= 0.54$ moles/ dm^3 / min.

$$\text{Average rate} = \frac{dc}{dt} = \frac{C_{30} - C_{50}}{50 - 30} = \frac{23.6 - 15.6}{20} = \frac{8}{20} = 0.4$$

Teacher : Find the time required for the ester concentration to decrease by 25% of their values at 5′, 10′ and 30′.

Student : Find out from the graph.

Teacher	:	Now plot $-\log(V_\infty - V_t)$ against time. Find the slope of this plot and what conclusion can be drawn from this.
Student	:	A straight line with decrease slope and intercept is obtained. $\text{Slope} = \frac{AB}{BC} = 0.0083$
Teacher	:	Can a mathematical expression be given for the obtained graph ? $-\log_{10}(a-x) = \frac{k\,t}{2.303} - \log(a)$
Teacher	:	What will be the rate constant or velocity constant of reaction ?
Student	:	$k = 2.303 \times \text{slope}$ $= 2.303 \times 0.083$ $= 0.0191\ \text{min}^{-1}$
Teacher	:	From the above exercise, infer the relationship between rate of reaction and concentration and predict the order of reaction.
Student	:	As the log of concentration of ester when plotted against time gives a straight line it is I order reaction.
Teacher	:	Similarly, calculate a set of average rates on lines similar to the above, from the plot $(V_t - V_0)$ against time. Find the time required for the acid concentrations to increase by 25% of their values 5′, 10′ and 30′.

Amino acids, Structure – Isoelectric Point – pH at isoelectric point – An enquiry approach

Teacher : Examine the given structure and give the IUPAC name.
$CH_3 - CH(NH_2) - COOH$

Student : 2-Amino-propanoic acid

Teacher : In common nomenclature, carbon directly attached to the functional group is called α – position. Give the general name for $R - CH_2 - CH(NH_2) - COOH$ as per common nomenclature.

Student : α - amino acids.

Teacher : Although amino acids ($H_2N - CH(R) - COOH$) are shown to possess an amino group and a carboxyl group, but the properties (both physical and chemical) are not consistent with the structure.

a) In contrast to amines and carboxylic acids, the amino acids are not-volatile crystalline solid which melt at fairly high temperature.
b) They are generally insoluble in organic solvents. [non-polar solvents] and highly soluble in water.
c) Their aqueous solution have dipole moment.
d) Acidity and basicity constants are low for $-COOH$ and $-NH_2$ groups.

Glycine : $K_a = 1.6 \times 10^{-10}$ and $K_b = 2.5 \times 10^{-12}$
[$H_2N - CH_2 - COOH$]
whereas most of carboxylic acids have K_a values about 10^{-5} and most of aliphatic amines have K_a values about 10^{-4}. What do these properties of amino acids indicate nature of compounds ?

Student : Properties of amino acids resemble with properties of electrovalent compounds.

Teacher : Given the structure of amino acids ($R - CH_2(NH_2) - COOH$) contains basic end as $-NH_2$ and acidic end as $-COOH$. If acid end ($-COOH$) transfers proton to the basic end, ($-NH_2$) what would be the resultant structure of amino acid ?

Student : $H_3N - CH(R) - COO^-$

Teacher : Can above structure explain the observed properties of amino acids?

Student : Yes.

Teacher : True, all the mentioned properties of amino acids are quite consistent with the dipolar-ion structure for the amino acid. This dipolar ion structure is called Zwitter ion.

$H_3N^+ - CHR - COO^-$
Examine the dipolar-ion structure of amino acid and identify acidic and basic ends.

Student : H_3N^+ is the acidic end.
- COO^- is the basic end.

Student : i) $H_3N^+ - CHR - COO^- + H_2O \rightarrow H_3O^+ + H_2N - CHR - COO^-$
Dissociation constant

$$k_a = \frac{[H_3O^+][H_2N - CHR - COO^-]}{[H_3N^+ - CHR - COO^-]}$$

ii) $H_3N^+ - CHR - COO^- + H_2O \rightarrow OH^- + H_3N - CHR - COOH$
Dissociation constant

$$k_b = \frac{[H_3N - CHR - COOH][OH^-]}{[H_3N^+ - CHR - COO^-]}$$

Teacher : What happens when a solution of amino acid present in the acid medium? Write the equation to represent the equilibrium that exists in the solution.

Student :

$$H_3N^+ - CHR - COO^- \underset{-H^+}{\overset{H^+}{\rightleftharpoons}} H_3N^+ - CHR - COOH \quad \text{(I)} \quad \text{(cation)}$$

Teacher : What happens when amino acid solution is present in basic medium and write the equation to represent its equation.

Student :

$$H_3N^+ - CHR - COO^- \underset{+H^+}{\overset{-H^+}{\rightleftharpoons}} H_2N^+ - CHR - COO^- \quad \text{(II)} \quad \text{(Anion)}$$

Teacher : When amino acid solution in acid medium is placed in electric field and ion of solution migrate towards which electrode ?

Student : Cation (I) ($H_3N^+ - CHR - COOH$) is migrated towards cathode.

Teacher : Suppose amino acid solution in basic medium is placed in electric field what happens and where the ion migrate in the electric field.

Student : Anion (II) [$H_2N - CHR - COO^-$] migrate towards anode

Teacher : If concentrations of anion (II) and cation (I) are exactly balanced then which electrode ion migrate in the electric field and why ?

Student : There will be no net migration because solution has exactly balanced concentration of cation and anion.

Teacher : Thus, particular $[H^+]$ ion of solution a particular amino acid does not migrate towards electrodes under the influence of electric field and it is called Isoelectric point.
Now, what will be pH of an amino acid at the isoelectric point ?
Now let us answer the following questions in order to arrive at the conclusion.
Write the dissociation equation of the cation H_3N^+ - CHR –COOH in solution.

Student : $H_3N^+ - CHR - COOH \longrightarrow H_3N^+ - CHR - COO^- + H^+$
Conj. Acid (CA) Dipolar ion (Di)

Teacher : Write the mathematical expression for the above equation.

Student : $$k_1 = \frac{[D_i][H^+]}{[CA]}$$

Teacher : Similarly write dissociation equation of dipolar ion (H_3N^+ -CHR – COO^-] in solution.

Student :

$H_3N^+ - CHR - COO^- \longrightarrow H_2N - CHR - COO^- \quad H^+$
(Di) conj. Base (CB)

Teacher : Write the mathematical expression K_2 for the above.

Student : $$K_2 = \frac{[CB][H^+]}{[D_i]}$$

Teacher : Evaluate concentration of conj. Base and concentration of conj. Acid from the expression.

Student : $$[CB] = \frac{[D_i]k_2}{[H^+]} \text{ and } [CA] = \frac{[Di][H^+]}{k_1}$$

Teacher : What will happen to concentration of conj. Base and conj .acid at isoelectric point ?

Student : [CB] =[CA] at isoelectric point.

Teacher : At isoelectric point equate the above expressions for [CA] and [CB].

Student : $\frac{[Di]\, k_2}{[H^+]} = \frac{[Di][H^+]}{k_2}$

Teacher : Evaluate for $[H^+]$ at isoelectric point. $[H^+]$ is called isoelectric point as $[H_i^+]$.

Student : $[H_i^+]^2 = K_1 + K_2$

Teacher : Now take log on both sides.

Student : $2 \log [H_i^+] = \log k_1 + \log k_2$.

Teacher : Now multiply (-) sign on both sides.

Student : - $2 \log [Hi^+] = - \log [H^+] - \log k_2$

Teacher : What are $- \log [H^+]$, $- K_1$ and $- \log k_2$ called ?

Student : $-\log [H^+] = pH$, $- \log k_1 = pk_1$, $- \log k_2 = pk_2$.

Teacher : Rewrite the expression on the basis of your answer.

Student : $2\ pH_i = pK_1 + pK_2$

Teacher : What is the pH of amino acid at the isoelectric point ?

Student : $pH_i = \frac{pk_1 + p\,k_2}{2}$

Teacher : Isoelectric point of amino acid, PI is the average value of both pKa values.
PI at the isoelectric point of glycine is 5.92. [$pK_1 = 2.34$, $pK_2 = 9.69$]

Therefore, at isoelectric point, amino acid may not be neutral. Some of them will be acidic and some of them will be basic. So they will depend on nature of amino acid.

Teacher : pKa value of some amino acids are given below. Calculate their PI.

Name	pK_1 (- COOH)	pK_2 ($- NH_3^+$)	pK_3 (-R)	PI
Alanine	2.34	9.69	--	--
Proline	1.99	10.96	--	--
Tyrosine	2.20	9.11	10.07	--
Aspartate	1.88	9.60	3.65	--
Glutamate	2.19	9.67	4.25	--
Lysine	2.18	8.95	10.57	

BIOLOGY

Addressing major pedagogic issues and concerns in Teaching at Higher Secondary Level:

In higher secondary, teacher has to enrich the content knowledge as well as initiate a constructivist learning process as it may help the student to focus on specific area at higher level of education (in college).

Keeping in views the above two purpose in mind, the teacher may have to combine many strategies to transact a particular concept from the textbook.

Learning strategies may be selected keeping in mind that teacher has to build up from a child world of 'known to unknown' and draw the child's attention from a concrete examples to an abstract one with reference to the concept.

A few content from the Textbook of XI and XII are taken and pedagogical methodology is explained.

Concept 1

XII TEXTBOOK (NCERT) Chapter 1, Page 16, 2nd para.

A teacher sets a target for herself before initiating the content.

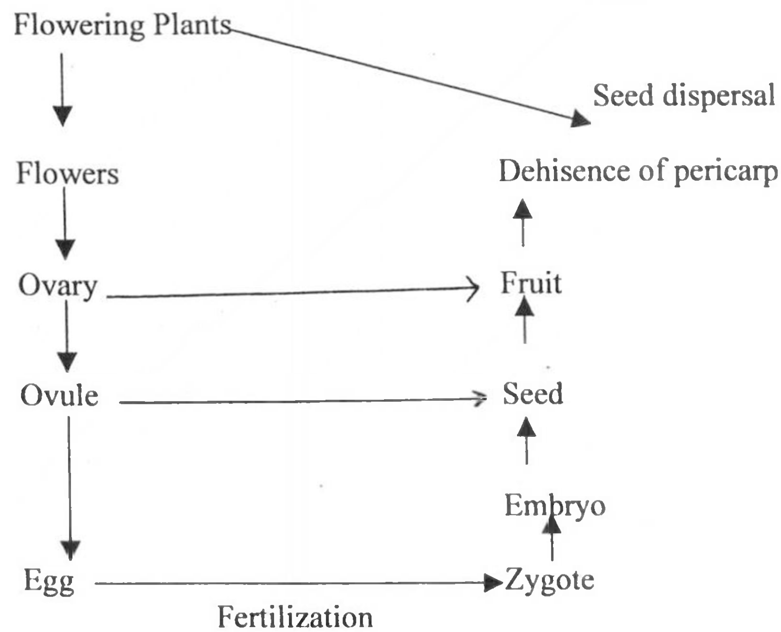
Target/ Aim :

1. Child should be able to distinguish flowering plants like mango tree, guava tree, coriander, etc. from non-flowering plants like pinus, cycas, mushroom, etc.
2. Child should be able to locate the position of a ovule, ovary and after fertilization, what happens to egg, ovule, ovary, petal, sepal of a flower.
3. The child understands the pericarp (fruit wall), seed (contain zygote and ovules) in some of the common fruits and vegetables used in everyday life.

Talking to a child helps in building up the Chapter from their ideas but time restrain may prevent it.

Step 1 : Simplify the Text Content

Flow Chart



Step 2: Some of the activities to grill the mind.

Activity 1 :

Arrange pine cone, cycas cone, mushroom, bongainvilla, Hibiscus, Bamboo, Sugarcane, grass.

Worksheet 1 : Ask student to categorise them under flowering plants and non-flowering grass.

Worksheet 2 : To locate the position of ovary in each flowering plant.

Worksheet 3 : Can they see ovule and egg inside the ovary? Yes / No.

Worksheet 4 : Select fruits like Guava, Mango, tomato, peas.

Mark pericarp and seed in each fruit.

While filling the worksheet, the teacher may come across various misconception of students.

Teacher may summarize by explaining simple known fruits to complex fruits like pineapple, custard apple, etc.

Concept 2 : Mutation

NCERT Textbook pg. 87, Chapter : Principles of inheritance and Variation

Teachers cannot demonstrate mutation. So she has to follow a strategy so that a child builds up spatial concept. One way to provide evidence to this theoretical concept is to make use of newspaper clippings and T V programmes of Nuclear bomb explosion in Hiroshima Nagasaki in II World War.

Step 1 : Simply the Content

Note Making :

1. Mutation is sudden, heritable change in genetic constitution of a person.
2. Mutation involving DNA sequence is called Point Mutation and that involving chromosome is called chromosal abberation (eg. Inversion, Duplication, Translocation, Deletion).
3. Mutation is caused by chemicals and radiations called mutagens.
4. All mutations are not harmful. Some of them are useful as they are source of evolution.

Step 2 : Activity 1

Reading newspapers, follow up process involving all the abnormalities which occurred in the people who suffered from mutation due to nuclear radiation.

Worksheet 1 : Grouping the symptoms under genotype / phenotype.

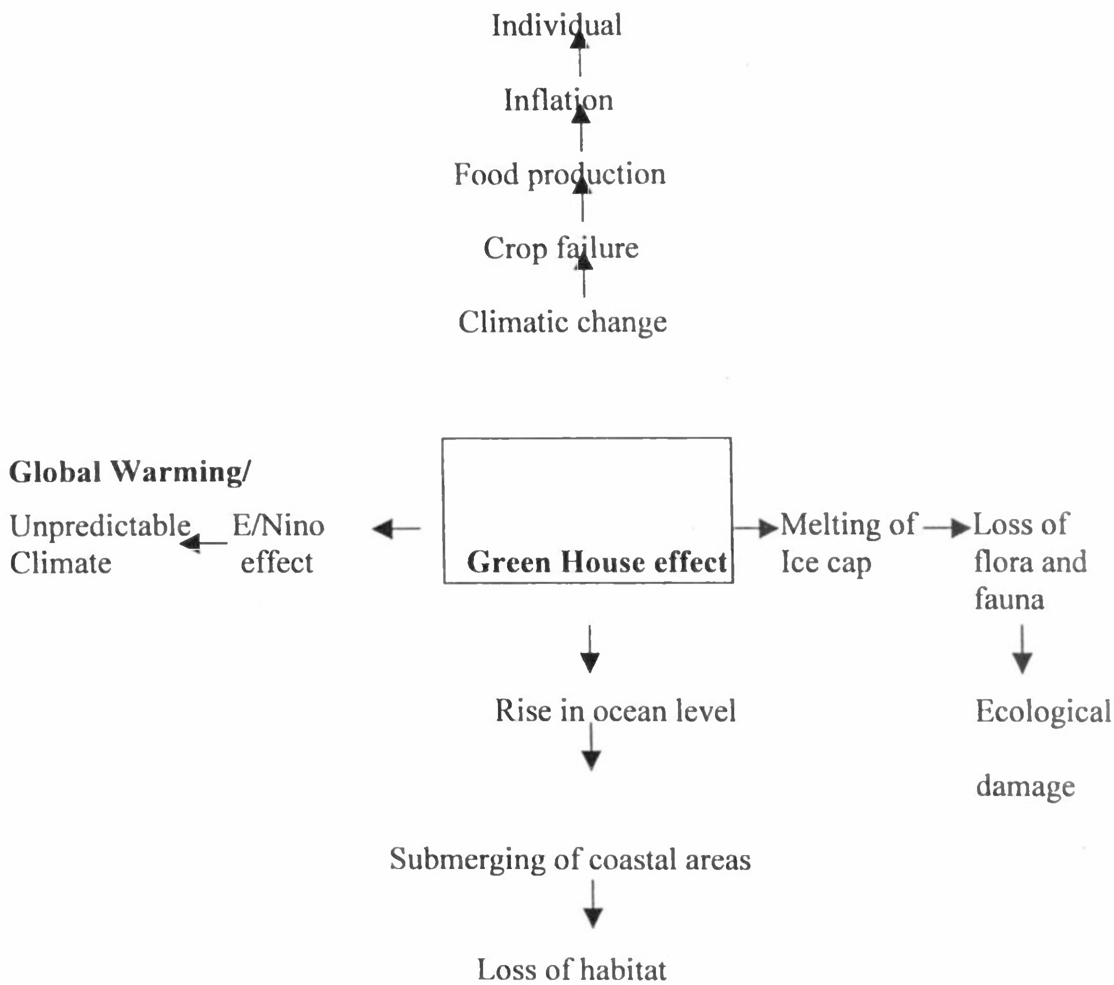
Worksheet 2 : Grouping the genotypic symptoms under point mutation and chromosomal aberrations.

After finishing the worksheet the child correlates the 'mutation', a phenomenon with real examples. He/She also gets an idea about mutagens and distinguish genotype, phenotypic characters.

Concept 4 : Green House Effect

Some of the concept which are related with environment, which in turn can affect people, can be dealt in the form of consequencing mapping.

Step 1 : Keeping green house effect as a center concept, its consequential effect is noted in the form of flow chart if the effect directly affects the individual person.



Step 2 :
Activity 1 :

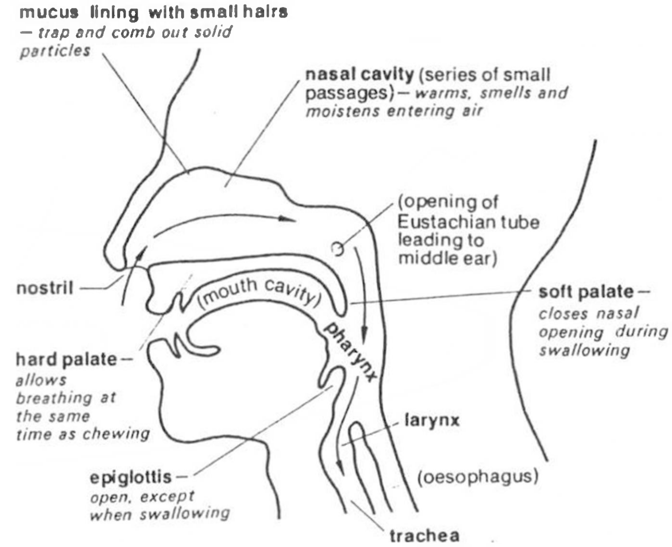
Worksheet

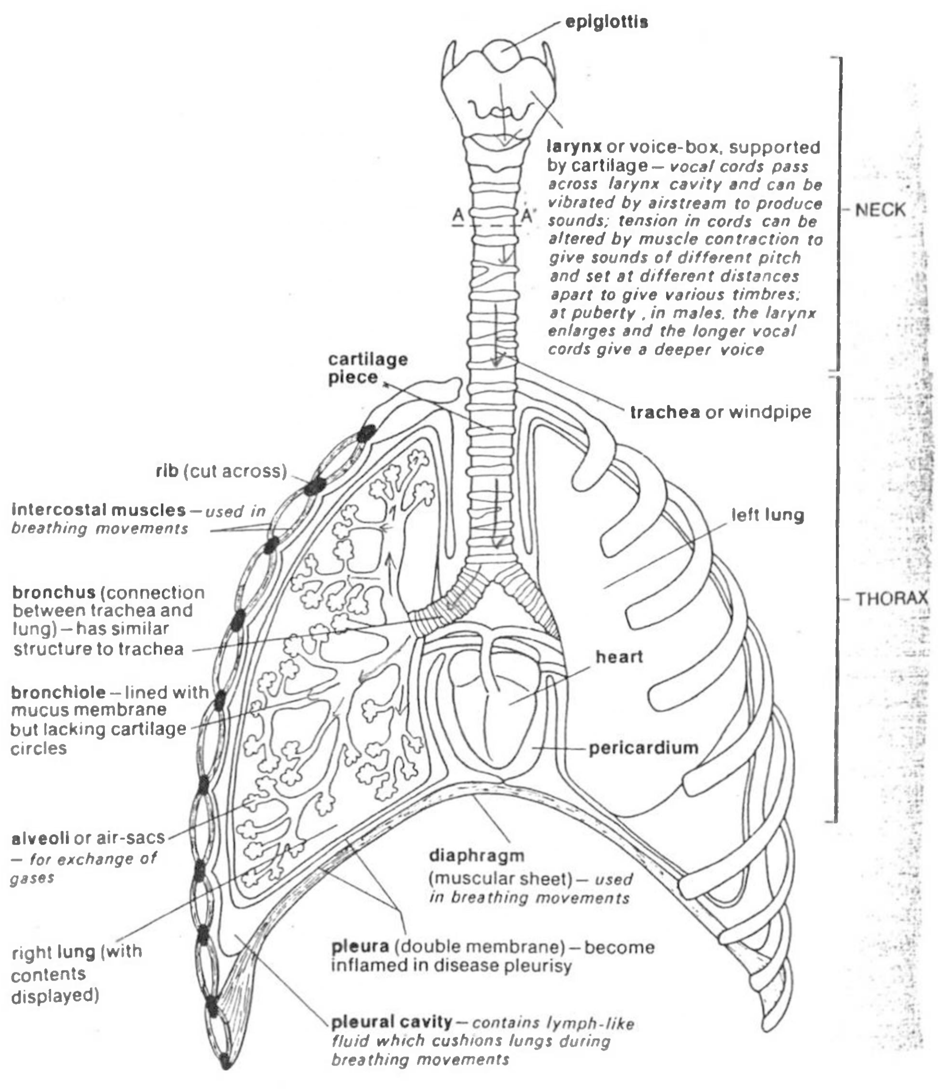
Statements	Agree / Disagree / Undecided
1. Man is responsible for Global warming.	
2. Burning of fossil fuel emit CO_2 which forms a layer in atmosphere.	
3. CO_2 absorbs the reflected radiation.	
4. CO_2 keeps the earth surface warm.	

Concept 4 : Breathing and exchange of gases
NCERT Textbook XI, page 269.

Diagrammatic Presentation Approach

Air passage in Human body





Worksheet for students :

1. Trace out the passage of air from its entry point to its target region. (List out the structures) through diagram.

Gases entering the respiratory tract along with breathing in and those gases which go out along with breathing out air.

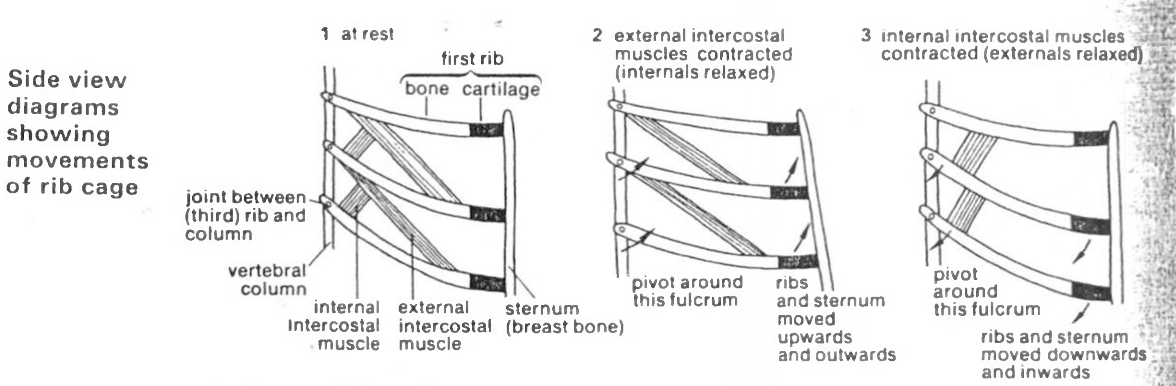
	Nitrogen	Oxygen	Carbon Dioxide	Water Vapour
Inspired Air	78%	20%	0.03%	Variable near 1%
Expired Air	75%	15%	4.0%	near 6%

Worksheet 2 :

- Though N_2 absorbed is 78%, but it is not circulated like O_2 and CO_2. Why ?
- Which gas will be more in alveoli after inspiration?
- How CO_2 percentage increases in expired air ?

Muscles attached to rib cage (intercoastal muscles) and diaphragm play an important role in inspiration and expiration.

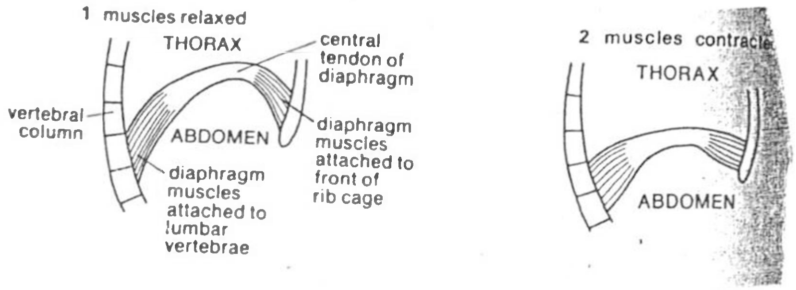
Similarly, movement of the rib cage upwards and outwards causes an increase in volume.



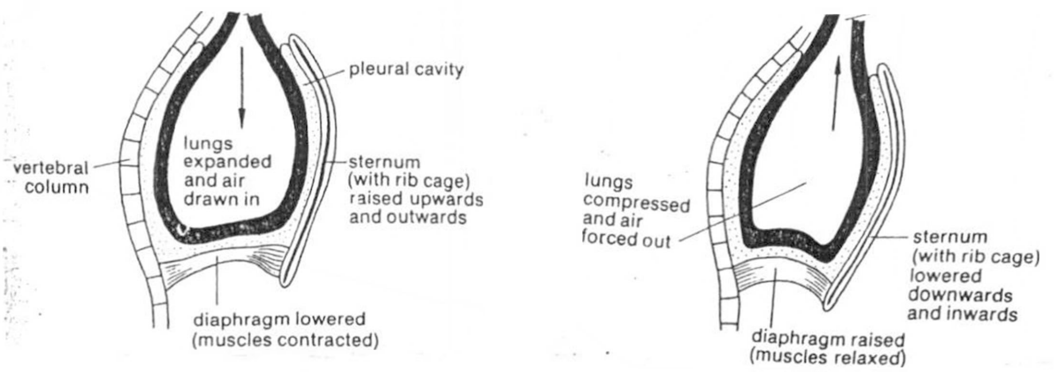
Ribs and Sternum move upward and outward increasing the volume of thoracic region causing low pressure inside resulting in inspiration (Fig. 2).

During expiration ribs and sternum move downwards and inwards reducing the volume of thoracic region resulting in expiration (Fig.3)

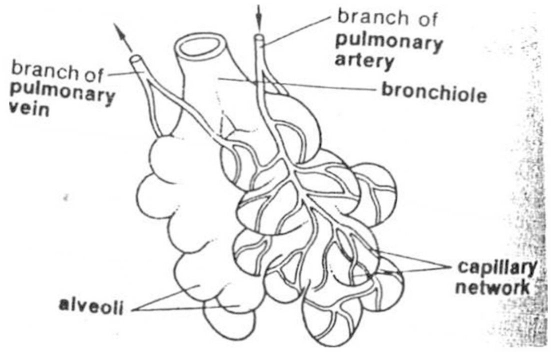
Role of Diaphragm in inspiration and expiration



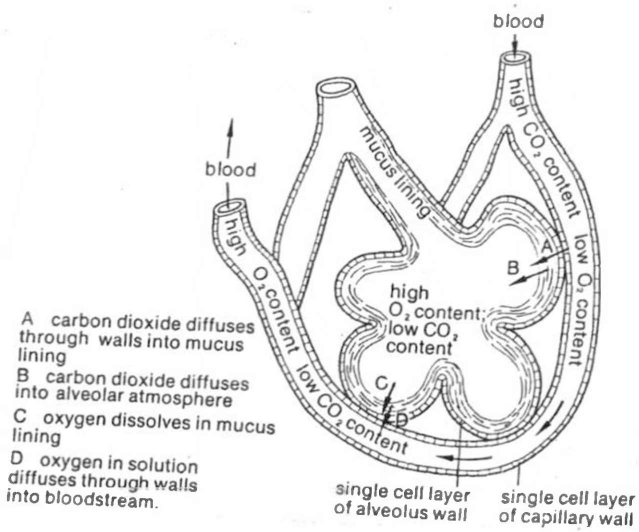
Condition of lungs after breathing in and after breathing out.



Air filled alveoli in association with blood capillaries



Place where exchange of gases takes place.



Worksheet :

- Do partial pressure of gases play a role in their diffusion?
- What is the role of mucus lining ?
- In the diagram pulmonary artery is represented by which vessel? Which vessel indicates pulmonary vein?

Concept 5 :

Mendel's laws of inheritance, spermatogenesis, oogenesis and meiosis I can be linked to each other by posing a suitable problem whose solution lies in the above mentioned concepts.

Methodology : Problem Solving Approach

Problem :
Why siblings sometimes look similar to each other but not resemble 100%?

Step 1 :
Teacher (or students search from the book): Explains genetic constitution of an organism. The process of meiosis where exchange of genetic material takes place in pachytene stage is explained to them.

Problem posed : Meiosis which occurs in father's and mother's body at the time of formation of sperm (spermatogenesis) and formation of egg (oogenesis) reaches the sibling.

Step 2 :
(Teacher's explanation or students search from the book.)
Genes of parents reaches the sibling through sexual reproduction.
As genes are inherited from parents, why siblings are not 100% resembling the parents.

Step 3 :
Teacher's explanation / Student search from textbook

i) Mendel's law of Dominance : (Page 75 NCERT textbook XII) which states that genes are inherited from parents, among a pair (alleles) only one expresses itself and another is suppressed.
ii) Mendel's law of segregation: Genes of the parents segregate from each other such a way that only one gene goes to the gamete.
iii) Mendel's law of independent assortment : In inheritance of two pair genes, the segregation of one pair is independent of others.

Teacher asks the student to select those statement which solves the first posed problem.

Worksheet for Children :

Children are asked to tick those point which solves their problem.

1. Genes are inherited from parents. Reason for looking similar.

2. Oogenesis is the formation of egg.

3. Exchange of genetic material occurs in Meiosis I. Reason for variation.

4. Independent assortment helps in procuring new combination of genes.

5. Genetic exchange involves many genes.

6. Genes might have come from their grand parents also.

MATHEMATICS

List of Resource Persons

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2. Dr D Basavayya
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CONTENTS

MATHEMATICS

Introduction

One of the most important activities promoting mathematics learning is problem learning. Problem solving is synonymous to mathematics. Problems posed in many real life situations motivate mathematical formulations and mathematical modelling. Mathematical models are restatements of problems. Rewording problems in mathematical form is the first important step towards solution. This being done, solution is the result of appropriate stages on – identification of key ideas, use of mathematical strategies in the form of suitable inputs as mathematical results, interpretation of the end result. A classroom teacher being a facilitator to learning mathematics, must himself be aware of these. Doing problems mechanically is unproductive unless it is done with insight. Illustrations of problem solving must bring to fore these aspects detailed above. Accordingly, the following samples drawn from different topics in mathematics of higher secondary level are provided. The 'why'? of each step in the solution is the link connecting a step with the next. Finally, the exercise of problem solving is an exercise in the formation of a logical chain. Role of diagrams be it line sketches, pictures drawn to the scale or otherwise or graphs, necessitated in the problem help to bring in clarity of thinking.

Illustrations :

Sets

Problem : In a locality with 60 families, 25 subscribe to English newspaper, 26 to Kannada paper and 26 to Telugu paper. 9, 11 and 8 subscribe to both English, Kannada; Kannada, Telugu and English, Telugu in order. 3 subscribe to all.

How many subscribe to (i) at least one paper, (ii) exactly one paper and (iii) no paper.

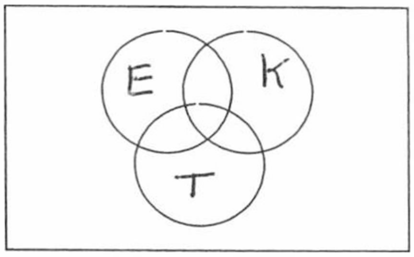
Solution : In the problem, we identify

i) three sets

ii) these are overlapping (not disjoint) and

iii) it is likely that not all are subscribers.

To make the data clear we draw (Venn) diagram.



Taking E = the set of subscribers for English paper

K = the set of the subscribers for Kannada.

T = The set of subscribers for Telugu.

Data : $n(E) = 25$, $n(K) = 26$ and $n(T) = 26$.

$n(E \cap K) = 9$, $n(K \cap T) = 11$ and $n(E \cap T) = 8$. $n(E \cap K \cap T) = 3$.

(i) the set of atleast one paper subscribers = $n(E \cup K \cup T)$.

We know : $n(E \cup K \cup T) = \Sigma n(E) - \Sigma n(E \cap K) + n(E \cap K \cap T)$

$= (25 + 26 + 26) - (9 + 11 + 8) + 3$

$= 77 - 28 + 3 = 52$

$\therefore$ No. of subscribers for at least one paper = 52 (i)

From Venn diagram

The set of subscribers to English only

$= n(E) - n(E \cap K) - n(E \cap T) + n(E \cap K \cap T)$

$= 25 - 9 - 8 + 3 = 28 - 17 = 11$

Similarly, the set of subscribers to Kannada only.

$= n(K) - n(K \cap E) - n(K \cap T) + n(E \cap K \cap T)$

$= 26 - 9 - 11 + 3 = 29 - 20 = 9$

and the set of subscribers to Telugu only.

$= n(T) - n(K \cap T) - n(E \cap T) + n(E \cap K \cap T)$

$= 26 - 11 - 8 + 3 = 29 - 19 = 10$

Hence the total no. of subscribers to only one paper

$= 11 + 9 + 10 = 30$ (ii)

iii) No. of subscribers to no paper

= 60 – No. of subscribers to atleast one paper.

$= 60 - 52 = 8$

$\therefore$ No. of those who do not subscribe to any paper = 8 (iii)

Suggestions :

1. **Hardspot in the illustration to be clearly stated. Eg. – writing the Venn diagram appropriately.**
2. **Shading the different regions in the Venn diagram.**
3. **Exercises – To reinforce learning.**

Range of a function

Problem : Find the range of $f(x) = \frac{x^2}{1+x^2}$, $x \in R$.

Solution : Rewriting $f(x) = 1 - \frac{1}{1+x^2}$, as $x \to \infty$, $f(x) < 1$.

Range of f(x) = the set of all possible values of f(x).

Since $f_{min} \leq f(x) < 1$.

The range of $f(x) = [\, f_{min}, 1)$.

$f(x) = 1 - \frac{1}{1+x^2}$. $\therefore f'(x) = \frac{2x}{(1+x^2)^2} = 0 \Rightarrow x = 0$. And

$$f''(x) = 2\left[\frac{(1+x^2)^2 \,.\, 1 - x \,.\, 2\,(1+x^2)\, 2x}{(1+x^2)^4}\right] = 2\frac{1-3x^2}{(1+x^2)^3}$$

At x = 0, f''(0) = +ve.

$\therefore$ f has min. at x = 0 and $f_{min} = 1 - \frac{1}{1+0} = 0$.

$\therefore$ The range of f(x) = [0,1).

Principle of Mathematical Induction

Problem : Prove, using Mathematical Induction, that the total number of subsets of a finite set with n elements is 2^n.

Solution : Let A be a finite set and n(A) = No. of elements of A = n.

To show that the total no. of subsets of A = 2^n.

The use of principles of Mathematical Induction involves two steps.

If P(n) is a statement involving a positive integer, then P(n) is true provided

i) P (1) is True and

ii) For any n = k, the truth of P(K) implies that of P(k + 1)

Accordingly, (i) and (ii) implies the truth of P(n) for all $n \in N$.

In the problem, P(n) = The total no. of subsets of A is 2^n, when n(A) = n. We first verify the steps (i) and (ii)

i) Let n = 1. Then A has only one element only. Hence the subsets of A are ϕ and A i.e. the total no. of subsets of A = 2 = 2^1.

ii) Let the statement be valid for any set A with K elements.

i.e. the total no. of subsets of A = 2^k.

Let A = $\{a_1, a_2, \ldots., a_k\}$.

Consider the set B = $\{a_1, a_2, \ldots., a_k, a_{k+1}\}$ with (k + 1) elements. We now count the subsets of B.

They are

a) those not involving a_{k+1} and

b) those involving a_{k+1}

a) The total no. of subsets not involving a_{k+1} = The total number of subsets of A

= 2^k by Induction hypothesis.

b) The subsets involving a_{k+1} are got by including a_{k+1} to each subset of A. Hence the total no. of subsets involving a_{k+1} = 2^k.

Hence the total no. of subsets of B (having (k + 1) elements = $2^k + 2^k = 2^{k+1}$.

Hence the second step is verified.

Therefore, by the Principle of Mathematical Induction, the total no. of subsets of A (when n(A) = n) is 2^n for all +ve integers n.

Suggestion :

1. Explanation as to why the problem is linked to Min/ Max. of a solution.

Explanation : In general, when the graph of a function is drawn, the range is formed by the interval with end points on absolute min and absolute max., even if the graph cannot be drawn (as in this problem).

Exercises : When the range can be got without the use of max/ min. ideas.

Suggestions : Why of this choice (i.e. the problem) caution to the teachers – that the Induction Method is not restricted to proving formulas or identities but even verbal statements and inequalities.

Complex Numbers

Problem : State the geometrical meaning for each statement given below :

i) $\quad \text{Arg}\left(\dfrac{z_1 - z_2}{z_3 - z_4}\right)$, z_k (k = 1,2,3,4) being complex numbers.

ii) $\quad \text{Arg}\left(\dfrac{z_1 - z}{z_2 - z}\right)$

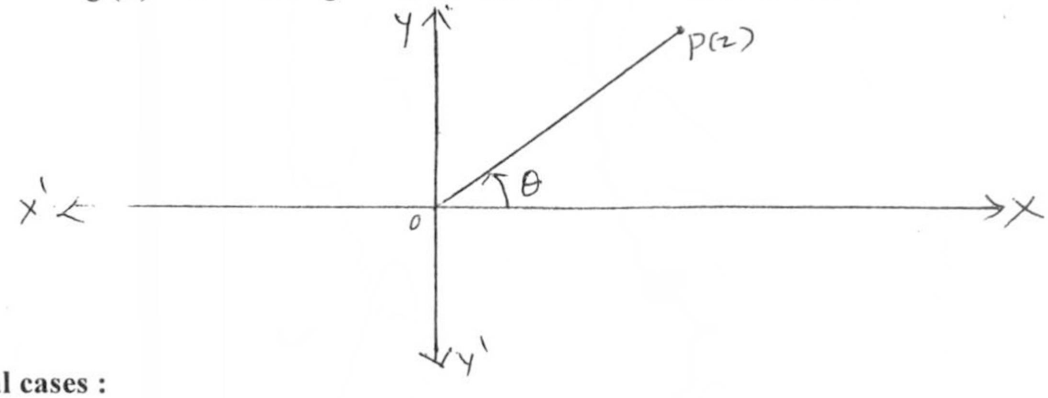
iii) $\quad \text{Arg}\left(\dfrac{z_1 - z}{z_2 - z}\right) = 0 \text{ or } \pi$

iv) $\quad \text{Arg}\left(\dfrac{z_1 - z_3}{z_2 - z_3}\right) = \text{Arg}\left(\dfrac{z_1 - z_4}{z_2 - z_4}\right)$

The geometrical meanings are very useful in solving some geometrical problems. Therefore, it is necessary to understand the meaning of Arg (Z) for any complex number.

For any complex no. Z represented by P of the complex plane

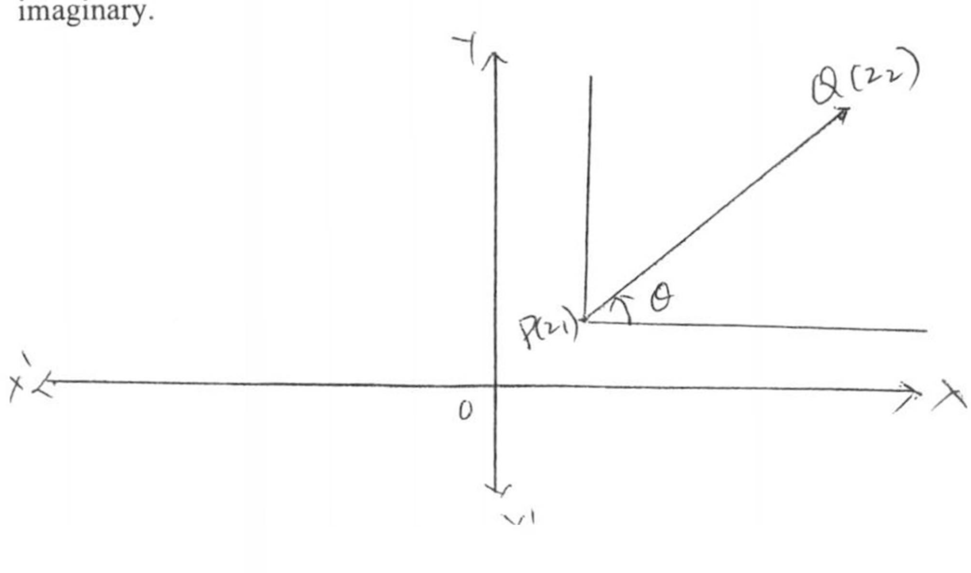
1. Arg (Z) = θ = the angle P $\hat{O}$ X mean and counter clockwise.



Special cases :

a) If Arg (z) = 0 or π, P(z) lies on the real axis so that Z is real.

b) If Arg (z) =$\pi/2$ or - $\pi/2$, P(2) lies on the imaginary axis so that Z is purely imaginary.

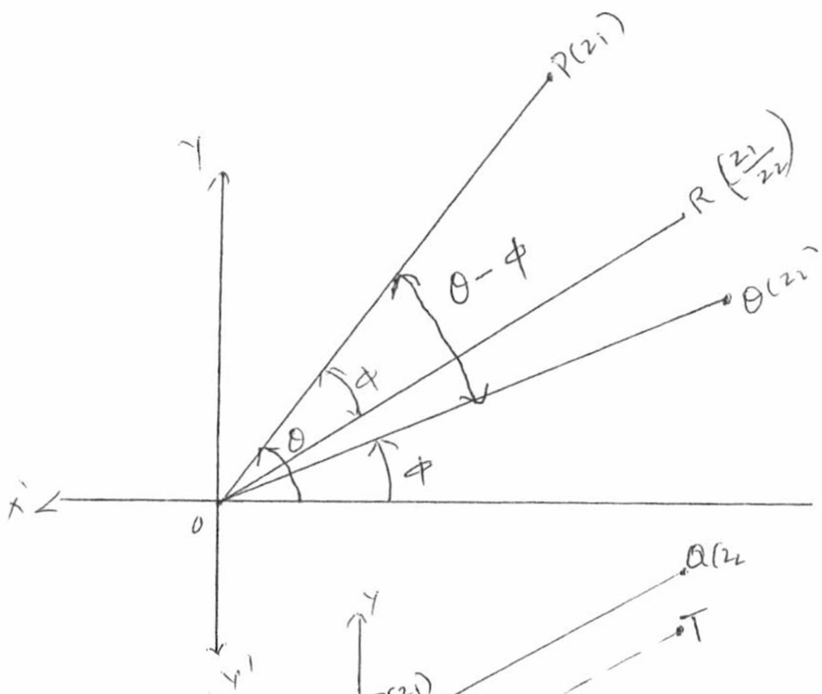


2. For any complex nos $P(z_1)$ and $Q(z_2)$ shifting the origin to P, $Q = (z_2 - z_1)$

$\therefore$ $\arg(z_2 - z_1) = \theta$

so that $\text{Arg}(z_2 - z_1)$ = the angle which the segment PQ makes with OX.

c) If $\text{Arg}(z_2 - z_1) = 0$ or π, the segment PQ is horizontal while if $\text{Arg}(z_2 - z_1) = \pi/2$ or $-\pi/2$, the segment PQ is vertical.

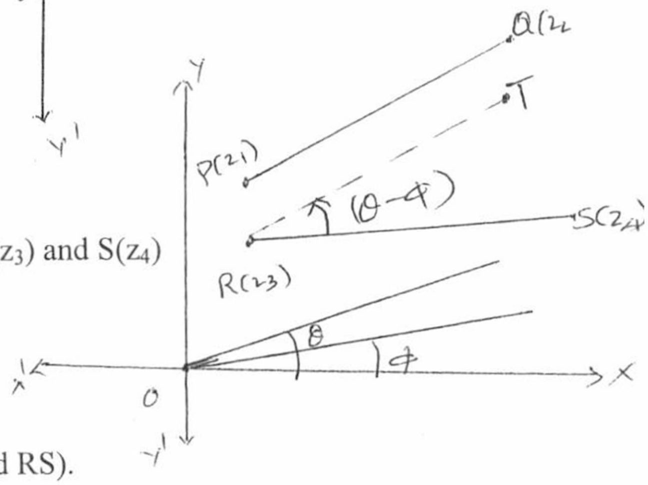


3. $\text{Arg}\left(\frac{z_1}{z_2}\right)$, $(z_2 \neq 0)$

$\text{Arg}\left(\frac{z_1}{z_2}\right) = \text{Arg}(z_1) - \text{Arg}(z_2)$

$= \theta - \phi$

$= P\hat{O}Q$ (OQ to OP).

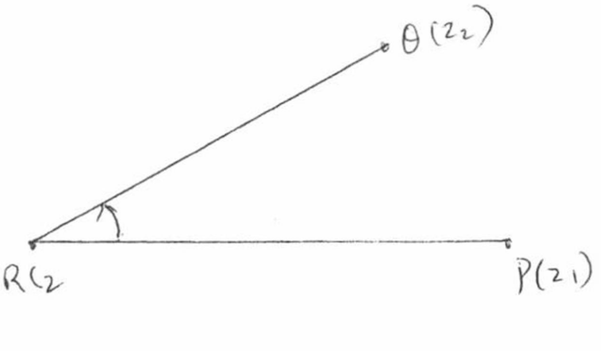


i) $\text{Arg}\left(\frac{z_1 - z_2}{z_3 - z_4}\right)$ Taking $P(z_1)$, $Q(z_2)$, $R(z_3)$ and $S(z_4)$

$\text{Arg}\left(\frac{z_1 - z_2}{z_3 - z_4}\right) = \text{Arg}(z_1 - z_2) - \text{Arg}(z_3 - z_4)$

$= \theta - \phi$ (the angle between the segments PQ and RS).

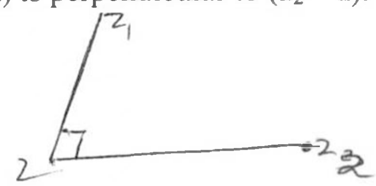
ii) $\text{Arg}\left(\frac{z_1 - z}{z_2 - z}\right)$, taking $P(z_1)$, $Q(z_2)$ and $R(z) = P\hat{O}Q$ (RP to RQ).



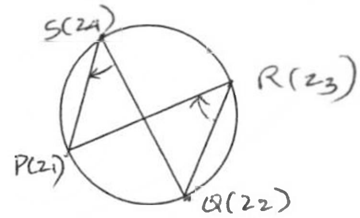
In particular,

i) If $\text{Arg}\left(\dfrac{z_1 - z}{z_2 - z}\right) = 0$ or π, z_1, z_2 and z are collinear.

ii) If $\arg\left(\dfrac{z_1 - z}{z_2 - z}\right) = \pi/2$ or $-\pi/2$, $(z_1 - z)$ is perpendicular to $(z_2 - z)$.



$\text{Arg}\left(\dfrac{z_1 - z_3}{z_2 - z_3}\right) = \text{Arg}\left(\dfrac{z_1 - z_4}{z_2 - z_4}\right) \Rightarrow P\hat{R}Q = P\hat{S}Q \Rightarrow$ P, Q, R , S are the vertices of a cyclic quadrilateral.

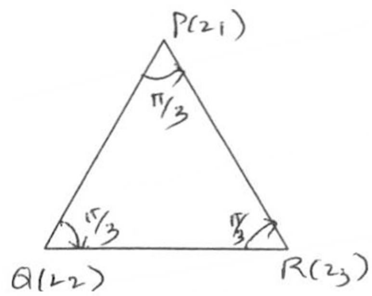


5. Prove that $P(z_1)$, $Q(z_2)$, $R(z_3)$ are the vertices of an equilateral triangle iff

a) $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$

b) $\dfrac{1}{z_1 - z_2} + \dfrac{1}{z_2 - z_3} + \dfrac{1}{z_3 - z_1} = 0$

a) Let PQR be an equivalent triangle.



$$\frac{z_3 - z_1}{z_2 - z_1} = \left|\frac{z_3 - z_1}{z_2 - z_1}\right| e^{i\pi/3}$$

$$\frac{z_1 - z_2}{z_3 - z_2} = \left|\frac{z_1 - z_2}{z_3 - z_2}\right| e^{i\pi/3}$$

But PQ = QR =RS $\Leftrightarrow$ $|z_2 - z_1| = |z_3 - z_1| = |z_3 - z_2|$

$$\therefore \quad \frac{z_3 - z_1}{z_2 - z_1} = \frac{z_1 - z_2}{z_3 - z_2} \Rightarrow (z_3 - z_1)(z_3 - z_2) = (z_2 - z_1)(z_1 - z_2)$$

$$\Rightarrow \quad z_3^2 - z_2z_3 - z_1z_3 + z_1z_2 = -z_1^2 - z_2^2 + 2z_1z_2$$

$$\Rightarrow \quad z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$$

Conversely, let $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$

$$\Rightarrow \frac{z_1 - z_2}{z_3 - z_2} = \left|\frac{z_1 - z_2}{z_3 - z_2}\right| = \frac{z_3 - z_1 + z_1 - z_2}{z_2 - z_1 + z_3 - z_2} = \frac{z_3 - z_2}{z_3 - z_1}$$

$\Rightarrow \hat{P} = \hat{Q} = \hat{R}$

$\therefore \Delta$ PQR is equilateral.

b) Let triangle PQR be equilateral.

Then, $|z_1 - z_2|^2 = |z_2 - z_3|^2 = |z_3 - z_1|^2 = a^2$ (say)

$$\Rightarrow (z_1 - z_2)\,(\overline{z_1 - z_2}) = a^2$$

$$\Rightarrow \frac{\overline{z_1 - z_2}}{a^2} = \frac{1}{z_1 - z_2}$$

$$\Rightarrow \sum \frac{1}{z_1 - z_2} = \frac{1}{a^2} \sum \overline{z_1 - z_2}$$

$$= \frac{1}{a^2} \sum (\bar{z}_1 - \bar{z}_2)$$

$$= \frac{1}{a^2} (\bar{z}_1 - \bar{z}_2 + \bar{z}_2 - \bar{z}_3 + \bar{z}_3 - \bar{z}_1)$$

$$\therefore \sum \frac{1}{z_1 - z_2} = 0$$

Ex: Prove the converse.

6. Problem : If α and β are different complex numbers, with $|\beta| = 1$, then find,

$$\left| \frac{\beta - \alpha}{1 - \bar{\alpha}\,\beta} \right|$$

Solution: $|\beta| = 1 \Rightarrow \beta \bar{B} = 1 \Rightarrow \beta = \frac{1}{\bar{\beta}}$ [$\because |\beta|^2 = \beta\,\bar{B}$.

$$\therefore \quad \left| \frac{\beta - \alpha}{1 - \bar{\alpha}\,\beta} \right| = \left| \frac{\beta - \alpha}{1 - \bar{\alpha}\left(\frac{1}{\bar{\beta}}\right)} \right| = |\bar{\beta}| \left| \frac{\beta - \alpha}{\bar{\beta} - \bar{\alpha}} \right| = 1$$

Since $|\bar{\beta}| = 1$ and $|\beta - \alpha| = |\bar{\beta} - \bar{\alpha}|$

7. **Problem:** w being a complex cube root of unity, find the value■ of

$$\frac{a + bw + cw^2}{b + cw + aw^2} + \frac{a + bw + cw^2}{c + aw + bw^2} + 1$$

Solution : First term = w × $\dfrac{a + bw + cw^2}{bw + cw^2 + aw^3}$ (multiply and divide by w)

$$= w\left(\frac{a + bw + cw^2}{a + bw + cw^2}\right) = w$$

$$\text{Second term} = w^2\left(\frac{a + bw + cw^2}{cw^2 + aw^3 + bw^4}\right) \text{ (multiply and divide by } w^2\text{)}$$

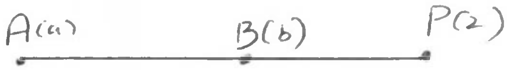
$$= w^2\left(\frac{a + bw + cw^2}{a + bw + cw^2}\right) = w^2$$

∴ G.E. = $w + w^2 + 1 = 0$.

8. **Problem** : If a,b,z are collinear points, then, $\begin{vmatrix} 1 & z & \bar{z} \\ 1 & a & \bar{a} \\ 1 & b & \bar{b} \end{vmatrix} = 0$

Solution :

Let A(a), B(b) and P (z) be collinear.



Then Arg $\left(\dfrac{z - a}{a - b}\right) = 0$ or π.

$\Rightarrow$ $\dfrac{z - a}{a - b}$ is real. (Why?) [∵ z is real iff Arg (z) = 0 or π]

$\Rightarrow$ $\dfrac{z - a}{a - b} = \dfrac{\bar{z} - \bar{a}}{\bar{a} - \bar{b}}$ (how?) [∵ z is real iff $z = \bar{z}$]

$\Rightarrow$ $(z - a)(\bar{a} - \bar{b}) = (a - b)(\bar{z} - \bar{a})$

$\Rightarrow$ $z\bar{a} - z\bar{b} - a\bar{a} + a\bar{b} = \bar{z}a - a\bar{a} - \bar{z}b + \bar{a}b =$

$\Rightarrow$ $(a\bar{b} - \bar{a}b) - z(\bar{b} - \bar{a}) + \bar{z}(b - a) = 0$

$\Rightarrow$ $\begin{vmatrix} 1 & z & \bar{z} \\ 1 & a & \bar{a} \\ 1 & b & \bar{b} \end{vmatrix} = 0$

9. **Problem :** Identify the locus of P(z) in the z-plane satisfying

a) A(a), B(b) being two given points of the z-plane

$|z - a| = 2|z - b|$

b) A(a), B(b) being two given points on the real axis of the z-plane,

$| z - a | + | z - b | = c$, c being a positive number.

Solution :

a) For given A(a) and B(b) and a variable point P(z) in the z-plane $| z - a |$ = Distance of P from A = PA and $| z - b |$ = Distance P from B = PB.

Taking $a = (x_1, y_1)$, $b = (x_2, y_2)$ and $z = (x, y)$

$AP = \sqrt{(x - x_1)^2 + (y - y_1)^2}$; $BP = \sqrt{(x - x_2)^2 + (y - y_2)^2}$

$| z - a | = 2 | z - b |$

$\Rightarrow$ $| z - a |^2 = 4 | z - b |^2$

$\Rightarrow$ $(x - x_1)^2 + (y - y_1)^2 = 4 [(x - x_2)^2 + (y - y_2)^2]$

$\Rightarrow$ reduces to an equation representing a circle.

$\therefore$ The locus of p(z) is a circle.

b) $| z - a | = AP$, $| z - b | = BP$.

$\therefore$ $| z - a | + | z - b | = c$

$\Rightarrow$ AP + BP = C (a constant)

By the known property of an ellipse, P lies on an ellipse with foci at A and B. Hence the locus of P(z) is an ellipse whose foci are A and B.

Trigonometric Equations/ Inverse Trigonometric functions

10. **Solve :** $\sin x + \sin 2x + \sin 3x = 0$ $(0 \le x \le \pi)$

Solution : Firstly, we break up the given equation to simpler equations. For that purpose, we convert the given equation using the **sum to product** formula.

The given equation is $\sin 3x + \sin x + \sin 2x = 0$

$\Rightarrow$ $2 \sin 2x \cos x + \sin 2x = 0$

$\Rightarrow$ $\sin 2x (2 \cos x + 1) = 0$

$\Rightarrow$ $\sin 2x = 0,\ 2 \cos x + 1 = 0$

$\Rightarrow$ $\sin 2x = 0,\ \cos x = - ½$

i) $\sin 2x = 0$

$x = 0$ is a solution

$\therefore 2x = n\pi + (-1)^n \alpha$

$= n\pi + (-1)^n = 0$

$2x = n\pi$

$\therefore n = \frac{n\pi}{2}, n \in Z$

The solution in $0 \le x \le \pi$

is $x = \pi/2$

$\cos x = - ½$

$x = 2\pi/3$ is a solution.

$\therefore x = 2n\pi \pm \alpha$

$x = 2n\pi \pm 2\pi/3$

$\therefore$ The solution: $0 \le n \le \pi$

i.e. $x = \frac{2\pi}{3}$

Then the solution in $[0, \pi]$

are $x = \pi/2$ and $2\pi/3$

11. Solve : $2 \text{Tan}^{-1} (\cos x) = \text{Tan}^{-1} (2 \text{cosec } x)$

Solution : If the equation can be rewritten in the form $\text{Tan}^{-1} (X) = \text{Tan}^{-1} (Y)$, the equation can be simplified.

But we know $2 \text{Tan}^{-1} (x) = \tan^{-1} \left(\frac{2x}{1-x^2} \right)$

$\therefore$ The given equation is $\text{Tan}^{-1} \left(\frac{2\cos x}{1-\cos^2 x} \right) = \text{Tan}^{1} (2 \text{cosec } x)$

$\Rightarrow$ $\frac{2\cos x}{1-\cos^2 x} = 2 \text{cosec } x$

$\Rightarrow$ $\frac{\cos x}{\sin^2 x} = \frac{1}{\sin x}$

$\Rightarrow$ $\cos x = \sin x \Rightarrow x = n\pi + \pi/4, n \in z$

12. A value of θ satisfying : $\tan(\pi \cos\theta) = \cot(\pi \sin\theta)$ is

(A) $\frac{1}{2}\sin^{-1}\left(\frac{3}{4}\right)$ (B) $-\frac{1}{2}\sin^{-1}\left(\frac{3}{4}\right)$ (C) $\frac{\pi}{4} - \sin^{-1}\left(\frac{1}{2\sqrt{2}}\right)$

(D) None of these

Solution : (D).

$$\tan(\pi \cos\theta) = \cot(\pi \sin\theta)$$

$\Rightarrow$ $\tan(\pi \sin\theta)\ \tan(\pi\cos\theta) = 1$

$\Rightarrow$ $\tan(\pi(\sin\theta + \cos\theta)) = \infty$

$\Rightarrow$ $\pi(\sin\theta + \cos\theta) = \pi/2$

$\therefore$ $\sin\theta + \cos\theta = ½$

$\therefore$ $\frac{1}{\sqrt{2}}\sin\theta + \frac{1}{\sqrt{2}}\sin\theta = \frac{1}{2\sqrt{2}}$

$\Rightarrow$ $\sin\left(\theta + \frac{\pi}{4}\right) = \frac{1}{2\sqrt{2}}$

$\Rightarrow$ $\theta = -\frac{\pi}{4} + \sin^{-1}\left(\frac{1}{2\sqrt{2}}\right)$

Differential Calculus

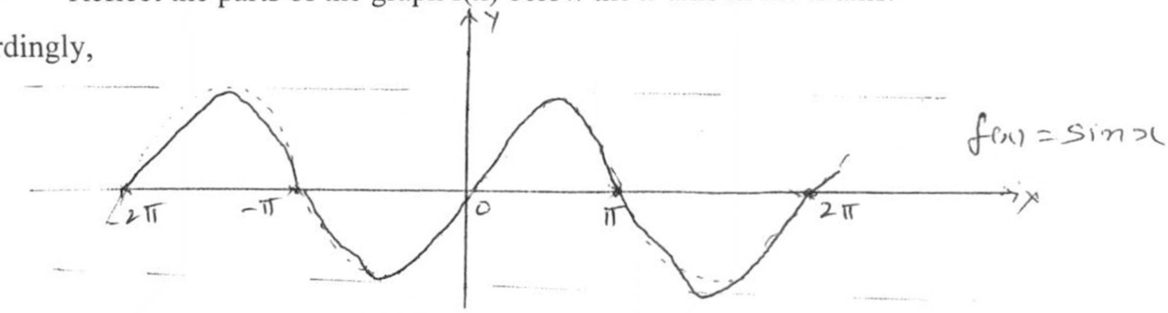
13. Draw the graph of f(x) = | sin x |

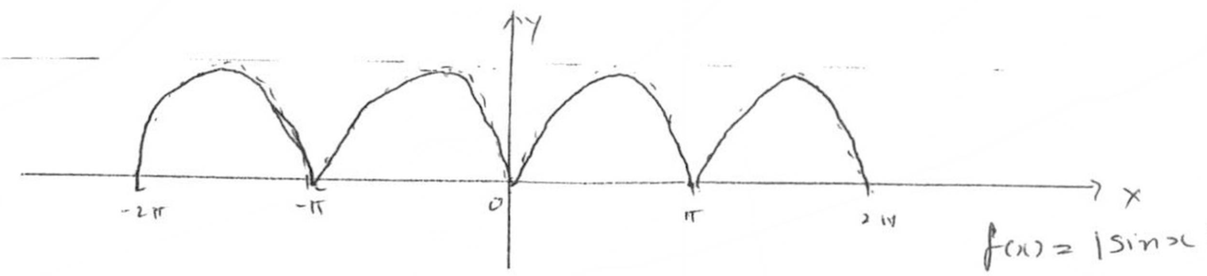
Solution : An important tip to draw the graph of | f(x) | knowing the graph of f(x).

i) Draw the graph of f(x).

ii) Reflect the parts of the graph f(x) below the x-axis in the x-axis.

Accordingly,



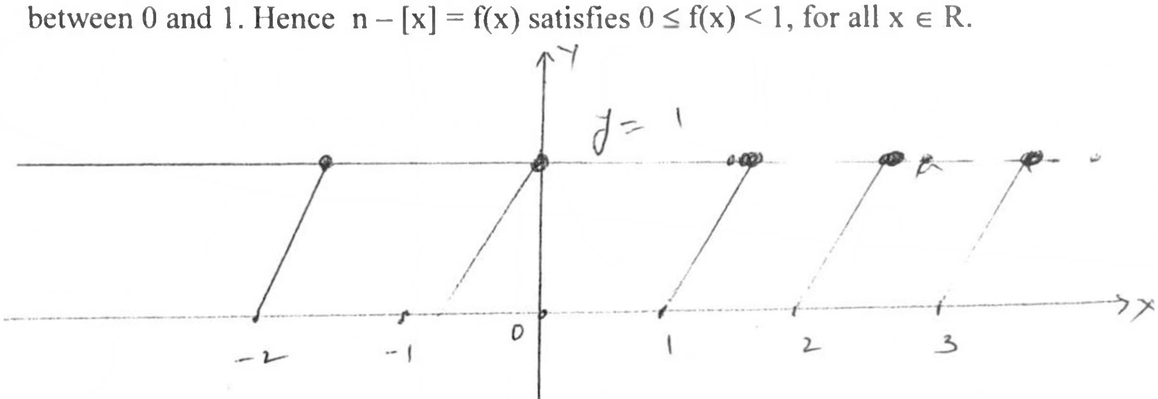


14. Draw the graph of f(x) = x – [x] where [x] is the greatest integer not exceeding x.

Solution :

x	[x]	f(x) = x – [x]*
0 ≤ 1 < 1	0	0 ≤ f(x0 < 1
1 ≤ x < 2	1	0 ≤ f(x) < 1
2 ≤ x < 3	2	0 ≤ f(x) < 1 etc.

*When the greatest integer in x is subtracted, we get x – [x]. this is a fraction between 0 and 1. Hence n – [x] = f(x) satisfies 0 ≤ f(x) < 1, for all x ∈ R.



Graph of f(x) = x – [x]. Observe that the points circled on y = 1 are not a part of the graph.

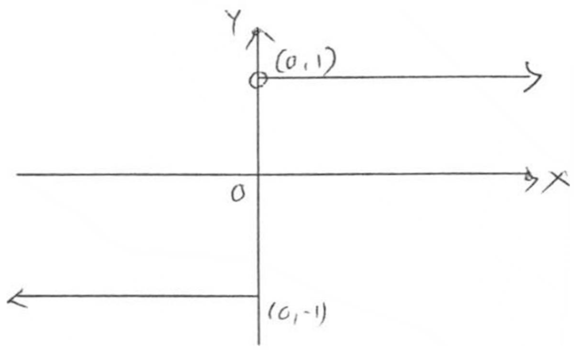
15. **Problem** : Find the discontinuities of $f(x) = \frac{|x|}{x}, x \neq 0$, f(0) =0.

$$\text{Recalling, } |x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

$$\therefore \ f(x) = \frac{|x|}{x} \begin{cases} \frac{x}{n} = 1, & \text{if } n > 0 \\ \frac{-x}{n} = -1, & \text{if } x < 0 \end{cases}$$

and f(0) = 0.

Accordingly, the graph of f(x) is



f(x) is also known as 'signum function'. From the graph, x = 0, is a discontinuity of f(x).

16. At what angle, the curve $y = \frac{x-7}{(x-2)(x-3)}$ crosses the x-axis ?

Solution : If the curve crosses the x-axis at P at an angle θ, then Tan θ = slope of the tangent P = $\left(\frac{dy}{dx}\right)_P$.

Consider $Y = \frac{(x-7)}{(x-2)(x-3)}$

Putting y = 0, x = 7. ∴ The curve crosses the x-axis at P(7,0).

$$y = \frac{x-7}{(x-2)(x-3)}$$

$$y' = \frac{(x-2)(x-3) - (x-7)(2x-5)}{[(x-2)(x-3)]^2}$$

$$= \frac{(x-2)(x-3) - y(x-2)(x-3)(2x-5)}{[(x-2)(x-3)]^2}$$

$$= \frac{1 - y(2x-5)}{(x-2)(x-3)}$$

$$\therefore y'_p = \frac{1-0}{5 \times 4} = \frac{1}{20} = Tan\,\theta$$

$$\therefore \theta = \text{Tan}^{-1}\left(\frac{1}{20}\right)$$

18. The perimeter of the biggest rectangle fixed in a semi circle of radian a so that a side is a part of the bonding diameter is (A) $3\sqrt{2}\,a$ (B) $4\sqrt{2}\,a$ (C) $5\sqrt{2}\,a$ (D) None of these.

Solution :

Taking P as a vertex of the rectangle as in the figure.

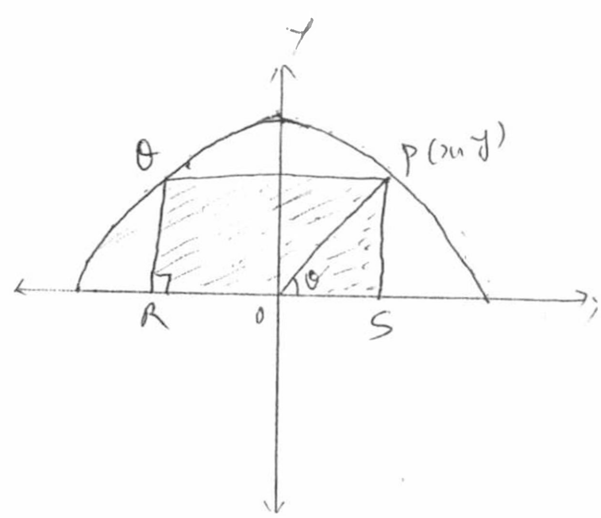
If OP = a makes θ as shown, then P (x,y) = (a cos θ, a sin θ)

$\therefore$ Area of the rectangle = S = 2xy = 2a cos θ a sin θ = a^2 sin 2 θ

$$\therefore \frac{ds}{d\theta} = 2a^2 \cos 2\theta = 0 \Rightarrow \theta = \frac{\pi}{4}$$

$$\therefore \frac{d^2 s}{d\theta^2} = -4a^2 \sin 2\theta = -4a^2 \text{ at } \theta = \frac{\pi}{4}$$

$$\therefore \text{When S is max, } \theta = \frac{\pi}{4}$$



Perimeter = 4x + 2y = 4 a cos θ + 2a sin θ

$$= 4a \cos\frac{\pi}{4} + 2a \sin\frac{\pi}{4}$$

$$= 6a \cdot \frac{1}{\sqrt{2}} = 3\sqrt{2}\,a$$

$\therefore$ Perimeter of the bigger rectangle = $3\sqrt{2}$ a

Solution is (A).

Using differentials find the approximate value of $\sqrt[3]{730}$.

Consider $Y = f(x) = \sqrt[3]{x}$.

If Δx is the error due to approximation in x and Δy is the error due to approximation in y,

Then $\Delta y = f(x + \Delta x) - f(x)$

$= \Delta x f'(x)$ Taking $x + \Delta x = 730$

and $x = 729 = 9^3$

$\Delta x = 1$

$\therefore \quad f(x + \Delta x) = f(x) + \Delta x f'(x)$ $\quad f'(x) = \frac{1}{3} x^{-2/3}$

$= 9 + \frac{1}{243}$ $\quad = \frac{1}{3}(9^{-2}) = \frac{1}{243}$

$\therefore \sqrt[3]{730} = 9 + \frac{1}{243}$ (approximately)

$= 9.004115$ (approximately)

19. An insect crawls along the curve $6y = x^3 + 2$. Find the points on the curve where the y-coordinate of the insect is changing 8 times as fast as the x-coordinate.

$6y = x^3 + 2$

Diff. w.r.t. time (t)

$6\frac{dy}{dt} = 3x^2 \frac{dx}{dt}$. $\quad$ But $\frac{dy}{dt} = 8\frac{dx}{dt}$

$\therefore \; 48\frac{dx}{dt} = 3x^2 \frac{dx}{dt} \Rightarrow x^2 = 16$

$\therefore \; x = \pm 4$

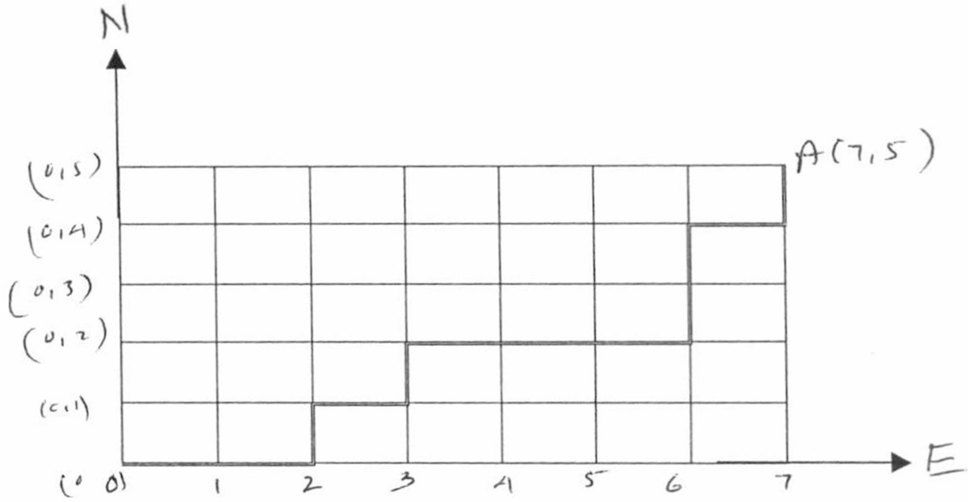
$\therefore \; 6y = \pm 64 + 2$

$\therefore \; 6y = 66 \Rightarrow$ $\quad y = 11$ or $6y = -62 \Rightarrow y = -\frac{31}{3}$

$\therefore$ The points on the curve are (4,11) and $(-4, -\frac{31}{3})$

20. In the picture aside, there is a grid formed by horizontal and vertical lines (due east and due north). If one wants to reach A (7,5) from O(0,0) traveling directions due east and north (or north and east) only, how many paths are there ?

Hint : A path indicated by red line segments.



Denoting a unit step due east by E and a unit step due north by N, the path shown in the diagram is – E E N E N E E E N N E N.

Solution : For a path from O to A, we need 7 unit steps due east and 5 unit steps due north.

Any permutation of these, gives a path from O toA.

Total No. of steps = 7 unit steps due east + 5 unit steps due north

= 12 unit steps.

Among these 12 steps, 7 steps are of one kind – due east and 5 steps are of another kind – due north.

$\therefore$ Total No. of permutations of the steps = $\dfrac{(7+5)!}{7!\,5!}$

$= \dfrac{12!}{7!\,5!} = 792$

Maxima and Minima of a function

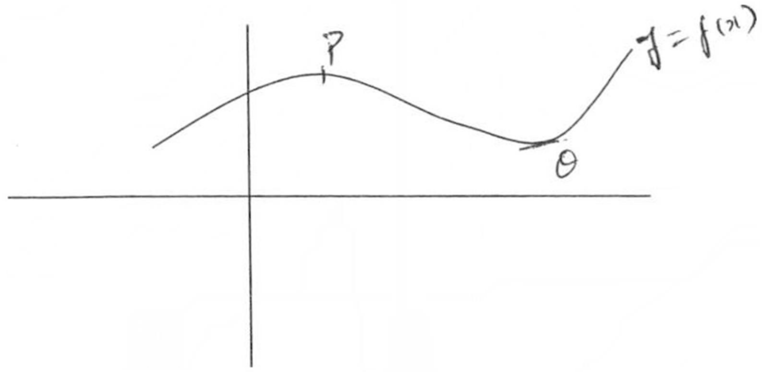
Problem : Majority of the teachers are not clear about the necessity of second derivative test in finding the maxima and minima of a function.

Illustration : Consider the problem of finding out the maxima and minima of a function $f(x) = x^2$.

It is obvious that this function takes all non-negative values and hence there is no specific maximum. But the minimum value of this function is 0. Hence the minimum of this function $f(x) = x^2$ is at $x = 0$.

The above method is convenient in such simple functions to find maxima or minima. Now what is the general method to find a maxima or minima of any given function ?

Consider the following graph of a certain given function.



Examine the values of the function around the point P. In the left side neighbourhood of P, the values of the function f(x) are increasing when x approaches P. That is the derivative of the function in this neighbourhood is positive. Similarly in the right side neighbourhood of P, the derivative of the function is negative as the function takes smaller values when x is away from P. That means in the neighbourhood of P, $f'(x) > 0$ on the left and $f'(x) < 0$ on the right and therefore, $f'(x)$ changes its sign from +ve to –ve by taking $f'(x)$ zero at the point P.

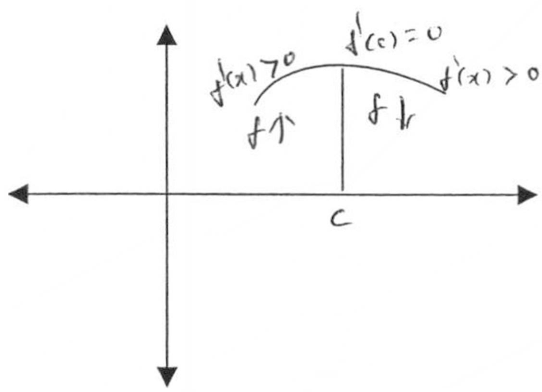
Similar observation can be made at the point Q that $f'(x)$ is zero.

Now one can put a condition that for a function to take a minimum or maximum, it is necessary to have $f'(x) = 0$ at that point.

Now the question is to verify whether $f(x)$ takes maximum or minimum when $f'(x) = 0$. Examine the changes in the values of $f'(x)$ at the neighbourhood of P. $f'(x)$ is positive on the left of P and negative on the right. That is the derivative of $f'(x)$ say $f'(x)$ is negative in the neighbourhod of P so that $f(x)$ takes maximum value at P. When we consider $f''(x)$ in the neighbourhood Q its value is positive where $f(x)$ takes minimum value. This leads to additional necessary condition $f''(x) < 0$ for maximum and $f''(x) > 0$ for minimum. From this we conclude that for a function $f(x)$ to have maximum of the value $x = a$ the necessary and sufficient conditions are $f'(x) = 0$ and $f''(x) < 0$ at $x = a$. Similarly for a function $f(x)$ to have minimum at $x = a$ the two necessary and sufficient conditions are $f'(x) = 0$ and $f''(x) > 0$ at $x = a$.

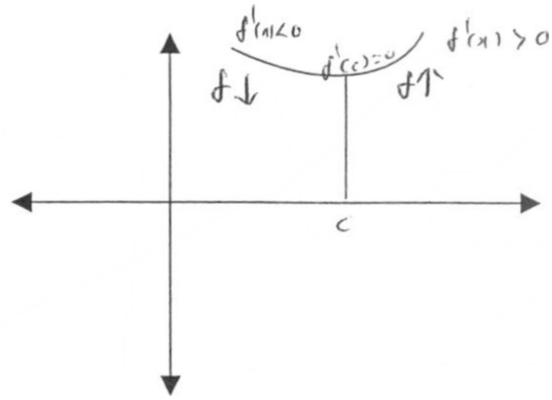
If in the case of a function $f(x)$, $f'(x) = 0$ and $f''(x) = 0$ at $x = a$ then what can we say about the maximum and minimum of this function at $x = a$?

It is obvious that this function is having neither maximum nor minimum at the point $x = a$ when $f''(x) = 0$ at this point. This is because the function $f(x)$ is neither increasing nor decreasing at the neighbourhood of the point with $x = a$. Take an example, $f(x) = x^3$. In this case, $f'(x) = 0$ and $f''(x) = 0$ at $x = 0$. Therefore, this function is having neither maximum nor minimum at $x = 0$. This can be verified by drawing the graph of this function $f(x) = x^3$. Such points where $f'(x) = 0$ and $f''(x) = 0$ are known as points of inflexion.



f(x) has a local max at x = c

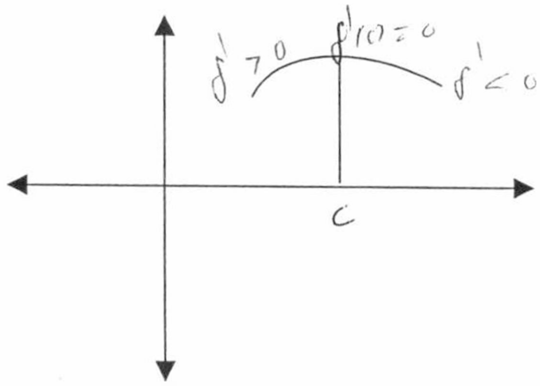
(1)



f(x) has a local min at x = c

(2)

Further,



Then f' takes +ve, 0 and –ve values.

As x changes from left of c to the right of c.

$\therefore$ f' is decreasing so that in the small neighbourhood of c, f'' is –ve.

$\therefore$ $f''(c) < 0$.

$\therefore$ $f'(c) = 0$, $f''(c) < 0$ for f to have a local max at $x = c$.

Exercise : Examine the maxima, minima or point of inflexion of a function $f(x) = (x-1)^3$.

Suggestion : The graphs and variations of different functions can be shown using 'The Geometer's sketchpad' software. Using this software a constructivism approach may be followed to explain the concept of extreme values of any given function.

Integral Calculus

If a function f is differentiable in an interval I, then the natural question arises that, given its derivative f' at each point of I, can we determine the function? Such functions are called **antiderivatives**, further the formula that gives all these antiderivatives, is called the '**indefinite integral**' of the function. The process of finding antiderivatives is called '**integration**'.

The development of integral calculus arises out of the efforts of solving the problems of the following types :

a) The problem of finding a function whenever its derivative is given.

b) The problem of finding the area bounded by the graph of a function under certain conditions.

These two problems lead to the two forms of the integrals i.e. indefinite and definite integrals, which together constitute the 'integral calculus'.

In this unit, we illustrate some techniques of solving the indefinite integral problems.

Techniques of Indefinite Integration

A. Change of variable (or substitution) method :

Illustration

1. $$I = \int \frac{2\,dx}{\sqrt{1-5x^2}}$$

Solution : Put $t = \sqrt{5}\,x$ then $dt = \sqrt{5}\,dx$ i.e. $dx = \frac{dt}{\sqrt{5}}$.

So we have

$$I = \int \frac{2\,dt}{\sqrt{5}\,(\sqrt{1-t^2}} = \frac{2}{\sqrt{5}} \qquad \int \frac{dt}{\sqrt{1-t^2}} = \frac{2}{\sqrt{5}}\,\sin^{-1}t + C$$

$$I = \int \frac{2}{\sqrt{3}}\,\sin^{-1}(\sqrt{5}\,x) + c$$

Note : Here the standard form used is $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + c$

2. $$I = \frac{x^3 \sin(\tan^{-1} x^4)}{1+x^8}\,dx$$

Solution : Put $\tan^{-1}(x^4) = t$ then $dt = \frac{4x^3}{1+x^8} \cdot dx$

$$\therefore \frac{dt}{4} = \frac{x^3\, dx}{1+x^8}$$

$$\therefore I = \int \sin t \frac{dt}{4} = \frac{1}{4} \int \sin dt = \frac{1}{4}(-\cos t) + c$$

$$= \frac{1}{4} - \cos(\tan^{-1} x^4) + c$$

Integrals of Trigonometric Functions

Illustration : Most of the integrals containing trigonometric functions are reducible to those containing sines and cosines only. So we reduce generally the trigonometric functions in terms of sine and cosine. Some times expressing sines and cosines in terms of tangent and cotangent also prove useful. Illustrative examples are given below:

$$I = \int \frac{1}{1+\tan x} dx$$

Here $\frac{1}{1+\tan x} = \frac{1}{1+\frac{\sin x}{\cos x}} = \frac{\cos x}{\cos x + \sin x} = \frac{2\cos x}{2(\cos x + \sin x)}$

$$= \frac{(\cos x + \sin x) + (\cos x + \sin x)}{2(\cos x + \sin x)}$$

$$= \frac{1}{2} + \frac{\cos x - \sin x}{2(\cos x + \sin x)}$$

Put $t = \cos x + \sin x$, then $dt = (\cos x - \sin x)\, dx$

So the integral

$$I = \int \frac{dx}{2} + \int \frac{1}{2}\frac{dt}{t} = \frac{x}{2} + \frac{1}{2}\log|t| + c$$

$$= \frac{x}{2} + \frac{1}{2} Log|Cos\, x + \sin x| + c$$

$$I = \int \sin^{-1}(\cos x) dx$$

Put $\cos x = t$ $\qquad$ $\sin x\, dx = dt$

$$dx = \frac{dt}{\sin x} = \frac{dt}{\sqrt{1-\cos^2 x}} = \frac{dt}{\sqrt{1-t^2}}$$

So, I = $\int \sin^{-1} t \dfrac{dt}{\sqrt{1-t^2}}$

Put $\sin^{-1}$ t = u, $\dfrac{1}{\sqrt{1-t^2}} dt$ = du

$$\therefore \; I = \int u \, du = \frac{4^2}{2} + c$$

$$= \frac{(\sin^{-1} t)^2}{2} + c'$$

$$= \frac{(\sin^{-1} c \cos x)\,)^2}{2} + c$$

3. $$\int \cos^6 x \, dx = \int \left(\frac{1+\cos 2x}{2}\right)^3 dx$$

$$= \frac{1}{8} \int (1 + 3\cos 2x + 3\cos^2 2x + \cos^3 2x \,) dx$$

$$= \frac{1}{8} \int (1 + 3\cos 2x + \frac{3\,(1+\cos 4x)}{2} + (1-\sin^2 2x)\,(\cos 2x)\, dy$$

$$= \frac{1}{8}\left[x + \frac{3\sin 2x}{2} + \frac{3}{2}\left(x + \frac{\sin 4x}{4}\right) + \frac{\sin 2x}{2} - \frac{\sin^3 2x}{3\,.2}\right] + c$$

$$= \frac{1}{8}\left(\frac{5}{2} x + 2\sin 2x + \frac{3}{8}\sin 4x - \frac{1}{6}\sin^3 2x \right) + \mathrm{c}$$

Integration of Rational Algebraic Functions

Illustration :

1. $$I = \int \frac{x+4}{x^3+3x^2-10x} dx$$

Resolving the integral into partial functions we have

$$\frac{x+4}{x^3+3x^2-10\,x} = \frac{x+4}{x(x-2)\,(x+5)} = \frac{-2}{5x} + \frac{3}{7(x-2)} - \frac{1}{35\,(x+5)}$$

So, using proper substitution we get

$$I = \frac{-2}{5} Log|x| + \frac{3}{7} Log\,|x-2| - \frac{1}{35} Log\,|\,x+5| + c$$

2. $$I = \int \frac{Cos\ x}{(1-\sin x)\,(2-\sin x)} dx$$

Put sin x = t cos x dy = dt

$$I = \int \frac{dt}{(1-t)(2-t)} = Log \left| \frac{2-\sin x}{1-\sin x} \right| + c$$

Integration by Parts

When we have product of functions to be integrate, this technique is useful.

Rule : $\int f\, g\, dx = f(x) \int g(x)\, dx - \left(\int g(x)\, dx\, f'(x) \right) dx$

Illustrations :

1. $$\int \frac{x \cos^{-1} x}{\sqrt{1-x^2}}\, dx = (\cos^{-1} x)\left(-\sqrt{1-x^2}\right) - \int \frac{-1}{\sqrt{1-x^2}} \left(-\sqrt{1-x^2}\right) dx$$

$$= \left(-\sqrt{1-x^2}\right) \cos^{-1} x - x + c$$

2. $$\int (\sin^{-1} x)^2\, dx = (\sin^{-1} x)^2\, (x) - \int x - \frac{2 \sin^{-1} x}{\sqrt{1-x^2}}\, dx$$

$$= \left[-\, x(\sin^{-1} x) - 2\left[-\sqrt{1-x^2} \sin^{-1} x + x + c \right] \right.$$

$$= x\,(\sin^{-1} x)^2 + 2\sqrt{1-x^2}\, \sin^{-1} x - 2x + c$$

II. Definite Integral

A definite integral is denoted by $\int_a^b f(x)\,dx$, where a is called the lower limit of the integral and b is called the upper limit of the integral. The definite integral is introduced either as the limit of a sum or if it has an antiderivative F in the interval [a,b], then its value is the difference between the values of F at the end points i.e. F(b) – F(a).

We shall illustrate these two cases separately by considering some examples.

A. Definite integral as the limit of a sum

Note : The definition of the definite integral can be used with profit to evaluate easily the limits of the sums of certain series, when the number of terms in the series tend to infinity. The method lies in identifying a definite integral equal to series. In fact,

$$\int_a^b f(x)\,dx = \lim_{h \to 0} h \sum f\,(a + nh) \quad \text{where } nh = b - a$$

$$\text{or } \lim_{n \to \infty} \left(\frac{b-a}{n}\right) \sum \left(f\left(a + \frac{r\,(b-a)}{n} \right)\right) = \int_a^b f(x)\,dx$$

If a = 0, and b = 1, we have

$$\lim_{n \to \infty} \frac{1}{n} \sum f\left(\frac{r}{n}\right) = \int_0^1 f(x)\,dx$$

Illustrations :

1. Show that $\lim_{n \to \infty} \left(\frac{1}{n+m} + \frac{1}{n+2m} + \dots + \frac{1}{n+nm} \right) = \frac{1}{m} \log\,(1+m)$

Solution: Look at

$$\lim_{n \to \infty} \left(\frac{1}{n+m} + \frac{1}{n+2m} + \dots + \frac{1}{n+nm} \right) = \lim_{n \to \infty} \frac{1}{n} \sum_{r=1}^{n} \frac{1}{1 + m/n}$$

$$= \int_0^1 \frac{1}{1+mx}\,dx$$

Put 1 + mx = t , then $dx = \frac{dt}{m}$.

When x = 0, t = 1 and when x = 1, t = 1 + m.

$$\therefore \quad I = \frac{1}{m} \int_1^{1+m} \frac{1}{t}\,dt$$

$$= \frac{1}{m} Log\ t \Big]_1^{1+m}$$

$$= \frac{1}{m} [\ Log\ (1 + m) - Log\, 1\ ]$$

$$= \frac{1}{m} [\ Log\ (1 + m) - 0\ ]$$

$$I = \frac{1}{m} Log\ (1 + m)$$

$$\lim_{n \to \infty} \left[\left(1 + \frac{1}{n}\right)\left(1 + \frac{2}{n}\right) \ldots\ldots \left(1 + \frac{n}{n}\right)\right]^{1/n} = \frac{4}{e}.$$

Let us consider

$$\lim_{n \to \infty} \left[\left(1 + \frac{1}{n}\right)\left(1 + \frac{2}{n}\right) \ldots\ldots \left(1 + \frac{n}{n}\right)\right]^{1/n} = \lim_{n \to \infty} x^{1/n}$$

By taking log we get,

$$\text{Log} \lim_{n \to \infty} x^{1/n} = \lim_{n \to \infty} Log\ x^{1/n}$$

$$= \lim_{n \to \infty} \frac{1}{n} Log\ x$$

$$= \lim_{n \to \infty} \left[\frac{1}{n} \sum_{r=1}^{n} Log \left(1 + \frac{r}{n}\right)\right]$$

$$= \int_0^1 Log\ (1+x)\ dx$$

By applying integration by parts, we have

$$I = (1 + x) \text{ Log } (1 + x)\ \ (1 + x)\ \Big]_0^1$$

$$= (2 \text{ Log } 2 - 2\ ) - (\text{Log } 1 - 1\ )$$

$$= \text{Log } 4 - 2 + 1$$

$$I = \text{Log } 4 - 1$$

$$\therefore \quad \text{Log} \lim_{n \to \infty} x^{1/n} = Log\ 4 - 1$$

$$\therefore \quad \lim_{n \to \infty} x^{1/n} = e^{(Log\, 4 - 1)} = \frac{4}{e}$$

$$\therefore \quad \lim_{n \to \infty} \left[\left(1 + \frac{1}{n}\right)\left(1 + \frac{2}{n}\right) \ldots\ldots \left(1 + \frac{n}{n}\right)\right]^{1/n} = \frac{4}{e}$$

Similarly, try this:

$$\lim_{n \to \infty} \left[\left(1 + \frac{1}{n^2}\right)^{2/n^2} \left(1 + \frac{2^2}{n^2}\right)^{4/n^2} \cdots\cdots \left(1 + \frac{n^2}{n^2}\right)^{\frac{2n}{n^2}} \right] = \frac{4}{e}$$

Exercise :

1. Show that

$$\lim_{n \to \infty} \left[\frac{\sqrt{n+1}\ \ \sqrt{n+2}\ + \ldots\ldots + \sqrt{2n}}{n\sqrt{n}} \right] = \frac{4}{3}\sqrt{2} - \frac{2}{3}.$$

Illustrates by using properties of definite integrals :

The following examples will illustrate the use of the properties of the definite integrals in solving problems.

1. Show that $\displaystyle\int_0^{\pi/2} Log \sin x\, dx = \frac{\pi}{2} Log\left(\frac{1}{2}\right)$

Solution: Let us consider :

$$\int_0^{\pi/2} Log\ \sin x\, dx \qquad \ldots \qquad (1)$$

By using the property $\displaystyle\int_0^a f(x)\,dx = \int_0^a f\ (a-x)\ dx$

We have,

$$I = \int_0^{\pi/2} Log \sin(\pi/2 - x)\ dx$$

$$I = \int_0^{\pi/2} Log \cos\ x\, dx \qquad \ldots \qquad (2)$$

$\therefore$ By adding (1) and (2) on both sides, we get

$$2I = \int_0^{\pi/2} (\,Log \sin x + Log \cos x\,)\ dx$$

$$= \int_0^{\pi/2} (\mathrm{Log}\ (\sin x \,.\, \cos x)\ dx$$

$$= \int_0^{\pi/2} Log\left(\frac{\sin\ 2x}{2}\right) dx$$

$$= \int_0^{\pi/2} Log \sin 2x \; dx - \int_0^{\pi/2} Log\, 2\, dx$$

Note that

$$\int_0^{\pi/2} Log \sin 2x \; dx = \int_0^{\pi/2} Log \; \sin x \; dx \text{ , we have}$$

$$2I = I - \text{Log } 2 \int_0^{\pi/2} dx$$

$$\therefore \quad I = (\text{ - Log } 2) \; (x \;]_0^{\pi/2}$$

$$= \frac{\pi}{2} (- Log \; 2)$$

$$I = \frac{\pi}{2} Log \left(\frac{1}{2} \right)$$

2. $$\int_0^1 \frac{Log \; (1+x)}{1+x^2} \; dx$$

Let $I = \int_0^1 \frac{Log \; (1+x)}{1+x^2} \; dx$

Put $x = \tan \theta$.

$dx = \sec^2 \theta \, d\, \theta$

If $x = 0$, then $\theta = 0$ and if $x = 1$ $\tan \theta = \frac{\pi}{4}$.

$$\therefore \quad I = \int_0^{\pi/4} \frac{Log \; (1+\tan \theta) \sec^2 \theta \; d\theta}{1+\tan^2 \theta} = \int_0^{\pi/4} Log \, (1+\tan \theta) \, d\theta$$

By using the property $\int_0^a f(x) \; dx = \int_0^a f(a-x) \; dx$, we have

$$I = \int_0^{\pi/4} Log \left(1 + \tan \left(\frac{\pi}{4} - \theta \right) \right) d\theta$$

$$= \int_0^{\pi/4} Log \left(1 + \frac{\tan \frac{\pi}{4} - \tan \theta}{1 + \tan \frac{\pi}{4} \,.\, \tan \theta} \right) d\,\theta$$

$$= \int_0^{\pi/4} Log \left(1 + \frac{\tan \theta \; + 1 - \tan \theta}{1 + \tan \theta} \right) d\theta$$

$$= \int_0^{\pi/4} Log\left(\frac{2}{1+\tan\theta}\right) d\theta$$

$$= \int_0^{\pi/4} Log\, 2\, d\,\theta - \int_0^{\pi/4} Log\ (1 + \tan\theta)\, d\,\theta$$

$$I = \int_0^{\pi/4} Log\ 2d\,\theta\ - I$$

$$\therefore \qquad 2I = \int_0^{\pi/4} Log\, 2\, d\,\theta = \text{Log } 2\ [\theta]_0^{\pi/4} = \frac{\pi}{4}\ Log\ 2$$

$$\therefore \qquad I = \frac{\pi}{8}.Log\, 2$$

$$\therefore \qquad \int_0^1 \frac{Log\ (1+x)}{1+x^2}\ dx = \frac{\pi}{8}\ Log\ 2$$

Exercises :

1. Show that $\displaystyle\int_0^{\pi} \frac{x\ d\,x}{a^2 \cos^2 x + b^2 \sin^2 x} = \frac{\pi^2}{2ab}$

2. $\displaystyle\int_0^{\pi/4} \frac{\sin^{3/2}\theta\ d\theta}{\sin^{3/2}\theta + \cos^{3/2}\theta} = \frac{\pi}{4}$

3. $\displaystyle\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}} = \frac{\pi}{12}$

B. As area function

Having known the different techniques of indefinite integration, we are in a better position to use the fundamental theorem of integral calculus in solving problem on area by integration.

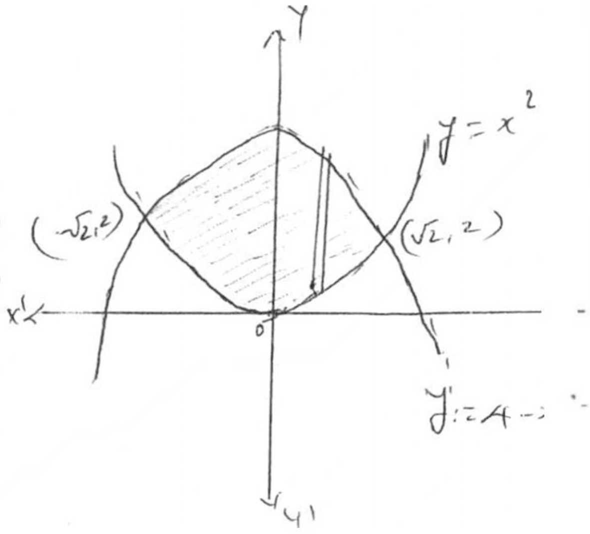
Here we will illustrate the techniques of finding the area of a region by integration.

1. Find the area of the region bounded by the two parabolas y = x^2 and dy = 4 – x^2.

The given curves are

$y = x^2$ .... (1)

$y = 4 - x^2$ .... (2)



which are as given in the figure.

Now for the points of intersection of (1) and (2),

$x^2 = 4 - x^2$

i.e. $2x^2 = 4$

i.e. $x = \pm\sqrt{2}$

and the corresponding values of y are 2 and 2 respectively. So the points are intersection of (1) and (2) are ($-\sqrt{2}$, 2) and ($\sqrt{2}$, 2). As is clear from the figure, for a given x in the region, the value of y ranges from x^2 which is the lower limit, to $4 - x^2$ which is the upper limit. So the length of the vertical strip in the figure is

(upper limit of y) = (lower limit of y) = $(4 - x^2) - x^2 = 4 - 2x^2$.

Therefore, the area of the region is

$$A = \int_{-\sqrt{2}}^{\sqrt{2}} 4 - 2x^2\, dx$$

$$= 2\int_{0}^{\sqrt{2}} (4 - 2x^2)\, dx \text{ as the region is symmetric about y-axis.}$$

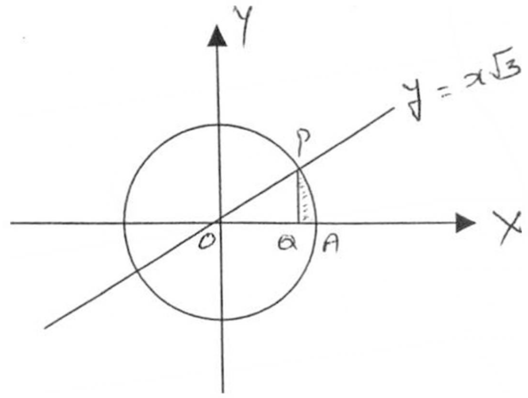
$$= 2\left[4x - \frac{2}{3}x^3\right]_0^{\sqrt{2}} = 2\left[4\sqrt{2} - \frac{2}{3}(\sqrt{2})^3\right] - 0$$

$$= 2\left[4\sqrt{2} - \frac{2}{3}(\sqrt{2})\right]$$

$$= 8\sqrt{2} - \frac{8}{3}\sqrt{2} = \frac{16}{3}\sqrt{2}$$

2. Show that the area in the first quadrant, enclosed by the x-axis, the line $x = y\sqrt{3}$ and the circle $x^2 + y^2 = 4$ is $\pi/3$

Solution : The equation of the circle is $x^2 + y^2 = 4 = 2^2$ (1)



So the center is O(0,0) and radius = 2. The point of intersection of (1) with x-axis curve (2,0) and (-2, 0). So the point A as shown in the figure is in the first quadrant and so is (2,0).

Now the given line $x = y\sqrt{3}$. That is $y = \frac{x}{\sqrt{3}}$ ... (2)

(1) and (2) gives

$3x^2 + x^2 = 12$, i.e. $4x^2 = 12$ i.e. $x = \pm\sqrt{3}$

As P is in the first quadrant, P is ($\sqrt{3}$, 1) and Q is ($\sqrt{3}$, 0). Here P is the point of intersection of 1 and 2 in the first quadrant and PQ $\perp$ x-axis. So OQ = 3, PQ = 1. Let Area of ΔOQP = A_1 and shaded area = A_2. Therefore, the required area

$A = A_1 + A_2$

A_1 = area of ΔOQP = ½ . OQ . PQ = ½ $\sqrt{3}$.1 = $\frac{\sqrt{3}}{2}$ sq. units. ... (3)

$A_2 = \int_{\sqrt{3}}^{2} y\,dx$ where $x^2 + y^2 = 4$

Now $x^2 + y^2 = 4 \Rightarrow y^2 = 4 - x^2 \Rightarrow y = \sqrt{4 - x^2}$ in the first quadrant.

So,

$$A_2 = \int_{\sqrt{3}}^{2} \sqrt{2 - x^2}\,dx$$

$$= \left[\frac{x}{2}\sqrt{4 - x^2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_{\sqrt{3}}^{2}$$

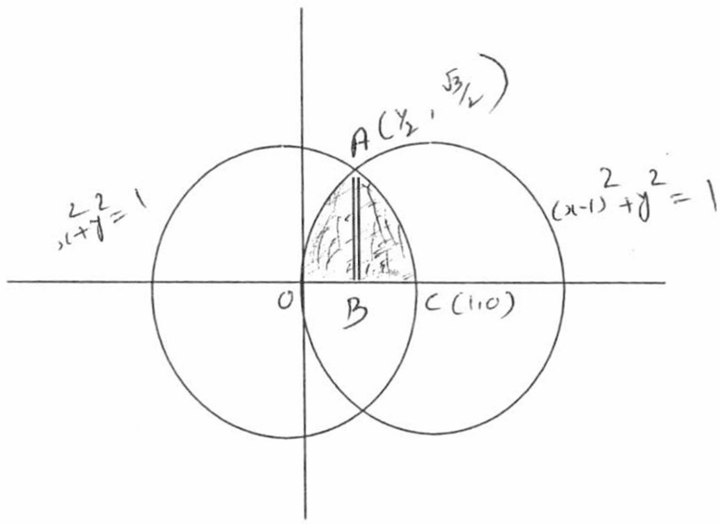
$$= [\,0 + 2\sin^{-1} 1\,] - \left[\frac{\sqrt{3}}{2}\sqrt{4-3} - 2\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right]$$

$$= 2\sin^{-1}(1) - \frac{\sqrt{3}}{2} - 2\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

From (3) and (4) we get,

$A = A_1 + A_2 = \frac{\sqrt{3}}{2} + \frac{\pi}{3} - \frac{\sqrt{3}}{2} = \frac{\pi}{3}$ sq. units.

3. Find the area of the region enclosed between two circles $x^2 + y^2 = 1$ and $(x - 1)^2 + y^2 = 1$ and the x-axis.



Solution :

$x^2 + y^2 = 1$ (1)

$(x - 1)^2 + y^2 = 1$ (2)

From the forms of the equations (1) and (2) it is clear that their centers are O(0,0) and C(1,0) and their radius = 1. Solving (1) and (2), their points of intersections are found to be A $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ in the first quadrant and $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ in the second quadrant.

So the required area is

A = Area of OACB

= 2 (area of OAB)

$=4 \int_0^{1/2} \sqrt{1-(x-1)^2}\, dx$ as for circle (2) $y = \sqrt{1-(x-1)^2}$

$= \sin^{-1}\left(-\frac{1}{2}\right) - \frac{\sqrt{3}}{4} - \sin^{-1}(-1)$

$= -\frac{\pi}{3} - \frac{\sqrt{3}}{4} + \frac{\pi}{2}$

$= \left(\frac{\pi}{2} - \frac{\pi}{3}\right) - \frac{\sqrt{3}}{4}$

$= \frac{\pi}{6} - \frac{\sqrt{3}}{4}$

Differential Equations

An equation involving an unknown function of one or more independent variables and the derivatives of the unknown function w.r.t. the independent variable(s) is called a differential equation. The following are some examples.

1. $$\frac{dy}{dx} = k$$

2. $$\frac{dy}{dx} = x + y$$

3. $$\frac{d^2 y}{dx^2} + x + \left(\frac{dy}{dx}\right)^2 = 0$$

4. $$\frac{\partial v}{\partial s} + \frac{\partial v}{\partial t} = 0$$

5. $$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

6. $$\left(\frac{dy}{dx}\right)^2 + \sqrt{\left(1 + \frac{d^2 y}{dx^2}\right)^2} = 0$$

7. $$\frac{dx}{dt} = y, \ \frac{dy}{dt} = -x$$

A differential equation involving ordinary derivatives of dependent variable w.r.t. the independent variable is called an ordinary differential equation.

In the above set Equation (1), (2) and (7) are ordinary differential equations. In these, y is dependent and x is independent variable. In (7), x and y are dependent and t is independent.

A differential equation involving partial derivatives of one dependent variable w.r.t. more than one independent variable is called a partial differential equation. In the above set, equations (4) and (5) are partial differential equations. In (4) v is the dependent variable and s and t are independent variables. In (5), z is dependent variable and x and y are independent variables.

The order of the highest ordered derivative in a differential equation is called the order of the equation.

In the earlier set of equations (1), (2), (4) and (7) are of first order and the remaining are of second order.

The degree of the highest order derivative in a differential equation which is free from radicals and fractions in its derivatives is called the degree of the equation. All the equations except (6) are of first degree and the equation (6) is of third degree.

Example : Find the order and degree of the equation,

$$\left(\frac{d^2 y}{dx^2}\right)^2 + k\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = 0$$

Solution : Here the highest order derivative is two. Therefore, this is second order equation. In this example, the degree of the second derivative i.e. $\frac{d^2 y}{dx^2}$ is two and hence the degree of the equation 2.

Always it is not possible to find the degree f a .....

Examine the following equations :

1. $\frac{d^2 y}{dx} + 2\left(\frac{dy}{dx}\right)^3 - y = 0$

2. $\left(\frac{dy}{dx}\right)^3 + \frac{dy}{dx} - \sin^2 y = 0.$

3. $\frac{d^2 y}{dx^2} + \sin\left(\frac{dy}{dx}\right) = x$

Here the equations (1) and (2) can be expressed as polynomials in y'', y' and hence we can find the degree of these equations 1 and 3 respectively. But the equation (3) cannot be put as a polynomial in $\frac{d^2 y}{dx^2}$, $\frac{dy}{dx}$ and hence it is not possible to find the degree of this equation.

To find out the degree of differential equation, just try to write the equation as polynomial equation in derivatives. Then the highest power of the highest order derivative involved is the degree of that equation.

Note :

1. y and its derivatives in the linear equation occur in first degree only.
2. A linear equation is necessarily of first degree.
3. No products of y and / or any of its derivatives are present.
4. No transcendental functions of y and/or its derivatives occur.

The differential equation with degree one is known as a linear equation.

An ordinary differential equation of nth order is given by

$$F\left(\frac{d^n y}{dx^n}, \frac{d^{n-1} y}{dx^{n-1}}, \ldots\ldots, y, x\right) = 0$$

Or $F(y^{(n)}, y^{(n-1)}, \ldots\ y^{(1)}, y, n) = 0$

Where $y^{(k)} = \dfrac{d^k y}{dx^k}$ and F is a real valued function or

$a_o(x)\ y^{(n)} + a_1(x)\, y^{(n-1)} + \ldots.+ a_n(x)y = b(x)$.

The above equation is said to be homogeneous equation if $b(x) = 0$.

It is said to be a linear equation with constant coefficients if all the coefficients $a_o(x)$, $a_1(x), \ldots, a_n(x)$ are constants. An equation which is not homogeneous is called non-homogeneous or inhomogeneous equation.

Example 1 : $y''' + 3x^2 y'' + 3xy' + 2y = 0$ is a linear homogeneous equation where

$$y''' = \frac{d^3 y}{dx^3},\ y'' = \frac{d^2 y}{dx^2},\ y' = \frac{dy}{dx}$$

Example 2 : $y'' + y' + xy = 0$ is a homogeneous linear equation with variable coefficients.

Solution of a Differential Equation

Consider an nth order ordinary differential equation

$F(y^{(n)}, y^{(n-1)}, \ldots, y^{(1)}, y, x) = 0$

Where F is the real function of x, y, y', ... $y^{(n)}$.

A real function y = f(x) is called a solution of the differential equation over some interval I when Y, $y^{(1)}$, $y^{(2)}$, ...$y^{(n)}$ are replaced by f(x), f'(x), f''(x)......$f^{(n)}(x)$ respectively then the above equation becomes an identity. That is the equation satisfies.

Some of the real life problems can be converted (explained) into a mathematical problem which involves solving of a differential equation to get the solution to that problem.

Consider the following problems :

Example 1 : A tank initially contains 50 litres of pure water. Starting at time t = 0 a brine containing 2 lb of dissolved salt per litre flows into the tank at the rate of 3 lt/ min. The mixture is kept uniform by stirring and the well stirred mixture simultaneously flows out of the tank at the same rate.

a) How much salt is in the tank at any time t > 0 ?

b) How much salt is present at end of 25 minutes ?

c) How much salt is present after a long time ?

Solution : First we consider the mathematical formulation of this problem.

Let x denote the amount of salt in the tank at time t.

$$\therefore \quad \frac{dx}{dt} = In - out$$

The brine flows in at the rate of 3 lt/ min, and each litre contains 2 lb of salt. Thus

In = (2 b / g Lt) 3 lt/ min = 6 lb/ min

Since the rate of outflow equals the rate of inflow, the tank contains x lb of salt at time t and so the concentration of salt at time t is $\frac{1}{50} x\ lb/lt$. Thus since the mixture flows out at the rate of 3 lb/ min, we have

$$\text{Out} = \left(\frac{x}{50}\ lb\ /\ Lt\right)(3\ Lt/\min) = \frac{3x}{50}\ lb/\min$$

Thus the differential equation for x as a function of t is

$$\frac{dx}{dt} = 6 - \frac{3x}{50}$$

Since initially there was no salt in the tank, we also have the initial condition,

$x(0) = 0$

(A) can be written as

$$\frac{dx}{100 - x} = \frac{3}{50}\, dt$$

Integrating and simplify, we obtain

$$x = 100 + ce^{-3t/50}$$

$\therefore$ $x = 100\,(1 - e^{-3t/50})$ ($\because x = 0$ at $t = 0$, $\therefore c = -100$)

This is the answer for question (or)

When t = 25

$$x(25) = 100\,(1 - e^{-3 \times 25/10}) = 78 \text{ lb}$$

This is the answer for question (b).

Similarly when $t \to \infty$, $x \to 100$ which gives the solution for question (C). This problem illustrates how a differential equation describes the problem.

Example 2 : What rate of interest, compounded annually, is equivalent to percent compounded continuously ?

(Formulate the problem as solving a differential equation and obtain the answer as 6.18 percent by solving that differential equation).

Example 3 : Newton's Law of Cooling states that the rate at which a body cool is proportional to the difference between the temperature of the body and that of medium in which it is situated. A body of temperature 80° F is placed at time t = 0 in a medium the temperature of which is maintained at 50° F. At the end of 5 min, the body has cooled to a temperature of 70° F.

a) What is the temperature of the body at the end of 10 min ?

b) When will the temperature o the body be 60° F ?

[Ans: a) 63.33° F b) 13.55 minutes]

Solution : Let θ denote the temperature of the body at any moment t and θ_o the temperature of the surroundings of the body. Then the rate of cooling is $\frac{d\theta}{dt}$ and this proportional to $(\theta - \theta_o)$. Then the cooling of the body is governed by the equation

$$\frac{d\theta}{dt} = -k\,(\theta - \theta_o),\ k > 0.$$

Or $\dfrac{d\theta}{\theta - \theta_o}$ = - k dt, k > 0 (A)

The solution of this d.e. is log' $(\theta - \theta_o) = - kt + C$ (B)

It is given that the initial temperature of the body, $\theta = 80^o$ F and temperature of medium or surrounding $\theta_o = 50^o$ when t = 0.

$\therefore$ (A) becomes

log (80 –50) = C $\Rightarrow$ C = log 30.

And log $(\theta$ -50) = - kt + log 30

But when t = 5, θ = 70,

$\therefore$ (c) becomes

$$\log (70 - 50) = - 5 \text{ kt} \log 30$$

$$\therefore \qquad K = \frac{\log 30 - \log 20}{5} = \frac{1}{5} \log\left(\frac{3}{2}\right)$$

and log θ.

Case (a) : When t = 10, what is the value of θ ? Substituting these values in (c) we get

$$\log (\theta - 50) = -\frac{1}{5} \log \left(\frac{3}{2}\right) 10 + \log 30$$

$$= - \log \left(\frac{3}{2}\right)^2 + \log 30$$

$$= - \log \left(\frac{9}{4}\right) + \log 30$$

$$= \log \left(\frac{30 \times 4}{9}\right) = \log \frac{40}{3}$$

Hence $\theta = 50 + \dfrac{40}{3} = 63.33^o$ F.

Case (b) : When $\theta = 60^o$F, what is the value of t? Again substituting these values in (c) we get

$$\text{Log} (60 - 50) = - \frac{1}{5} \log \left(\frac{3}{2}\right) t + \log 30$$

$$\log(10) = -\frac{1}{5}\left(\log\frac{3}{2}\right)t + \log 30$$

$$\therefore \quad t = \frac{\log 30 - \log 10}{\frac{1}{5}\log\left(\frac{3}{2}\right)} = 5\,\frac{\log 3}{\log\frac{3}{2}}$$

$$= 5\,\frac{\log 3}{\log 3 - \log 2}$$

$$=$$

Example 4 : A body of mass m kg falls from rest in a medium for which the resistance is proportional to the square of the velocity. If the terminal velocity is 50 ms^{-1} find

a) the velocity at the end of 2 seconds and

b) the time required for the velocity to become 30 ms^{-1}.

Ans: $m\frac{du}{dt} = mg - ku^2$, a) v = 18.5 ms^{-1} b) t = 3.5 sec.

Example 5 : Find the family of curves for which the length of the part of the tangent between the point of contact (x,y) and y axis is equal to the y-intercept of the tangent.

Example 6 : A fire suddenly develops in a theatre which is packed to capacity with 3000 people. The people rush to the exits; the rate at which they go out at any time t is proportional to the number of people still in the theatre. Set up the differential equation needed to determine how long it will take a given number (say N) of people to get out of the theatre when the constant of proportionality is 0.2.

Probability

Problem : Teachers generally are not clear about the use of various definitions of Probability viz. Mathematical, Statistical and Axiomatic.

Explanation : The limitations of Mathematical definition viz. finite number of possible outcomes and equally likely cases should be explained with examples. Next consider the examples violating the above limitations and explain the unsuitability of the definition. To overcome this problem, discuss the statistical probability definition with proper examples. Then discuss limitations of this definition with examples where it may not be possible to control the identical conditions of the experiment. Finally explain the use of axiomatic probability definition. Also show that mathematical probability definition is a special case of axiomatic probability.

Problem: Teachers find difficulty in solving problems involving continuous sample space liker in the following example.

Find the probability of a randomly selected point in a unit square to lie on its diagonal. Therefore it is necessary to discuss the problem solving technique in such situations.

Statistics

Problem: Identification of lower limit to find the Median or more of a frequency distribution with continuous or discrete class intervals.

Example: Find the median of the following distribution

a)

class interval	0 – 10	11-20	21-30	31-40
frequency	5	6	12	7

b)

class interval :	0 - 10	11 –20	21 – 30	31-40
frequency	5	6	12	7

Discussion: In (a) the median class is 20-30

$$\text{and hence Median} = 20 + \frac{\frac{30}{2} - 11}{12} . 10$$

$$= 20 + \frac{10}{3} = 23.33$$

In (b) the median class is 21-30 and hence the median is given by

$$21 + \frac{\frac{30}{2} - 11}{12} . 9$$

$$= 21 + 3 = 24$$

In case (b) if we consider the class boundaries say –0.5-10.5 10.5-20.5, 20.5-30.5 30.5040.5 we get median as $20.5 + \frac{\frac{30}{2} - 11}{12} . 10 = 20.83$

Now the confusion among the students and teachers is which method is correct.

For all practical purposes, when the nature of the data has not been specified the first two methods are appropriate. Otherwise the last method should be considered generally for computation.

Linear Inequalities

Problem 1: Teachers find it difficulty in solving the linear inequalities involving modules sign.

Example 1 : Solve $| x - 3 | \leq 2$

Solution : The given inequality can be written as

$3 - 2 \leq x \leq 3 + 2$

i.e. $1 \leq x \leq 5$

Hence [1,5] is the solution.

Hint: $| x - A | \leq \in$ can be written as $A - \in \leq x \leq A + \in$

Example 2 : Solve $| x - 6 | \leq 4$

$x \leq 6 - 4$ or $x \geq 6 + 4$

$x \leq 2$ or $x \geq 10$

$\therefore$ Solution set is $(-\infty, 2] \cup [10, \infty)$

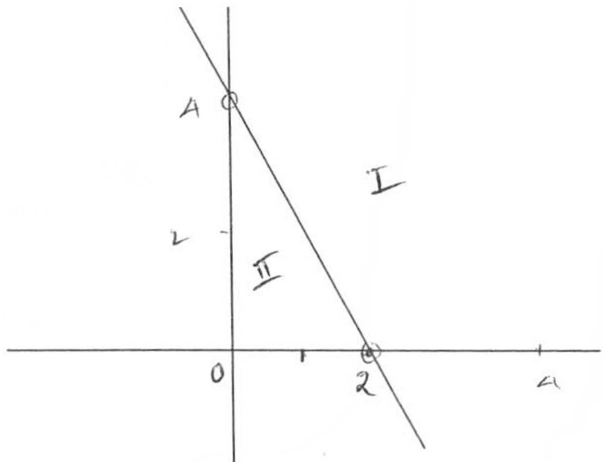
Graphical Solution :

Hint : The inequality $| x - A | \geq \in$ can be written as $x \leq A - \in$ or $x \geq A + \in$.

Problem 2 : Teachers normally find difficulty in solving inequalities in two variables graphically. This is because of no clarity in identifying the half plane corresponding to the given inequality.

Example : Solve $3x + 2y > 6$

Solution : Consider the equation 3x + 2y = 6 corresponding to the above inequality. Then represent this equation graphically as shown below.



The line 3x + 2y = 6 divides Cartesian plane into two half planes. In the given problem, strict inequality is given and therefore, either half plane I or half plane II (excluding the line). We observe that (0,0) is in plane I and this set is not satisfying the inequality. Therefore the solution is any point in plane II.

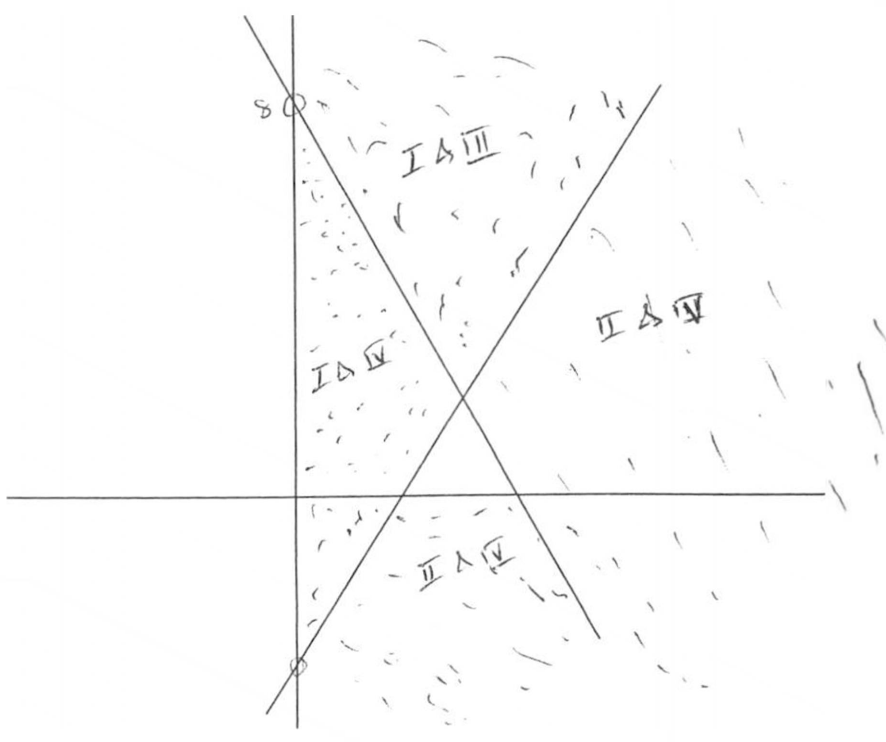
Hint : In solving the problems like this, first identify the half planes and examine whether (0,0) is a solution or not. If (0, 0) is the solution on the plane containing this (0,0) is the solution plane otherwise the other plane not containing this (0,0) is the solution plane.

Example : Solve the following system of linear inequalities.

$$x - y \leq 3$$

$$2x + y \geq 8$$

Solution : Consider the equations x – y = 3 and 2x + y = 8 corresponding to the above inequalities. Now represent these two equations graphically as given below.



These two lines together divide the Cartesian plane into four sub-planes viz. (I & IV), (I and III), (II and IV) and (II and III) as indicated in the figure. We have to examine now among these four subplanes which are the solution planes. Examine whether (0,0) is a solution or not. It is obvious that (0,0) is a not a solution. Therefore, the sub-plane (II and IV) cannot be the solution plane. Consider a point (0, -4) in plane (II and IV0. It is very clear that this pair is not satisfying the inequalities and hence the plane (II and IV) is also not a solution plane. Now consider a point (6,0) in the plane (II and III) and observe that this point is also not satisfying the given inequalities and so this plane is also not the solution plane. Therefore, the left-over sub-plane (I and III) is the solution plane, because a point (0,9) in this plane is a solution. The above procedure is applicable even in the case of three or more number of inequalities in two variables after identifying the sub planes obtained by the corresponding equations.

Hint : One should be careful in choosing a point in the sub-planes and examining this point is a solution or not.

Example 3 : Solve the system of inequalities

$$x + y < 3$$

$$2x + y \geq 8$$

$$y \geq 4$$

$$x \geq 0$$

Here the solution set is empty.

Problem : Concept of Random variable

Explanation : Scientific theories on models are our way of depicting and explaining how observations come about. Such theories are simplified statements containing essential features and make for easier comprehension and communication. In statistics, we use a mathematical approach since we quantify our observations. Random variable is the result of such mathematical approach dealing with the probabilities assigning the different events of a random experiment. The set of possible outcomes for a random experiment can be described with the help of real-valued variable by assigning a single value of this variable to reach outcome. For a two coin experiment, the outcomes are two tails, a tail and head, a head and tail or two heads. The sample space can be represented as (TT, TH, HT, HH). Here we express the outcomes by using the number of heads and so assigning the values (0, 1,1,2) respectively to those outcomes. Therefore, the outcomes of this experiment can be denoted by the different values of the real-valued variable viz. 0,1,2.

Any function or association that assigns a unique real value to each sample point is called a chance or random variable. The assigned values are the values of the random variable.

Random variables are symbolized by capital letters, most often X and their values by lower case letters. The outcome of a random experiment determines a point i.e. the sample space, called the domain of the random variable and the function transform each sample point to one of a set of real numbers. This set of real numbers is called the range of the random variable.

If the sample space is discrete, then the outcomes will be denoted by certain discrete values. The random variable associated with a discrete sample is known as discrete random variable. Similarly, the random variable associated with continuous sample space is known as continuous random variable, for example, selecting a point randomly on a line segment ‘l’.

Problem : Distribution of a random variable.

Explanation : The association o probabilities with the various values of a discrete random variable is done by reference to the probabilities in the sample space and through a system of relationships or a function is called a probability set function or simply, a probability function.

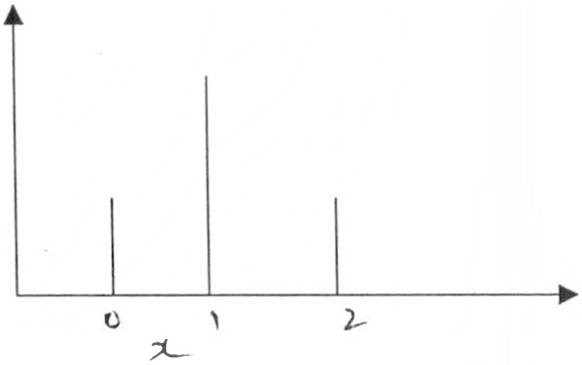
Let us discrete random variable X assume the values $x_1, x_2, \ldots, x_n$. Then the system of relations can be written as

$$P\,(X = x_i) = P_i$$

This is read as 'the probability that the random variable X takes the value x_i is p_i'. The set of ordered pairs (x_i, p_i) constitutes a probability function with numerical values to be provided for the x_i and p_is such that $p_i \geq 0$ for all i and $\sum_i P_i = 1$.

A discrete probability function is a set of ordered pairs of values of a random variable and the corresponding probabilities. For a two coin experiment, X takes the values 0,1,2 with the probabilities $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$ respectively.

Sometimes probability function can be represented by a graph or a mathematical function. In case of above example, the X values and the corresponding probabilities can be represented with the help of the following graph.



Suppose X assume the values 1 and 0 with the probability p and 1 – p respectively. This information can be given with the help of the following function p(x) defined by

$$p(x) = p^x (1 - p)^{1-x}, x = 0, 1.$$

This type of function which gives the probabilities of the different values assumed by a random variable is known as probability mass function or simply probability function. Therefore, a function p(x) is said to be probability function of random variable or a distribution if

i) $p(x) \geq 0$ for all x and ii) $\sum_x p(x) = 1$

where p(x) denoted the probability of the events that the random variable X assumes the value x.

Probability density function (39 – 40)

Distribution function (41 -)

Problem : Independent events and Independent experiments

Explanation : Consider tossing of two coins experiment. Here the event A, say of getting head on first coin and the event B of getting head on second coin are independent events.

Now consider two experiments. One is drawing two balls from a bag containing 5 white and 3 black balls and another is drawing of two balls from another bag containing 6 white balls and 5 black balls. Here these two experiments are independent because the outcome of drawing two balls from the first is no way influenced by the outcome of drawing two balls from the second bag.

Problem : Confusion between 'Mutually exclusive events' and 'Independent events'.

Explanation : Events are said to be mutually exclusive or incompatible if the happening of any one o them precludes or excludes the happening of all others. For example, in tossing of a coin experiment, the events of getting head and tail are mutually exclusive, because both cannot occur simultaneously.

Events are said to be independent if the probability of happening of any event does not affect the probability of happening of the other events. For example, in tossing of two coins experiment, the probability of getting head on the first coin has no effect on the probability of getting head on the second coin. Therefore, the event of getting head on the first coin and the event of getting head on the second coin are independent.

Mathematically, two events A and B are said to be mutually exclusive when $P(A \cap B) = 0$. Similarly two events A and B are said to be independent when $P(A \cap B) = P(A) . P(B)$.

From the above definitions, it is clear that independence does not imply mutually exclusiveness and vice-versa.

In tossing of a single coin experiment, the event of getting head and the event of getting tail are mutually exclusive but they are not independent, because

$P(H \cap T) = 0$

And $P(H \cap T) \neq P(H).\ P(T)$

$\therefore$ P(H) = ½ and P(T) = ½

where H : event of getting Head

and T : event of getting Tail.

Now consider the experiment of tossing two coins and the events A and B as

A : Events of getting Head on first coin

B : Event of getting Head on Second Coin

Here $P(A \cap B) = \frac{1}{4} \neq 0$	HH
And $p(A \cap B) = \frac{1}{4} = \frac{1}{2} . \frac{1}{2}$	HT
$= P(A).\ P(B)$	TH
	TT

Therefore, A and B are independent	P(A) = ½
But they are not mutually exclusive.	P(B) = ½
	$A \cap B = \frac{1}{4}$

More precisely, the concept of independency in case of n events can be explained as follows :

If there are three or more than three events, we will have the situation where every pair of these events are independent or the situation where the events in every set of events are independent. In the first case, we call the events as pairwise independent and in the second case, we call as complete or mutually independent events. Mathematically, the events $E_1, E_2, \ldots E_n$ are pairwise independents

If $P(E_i \cap E_j) = P(E_i) . P(E_j)\ \ \forall$ i, j.

Similarly, the events $E_1, E_2, E_3 \ldots\ldots E_n$ are complete or mutually independent if

$P(E_1 \cap E_2 \cap \ldots\ldots E_k) = P(E_1) . P(E_2) \ldots\ldots P(E_k)$ for all k = 2, ….n.

One can prove that pairwise independent events need not be mutually independent. Consider the following example.

Mrs Rao has a jewel box containing four ornate rings. One ring has a diamond and an emerald another a diamond and topaz; another an emerald and a topaz; and the other five pearls. She will select one ring a random from her box. Show that the events

A : she will select a ring with a diamond

B : she will select a ring with an emerald

And C : She will select a ring with a topaz

And pairwise independent but not mutually independent.

In this problem $P(A) = \frac{1}{2}$

$P(B) = \frac{1}{2}$

$P(C) = \frac{1}{2}$

$P(A \cap B) = \frac{1}{4}$, $P(A \cap C) = \frac{1}{4}$, $P(B \cap C) = \frac{1}{4}$.

Hence, $P(A \cap B) = P(A) . P(B)$, $P(B \cap C) = P(B). P(C)$ and $P(A \cap C) = P(A) . P(C)$.

A, B, C are pairwise independent.

But we compute

$$P(A \cap B \cap C) = 0 \neq P(A) . P(B) . P(C)$$

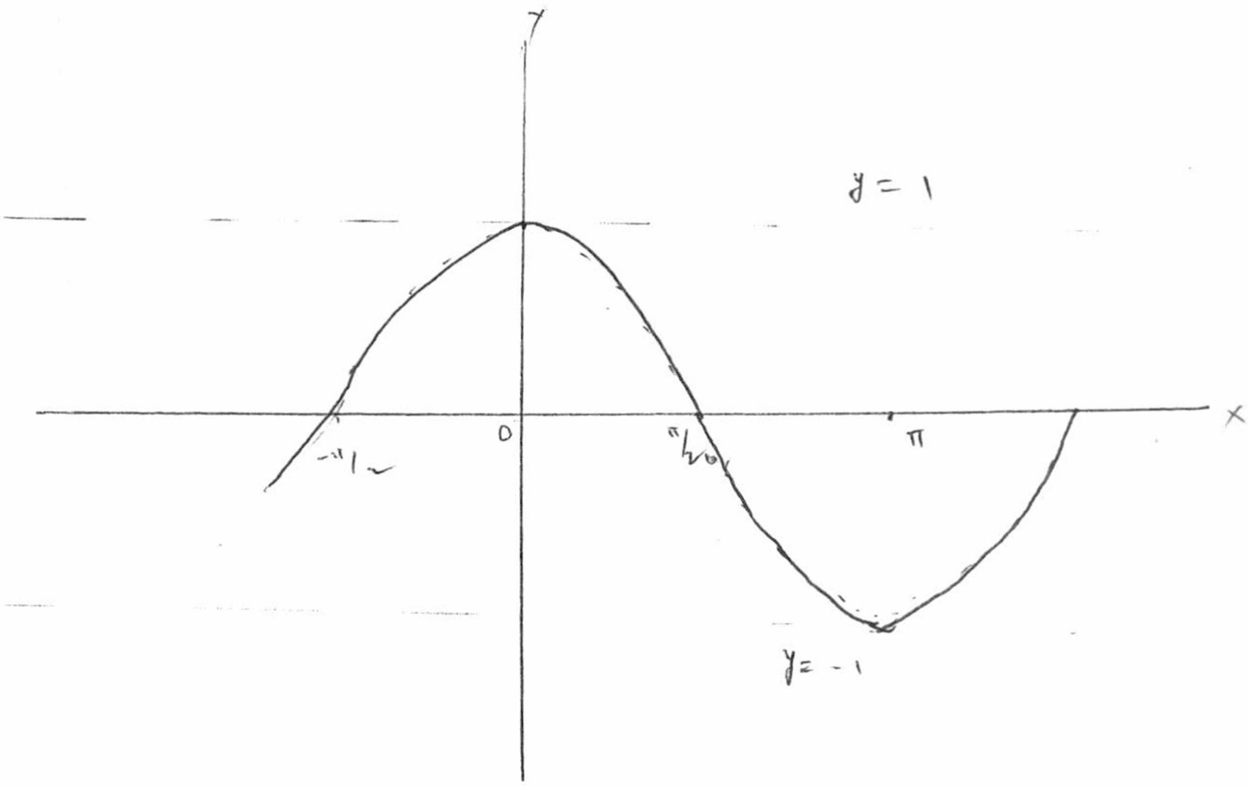
And draw the conclusion that A, B, C are not complete independent. Similarly, we can give examples for three events A, B, C for pairwise independent, $P(A \cap B) C = P(A) . P(B) . P(C)$ is not sufficient. That is, $P(A \cap B \cap C) = P(A) . P(B) . P(C)$ we cannot say they are pairwise independent.

Graphs of Trigonometric functions – Graph and Periodicity

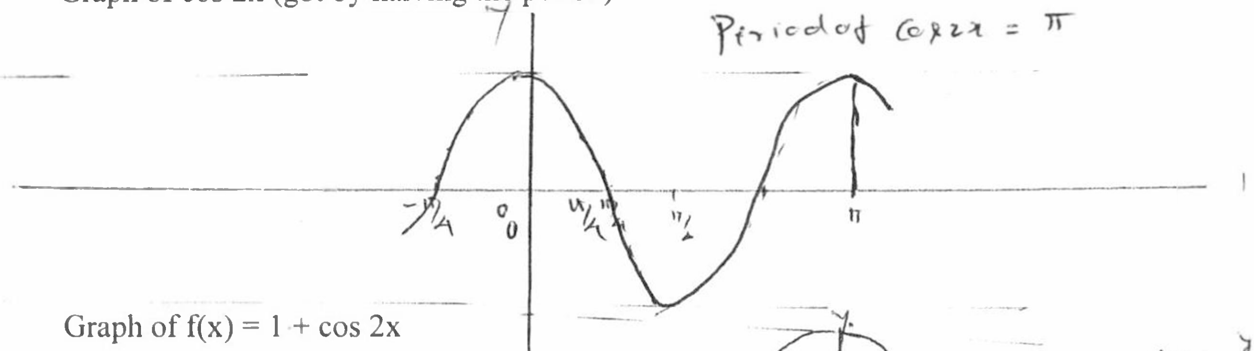
Problem : Find the graph of $f(x) = 2\cos^2 x$. What is the period of $f(x)$?

Solution : $f(x) = 2\cos^2 x = 1 + \cos 2x$.

a) Graph of Cos x.

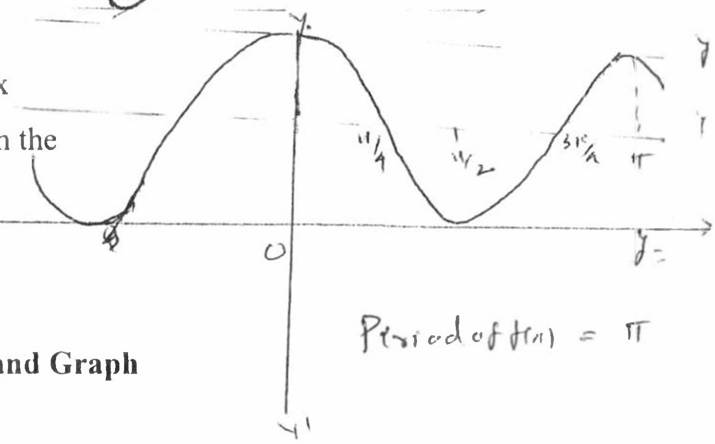


b) Graph of cos 2x (got by halving the period)



c) Graph of f(x) = 1 + cos 2x

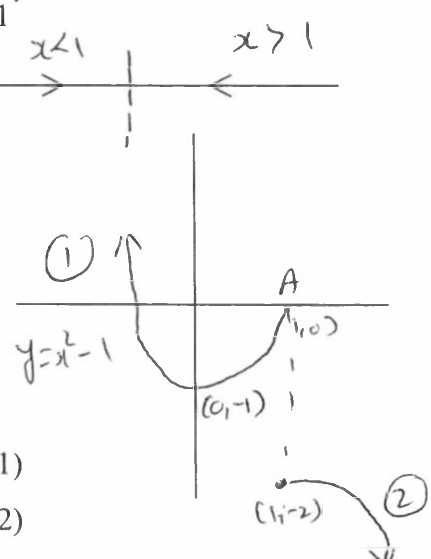
(got by translating the graph of cos 2x through a unit distance upwards i.e. in the +ve direction of the y-axis).



Topic : Limit Continuity / Discontinuity and Graph

22. Find $\underset{x \to 1}{Lim}\, f(x)$, given $f(x) = \begin{cases} x - 1, & x \le 1 \\ -x^2 - 1, & x > 1 \end{cases}$.

Explain the conclusion graphically.



$\underset{x \to c}{Lt}\, f(x)$ exists, if $\underset{x \to c-0}{Lt}\, f(x) = \underset{x \to c+0}{Lt}\, f(x)$

$\underset{x \leftarrow 1-0}{Lt}\, f(x) = \underset{x \to 1}{Lt}\,(x^2 - 1) = 0$

$\underset{x \leftarrow 1+0}{Lt}\, f(x) = \underset{x \to 1}{Lt}\,(-x^2 - 1) = -2$

Since $0 \neq -2$, $\underset{x \to 1}{Lt}$ f(x) does not exist.

The graphs of $y = x^2 - 1$ is a parabola. (1)

The graph of $y = -x^2 - 1$ is a parabola. (2)

The graph of f(x) consists of the two parts of parabolas as shown. At x = 1, the graph of f(x) becomes discontinuous. This explains why the limit at x = 1 does not exist.

23. Find the coefficient of x^{49} in the expansion of $\log_e(1 + 5x + 6x^2)$. When is the expansion valid ?

We know $\log_e(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + \ldots$ to ∞, $|x| < 1$ $(-1 < x \leq 1)$

$1 + 5x + 6x^2 = (1 + 2x)(1 + 3x)$

$\therefore \log_e(1 + 5x + 6x^2) = \log_e(1 + 2x)(1 + 3x)$

$= \log_e(1 + 2x) + \log_e(1 + 3x)$

$$= \left[2x - \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} - + \ldots + \frac{(2x)^{49}}{49!} - + \ldots\right]$$

$$+ \left[3x - \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} - + \ldots + \frac{(3x)^{49}}{49!} - + \ldots\right]$$

$\therefore$ The term containing $x^{49} = \frac{2^{49}\,x^{49}}{49!} + \frac{3^{49}\,x^{49}}{49!}$.

$= \left(\frac{2^{49} + 3^{49}}{49!}\right)x^{49}$ $\qquad\therefore$ The coefficient of $x^{49} = \frac{2^{49} + 3^{49}}{49!}$

The series expansion for $\log(1+ 2x)$ is valid for $|2x| < 1 \Rightarrow x \leq ½$

And the series expansion for $\log(1 + 3x)$ is valid for $|3x| < 1 \Rightarrow x \leq 1/3$.

Since $x \leq 1/3 \Rightarrow x < ½$

The expansion is valid for $|x| < 1/3$ and $x = 1/3$ also.

Binomial Theorem

24. The coefficient of (r – 1)th term, rth term and (r + 1)th term in the expansion of $(x + 1)^n$ are as 1 : 3 : 5, find the values of n and r.

Let T_{r+1} = rth term of $(x + 1)^n$ expansion.

$\therefore\ T_{r+1} = {}^nC_r \ . x^{n-r}$.

$\therefore$ The coefficient is $T_{r+1} = {}^nC_r$.

Hence the coefficient on (r –1)th term = ${}^nC_{r-2}$

Hence the coefficient of rth term = ${}^nC_{r-1}$

Hence the coefficient of (r+1)th term = ${}^nC_r$

${}^nC_{r-2}\ ,\ {}^nC_{r-1}\ ,\ {}^nC_r \ = 1:3:5$

$$\Rightarrow \frac{n}{r-2 \quad n-r+2} = \frac{n}{r-1 \quad n-r+1} = \frac{n}{r \quad n-r} = 1:3:5$$

$$\Rightarrow \frac{1}{(n-r+2)\,(n-r+1)} = \frac{1}{(r-1)\,(n-r+1)} = \frac{1}{r\,(r-1)} = 1:3:5$$

$\Rightarrow$ (n – r + 2) (n – r + 1) = 3 (r – 1) (n – r + 1) = 5 r (r – 1)

$\Rightarrow$ 3 – r + 2 = 3 (r – 1) ; 3 (n – r + 1) = 5x

$\Rightarrow$ n – r + 2 = 3 r – 3 ; 3n – 8r + 3 = 0

$\Rightarrow$ n – 4 r = -5; 3n – 8 r = -3

$\Rightarrow$ n = 7, r = 3

Vectors

25. In the notation $\vec{a}\,.\,\vec{b}$ = dot (or scalar) product and $\vec{a} \times \vec{b}$ = cross (or vector) product of two vectors $\vec{a}$ *and* $\vec{b}$ which of the following are meaningful. Give suitable explanation for the meaningful product.

i) $(\vec{a} \times \vec{b})\,.\,\vec{c}$

ii) $\vec{a} \times (\vec{b}\,.\,\vec{c})$

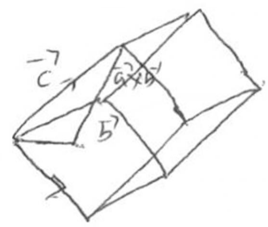
iii) $\vec{a} \times (\vec{b} \times \vec{c})$

iv) $(\vec{a}\,.\,\vec{b})\,.\,\vec{c}$

(i) $(\vec{a} \times \vec{b})\,.\,\vec{c}$ is meaningful, this being the dot product of two vectors.

$|(\vec{a} \times \vec{b})\,.\,\vec{c}|$

when θ is the angle between $\vec{a} \times \vec{b}$ and $\vec{c}$. $\vec{a} \times \vec{b}$ is normal to the plane $\vec{a} \times \vec{b}$.



$= | \vec{a} \times \vec{b} | \; | \vec{c} | \cos\theta$

= (Area of the base of the box) × height of the box $\vec{a}$

= Volume of the box in the coinitial edges as $\vec{a}$, $\vec{b}$, $\vec{c}$.

$\therefore$ $|(\vec{a} \times \vec{b}) . \vec{c}|$ = The volume of the box formed by $\vec{a}$, $\vec{b}$, $\vec{c}$.

As adjacent sides.

ii) $\vec{a} \times (\vec{b} . \vec{c})$ is meaningless since X is defined for multiplication of **two vectors.** ($\vec{b} . \vec{c}$ is not a vector).

iii) $\vec{a} \times (\vec{b} \times \vec{c})$ is meaningful. This is a vector.

$\vec{b} \times \vec{c}$ is perpendicular to $\vec{b}$ *and* $\vec{c}$.

$\therefore$ $\vec{a} \times (\vec{b} \times \vec{c})$ is perpendicular to $\vec{a}$ *and* $\vec{b} \times \vec{c}$.

$\Rightarrow$ $\vec{a} \times (\vec{b} \times \vec{c})$ is ……………..lying in the planes of $\vec{b}$ and $\vec{c}$.

iv) $\vec{a} . (\vec{b} . \vec{c})$is meaningless since product is defined as a product for two vectors ($\vec{b} . \vec{c}$) is not a vector).

Matrices and Determinants

26. Examine the system.

$$\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 4$$

$$\frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 1$$

$\dfrac{6}{x} + \dfrac{9}{y} - \dfrac{20}{z} = 2$, for consistency and solve the system if the solution exists.

Taking $\dfrac{1}{x} = 4, \; \dfrac{1}{y} = v, \; \dfrac{1}{z} = w,$

the system of equations is $2u + 3v + 10w = 4$

$4u - 6v + 5w = 1$

$6u + 9v - 20w = 2$

$\Rightarrow A X = B$

when $A = \begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}$, $X = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$, $B = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$

$$|A| = \begin{vmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{vmatrix} = 2\,(120-45) - 3\,(-80-30) + 10\,(36+36)$$

$= 150 + 330 + 720 = 1200 \neq 0$ $\qquad\qquad \therefore\ |A| \neq 0$

Hence the system is consistent so that the solution exists and is unique.

Using the matrix method, $A^{-1} = \frac{1}{|A|}(Adj\ A)$

$$\text{Adj } A = \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$$ [Calculate the cofactors A_{ij}]

$$\therefore\ A^{-1} = \frac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$$

$$\therefore\ X = A^{-1}B = \begin{bmatrix} 1/2 \\ 1/3 \\ 1/5 \end{bmatrix}$$ [Check the calculations].

$$\Rightarrow \begin{bmatrix} 1/x \\ 1/y \\ 1/z \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/3 \\ 1/5 \end{bmatrix} \Rightarrow (x, y, z) = (2, 3, 5)$$

27. If A is a 3×3 non-singular matrix, then $|\text{Adj } A|$ is equal to

(A) $|A|$ (B) $|A|^2$ (C) $|A|^3$ (D) $3|A|$

Solution (B) : We know for an $n\times n$ non-singular matrix A

$|\text{Adj } a| = |A|^{n-1}$

Here A is a 3×3 non-singular matrix $\qquad \therefore\ |\text{Adj } A| = |A|^2$

28. Examine whether the lines $\frac{x-4}{1} = \frac{y-3}{4} = \frac{z-2}{5}$ and $\frac{x-3}{1} = \frac{y-2}{-4} = \frac{z}{5}$ are coplanar or skew lines.

If the lines pass through (x_1, y_1, z_2), (x_2, y_2, z_2) respectively and have ....(l_1, m_1, n_1), (l_2, m_2, n_2)

They are coplanar if $\begin{vmatrix} x_1 - x_2 & y_1 - y_2 & z_1 - z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$

Accordingly, $\begin{vmatrix} 4-3 & 3-2 & 2-0 \\ 1 & 4 & 5 \\ 1 & -4 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 2 \\ 1 & 4 & 5 \\ 1 & -4 & 5 \end{vmatrix}$

$= 1\ (20 + 20) - 1\ (5 - 5)\ + 2\ (-4 -4)$

$= 40 - 16 = 24 \neq 0$

Hence the lines are skew lines.

Caution : In such problems, one may be tempted to find the equation of the plane containing the given lines without testing the coplanarity condition.

If the problem had been – find the plane containing the given lines, if we write the equation of the plane without verifying the coplanarity condition, we get the equation of the plane as

$$\begin{vmatrix} x-4 & y-3 & z-2 \\ 1 & 4 & 5 \\ 1 & -4 & 5 \end{vmatrix} = 0$$

$\Rightarrow$ $(x - 4)\ 40 - (y - 3)\ 0 + (z - 2)\ 8 = 0$

$\Rightarrow$ $(x - 4)\ 5 + (z - 2) = 0$

$\Rightarrow$ $5x + z - 22 = 0$

Putting $x = 3$, $y = 2$, $z = 0$,in this equation

$5 \times 3 + 0 - 22 = 15 - 22 = -7 = 0$ which is absurd.

$\therefore$ (3,2,0) a point on the other line does not lie in the plane.

Therefore, the equation of the plane we have got does not contain both the lines.

Binary Operation

29. If * is a binary operation on a set A, then in the following cases which property (or properties) holds? Match the statements under I correct with those under II.

	I		II
1.	$a*b = a + b + 1$, $a, b \in Z$	a)	* is commutative.
2.	$a*b = a - ab + b$, $a * b \in Z$	b)	* is Associative.
3.	$a*b = a^b$, $a, b \in N$	c)	This identity exists w.r.t. *
		d)	The inverse element for all elements w.r.t. * exist.

Answers :

1. (a), (b), (c),(d)
2. (a), (b),(c)
3. None of the properties

Solution:

1. $a*b = a + b + 1$ $\therefore a * b = b * a$

$b * a = b + a + 1$

$\therefore$ * is a commutative property on Z. (a)

$(a * b) * c = (a + b + 1) * c = a + b + 1 + c + 1 = a + b + c + 2$

$a * (b * c) = a * (b + c + 1) = a + b + c + 1 = a + b + c + 2$

$\therefore$ $(a * b) * c = a * (b * c)$

$\therefore$ * is an Associative Property on Z. (b)

Let $a * e = a \ \forall \ a \in Z$

$\Rightarrow$ $a + e + 1 = a \Rightarrow e = -1 \in Z$

$\therefore$ The identity element w.r.t. * is $-1 \in Z$. (C)

Let a^{-1} : $a * a^{-1} = e = -1$

$\therefore$ $a + a^{-1} + = -1 \Rightarrow a^{-1} = -2 - a \in Z$

$\therefore$ a^{-1} w.r.t. * exists for all $a \in Z$ (d)

2. $a * b = a - ab + b$ $\therefore a * b = b * a$

$b * a = b - ba + a$

$\therefore$ * is a commutative property on Z.

$(a * b) * c = (a - ab + b) * c$

$= a - ab + b + c - (a - ab + b)\, c$

$= a + b + c - ab - bc - ca + abc$

Similarly, $a * (b * c) = a + b + c - ab - bc - ca + abc$

$\therefore$ * is Associative on Z (b)

Let $e : a * e = a$

$\Rightarrow \quad a - ae + e = a$

$\Rightarrow \quad e = 0 \in Z$

$\therefore$ The identity element w.r.t. * is $0 \in Z$.

Let a^{-1} $(a * a^{-1}) = e = 0$

$\Rightarrow \quad a - a\,a^{-1} + a^{-1} = 0$

$\Rightarrow \quad a^{-1} = \dfrac{a}{a-1} \notin Z$ for all elements of Z.

$\therefore$ Not all elements $a \in Z$ have a^{-1} w.r.t. *

3. $a * b = a^b$; $b * a = b^a$

$a * b \neq b * a$ $\quad\therefore$ * is not commutative.

$(a * b) * c = (a^b) * c = (a^b)^c = a^{bc}$

$a * (b * c) = a * (b^c) = a^{(b^c)}$ $\quad\therefore$ * is not associative.

If $e : a * e = a = e * a$

Then $a^e = e^a = a$

No such e exists.

$\therefore$ Identity element w.r.t. * does not exist.

Hence Inverse element $\hat{a}$ for 'a' does not exist.

30. If $f(x) = \dfrac{x-1}{x}$

Find the least +ve integer n such that $f^n(x) = x$

When $f^2 = f \circ f$, $f^3 = f \circ f^2$, etc.

Solution :

$$f^2(x) = (f \circ f)(x) = \frac{\dfrac{x-1}{x} - 1}{\dfrac{x-1}{x}} = \frac{1}{1-x}$$

$$f^3(x) = \frac{1}{1 - \frac{x-1}{x}} = x$$

Then $f^{3n}(x) = x$ for all positive integer n.

The least n : $f^n(x) = x$ if $n = 3$.

THE HISTORY OF SINE AND COSINE FUNCTIONS

The word Sine is originated from the Latin term 'SINUS', which meant 'bay' or 'bosom of garment' . This is a translation of the Arabic word "jaib" meaning the same thing. It is not known how this Arabic term originated. Some believe that it come from the Hindu (Sanskrit) word 'jiva' (the first meaning of which is bowstring; in geometry it meant chord of an arc).But Sine in Hindu terminology is designated by 'ardha-jiva' which means 'half chord'.

The name 'Cosine' appeared only at the beginning of the 17^{th} century as a contraction of the term 'complement sinus' (Sine of the complement), which indicated that the cosine of an angle A is the sine of the complementary angle.

The term 'tangent' and 'secant' (which are translated from the Latin, mean 'contracting and 'cutting') were introduced in 1583 by the German Scholar Finck.

Literal symbolism, which in Algebra come in at the end of the 16^{th} century, was established in Trigonometry only in the middle of 18^{th} century thanks to the efforts of the great Euler (1707 – 1783), who gave trigonometry its modern aspect. The quantities sinx, cosx etc. were regarded by him as functions of a number x, the radian measure of the appropriate angle. Euler assigned to the number x all possible values; positive, negative and even complex numbers also. He also introduced the inverse trigonometric functions.

FIELDS MEDALS IN MATHEMATICS

The Fields Medals are commonly regarded as mathematics' closest analog to the Nobel Prize (Which does not exist in mathematics), and are awarded every four years by the International Mathematical Union to one or more outstanding researchers. **"Fields Medals"** are more properly known by their official name, ***"International medals for outstanding discoveries in mathematics".***

The Field Medals were first proposed at the 1924 International Congress of Mathematicians in Toronto, where a resolution was adopted stating that at each subsequent conference, two gold medals should be awarded to recognize outstanding mathematical achievement. ***Professor J.C. Fields***, a Canadian mathematician who was secretary of the International Congress of Mathematicians held in Toronto, later donated funds establishing the medal which were named in his honor. Consistent with Fields' wish that the awards recognize both existing work and the promise of future achievement, it was agreed to restrict the medals to **mathematicians not over forty** at the year of the Congress. In 1966 it was agreed that, in light of the great expansion of mathematical research, **up to four medals** could be awarded at each Congress.

Each medal carries with it a cash prize of 1500 Canadian dollars. The first two such medals were presented at the Oslo Congress in 1936. After an interruption caused by War, two medals have been presented at each of the Congresses in 1950, 1954, 1958, 1962, 1974 and 2002; four medals at each of the Congresses in 1966, 1970, 1978, 1990, 1994, 1998 and 2006; and three medals at each of the Congresses in 1982 and 1986.

The Fields Medal is made of gold, and shows the head of Archimedes (287-212 BC) together with a quotation attributed to him : *"Transire suum pectus mundoque potiri "* ***("Rise above oneself and grasp the world").*** The reverse side bears the inscription: *"Congregati ex toto orbe mathematici ob scripta insignia tribuere"* ***("the mathematicians assembled here from all over the world pay tribute for outstanding work")***

<u>Why no Nobel prize for Mathematics</u> ?

Nobel prizes were created in the will of the Swedish chemist and inventor of dynamite Alfred Nobel, but Nobel, who was an inventor and industrialist, did not create a prize in mathematics because he was not particularly interested in mathematics or theoretical science. In fact, his will speaks of prizes for those "inventions or discoveries" of greatest practical benefit to mankind. While it is commonly stated that Nobel decided against a Nobel Prize in math because of anger over the romantic attentions of a famous mathematician (often claimed to be Gosta Mittag-Leffler) to a woman in his life, there is no historical evidence to support the story. Furthermore, Nobel was a lifelong bachelor, although he did have a Viennese woman named as his mistress (Lopez-Ortiz).

Note : In the year 1997, film "**GOOD WILL HUNTING",** fictional MIT professor Gerald Lambeau (played by Stallan Skarsgard) is described as having been awarded a Fields medal for his work in combinatorial mathematics.

LIST OF FIELDS MEDAL WINNERS IN MATHEMATICS

Sl.No	YEAR	WINNERS
1)	1936	**a) Lars Valerian Ahlfors (Hardvard University)** Sub: **Riemann Surfaces of Inverse Functions** b) Jesse Douglas (Massachussetts Institute of Technology) **Sub:** Work on the Plateau problem
2)	1950	a) Laurent Schwartz (University of Nancy) **Sub:** Theory of Distributions b) Atle Selberg (Institute of Advanced Study, Princeton **Sub:** Elementary proof of prime number theorem
3)	1954	a) Kunihiko kodaira (Princeton University) **Sub:** Harmonic integrals & Algebraic varieties b) Jean-Pierre Serre (University of Paris) **Sub:** Cohomology & Sheaf Theory
4)	1958	a) Klaus Friedrich Rot (University of London) **Sub:** Analytic Number Theory b) Rene Thom (University of Strasbourg) **Sub:** Cobordism Theory in Differential Topology
5)	1962	a) Lars V.Hormander (University of Stockholm) **Sub:** Linear Partial Differential Operators b) John Willard Minor (Princeton University) **Sub:** Differential Topology
6)	1966	a) Michael Francis Atiyah (Oxford University) **Sub:** Index Theorem for Elliptic Operators b) Paul Joseph Cohen (Stanford University) **Sub:** Foundations of Mathematics- Forcing c) Alexander Grothendieck (University of Paris) **Sub:** Algebraic Geometry- Schemes d) Stephen Smale (University of California, Berkelery) **Sub:** Dynamic Systems-Structural Stability
7)	1970	a) Alan Baker (Cambridge University) **Sub:** Analytic Number Theory- Transcedental Numbers b) Heisuke Hironaka (Harvard University) **Sub:** Algebraic Geometry- Resolution of Singularities c) Serge P. Novikov (Moscow University) **Sub:** Topological Invariance of Pontrjagin class d) John Griggs Thompson (Cambridge University) **Sub:** Finite Simple Groups

8) 1974 a) Enrico Bombieri (University of Pisa)
Sub: Number Theory & Algebraic Geometry
b) David Bryant Mumford (Hardverd University)
Sub: Algebraic Geometry

9) 1978 a) Pierre Rene' Deligne (Institute des Hautes Etudes Scientifiques)
Sub: Weil's Conjecture on Riemann Hypothesis
b) Charles Louis Fefferman (Princeton University)
Sub: Multi Dimensional Complex Analysis
c) Gregori Alexandrovitch Margulis (Moscow University)
Sub: Structure of Lie Groups
d) Daniel G. Quillen (Massachusettes Institute of Technology)
Sub: **Serre's Conjecture in Algebraic K-Theory**

10) 1982 a) Alain Connes (Institut des Hautes Etudes Scientifiques)
Sub: Operator Algebras & Applications
b) William P. Thurston (Princeton University)
Sub: Low dimensional Manifolds
c) Shing – Tung Yau (Institute for Advanced Study, Princeton)
Sub: Differential Geometry & Partial Differential Equations

11) 1986 a) Simon Donaldson (Oxford University)
Sub: Exotic 4-dimensional Manifolds
b) Gerd Faltings (Princeton University)
Sub: Mordell's Conjecture in Arithmetic Algebraic Geometry
c) Michael Freedman (University of California, San Diego)
Sub: 4-dimensional Poincare conjecture

12) 1990 a) Vladimir Drinfeld (Phys. Inst. Kharkov)
b) Vaughan Jones (University of California, Berkeley)
c) Shigefumi Mori (University of Kyuto?)
d) Edward Witten (Institute for Advanced Study, Princeton)

13) 1994 a) Pierre-Louis Lions (Universite de Paris-Dauphine)
b) Jaen-Christophe Yoccoz (Universite de Paris-Sud, Orsay, France)
c) Jean Bourgain (Institute for Advanced Study, Princeton)
d) Efim Zelmanov (University of Wisconsin)

14) 1998 a) Richard E. Borcherds (Cambridge University)
b) W.Timothy Gowers (Cambridge University)
c) Maxim Kontsevich (IHES Bures-sur-Yvette)
d) Curtis T.McMullen (Hardvard University)

15) 2002 a) Laurent Lafforgue (Institut des Hautes Etudes Scientfiques, Bures-Sur- Yvette, France)
b) Vladimir Voevodsky (Institute for Advanced Study Princeton)

16) 2006 a) Andrei Okounkov (Princeton University)
b) Grigori Perelman (Russia) (declined award)
c) Terence Tao (University of California, Los Angeles)
d) Wendelin Werner (Universite de Paris-Sud, Orsay, France)