SELF-LEARNING MATERIALS
IN
MATHEMATICS

Content Enrichment Programme in Mathematics through Telemode
for Secondary School Teachers of Karnataka

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Self Learning Materials in Mathematics for Training of Secondary School Teachers by Telemode

Units: Selected units of X class Mathematics of Karnataka

Addressed to: A Mathematics Teacher

Format Used - Concepts and Style

Unit: Title

Unit: Introduction

Format used for each subunit of a unit:

1. Concepts/Definitions/Notations/Terminology and Results
2. Explanations
3. Examples
4. Verifications/Proofs
5. Problem solving (with special reference to teaching strategies)
6. Activities (including enrichment activities, if any)
7. Exercises (for self-assessment)

CONTENTS

UNIT NO. 1: PERMUTATIONS AND COMBINATIONS

Unit Introduction

In daily life we come across situations where order or arrangement is important. Formation of telephone numbers, words using letters, sentences using words - are some examples of this sort. When arrangement matters, the idea is called permutation. In contrast, in some other situations, the presence of the objects rather than the order in which they occur is important. Combination is the name given to a selection of objects with no importance to the order of the objects. Joining a point A to B or B to A we get the same straight line. When it is necessary to select three candidates for a job from among a set of aspirants for the job, the selection tells you who are in and in what order is unimportant.

The difference between the two concepts - permutation and combination is subtle but very crucial. Both teachers and students feel confused and are indecisive about, which is involved, when they have to solve a problem related to these ideas. In this direction, the difference explained must be clearly understood to ensure the solution of the given problem.

Combinatorial mathematics (or combinatorics) deals with many types of arrangements apart from those without repetitions. The result of this branch are applied to solve problems in probability, genetic engineering and life sciences.

Subunit No. 1.1: A Fundamental Counting Principle

I. Concept

You know that daily we do many things like - taking food, dressing up, travelling, etc. Suppose we have two activities to be done in some order and know that the activities can be done in m and n ways respectively (say), then we can find the total number of ways of doing both the activities one after the other using a Fundamental Counting Principle.

A Fundamental Counting Principle (or the Principle of Counting)

If the activity A can be done in m ways and after it, another activity B can be done in n ways, then the two activities A and B, in order, can be done in m x n ways.

The principle can be extended for more than two activities also.

II. Explanations

Activity B →	1	2	3	4	...	n	No.of ways
Activity A ↓							
1	(1,1)	(1,2)	(1,3)	(1,4)	...	(1,n)	n
2	(2,1)	(2,2)	(2,3)	(2,4)	...	(2,n)	n
3	(3,1)	(3,2)	(3,3)	(3,4)	...	(3,n)	n
. . .							
m	(m,1)	(m,2)	(m,3)	(m,4)	...	(m,n)	n
No. of ways	m	m	m	m	...	m	m x n

Here (1,2) means doing A by method 1 and doing B by method 2.

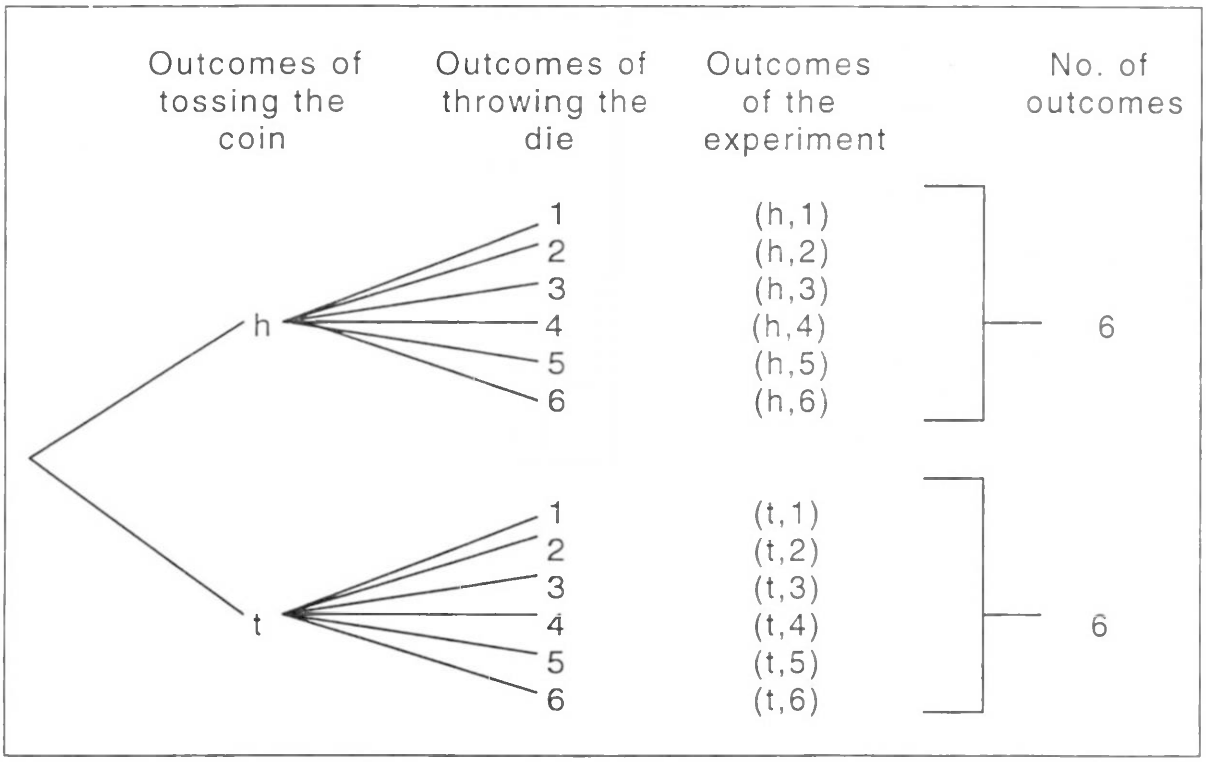
An Illustration

A coin is tossed. Afterwards, a die is thrown. Let us find the total number of outcomes of this experiment.

A coin has two faces: h (head) and t (tail)

A die has six faces : 1, 2, 3, 4, 5 and 6

If the outcome is head (of the coin) followed by four of the die, then we denote it by (h,4).



Total number of outcomes of the experiment = 12

Observe that 2x6 = 12

Hence the total number of outcomes of the experiment =
(the number of outcomes of tossing the coin)
x (the number of outcomes of throwing the die)

III. Examples

Study each example given below and identify the principle of counting

(i) A and B are connected by five routes and B and C by four routes. Then the number of ways of travelling from A to C enroute B is 5x4 = 20.

(ii) I throw a die twice. The total number of outcomes is 6x6 = 36.

(iii) The total number of 2-digit numbers in which the digits are non-repeating is 9x9 = 81.

(iv) The total number of two-digit numbers is 9x10 = 90.

(v) The total number of three digit numbers is 9x10x10 = 900.

(vi) The total number of 3-digit numbers with no digit repeated = 9x9x8 = 648

(vii) In a class there are 14 boys and 11 girls. The number of ways one boy and one girl pair can be formed is 14x11 = 154.

(viii) In a restaurant, 5 types of sweets, 4 types of snacks and 6 types of drinks are available. The number of ways a customer can order for a sweet, a snack and a drink is 5x4x6 = 120.

IV. Proof of the Counting Principle

There are m ways of doing the first activity. Corresponding to each way of doing the first activity, there are n ways of doing the second activity (because the second activity can be done in n ways). Hence the total number of ways of doing the two activities one by one is mxn.

V. Problem Solving

Problem solving is one of the most important learning activities. 'The proof of the pudding is in its eating' - they say. Likewise, problem solving ensures cent per cent learning. A few problems are solved below to illustrate the techniques of problem solving besides the use of the principles learnt in problem solving. Study each problem and its solution carefully.

1. A child has 5 red marbles and 4 blue marbles. In how many ways can the child pair a red marble with a blue marble.

Solution: To form a pair, you have to take one red marble from 5 red marbles. This can be done in 5 ways. After this, a blue marble has to be taken. This is possible in 4 ways ? Therefore the total number of ways of forming the pairs is 5x4 = 20 (how ?).

2. Find the total number of ways in which 3 players can score 3 goals in a foot ball match.

Solution: Let the players be A, B and C. The first of the 3 goals scored by them, can be scored in 3 ways. After this the second goal can also be scored in 3 ways since nothing prevents the same players from scoring both the goals. Then the first and second goals can be scored in 3x3=9 ways (how ?). Afterwards, the third goal also can be scored in 3 ways. Hence the three players can score the 3 goals in 9x3=27 ways (how ?).

Tail end question: How many times the counting principle has been used in the solution ?

3. A student has to solve 3 questions taking one from each of the parts A, B and C of a question paper. A has 4 questions. B has 6 questions and C has 5 questions. If the student can solve all of them, in how many ways can he answer the question paper ?

Solution: A question from part A may be chosen in 3 ways. After answering the part A, the number of ways of choosing part B question is 6. Therefore the number of ways of answering part A and part B = 3x6 = 18. The part C can be answered in 5 ways. Therefore the question paper can be answered in 18x5 = 90 ways.

4. An idol has four hands. There are four weapons of different types to decorate the hands of the idol and three crowns. Find the number of ways of decorating the idol.

Solution: There are 4 weapons. A hand can be adored by any of the four weapons, hence in 4 ways. After one weapon is given to one hand, there are 3 weapons and 3 hands. Another hand can be decorated in 3 ways. Likewise, the other two hands can be decorated in 2 ways and one way. Further there are 3 crowns. Hence the idol's head can be decorated in 3 ways.

Hence the number of ways of decorating the idol = 4x3x2x1x3 = 72.

Tail end question: How many time the counting principle has been used in the solution ?

5. A, B, C are three stations on a railway route. How many types of tickets are needed so that a passenger can travel from any station to any other station.

Solution: Each station needs two types of tickets (why ?). There are 3 stations in all. Hence the number of types of tickets required = 2x3 = 6.

VI. Activities

(i) List the daily life situations where the fundamental counting principle can be used.

(ii) Prove the counting principle extended to three activities.

VII. Exercise

(i) A coin is tossed 4 times in succession. Find the total number of outcomes, without listing all the outcomes.

(ii) A family has 3 members - husband, wife and a child. Husband has 4 dresses, wife has 6 dresses and the child has 9 dresses. In how many different ways, can the family members dress themselves up.

(iii) 0, 1, 2, 3 are written on four cards. Using these how many 3-digit numbers can be formed ?

(iv) In how many ways three places can be filled by 3 English letters: (a) if no letter is repeated ? and (ii) otherwise ?

(v) A factory produces some items. Its manufacture involves 3 processes A, B and C. There are 2 plants for A, 4 plants for B and 3 plants for C. In how many ways an item can be manufactured ?

(vi) A thief runs away from a place A to B and then to C after which takes a route beyond C to escape. While A and B are connected by 4 routes, B and C by 5 routes, there are 5 routes beyond C. In how many ways he will have escaped ?

(vii) There are 4 bridges across the river A and 7 bridges across the river B. In how many ways can one cross both the bridges once each ? In how many ways can he cross the bridges in succession and come back to the starting point ?

SUBUNIT NO. 1.2: PERMUTATIONS

I. Concept/Notations/Formulas

a. **Permutation**: An arrangement of objects in some order is called a permutation of the objects.

b. **An r-permutation** is a permutation of r objects.

c. **^nP_r or P(n,r)** denotes the total number of r-permutations out of n objects.

d. **$\lfloor\underline{n}$ or n!** denotes the product of the first n natural numbers 1, 2, 3, ..., n.

(i.e. $\lfloor\underline{n} = 1 \times 2 \times 3 \times \ldots \times (n-1) \times n$)

e. (i) $^nP_r = n(n-1) \ldots (n-r+1)$

(ii) $^nP_n = n(n-1) \ldots 3.2.1 = \lfloor\underline{n}$

(iii) $^nP_r = \lfloor\underline{n} / \lfloor\underline{n-r}$ $(r \leq n)$

(iv) $\lfloor\underline{0} = 1$

(v) $\lfloor\underline{n} = n\lfloor\underline{n-1}$

II. Explanations

(a) Objects of any kind which are arranged in some order is called a permutation of the objects. For example, '321' is a permutation of the digits 1, 2, 3. Likewise 'WORD' is a permutation of the letters W, O, R, D.

$$\boxed{\text{Permutation Means Arrangement}}$$

(b) Given a permutation, find how many objects make the permutation. Suppose there are r objects in the permutation. Then we call it as an r-permutation.

For example, the word 'STUDY' is a permutation of 5 letters. It is a 5-permutation of letters.

(c) Given n objects, the total number of r-permutations of these n objects is denoted by ^nP_r or P(n,r).

For example, consider the digits 1, 2, 3; the 2-permutations of these digits are - 12, 21, 13, 31, 23, 32 - six in all. This is denoted by $^3P_2$. Thus $^3P_2 = 6$.

Total number of r-permutations out of n objects = ^nP_r

(d) The product of the first n-natural numbers is 1x2x3x...x(n-1)xn. This is denoted by $\underline{|n}$ or n! and read as factorial n (or n factorial).

When it is necessary to find the value of $\underline{|n}$ it is convenient to write $\underline{|n}$ = nx(n-1)x...x2x1.

For example: $\underline{|5}$ = 5x4x3x2x1 = 120.

(e) To calculate ^nP_r, we need some formulas since it is not always possible to list all the permutations and count them. It is neither necessary to list all the permutations.

III. Examples

(i) WORD is a permutation of the 4 letters W, O, R, D. It is a 4-permutation.

(ii) 35617 is a permutation of the 5 digits 1, 3, 5, 6, 7. It is a 5-permutation.

(iii) Given the letters G, O, D, the two-permutations are GO, OG, GD, DG, OD, DO while the three-permutations are GOD, GDO, OGD, ODG, DGO and DOG.

(iv) EDUCATION is a 9-permutation of the letters of the word.

(v) 1986 and 9168 are two different 4-permutations of 1, 9, 8, 6. Likewise TEACH and CHEAT are two different 5-permutations of the letters in CHEAT.

(vi) Given a, b, c, the 3-permutations of the letters are abc, acb, bca, bac, cab and cba - six in all. Each is different from the others as permutations.

IV. Proofs/Verifications and Results

You know that formulas make calculations easy. In the absence of the formulas, calculations are tedious and time consuming. Thus to find the total number of r-permutations out of n-objects (which is denoted by ${}^nP_r$) we need formulas for ${}^nP_r$.

An example: Let us find the total number of 3-digit numbers formed by digits 1, 2, 3, 4, without repetition of any digit in the number formed.

Each 3-digit number is a 3-permutation of the digits. Let us now find ${}^4P_3$ which is the required number. The formation of a 3-digit number from 1, 2, 3, 4 is the same as filling the 100th, 10^{th}

and unit places of a 3-digit number. In how many ways can this be done ?

100th place	10th place	Unit place

There are 4 digits (given). Therefore the first place (i.e. the 100th place) can be filled in 4 ways. After this is done in one of the ways, the second place (i.e. the 10th place) can be filled in 3-ways. Therefore, the first and the second places can be filled in 4x3 (=12) ways (by the counting principle) when the first and second places are filled (by one digit each), the third place (i.e. the unit place) can be filled in 2 ways. Hence the total number of ways of filling the 3 places = The total number of 3-digit numbers which can be found = 4x3x2.

$$\text{Hence } {}^4P_3 = \underbrace{4 \times 3 \times 2}_{\text{3 factors}}$$

Note:

(i) On the L.H.S. 4 is at the top of P and the first factor on the R.H.S. is 4.

(ii) On the L.H.S., 3 is at the bottom of P and the number of factors on the R.H.S. is 3.

(iii) Successive factors starting with 4 (i.e. the first factor) are got by decreasing each factor by 1.

A generalisation of the above is

$$^nP_r = n(n-1)(n-2) \ldots \text{ r factors} \qquad (1)$$

The last factor in (1) on the R.H.S. is got as follows:

n, n-1, n-2, ..., form an A.P.

with first term n, C.D. = -1

$\therefore$ tr = n + (r - 1)(-1) = n - r + 1

is the rth factor (or the last factor) = n - r + 1

$\therefore$ (1) becomes $^nP_r = n(n-1)(n-2) \ldots (n-r+1)$ (2)

Note: ^nP_r has no meaning unless

(a) n and r are natural numbers and
(b) $r \leq n$

For example, $^3P_{-1}$, $^3P_4$, $^{-3}P_2$, $^{1/3}P_{1/2}$ have no meaning.

More formulas

A special case of (2)

In (2), putting r = n, (2) becomes

$^nP_n = n(n-1) \ldots (n-n+1) = n(n-1) \ldots 2.1$

But $n(n-1) \ldots 2.1 = \underline{|n}$

Hence $^np_n = \underline{|n} = 1 x 2 x \ldots x (n-1)n$ (3)

Another formula for ^nP_r (in terms of factorials)

$\underline{|n} = n(n-1)\ldots(n-r+1)\ (n-r)(n-r-1) \ldots 3.2.1$

$= \quad {}^nP_r \quad x \quad \underline{|n-r}$

$\therefore$ ^nP_r x $\underline{|n-r} = \underline{|n}$

$$\therefore\ ^nP_r = \frac{\underline{|n}}{\underline{|n-r}} \qquad (4)$$

(r = 1, 2, 3, 4, 5, ..., n)

Meaning of $\underline{|0}$

$\underline{|0}$ cannot be defined the same way as $\underline{|n}$ is defined for any natural number n, because it makes no sense. It does not mean that $\underline{|0}$ is meaningless. Let us find the value of $\underline{|0}$.

In (4), putting r = n, we get

$$^{n}P_{r} = \frac{\underline{|n}}{\underline{|n-r}} = \frac{\underline{|n}}{\underline{|0}}$$

But $^{n}P_{n} = \underline{|n}$

$$\text{Hence } \underline{|n} = \frac{\underline{|n}}{\underline{|0}}$$

$$\therefore\ \underline{|0} = 1 \qquad (5)$$

A recurrence formula for $\underline{|n}$

$\underline{|n} = n\underline{|n-1}$

$\underline{|n} = n(n-1)(n-2) \ldots 3.2.1$

$$\therefore\ \underline{|n} = n\underline{|n-1} \qquad (6)$$

In some problems this formula is very useful.

Tips to Teach

1. While there are 3 formulas for $^{n}P_{r}$, while teaching give the clues as to which formula is more convenient to me in a given problem.

Accordingly

(a) The formula (1) is more suitable for finding the values of $^{n}P_{r}$ in numerical examples. The advantages are (i) it cuts down calculations and (ii) does not tax the memory.

Example to find $^4P_2$

$^4P_2 = 4 \times 3$ Start from 4 (=n) and go 2 factors (=r)

$= 12$ decreasing successively by 1.

Otherwise

$$4P2 = \frac{\underline{|4}}{\underline{|4-2}} = \frac{\underline{|4}}{\underline{|2}} = \frac{4 \times 3 \times 2 \times 1}{2 \times 1} = 4 \times 3 = 12$$

2. Emphasise that $\underline{|0} = 1$ and reinforce the point by using it in some problems.

eg: Find n ($\neq 1$) such that $\underline{|n} = 1$.

$\underline{|n} = 1$ But $1 = \underline{|0}$ and $1 = \underline{|1}$

Since $n \neq 1$, $n = 0$

V. Problem Solving

Following problems and their solutions are discussed to draw your attention to the

(a) role of operating principles/concepts/formulas developed in the material, and

(b) logic which decides the sequence of steps leading to the solution.

In addition,

(c) short cuts to solution whenever possible are suggested to help students solve problems by easy methods.

1. Find the value of $^{10}P_5 + {}^5P_3 + {}^3P_1$

Solution: $^{10}P_5 = 10 \times 9 \times \ldots$ 5 factors

$= 10 \times 9 \times 8 \times 7 \times 6 = 30240$ (i)

$$^{5}P_{3} = 5 \times 4 \times 3 = 60 \qquad \text{(ii)}$$

$$\text{and} \quad ^{3}P_{1} = 3 \qquad \text{(iii)}$$

Hence $^{10}P_{5} + {}^{5}P_{3} + {}^{3}P_{1} = 30240 + 60 + 1 = 30301$

Tail end question: Which formulas are used and why ?

2. Find r: (i) $^{6}P_{r} = 240$; (ii) $^{12}P_{r} = 1320$

Strategy: Express the given value as product of consecutive natural numbers.

(i) $240 - 16 \times 15 = {}^{16}P_{2} \qquad \therefore {}^{16}P_{r} = {}^{16}P_{2} \qquad \therefore r = 2$

(ii) $1320 = 12 \times 110 = 12 \times 11 \times 10 = {}^{12}P_{3}$

$$\therefore {}^{12}P_{r} = {}^{12}P_{3} \qquad \therefore r = 3$$

Teaching Tips

(a) This method (of expressing the value as the product of consecutive natural numbers) is easier.

(b) It is always possible to express the value of $^{n}P_{r}$ as the product of consecutive natural numbers.

This being done, the number of factors in the product is r, while n is the biggest of the factors.

Tail end question: Try to solve the problems by any other method, if you can.

3. Find n: (a) $^{n}P_{2} = 182$; (b) $^{n}P_{3} = 720$; (c) $^{n}P_{4} = 1680$

Solution: (a) $182 = 14 \times 13 = {}^{14}P_{2}$

$$\therefore {}^{n}P_{2} = {}^{14}P_{2} \qquad \therefore n = 14$$

(b) $720 = 10 \times 9 \times 8 = {}^{10}P_3$

$\therefore {}^{n}P_3 = {}^{10}P_3 \qquad \therefore n = 10$

(c) $1680 = 8 \times 7 \times 6 \times 5 = {}^{8}P_4$

$\therefore {}^{n}P_4 = {}^{8}P_4 \qquad \therefore n = 8$

Note: In each problem, the value is expressed as the product of consecutive natural numbers. Then n is the biggest factor and r is the number of factors.

4. Find $\dfrac{\underline{|5} - \underline{|3}}{\underline{|5} + \underline{|3}}$

$\underline{|5} - \underline{|3} = 5 \times 4 \times 3 \times 2 \times 1 - 3 \times 2 \times 1 = 120 - 6 = 114$

$\underline{|5} + \underline{|3} = 120 + 6 = 126$

$$\therefore \frac{\underline{|5} - \underline{|3}}{\underline{|5} + \underline{|3}} = \frac{114}{126} = \frac{19}{21}$$

Tips to Teach: While teaching, to find $\underline{|n}$ teach to write $\underline{|n}$ as $n(n-1) \ldots \times 3 \times 2 \times 1$ rather than as $1 \times 2 \times 3 \times \ldots \times (n-1) \times n$. The advantage is - when $\underline{|n}$ is written as $n(n-1) \ldots \times 2 \times 1$ the last factor is obvious (namely 1).

5. Find the total number of permutations of the letters of the word -

(a) EQUATION, (b)

Solution: (a) In the given word, how many letters are there ? 8

What is the total number of permutations of 8 letters ?

$${}^{8}P_8 = \underline{|8}$$

Hence the total number of permutations of the letters of EQUATION is $\underline{|8} = 8x7x6x5x4x3x2x1 = 40320$.

(b) There are five letters. Therefore the total number of permutations of the letters of the word = $\underline{|5} = 5x4x3x2x1 = 120$.

6. Show that: (a) $\underline{|6} = 6x\underline{|5}$

(b) $\underline{|6} = 6x5x\underline{|4}$

Generalise the result.

Solution

$\underline{|6} = 6x5x4x3x2x1 = 6 \times \underline{|5}$ (i)

$\underline{|6} = 6x5x4x3x2x1 = 6 \times 5 \times \underline{|4}$ (ii)

Noting that in both (i) and (ii)

$\underline{|6} = 6 \times \underline{|6-1}$

and $\underline{|6} = 6 \times 5 \times \underline{|6-2}$

In general, $\underline{|n} = n(n-1) \ldots (n-r) \times \underline{|n-(r-1)}$

or $\underline{|n} = n(n-1) \ldots (n-r) \times \underline{|n-r+1}$

7. How many 3 digit numbers can be formed using (a) 1, 2, 3, 4; (b) 0, 1, 2, 3, without repeating any digit ?

Solution

(a) A 3-digit number is a 3 permutation out of the 4 given digits. Hence the number of 3 digit numbers formed = $^4P_3 = 4x3x2 = 24$.

(b) A three-digit number has a non-zero digit in the 100^{th} place always. Hence while counting 3-digit numbers we have to leave out those with 0 in the 100^{th} place.

Accordingly, total number of 3 permutations of 4 digits = ${}^4P_3 = 24$

and total number of 3 permutations out of 4 digits with 0 in the 100th place = ${}^3P_2 = 6$

Hence, the total number of 3-digit numbers = 24 - 6 = 18

Aliter: Let us form all possible 3 digit numbers using the given digits.

100th place	10th place	Unit place

Any digit except 0 can occupy the 100th place. Therefore, the first place can be filled in 3 ways (how ?). After this second place can be filled in 3 ways (how ?) and the third place can be filled in 2 ways.

Hence the total number of ways of filling the places =3x3x2=18

$\therefore$ The total number of 3 digit numbers = 18

Tail end question: Which principle is used in the solution for (b) in the activity (the second method).

8. Asha, Mary, Shakila, Vijaya are to speak in a debate, one after the other. How many different list of speakers can be drawn ?

   In how many (a) Asha is the first speaker

   (b) Asha speaks first and Vijaya last

   (c) Mary and Shakila do not speak one after the other (i.e. in succession)

Solution

The number of lists which can be drawn

$= {}^4P_4 = \underline{|4} = 4 \times 3 \times 2 \times 1 = 24$

(a) Here Asha is the first speaker, the other three places can be filled by the other 3 speakers in $\underline{|3} = 3 \times 2 \times 1 = 6$ ways.

Hence the number of lists in which Asha is the first speaker = 6

(b) When Asha is the first speaker and Vijaya is the last to speak, the other two places can be filled in $\underline{|2} = 2$ ways. Therefore, the number of lists when Asha and Vijaya are the first and last speakers respectively is 2.

(c) Here, we first find the number of lists in which Shakila and Mary speak one after the other (i.e. in succession). We take the two speakers as one speaker (as a strategy). Then the number of speakers is 3 and hence can be permutated in $\underline{|3} = 3 \times 2 \times 1 = 6$ ways. Shakila and Mary may be permuted among themselves in $\underline{|2} = 2$ ways. Hence the total number of lists when Shakila and Mary speak in succession = 6 x 2 = 12 (how ?)

Therefore the number of lists in which the two do not speak in succession = Total number of lists - Number of lists in which they speak in succession = 24 - 12 = 12.

VI. Activities

List the life situations in which permutation is involved.

VII. Exercises

1. Find the values of (a) ${}^{15}P_2$, (b) ${}^{10}P_4$, (c) ${}^{6}P_3 + {}^{3}P_2 - {}^{2}P_1$,

(d) $\dfrac{{}^{5}P_3 - {}^{3}P_2}{{}^{5}P_3 + {}^{3}P_2}$, (e) $\underline{|5} + \underline{|4} + \underline{|3}$, (f) $\dfrac{\underline{|6} - \underline{|0}}{\underline{|6} + \underline{|0}}$, (g) $\underline{|10} - 2\underline{|5}$

2. Find n given (a) ${}^{n}P_2 = 132$, (b) ${}^{n}P_3 = 504$, (c) $\underline{|n} = 120$.

3. Find r given (a) ${}^{13}P_r = 156$, (b) ${}^{10}P_r = 720$

4. Find the number of permutations of the letters of the words, without repetition of any letter (a) MYSOREAN, (b) COMPANY, (c) GRATEFUL, (d) SECTIONAL.

5. How many numbers can be formed using the digits 2, 3, 5, 7, 8 each once ? How many of them lie between 20,000 and 50,000 ?

6. In how many ways five friends can sit in a row ? In how many of these two friends A and B are side by side ?

7. Find the values of n, if (a) ${}^{2n}P_3 = 2 \times {}^{n}P_4$, (b) ${}^{n}P_4 = 42 \times {}^{n}P_2$.

8. Five mathematics books and four science books are to be arranged in a row on a bookshelf. Find the total number of arrangements if books on the same subjects must be together.

9. How many expressions can be formed using 2, 5, 3, 4 and +, -, x, in a meaningful way ?

10. Show that (a) $\underline{|2n} = 2^n \times \underline{|n} \times (1 \times 3 \times 5 \times \ldots \times (2n-1))$

(b) ${}^{n}P_r = n \times {}^{n-1}P_{r-1}$

SUBUNIT NO. 1.3: COMBINATIONS

I. Concept/Notations/Formulas

(a) A selection of objects without any regard for the order or the arrangement is called a combination of the objects.

(b) A combination of r objects is called as r-combination of the objects.

(c) Given n objects, the total number of r-combinations of objects taken out of the n objects is denoted by ${}^nC_r$ or $C(n, r)$ or $\binom{n}{r}$.

(d) Formula for ${}^nC_r$

(i) $${}^nC_r = \frac{{}^nP_r}{\underline{|r}}$$

(ii) $${}^nC_r = \frac{n(n-1)(n-2) \ldots (n-r+1)}{\underline{|r}}$$

(iii) $${}^nC_r = \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}$$ $(r \leq n)$ when n, r are whole numbers.

(e) Properties

(i) ${}^nC_r = {}^nC_{n-r}$ or ${}^nC_x = {}^nC_y \Rightarrow x=y$ or $x+y = n$

(ii) ${}^nC_0 = {}^nC_n = 1$

(iii) ${}^nC_1 = {}^nC_{n-1} = n$

(iv) ${}^nC_r = \binom{n}{r}\,{}^{n-1}C_{r-1}$

(v) ${}^nC_r = {}^nC_{r-1} = {}^{n+1}C_r$

II. Explanations

(a) When some objects are chosen from a set of objects the set of chosen objects is called a combination of the objects. Here, the order of the objects chosen does not matter.

To cite an example, suppose there are ten books of which we pick three books, the selection we made is called a combination of the three chosen books.

In a combination it is the presence of the objects, rather than the order of selection of the objects, which is important.

Combination means Selection

When we select three letters a, b, c out of four letters a, b, c, d the possible selections are abc, acd, abd, bcd. We note that abc, acb, bca, bac, cab, cba are the same as a combination of a, b and c, eventhough these are six different permutations.

(b) A selection of three objects is called a 3-combination of the objects. More generally, a selection of r objects is called an r-combination of the objects.

abc, acd, abd, bcd are distinct 3-combinations out of a, b, c, d while ab, ac, ad, bc, bd, cd are distinct 2-combinations out of a, b, c, d, abcd is a single 4-combination out of the letters a, b, c, d and a, b, c, d are 1-combination out of the four letters.

(c) In the above, count how many 3-combinations are there out of a, b, c, d, how many 2-combinations and how many 1-combinations.

These are denoted by $^4C_3$, $^4C_2$ and $^4C_1$ respectively. It is clear in the example that

$^4C_3 = 4$, $^4C_2 = 6$, $^4C_1 = 4$, $^4C_4 = 1$.

More generally, given n-objects, the total number of r-combinations out of the given n-objects is denoted by ^nC_r or $C(n,r)$ or $\binom{n}{r}$.

$\boxed{\therefore \text{ Total number of r-combinations out of n-objects} = {}^nC_r}$

(d) In order to find the value of ^nC_r we require formulas for ^nC_r, to facilitate calculations easy. Knowledge of some properties are required in problem solving.

III. Examples

1. Suppose there are 10 candidates for 4 jobs in an office, any selection of 4 candidates for the jobs is a 4-combination out of 10 candidates.

   The total number of ways of selecting 4 candidates out of 10 candidates in $^{10}C_4$.

2. If there are 20 players of whom a team of cricket 11 is to be formed. Then any team so formed is a 11-combination and the total number of ways of forming the team is $^{20}C_{11}$.

3. Suppose 15 points are marked and the points are joined in pairs. How many straight lines are got ? How many triangles with the given points and vertices are got ? Each straight line formed by joining two points is a 2-combination of the points and each

triangle with vertices among the given points is a 3-combination of the points. Then the total number of straight lines got is ${}^{15}C_2$ and the total number of triangles got is ${}^{15}C_3$.

4. Given a set of 6 elements, a subset of 4 elements is a 4-combination of the elements and the total number of subsets with four elements is ${}^{6}C_4$.
5. Given ten distinct numbers, the number of products of 3 of the given numbers which can be formed in ${}^{10}C_3$. Here each product is got from a 3-combination of the given numbers.

IV. Verifications/Proofs of Formulas and Results

(i) Proof of ${}^{n}C_r = \dfrac{{}^{n}P_r}{\underline{\lfloor r}}$

An example: Consider 4 letters a, b, c, d, we find the total number of 3-combinations out of the given 4-letters (i.e. the value of ${}^{4}C_3$).

Let $x = {}^{4}C_3$ = The total number of 3-combinations out of 4 letters.

Let us calculate the total number of 3-permutations in two ways. Consider a 3-combination. It has 3 letters. Here it gives rise to $\underline{\lfloor 3}$ 3-permutations. Hence n combinations give n x $\underline{\lfloor 3}$ permutations.

$\therefore$ The total number of 3-permutations out of 4 letters $= {}^{4}P_3$

$$= x \times \underline{\lfloor 3}$$

But this is also equal to ${}^{4}P_3$

$$\therefore\ x \times \underline{\lfloor 3} = {}^{4}P_3 \Rightarrow x = \frac{{}^{4}P_3}{\underline{\lfloor 3}}$$

$$\therefore {}^nC_r = \frac{{}^nP_r}{\lfloor\underline{r}}$$

General Proof

(i) Consider n-objects. We denote the total number of r-combinations out of these by ${}^nC_r$ while ${}^nP_r$ denotes the total number of r-permutations.

Taken an r-combination. It has r objects. Hence it can be permuted into $\lfloor\underline{r}$ permutations. Therefore ${}^nC_r$ combinations gives ${}^nC_r \times \lfloor\underline{r}$

Therefore ${}^nC_r \times \lfloor\underline{r} = {}^nP_r$

$$\therefore {}^nC_r = \frac{{}^nP_r}{\lfloor\underline{r}} \qquad (r \leq n) \qquad (1)$$

Note: In general ${}^nC_r \leq {}^nP_r$

(ii) Since ${}^nP_r = n(n-1)(n-2) \ldots (n-r+1)$

$$\therefore \Rightarrow {}^nC_r = \frac{n(n-1)(n-2) \ldots (n-r+1)}{\lfloor\underline{r}} \qquad (2)$$

(iii) Since ${}^nP_r = \dfrac{\lfloor\underline{n}}{\lfloor\underline{n-r}}$

$$\therefore \Rightarrow {}^nC_r = \frac{\lfloor\underline{n}}{\lfloor\underline{r}\ \lfloor\underline{n-r}} \qquad (3)$$

Properties of ${}^nC_r$

(i) In (3), replace r by (n - r)

$$\text{Then } {}^nC_{n-r} = \frac{|\underline{n}}{|\underline{n-r}\ |\underline{n-(n-r)}} = \frac{|\underline{n}}{|\underline{n-r}\ |\underline{r}} = {}^nC_r$$

$\therefore\ {}^nC_r = {}^nC_{n-r}$ (4)

Hence if ${}^nC_x = {}^nC_y$, then either $x = y$ or $y = n - x$ so that $x + y = n$

$\therefore\ {}^nC_x = {}^nC_y \Rightarrow x = y$ or $x + y = n$

(ii) In (3), taking $r = 0$

$$\therefore\ {}^nC_0 = \frac{|\underline{n}}{|\underline{n-0}\ |\underline{0}} = \frac{|\underline{n}}{|\underline{n}\ |\underline{0}} = \frac{1}{1} = 1$$

$$\text{Taking } r=n,\ {}^nC_n = \frac{|\underline{n}}{|\underline{n-n}\ |\underline{n}} = \frac{1}{|\underline{0}} = 1$$

$\therefore\ {}^nC_0 = {}^nC_n = 1$ (5)

(iii) In (3), taking $r = 1$,

$${}^nC_1 = \frac{|\underline{n}}{|\underline{1}\ |\underline{n-1}} = \frac{n|\underline{n-1}}{|\underline{n-1}} \text{ (how ?)}$$

$\therefore\ {}^nC_1 = n$

In (3) again taking $r = n-1$

$$\text{Then } {}^nC_1 = \frac{|\underline{n}}{|\underline{n-1}\ |\underline{n-(n-1)}} = \frac{|\underline{n}}{|\underline{n-1}\ |\underline{1}} = \frac{n|\underline{n-1}}{|\underline{n-1}} = n$$

$\therefore\ {}^nC_1 = {}^nC_{n-1} = n$ (6)

$${}^nC_r = \frac{n(n-1)(n-2)\ \dots\ (n-r+1)}{|\underline{r}}$$

$$= \frac{n}{r}\left[\frac{(n-1)(n-2)\ \ldots\ (n-(r-1))}{\underline{|r-1}}\right]$$

$$\therefore\quad {}^{n}C_{r} = \left(\frac{n}{r}\right)^{n-1}C_{r-1} \qquad \text{(how ?)}$$

(v) Verify ${}^{n}C_{r} + {}^{n}C_{r-1} = {}^{n+1}C_{r}$

Taking $n = 10,\ r = 6$

$$\text{L.H.S.} = {}^{10}C_{6} + {}^{10}C_{5} = \frac{\underline{|10}}{\underline{|6}\ \underline{|4}} + \frac{\underline{|10}}{\underline{|5}\ \underline{|5}} \qquad \text{(how ?)}$$

$$= \underline{|10}\left[\frac{1}{6\underline{|5}\ \underline{|4}} + \frac{1}{5\underline{|4}\ \underline{|5}}\right] = \frac{\underline{|10}}{\underline{|5}\ \underline{|4}}\left[\frac{1}{6} + \frac{1}{5}\right]$$

$$\therefore\ \text{L.H.S.} = \frac{\underline{|10}}{\underline{|5}\ \underline{|4}} \times \frac{11}{6 \times 5}$$

$$= \frac{11\underline{|10}}{6\underline{|5} \times 5\underline{|4}} = \frac{\underline{|11}}{\underline{|6} \times \underline{|5}} \qquad \text{(how ?)}$$

$$\text{R.H.S.} = {}^{10+1}C_{6} = {}^{11}C_{6} = \frac{\underline{|11}}{\underline{|6}\ \underline{|11-6}} = \frac{\underline{|11}}{\underline{|6}\ \underline{|5}}$$

$$\therefore\ \text{L.H.S.} = \text{R.H.S.}$$

Aliter

$${}^{10}C_{6} = {}^{10}C_{4} = \frac{10 \times 9 \times 8 \times 7}{1 \times 2 \times 3 \times 4} = 210$$

$${}^{10}C_{5} = \frac{10 \times 9 \times 8 \times 7 \times 6}{1 \times 2 \times 3 \times 4 \times 5} = 252$$

$$^{11}C_6 = {}^{11}C_5 = \frac{11\times10\times9\times8\times7}{1\times2\times3\times4\times5} = 462$$

Clearly 210 + 252 = 462

or $^{10}C_6 + {}^{10}C_5 = {}^{11}C_6$

V. Problem Solving

While teaching problem solving, draw the attention of the students to -

(a) the key idea to solve the problem

(b) the technique you use and why

1. Find the values of

(a) $^{8}C_3$ (b) $^{6}C_2$ (c) $({}^{10}C_3 + {}^{5}C_2)$ and (d) $^{9}C_3 - {}^{7}C_3$

Solution: Key idea - Formulator $^{n}C_r$

Tips to Teach: Though you can use any of the formula for $^{n}C_r$ to reduce the calculations, we use

$$^{n}C_r = \frac{n(n-1)(n-2)\ \dots\ (n-r+1)}{\underline{|r}}$$

Accordingly, (a) $^{8}C_3 = \dfrac{8\times7\times6}{1\times2\times3} = 56$

(b) $^{6}C_2 = \dfrac{6\times5}{1\times2} = 15$

(c) $^{10}C_3 + {}^{5}C_2 = \dfrac{10\times9\times8}{1\times2\times3} + \dfrac{5\times4}{1\times2}$

$= 120 + 10 = 130$

$$\text{(d) } {}^9C_3 + {}^7C_3 = \frac{9\times8\times7}{1\times2\times3} - \frac{7\times6\times5}{1\times2\times3}$$

$$= 84 - 35 = 49$$

Draw the attention to

1. In using the formula $^nC_r = \dfrac{n(n-1)(n-2)\ \dots\ (n-r+1)}{\underline{|r}}$

(a) On the R.H.S. of the formula, n is the first factor and there are r-factors in the numerator.

(b) The factors in the numerator are got by decreasing 1 each time.

(c) The denominator $\underline{|r} = 1\times2\times\ldots\times r$

2. Find the value of (a) $^{100}C_2$ (b) $^{100}C_{99}$

Solution: (a) $^{100}C_2 = \dfrac{100 \times 99}{1 \times 2} = \dfrac{9900}{2} = 4950$

(b) $^{100}C_{99} = {}^{100}C_{100-99} = {}^{100}C_1 = 100$

$(\because {}^nC_r = {}^nC_{n-r})$

Tips to Teach: When n is big and r is close to n, you can avoid unnecessary calculations, using the property $^nC_r = {}^nC_{n-r}$

3. Find n, if $^nC_3 = {}^nC_5$. Hence find $^nC_4$.

Solution: We know that $^nC_x = {}^nC_y \Rightarrow x = y$ or $x+y = n$

Given $^nC_3 = {}^nC_5$, since $3 \neq 5$, $n = 3+5 = 8$

$\therefore n = 8$ $\qquad \therefore {}^nC_4 = {}^8C_4 = \dfrac{8\times7\times6\times5}{1\times2\times3\times4} = 70$

4. Find r, if (a) ${}^{10}C_r = {}^{10}C_{12-2r}$ (b) ${}^{10}C_r = {}^{10}C_{12-5r}$

Solution: (a) ${}^{n}C_x = {}^{n}C_y \Rightarrow x = y$ or $x + y = n$

Using this, either $r = 12 - 2r$ or $r + (12 - 2r) = 10$

$r = 12 - 2r$ $\quad\quad$ $r + 12 - 2r = 10$

$\Rightarrow 3r = 12$ $\quad\quad$ $\Rightarrow 12 - r = 10$

$\therefore r = 4$ $\quad\quad$ $\therefore r = 2$

Hence r = 2 or 4

(b) Using the same property,

either $r = 12 - 5r$ or $r + 12 - 5r = 10$

$\therefore 6r = 12$ $\quad\quad$ or $12 - 4r = 10$

or $r = 2$ $\quad\quad$ $4r = 2$

or $r = ½$

Since r cannot be a fraction as ½, r = 2

5. Ten points marked on a paper are joined in pairs. How many straight lines are got (a) if no three points are collinear, (b) if six points are collinear.

Solution

Key idea: A line is a 2-combination of points.

(a) Ten points are given. 2 points joined give a line.

$$\therefore \text{The total number of lines} = {}^{10}C_2 = \frac{10 \times 9}{1 \times 2} = 45$$

(b) In this case, the six collinear points give only one line. If the six points were not collinear, we get $^6C_2 = \frac{6 \times 5}{1 \times 2} = 15$

$\therefore$ The total number of lines = 45 - 15 + 1 = 31 lines

6. A polygon has 90 diagonals. How many vertices are there for the polygon.

Solution

Key idea: When all straight lines are got by joining the vertices of the polygon, the diagonals are those other than the sides.

Let the polygon have n vertices.

Therefore it has n sides.

Total number of lines got = Number of sides + Number of diagonals

$\therefore\ ^nC_2 = n + 90$

$$\therefore \frac{n(n-1)}{1 \times 2} = n + 90$$

$\therefore n^2 - n = 2n + 180$

$\therefore n^2 - 3n - 180 = 0$

$\therefore n^2 - 15n + 12n - 180 = 0$

$\therefore n(n-15) + 12(n-15) = 0$

$\therefore (n+12)(n-15) = 0$

$\therefore n = -12$ or $n = 15$

n being the number of sides of the polygon, cannot be negative.

i.e. $n \neq -12$ $\quad \therefore n$ = The number of vertices of the polygon = 15

7. Find n, if (a) ${}^{n}C_2 = 21$ (b) ${}^{n}C_3 = 120$ (c) ${}^{2n}C_3 = 24 \times {}^{n}C_4$

Solution

(a) ${}^{n}C_2 = 21$

$$\Rightarrow \frac{n(n-1)}{1 \times 2} = 21 \qquad \therefore n(n-1) = 42$$

or $n(n-1) = 7 \times 6$

$\therefore n = 7$

Strategy used: Expressing 42 as the product of two consecutive positive integers. Then the bigger number is n.

(b) ${}^{n}C_3 = 120$

$$\Rightarrow \frac{n(n-1)(n-2)}{1 \times 2 \times 3} = 120$$

$\Rightarrow n(n-1)(n-2) = 720 = 10 \times 9 \times 8$

$\therefore n = 10$

(c) ${}^{2n}C_3 = 24 \times {}^{n}C_4$

$$\Rightarrow \frac{2n(2n-1)(2n-2)}{1 \times 2 \times 3} = 24 \times \frac{n(n-1)(n-2)(n-3)}{1 \times 2 \times 3 \times 4}$$

$\Rightarrow 4n(2n-1)(n-1) = 6n(n-1)(n-2)(n-3)$

$\Rightarrow 4(2n-1) = 6(n-2)(n-3)$

$\Rightarrow 4n - 2 = 3n^2 - 15n + 18$

$\Rightarrow 3n^2 - 19n + 20 = 0$

$\Rightarrow 3n^2 - 15n - 4n + 20 = 0$

$\Rightarrow 3n(n-5) - 4(n-5) = 0$

$\Rightarrow (3n - 4)(n - 5) = 0$

$\therefore n = 4/3$, $n = 5$

Since n is always an integer, $n \neq 4/3$

$\therefore n = 5$

8. A team of cricket eleven has to be formed from 20 players of whom Ajit is one. In how many ways the team can be formed. In how many ways the team can be formed so as to (a) include Ajit. (b) exclude Ajit.

Solution

The total number of ways of forming the team of eleven = ${}^{20}C_{11}$

(a) If the team is to contain Ajit, the choice has to made from the other 19 players to select 10 players.

This can be done in ${}^{19}C_{10}$

$\therefore$ The total number of ways of forming the team to contain Ajit=${}^{19}C_{10}$

(b) If the team does not contain Ajit, such a team is to be formed from 19 players (excluding Ajit) selecting all the 11 of the team. This can be done in ${}^{19}C_{11}$.

$\therefore$ The total number of ways of forming the team excluding Ajit=${}^{19}C_{11}$

Note: ${}^{19}C_{10} + {}^{19}C_{11} = {}^{20}C_{11}$ because any team formed either includes Ajit or does not include Ajit.

9. There are 25 vacant seats in a double decker bus. 18 in the lower and 7 in the upper deck. Six persons refuse to go up while 3 persons insist on going up. Find the total number of ways in which 25 persons can be accommodated in the bus.

Solution

There are 18 vacant seats in the lower deck (L)

and 7 vacant seats in the upper deck (U)

Six are to be accommodated in lower only as they refuse to go up. Therefore this leaves behind only 12 vacant seats in lower. Among 25 passangers, 6 are already in lower. While 3 insist on going up. Hence there are 25 - 6 - 3 = 16 persons from whom the choice has to be made for 12 vacant seats in lower. Therefore this can be done in ${}^{16}C_{12}$.

Hence the number of ways of accommodating the passengers is

$$ {}^{16}C_{12} = {}^{16}C_4 = \frac{10 \times 15 \times 14 \times 13}{1 \times 2 \times 3 \times 4} = 1820 $$

VI. Activities

List life situations involving the concept of combination.

VII. Exercises

1. Find the value of

(a) ${}^{8}C_3$ (b) ${}^{11}C_4$ (c) ${}^{10}C_6 + {}^{6}C_2$ (d) ${}^{5}C_2 + {}^{4}C_3 - {}^{3}C_1$

(e) $({}^{6}C_4 \div {}^{5}C_3) \times {}^{6}C_1$ (f) ${}^{5}C_5 \times {}^{3}C_3 + {}^{4}C_2 \times {}^{3}C_2 - {}^{2}C_1 \times {}^{3}C_1$

2. (a) Find n: (i) ${}^{n}C_{2} = 105$ (ii) ${}^{n}C_{3} = 84$ (iii) ${}^{n}C_{5} = {}^{n}C_{4}$

   (b) Find r: (i) ${}^{9}C_{r} = {}^{9}C_{r-1}$ (ii) ${}^{10}C_{r} = {}^{10}C_{r-2}$ (iii) ${}^{15}C_{r+3} = {}^{15}C_{2r-3}$

   (c) If ${}^{n}C_{13} = {}^{n}C_{8}$, find ${}^{n}C_{10}$.

3. Show that ${}^{2n}C_{n} = 1.3.5...(2n-1) \times 2^{n}/\underline{|n}$

4. Find n and r given ${}^{n}P_{r} = 3024$. ${}^{n}C_{r} = 126$.

5. Show that the product of any three consecutive integers is divisible by 6. [Hint: Use ${}^{n}C_{3} = \dfrac{{}^{n}P_{3}}{\underline{|3}}$]

6. Given 12 points, how many lines can be got by joining them in pairs if (a) no three points are collinear ? (b) 5 points are collinear ?

7. Given 12 points, no three of which lie in a straight line are joined in pairs. how many triangles with vertices as given points are got?

8. A polygon has 54 diagonals, how many sides are there for the polygon.

9. A bag has 6 silver coins and 8 gold coins. In how many ways 6 coins can be drawn so as to have (a) equal number of silver and gold coins ? (b) number of gold coins is more than the number of silver coins and there is at least one silver coin in the draw ?

10. Committee of 5 is to be formed selecting teachers and students of a school. There are 4 teachers and 6 students willing to work on the committee. In how many ways the committee can be formed. If the committee must have atleast one teacher and one

student, in how many ways selection can be made so that the number of students exceed the number of teachers in the committee.

A scheme of finding the values of ^nC_r for various values of n and r

PASCAL'S TRIANGLE

n=1	1							
n=2	1	2	1					
n=3	1	3	3	1				
n=4	1	4	6	4	1			
n=5	1	5	10	10	5	1		
n=6	1	6	15	20	15	6	1	
n=7	1	7	21	35	35	21	7	1

You can continue this pyramid.

In the above

(1) Each row-entries give the values of ^nC_r for that value of n shown and r = 0, 1, 2, ..., (n-1), n.

(2) To get entries from one row to another (next) row

(a) Start with 1.

(b) Add the adjacent numbers to get the next numbers.

(c) The moment number reaches the biggest value. The values are repeated in the reverse order.

UNIT NO. 2: PROBABILITY

Unit Introduction

Certain experiments are such that the result (or outcome) of the experiment is predictable before the experiment is conducted. In such cases, the conclusion of the experiments are foregone or determined. Preparation of a gas in a laboratory, verifying a physical law are experiments of this type. An experiment whose outcome can be predicted with hundred per cent certainity is called a deterministic experiment.

In contrast, there are experiments whose outcome in a trial cannot be predicted with certainity. These experiments may result in many possible outcomes. For example, a fair coin which is tossed - the outcome are head(h) or tail(t) or a die numbered 1 to 6 thrown with possible outcomes 1 to 6. There are examples which are of interest in probability theory. An experiment with many possible outcomes whose outcome cannot be predicted with certainty is called a random experiment or a probabilistic experiment (also known as stochastic experiment). What prevents one predicting the outcome of a random experiment is the uncertain factor called the chance factor. The likelihood of a certain event taking place in a random experiment can be estimated. Accordingly, we determine the chance factor for an event to succeed and call it as the probability of the event.

In this unit on probability we discuss some of these aspects. Historically, probability theory began with the games of chances on cards. Pascal and Fermat are considered as founding father of the theory though many mathematicians enriched the subject by their contributions from time to time. Probability theory finds its applications in many branches of knowledge - Genetic engineering, Insurance, actureal mathematics - to name a few.

SUBUNIT NO.2.1: RANDOM EXPERIMENTS AND EVENTS

I. Concepts/Notations/Terminology

a. **Random Experiment**: An experiment whose outcome cannot be predicted with hundred per cent certainty.

b. **Sample Space**: For a random experiment, when we consider all possible outcomes, the set of these possible outcomes of the random experiment is called the sample space of the random experiment. We denote it by S.

c. **Event**: An event in a random experiment is any subset of the sample space S. Denoting an event by E, $E \subset S$.

d. **Elementary Event**: A subset of the sample space with a single element is an elementary event.

If $S = \{e_1, e_2, \ldots, e_n\}$, then $\{e_1\}$, $\{e_2\}$ ... $\{e_n\}$

are all elementary events. Then S is the union of all the elementary events e_i or it can also be defined as the set of all elementary events of the experiment.

e. **Favourable Elementary Event to a given Event**: Given an event E and an elementary event e, e is said to be favourable to E, if E has taken place (i.e. E is a success) when the outcome of the trial is e.

   Therefore E = The subset of S, consisting of all the elementary events favourable to E.

f. **Impossible and Certain (Sure) Events**: In a random experiment, an event which can never take place is called an impossible event, while an event which is bound to take place (with certainity) is called a certain (or Sure) event.

g. **Mutually Extensive Events**: Two events E and F which cannot take place simultaneously are called mutually exclusive events.

h. **Complementary Events**: Given an event E, the event 'E does not occur' is called the complement of E denoted by E'. The event E and E' are mutually complementary and hence they are called complementary events.

II. Explanations

a. If you toss a fair coin, the outcome is head or tail. However, you cannot predict the outcome of a trial. This is an example of a random experiment. In such an experiment while there are many possible outcomes, the outcome cannot be predicted with hundred per cent certainity.

b. In a random experiment, the set of all possible outcomes is the sample space S of the random experiment. Then if the experiment is tossing a coin repeatedly twice, then S={hh,ht,th,tt}.

c. In the random experiment of tossing a coin twice, the event head appears once is E = {ht,th} $\subset$ S. More generally, any subset of the sample space of an experiment is an event.

Note: For a given random experiment, the sample space S may be a finite set or an infinite set. Here, however, S is always taken as a finite set. The number of elements of S being finite, it is denoted by n(S) and is called the order of S.

d. An event being a subset of the sample space S, if the event is a set of single element (then it is called a singleton set), then it is called an elementary event. The sample space S of tossing a coin is made of two elementary events {h} and {t}, since S = {h,t}. Hence the elementary events are assumed to be equally likely.

e. Consider the random experiment of throwing a die numbered 1 to 6. Consider the event E = an even number appears. Clearly E = {2,4,6}. E takes place if the outcome is 2 or 4 or 6. In other words the appearance of each of these elements of S means the success of E. Therefore 2, 4 and 6 are each said to be favourable to E.

f. In the above experiment, the event '0 occurs' cannot occur at all. In other words no element of S is favourable to E. Consequently, the subset representing this event is the null set ϕ. Therefore the null set $\phi(\subset S)$ represents the impossible event.

On the other hand, the event 'a natural number less than 7 occurs' is bound to occur since an outcome is one of 1, 2, 3, 4, 5, 6. In this case all elements of S are favourable to the event. Therefore the subset denoting this event is S itself. Therefore the set S ($\subset$ S) represents the certain (or Sure) event.

g. In the experiment - 'tossing a coin thrice', consider

E = head appears once

and F = head appears at least twice

S = {hhh,hht,hth,thh,htt,tht,tth,ttt}

Then E = {htt,tht,tth}

and F = {hhh,hht,thh,hth}

It is evident that if E occurs, then F does not occur and vice versa. In other words, E and F do not occur simultaneously. These events are mutually exclusive events. Further $E \cap F = \phi$.

In the above experiment, consider the events

E = head appears once and

F = head appears in the first toss.

Hence E = {htt,tht,tth}

and F = {hhh,hht,hth,htt}

If the outcome is htt, both E and F have occurred. Then E and F are not mutually exclusive.

h. Given an event E, the event 'E does not occur' is called the complement of E. For example, when a die is thrown, suppose E = An even number occurs.

Then E' = An event number does not occur

i.e. E' = An odd number occurs

It is evident that the complement of E' is E. Then E and E' are complementary events.

Note that E, E' being subsets of S,

(a) $E \cup E' = S$ (b) $E \cap E' = \phi$ so that E and F are mutually exclusive. These properties help to test whether two given events are complementary or not.

An Important Point to be Remembered

Complementary events are always mutually exclusive but not conversely.

III. Examples

1. In the random experiment of throwing a die numbered 1 to 6.

   (i) S = The sample space = {1, 2, 3, 4, 5, 6}

   (ii) E = {1, 2}, F = {1, 3, 4}, G {2, 4, 5, 6} are all events.

   (iii) The event 'the number appearing in 7' is impossible while the event 'the number appearing in a positive integer less than or equal to 6' is a certain event.

(iv) Let E = The number appearing is a prime number = {2, 3, 5}

and F = The number appearing is a composite number

= {4,6}

E and F are mutually exclusive

Let E = The number is even = {2, 4, 6}

and F = The number is a multiple of 3 = {3, 6}

If the outcome is 6. E and F both have occurred. Simultaneously items E and F are not mutually exclusive.

(v) If E = The number is less than or equal to 4

= {1, 2, 3, 4}

Then E' = The number is greater than 4

= {5, 6}

E and E' are complementary events.

2. In the random experiment - A die is tossed; then according as an odd number appears or even number appears in the first toss (of the die), the die is again tossed or a coin is tossed.

For this experiment, the sample space is

(i) S = {(1,1),(1,2),(1,3),(1,4),(1,5),(1,6)

(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)

(5,1),(5,2),(5,3),(5,4),(5,5),(5,6)

(2,h),(2,t),(4,h),(4,t),(6,h),(6,t)}

ii. E = {(1,1),(2,h)}, F = {(3,1),(3,2),(3,4)} are events.

iii. S is made up of 24 elementary events.

iv. The event 'the outcome is an even number followed by any number' is impossible. The event 'the outcome consists of a number followed by either a number or head or tail' is a certain event.

v. E = An odd number followed by a number

F = An even number followed by head

are mutually exclusive.

vi. If E = An odd number followed by a number

then E' = An event number followed by head or tail

3. From an urn containing four balls - Red (R), Blue (B), White (W) and Yellow (Y), two balls are drawn randomly.

i. Here S = {RB, RW, RY, BW, BY, WY}

ii. E = One of the balls drawn is Red = {RB, RW, RY}

F = None of the balls is Red = {BW, BY, WY}

are events.

iii. The event 'both balls are of the same colour' is impossible but the event 'the balls are of different colours' is a certain event.

iv. The events E and F in (ii) are mutually exclusive and complementary as well.

IV. Verify the following properties

(a) If E and F are mutually exclusive, then $E \cap F = \phi$.

(b) Complementary events are mutually exclusive.

(c) Give an example that mutually exclusive events need not be complementary.

(d) Impossible and certain events are complementary. Consider the experiment of throwing a die numbered 1 to 6.

$S = \{1, 2, 3, 4, 5, 6\}$

(i) Consider $E = \{2, 3\}$ and $F = \{4, 6\}$

E = The event 'the number appearing is a prime no. ≤ 3

and F = The event 'the number appearing is an even no. ≥ 4.

Clearly E and F are mutually exclusive. As subset of S, E and F are disjoint, i.e. $E \cap F = \phi$.

b. In the same experiment, E and F given in (a) are mutually exclusive but not complementary, because, the event 'one appears' excludes both E and F.

c. By definition of an impossible event ϕ, no elementary event is favourable to ϕ. Hence ϕ' means every elementary event is favourable. $\therefore \phi' = S$ = Certain event.

IV. Problem-Solving

1. A family of two children is visited randomly and the sex of the children recorded. For this random experiment, write the sample space S. Give an example of (a) an impossible event, (b) a certain event, (c) mutually exclusive events, (d) events which are not mutually exclusive and (e) complementary events.

Solution

The possible outcomes are

boy, boy → bb, boy, girl → bg

girl, boy → gb, girl, girl → gg

∴ S = {bb, bg, gb, gg}

a. The event 'the children are neither of the same sex nor of opposite sex' is an impossible event.

b. The event 'the children are either of the same sex or of opposite sex' is a certain event.

c. E = both are boys; F = both are girls are mutually exclusive.

d. E = At least one is a boy; F = Children are of opposite sex are not mutually exclusive.

e. If E = Both are of the same sex and F = Children are of opposite sex, then E and F are complementary events.

2. A cube has 1-red, 2-blue and 3-white faces. Write the sample space for the experiment of tossing the cube.

   Solution: S = {R, B, W} when R = Red face, B = Blue face and W = White face.

3. A die numbered one to six is tossed twice write the sample space S. Also write the following events as subsets of S.

   (a) A = Sum of nos. is 10

   (b) B = Product is 6

   (c) C = The nos. are successive integers

Solution

$S = \{(a,b)|a,b=1,2,3,4,5,6\}$

(a) $A = \{(4,6), (6,4), (5,5)\}$

(b) $B = \{(1,6), (2,3), (3,2), (6,1)\}$

(c) $C = \{(1,2), (2,3), (3,4), (4,5), (5,6)\}$

4. A coin is tossed repeatedly thrice. Write S and the events (a) 'there are two heads' and (b) 'alternate faces occur' as subsets of S. (c)

   Give an example of mutually exclusive events.

Solution

S = {hhh,hht,hth,thh,tth,tht,htt,ttt}

where h = head and t = tail

(a) The event = 'there are two heads'

= {hht,hth,thh}

(b) The event = 'alternate faces occur'

= {hth, tht}

(c) E = {hhh,ttt}, F = {hth, tht} are mutually exclusive.

5. Numbers 1, 2, 3 are written on three cards and are arranged to read a 3-digit number, randomly. Write the sample space and write the events -

   A = The number formed is odd

   B = The number formed is between one hundred and two hundred
   as subsets of S. Give an example of a certain and an impossible event.

S = {123, 132, 213, 231, 312, 321}

$A = \{123, 213, 231, 321\}$

$B = \{123, 132\}$

The event - the number formed is a multiple of 3 is a certain event, because any event is formed by 1, 2, 3 whose sum is a multiple of 3.

The event 'the number formed is less than 123' is impossible, because the smallest no. with digits 1, 2, 3 is 123.

6. Two balls are drawn from a bag containing four balls of different colours. How many elements are there in S. Give an example of (a) an impossible event, (b) a certain event.

Solution

The number of ways of drawing two balls out of four balls

$$= {}^4C_2 = \frac{4 \times 3}{1 \times 2} = 6$$

$\therefore n(S)$ = The number of elements in the sample space S

$= 6$

The event 'both balls are of the same colour' is impossible while the event 'the balls are of different colours' is a certain event.

7. From a pack of 52-playing cards, 2 cards are drawn. Find n(S). Give examples for the following events (a) two events which are not mutually exclusive, (b) two events which are mutually exclusive but not complementary, (c) two complementary events.

Solution

2-cards out of 52-cards can be drawn in $^{52}C_2$ ways. Hence

$$n(S) = {}^{52}C_2 = \frac{52 \times 51}{1 \times 2}$$

$\therefore$ n(S) = 1326

(a) A = One of the cards is red

B = One of the cards is a king

A and B are not mutually exclusive, i.e. they can occur simultaneously. If one of the cards drawn is a red king, then both A and B have taken place.

(b) A = Both are red cards

B = Both are black cards

Hence A, B are mutually exclusive but not complementary (Why ?)

(c) A = Both are of the same colour

B = The cards are of different colours.

A and B are complementary events.

V. Activities

(i) List daily life situations involving the chance (randomness) factor.

(ii) Construct examples of random experiments.

VI. Exercises

(i) Give five examples for random experiments.

(ii) Give examples for the following types of events if the random experiment is tossing a 25 ps. 50 ps and 1 Re coins simultaneously, explaining the notations used.

(iii) A die numbered 1 to 6 is thrown. For this experiment, write the sample space S in the rule-method. Express the following events as subsets of S.

(a) A = The sum of the numbers is 8.

(b) B = The product of the numbers is 8.

(c) C = The difference of nos. is 3.

Among these which events are mutually exclusive and which are not mutually exclusive.

4. Twelve cards of which 4 are white, 3 are red and 5 are blue of the same size are shuffled very well. Then 3 cards are blindly drawn. Name the events mentioned below.

(a) The cards are of different colours.

(b) One of the cards drawn is black.

(c) Each card has one of the colours, White, Red, Blue.

In the above experiment give examples for

(i) mutually exclusive events

(ii) events which are not mutually exclusive

(iii) complementary events

5. Define the terms in your own words and give one example for each of them.

(a) Random experiment

(b) Sample space of a random experiment

(c) Event

(d) Elementary event

(e) Mutually exclusive events and

(f) Complementary events.

SUBUNIT NO. 2.2: PROBABILITY OF AN EVENT

I. Concepts/Notations/Formulas

In what follows, the assumptions made are -

(a) The sample space of the random experiment is finite. If S has s elements, the number of elements of S is denoted by n(S) = s. The elements of S are taken as e_i, i = 1, 2, ... , s.

(b) The elementary events are equally likely. This means likelihood of any elementary event occurring is equal to the likelihood of any other elementary event.

Definition (and Notation) of the Probability of an Event

1. Let A be an event in a random experiment whose sample space is S. Then

$$P(A) = \frac{\text{The total number of elementary events favourable to A}}{\text{The total number of elementary events}}$$

$$P(A) = \frac{n(A)}{n(S)}$$

2. Properties of P(A)

(i) $0 \leq P(A) \leq 1$

(ii) $P(e_i) = i/n(S)$ for all i

(iii) P(impossible event) = 0

(iv) P(certain event) = 1

(v) $P(A) + P(A') = 1$ or $P(A') = 1 - P(A)$

II. Explanations

The assumptions stated, namely that S is finite and that all the elemenary events are equally likely are the basis on which the probability of an event is defined. Because of the second assumption, the random experiment is called a fair experiment.

(i) The probability of an event A is defined as the ratio of the number of elementary events favourable to A to the total number of elementary events.

$$\text{Accordingly, } P(A) = \frac{n(A)}{n(S)}$$

(ii) Since $n(A) \geq 0$ and $n(A) \leq n(S)$,

$0 \leq n(A) \leq n(S)$

Dividing this inequality by $n(S)$, we get

$$0 \leq \frac{n(A)}{n(S)} \leq 1$$

or $\boxed{0 \leq P(A) \leq 1}$

(iii) If A is an impossible event, then no elementary event is favourable to A. This means $n(A) = 0$

$\therefore \quad P(A) = 0$

Hence, $\boxed{P(\text{impossible event}) = 0}$

(iv) If A is a certain event, then every elementary event is favourable A, so that $n(A) = n(S)$.

$$\therefore \quad P(A) = \frac{n(A)}{n(S)} = 1$$

Hence P(certain event) = 1

(v) Given an event A, its complementary event 'not A' is A'. The sum of the probabilities of A and A' is 1. Equivalently the probability of A' = 1 - Probability of A.

i.e $\boxed{P(A') = 1 - P(A)}$

III. Examples

1. Let the experiment be 'tossing a coin'.

Then S = {h, t} Then n(S) = 2

Then $P(\text{head appears}) = \dfrac{1}{n(S)} = \dfrac{1}{2}$

Then $P(\text{tail appears}) = \dfrac{1}{n(S)} = \dfrac{1}{2}$

2. Suppose a coin is tossed twice (equivalently, two dissimilar or distinguishable coins are tossed).

Then S = {hh, ht, th, tt}. Then n(S) = 4.

(i) $P\ (\text{both are heads}) = \dfrac{1}{n(S)} = \dfrac{1}{4}$

(ii) $P\ (\text{both faces are identical}) = \dfrac{2}{n(S)} = \dfrac{2}{4} = \dfrac{1}{2}$

because, the event 'both faces are identical' = {hh, tt} so that n (the event) = 2.

(iii) $P\ (\text{atleast one head}) = \dfrac{3}{n(S)} = \dfrac{3}{4}$

because the event atleast one head = {hh,ht,th} so that n (the event) = 3.

3. A box has 2 silver coins, 3 gold coins and 1 copper coin. Then $n(S)$ = The number of coins in the box = 6.

n(silver coins) = Number of silver coins = 2

n(gold coins) = Number of gold coins = 3

n(copper coins) = Number of copper coins = 1

Let A = Drawing a silver coin

B = Drawing a gold coin

and C = Drawing a copper coin

$$\text{Then, } P(A) = \frac{n(A)}{n(S)} = \frac{2}{6} = \frac{1}{3}$$

$$P(B) = \frac{n(B)}{n(S)} = \frac{3}{6} = \frac{1}{2}$$

$$P(C) = \frac{n(C)}{n(S)} = \frac{1}{6}$$

4. In the same example (as in (iii)) suppose two coins are drawn randomly.

Let A = both coins are silver

B = both coins are of gold

and C = both coins are copper

Let us find $P(A)$ and $P(B)$.

Hence $n(S)$ = Total number of ways of drawing 2 coins out of 6 coins

$$= {}^6C_2 = \frac{6\text{x}5}{2} = 15$$

To find P(A)

n(A) = Number of ways of drawing 2 silver coins out of 2 silver coins

$$= {}^{2}C_2 = \frac{2}{2} = 1$$

$\therefore$ n(A) = 1

$$\therefore \quad P(A) = \frac{n(A)}{n(S)} = \frac{1}{15}$$

To find P(B)

n(B) = Number of ways of drawing 2 gold coins out of 3 gold coins

$$= {}^{3}C_2 = 3$$

$$\therefore \quad P(B) = \frac{n(B)}{n(S)} = \frac{3}{15} = \frac{1}{3}$$

To find P(C)

Since there is only one copper coin in the box, n(C) = 0.

$$\therefore \quad P(C) = \frac{n(C)}{n(S)} = \frac{0}{15} = 0$$

Note: In this example, since P(C) = 0

C is an impossible event.

(v) A die numbered 1 to 6 is tossed. Let us find the probability of the events -

A = The number appearing is even

B = The number appearing is odd

C = The number appearing is a prime number

D = The number appearing is greater than 4

E = The number appearing is less than or equal to 6

F = The number appearing is a negative number

Hence n(S) = 6 because S = {1, 2, 3, 4, 5, 6}

A = {2, 4, 6} $\quad \therefore n(A) = 3$

$$\therefore \quad P(A) = \frac{n(A)}{n(S)} = \frac{3}{6} = \frac{1}{2}$$

B = {1, 3, 5} $\quad \therefore n(B) = 3$

$$\therefore \quad P(B) = \frac{n(B)}{n(S)} = \frac{3}{6} = \frac{1}{2}$$

Note: $P(A) = P(B) = \frac{1}{2}$ even though A and B are different events.

n(C) = 3 since C = {2, 3, 5}

$$\therefore P(C) = \frac{3}{6} = \frac{1}{2}$$

D = {5, 6} $\therefore n(D) = 2$

$$\therefore P(D) = \frac{n(D)}{n(S)} = \frac{2}{6} = \frac{1}{3}$$

E = S $\quad \therefore n(E) = n(S) = 6$

$$\therefore P(E) = \frac{6}{6} = 1$$

$\therefore$ E is a certain event.

F is an impossible event, since 0 cannot appear.

$\therefore P(F) = 0$

(vi) A coin is tossed thrice. Let us find the probability of the events -

A = Two heads appear

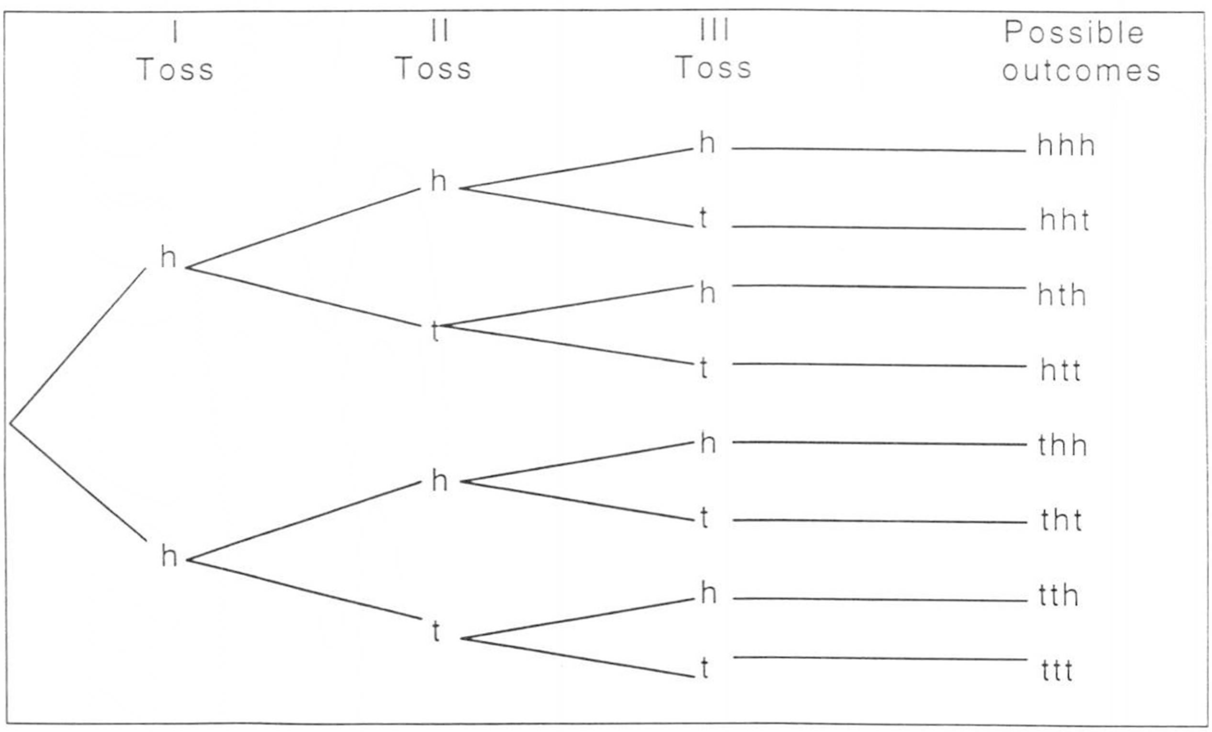
B = Two heads appear consecutively

C = Atleast two heads appear

D = Utmost one head appears

$n(S)$ = Number of possible outcomes = $2^3 = 8$

A way to list all the outcomes - The tree-diagram



Total number of outcomes = 8

A = {hht, hth, thh} $\therefore n(A) = 3$

$$\therefore P(A) = \frac{n(A)}{n(S)} = \frac{3}{8}$$

$B = \{hht, thh\} \quad \therefore n(B) = 2$

$$\therefore P(B) = \frac{n(B)}{n(S)} = \frac{2}{8} = \frac{1}{4}$$

$C = \{hht, hth, thh, hhh\} \quad \therefore n(C) = 4$

$$\therefore P(C) = \frac{n(C)}{n(S)} = \frac{4}{8} = \frac{1}{2}$$

$D = \{htt, tht, tth, ttt\} \quad \therefore n(D) = 4$

$$\therefore P(D) = \frac{n(D)}{n(S)} = \frac{4}{8} = \frac{1}{2}$$

Note: $P(C) + P(D) = \frac{1}{2} + \frac{1}{2} = 1$

So, C and D are complementary events.

7. A die numbered 1 to 6 is cast twice. Let us find the probability of the events -

   A = The same number appears both times.

   B = The number in the first throw is less than that in the second throw.

   C = The sum of the numbers is greater than 12.

   D = The product of the numbers is less than or equal to 36.

Here, $S = \{(a,b) | a,b = 1, 2, 3, 4, 5, 6\}$

$n(S) = 36$

$A = \{(x,x) | x = 1, 2, 3, 4, 5, 6\} \quad \therefore n(A) = 6$

$$\therefore P(A) = \frac{n(A)}{n(S)} = \frac{6}{36} = \frac{1}{6}$$

$B = \{(x,y) | x < y\}$

When $x = 1$, y takes 2, 3, 4, 5, 6 $\therefore$ There are 5 such pairs.

When $x = 2$, y takes 3, 4, 5, 6 $\therefore$ There are 4 such pairs.

When $x = 3$, y takes 4, 5, 6 $\therefore$ There are 3 such pairs.

When $x = 4$, y takes 5, 6 $\therefore$ There are 2 such pairs.

When $x = 5$, y takes 6 $\therefore$ There is one such pair.

$\therefore$ The number of favourable elementary events: $x<y$ is $1+2+3+4+5=15$

$\therefore n(B) = 15$

$$\therefore P(B) = \frac{n(B)}{n(S)} = \frac{15}{36} = \frac{5}{12}$$

C is an impossible event since $xy \leq 12$ for all x & y.

$\therefore P(C) = 0$

D is a certain event so that $P(D) = 1$.

8. Proof of $P(A) + P(A') = 1$

Let A be an event of a random experiment with sample space S.

$n(A')$ = The total number of elementary events favourable to A'

= Total number of elementary events - Total number of elementary events favourable to A.

$n(A') = n(S) - n(A)$, because if an elementary event is favourable to A', then it is not favourable to A.

$$\therefore P(A') = \frac{n(A')}{n(S)} = \frac{n(S) - n(A)}{n(S)}$$

$$= 1 - \frac{n(A)}{n(S)} = 1 - P(A)$$

$P(A') = 1 - P(A)$ or $P(A) + P(A') = 1$

IV. Problem Solving

I. A bag contains 4 red and 6 blue marbles. Find the following -

a. When a marble is randomly drawn

(i) P (red marble)
(ii) (blue marble)

b. When two marbles are drawn

(i) P (both are red)

(ii) P (both are blue)

(iii) P (both are of the same colour)

(iv) P (marbles drawn are of different colours)

Solution

a. When one marble is drawn

The bag contains 6 marbles of which 4 are red and 2 are blue.

$\therefore$ $n(S) = 6 \quad n(B) = 2$

$$\therefore P(\text{getting a red marble}) = \frac{4}{6} = \frac{2}{3}$$

$$\& \; P(\text{getting a blue marble}) = \frac{2}{6} = \frac{1}{3}$$

b. When two marbles are drawn

$n(S)$ = Total number of ways of drawing two marbles out of six months.

$$= {}^6C_2 = \frac{6 \times 5}{1 \times 2} = 15 \quad \therefore n(S) = 15$$

(i) When both are red marbles

Denoting the event by A,

$$n(A) = n\ (\text{getting both red marbles}) = {}^4C_2 = \frac{4 \times 3}{1 \times 2} = 6$$

$$\therefore P(A) = \frac{n(A)}{n(S)} = \frac{6}{15} = \frac{2}{5}$$

(ii) When both are blue marbles

Denoting the event by B,

$$n(B) = n\ (\text{getting both blue marbles}) = {}^2C_2 = 1$$

$$\therefore P(B) = \frac{n(B)}{n(S)} = \frac{1}{15}$$

(iii) When both are of the same colour

Denoting the event by C,

C = getting both red or both blue

$$n(C) = {}^4C_2 + {}^2C_2 = 6 + 1 = 7$$

$$\therefore P(C) = \frac{n(C)}{n(S)} = \frac{7}{15}$$

(iv) When the marbles are of different colours

Denoting the event by D, D is the complement of C.

i.e. $P(D) = P(C') = 1 - P(C)$

$$= 1 - \frac{7}{15} = \frac{8}{15}$$

2. What is the probability that in a randomly selected year, there are 53 Sundays. If the year is (a) a non-leap year, (b) a leap year.

[A leap year has 366 days, i.e. 52 weeks and two days. A non-leap year has 365 days, i.e. 52 weeks and one day.]

a. When the year is a non-leap year

There is one extra day over 52 weeks.

This extra day can be any one of the 7 week days.

Hence P(53 Sundays) = $\frac{1}{7}$

b. When the year is a leap year

There are two extra days over 52 weeks and these extra days must be consecutive week days, i.e. SnM, MT, TW, WTh, ThF, FS, SSn, where,

M = Monday, T= Tuesday, etc., S = Saturday and S_n = Sunday

$\therefore n(S) = 7$

For 53 Sundays, two elementary events (viz. SnM and SSn) are favourable.

$\therefore$ P (53 Sundays) = $\frac{2}{7}$

3. From a pack of 52 playing cards a card is drawn randomly. Find the probability of the following events -

A = The card is a jack

B = The card is a queen

C = The card is a spade card

D = The card is a heart card

Solution

In a pack of 52 playing cards, there are 4 jacks, 4 queens, 13 spades and 13 hearts.

$$\therefore P(A) = \frac{4}{52} = \frac{1}{13} \; ; \; P(B) = \frac{4}{52} = \frac{1}{13}$$

$$P(C) = \frac{13}{52} = \frac{1}{4} \; ; \; P(D) = \frac{13}{52} = \frac{1}{4}$$

4. In the above example, two cards are randomly drawn. Find the probabilities of the events.

A = Both are red cards

B = Both are kings

C = One is an ace and the other a queen

D = One is a diamond and the other a club

Solution

In a pack of 52 playing cards, 26 red cards are there. There are 4 kings, 4 aces, 13 diamonds and 13 clubs.

$$P(A) = \frac{{}^{26}C_2}{{}^{52}C_2} = \frac{26 \times 25}{52 \times 51} = \frac{25}{102}$$

$$P(B) = \frac{{}^{4}C_2}{{}^{52}C_2} = \frac{4 \times 4}{52 \times 51} = \frac{1}{221}$$

$$P(C) = \frac{{}^{4}C_1 \times {}^{4}C_1}{{}^{52}C_2} = \frac{4 \times 4}{(52\times51)/(1\times2)} = \frac{8}{663}$$

$$P(D) = \frac{{}^{13}C_1 \times {}^{13}C_1}{{}^{52}C_2} = \frac{13 \times 13}{(52\times51)/(1\times2)} = \frac{2\times13}{51} = \frac{26}{51}$$

V. Ativities (for enrichment)

List the names of mathematicians who contributed to the growth of probability theory.

VI. Exercises

1. A pair of distinguishable dice is thrown. Find the probabilities of the events.

    A = The numbers are consecutive

    B = The numbers are even

    C = The numbers are unequal

2. A die is thrown once. Find the probabilities of the events -

    A = The number is divisible by 3

    B = The number is an even prime number

    C = The number is an odd composite number

    D = The number is divisible by 1

3. A purse has 3 five-Re coins, 4 two-Re coins and 3 one-Re coins.

    A coin is randomly taken from the purse. Find the probability that the coin taken is

(a) a five-Re coin

(b) not a two-Re coin

(c) a one-Re coin or a two-Re coin

4. In the above example, two coins are randomly drawn. Find the probability of drawing.

   (a) both 5-Re coins

   (b) one 5-Re coins and one 2-Re coin

   (c) a sum less than 5 rupees

5. Five cards numbered 1, 2, 3, 4, 5 are shuffled well and two cards are randomly drawn.

   Find the probability that

   (a) the product of the numbers exceeds 20

   (b) the sum of the numbers is atleast 3

   (c) the numbers are consecutive

6. A coin is tossed thrice. Find the probability that

   (a) the same face is not shown up in all the tosses

   (b) heads not consecutive

   (c) neither heads nor tails are consecutive

7. Eight points are marked on a paper and one of these points is A. When a child is asked to join any two points, what is the probability that

   (a) the child does not join A with any point

   (b) the child joins a point to A

8. A polygon has 6 sides and a child is asked to tick a side or a diagonal of the polygon. What is the probability that the child ticks. (a) a side, (b) a diagonal.

9. Four friends randomly stand in a line. A and B are among them. What is the probability that

   (a) A and B occupy the end positions

   (b) A and B are side by side

   (c) A and B take alternate positions

10. Srividya is one among five badminton players. Three players are selected from this group of players. Find the probability that (a) Srividya is selected, (b) Srividya is left out.

11. A triangle is drawn randomly. What is the probability that the triangle is (a) right angled triangle, (b) an acute angled triangle.

12. A die has one red, two blue and three white faces. When it is thrown once, find the probability that the face shown up is (a) red, (b) blue, (c) white.

13. When the die is thrown twice, find the probability that the faces shown up are

   (a) both times red, (b) first blue and second white,

   (c) not white both times.

14. A basket has 12 eggs of which 5 are rotten. Three eggs are randomly taken. Find the probability that

   (a) all the eggs are good

(b) one egg is rotten

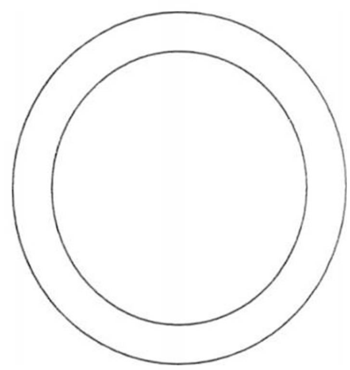
(c) atleast one egg is good

15. A person randomly shoots two arrows at a target consisting of two concentric circles. Each time the arrow hits one of the regions into which the plane of the cirlces is divided. Find the probability that

(a) both times the arrows hit the same region

(b) different regions

(c) consecutive regions



UNIT NO. 3: VECTORS

Unit Introduction

Like many fundamental ideas in Mathematics, notion of vectors developed from investigation into problems encountered in Physics, Kinematics, Engineering, etc. Vector methods is a powerful tool in the hands of applied scientists providing economy in computation and also elegant procedures, to solve physical problems.

The origin of notion of vectors dates back to seventeenth century. The works of Mathematicians like H.G. Grassman (1809-1877) and W.K. Hamilton (1805-1865) influenced very much the growth of vector analysis. A concise account of vectors was first found in the book "Elements of Vector Analysis" written by American Mathematician Willard Gibbs (1839-1903). However much of the credit for demonstrating the applications of vectors is due to English physicist O. Heaviside (1850-1925).

The word vector is derived from the Latin word 'Veho' which means 'I carry'.

3.1 Scalar Quantities

Concept and Explanations

In our daily life we come across various physical quantities which can be fully described using a single number. The physical quantities like, length of a rope, mass of a body, volume of a liquid in a container, our body temperature, etc. are quantified (measured) by assigning a single number to them.Such quantities are completely described by these single members called their magnitudes. Such of those quantities which are completely described by their magnitudes are called scalar quantities or simply scalars.

Examples

Mass of an object is 500 gms. Here the number 500 gms completely describes the mass. Hence mass of an object is a scalar quantity. Density, Energy, work, specific heat are some more examples for scalar quantities.

A scalar has only magnitude and no direction. Magnitude of a given scalar quantity may change with the choice of units. For example, if the height of a man is 1.7 meters, in the unit of meters, magnitude of height of man is 1.7 whereas in the unit of centimetres, the magnitude is 170.

Scalar quantities are represented by numbers and hence can be added, subtracted, multiplied, divided, etc. just like numbers and hence these operations follow arithmetic laws.

3.2 Vector Quantities

Concepts and Explanations

Contrary to scalar quantities, there are certain other physical quantities which cannot be described by magnitude alone. Their measurements involve both magnitude as well as direction. That is,a single member is not sufficient to describe completely (characterise) such physical quantities. For example, if we say a car is moving with a speed of 40 kms/hr, we cannot describe the movement of the car completely unless we specify the direction in which it is moving. Similarly to describe a force we need to know its magnitude as well as the direction of application of the force (we know that the effect of a force not only depends on its magnitude but also on the direction of its application). Thus both magnitude and direction are required to describe these physical quantities, they are called Vector Quantities or Vectors. In this unit, we shall study Vectors.

Examples

Displacement is a vector quantity. To describe displacement completely, we should know the distance by which it is displaced as well as the direction in which it is displaced. Hence displacement has both magnitude and direction. Hence displacement is a vector quantity. Velocity, momentum, acceleration are some more examples of vector quantities.

Activity

List 20 physical quantities and classify them into scalar and vector quantities.

3.3 Representation of Vectors

Concept and Explanations

A line segment with an arrow head (see figure) is called a directed line segment. Vector quantities are represented by directed line segments. Length of the segment gives the magnitude and the arrowhead indicates the direction.

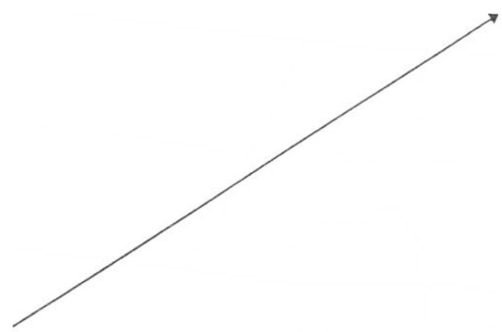


Fig. 1

Suppose an insect is moving along a straight line. Let P and Q be two positions of the insect at two different instances of time. Then, the length (magnitude) of PQ is a scalar. If the insect had moved from P towards Q, then the shift or displacement PQ is a vector quantity. It is represented by $\overrightarrow{PQ}$ to indicate that the magnitude of displacement is the length PQ and that the direction of displacement is from P to Q along PQ. In case the insect moves from Q towards P, then the displacement is given by Vector $\overrightarrow{QP}$. The vectors $\overrightarrow{PQ}$ and $\overrightarrow{QP}$ are different only because their directions are different. Both of them have the same length (or magnitude) denoted by $|\overrightarrow{PQ}|$ or $|\overrightarrow{QP}|$ equal to the length of the line segment PQ. For the vector $\overrightarrow{PQ}$, P is called its initial point and Q its terminal point.

Magnitude of $\overrightarrow{PQ}$ is also denoted by PQ and magnitude of $\overrightarrow{QP}$ is also denoted by QP. Single letters, generally small English letters are also used to represent vectors. In the figure, $\overrightarrow{PQ}$ is represented by $\vec{a}$. Hence $|\overrightarrow{PQ}|$ can be written as $|\vec{a}|$ or simply as a.

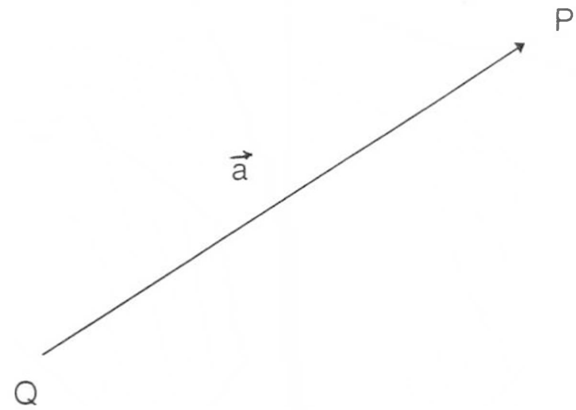


Fig. 2

Note: 1. Magnitude of a vector is always a non-negative real number. For any vector $\overrightarrow{AB}$ we have $|\overrightarrow{AB}| \geq 0$. $|\overrightarrow{AB}| = 0$ means length of vector $\overrightarrow{AB} = 0$. But the length of the vector $\overrightarrow{AB}$ = AB. Therefore AB = 0. Hence A = B. This can happen when initial point A coincides with the terminal point B of vector $\overrightarrow{AB}$.

2. In representing a given vector diagrammatically, any point in the plane can be taken to be its initial point. Choice of the initial point does not matter when a vector is drawn as a directed line segment. In the figure given below, all the four directed line segments have

the same magnitude and same direction but different initial points. But still all of them represent the same vector $\vec{a}$. That is, $\overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{EF} = \overrightarrow{GH} = \vec{a}$.

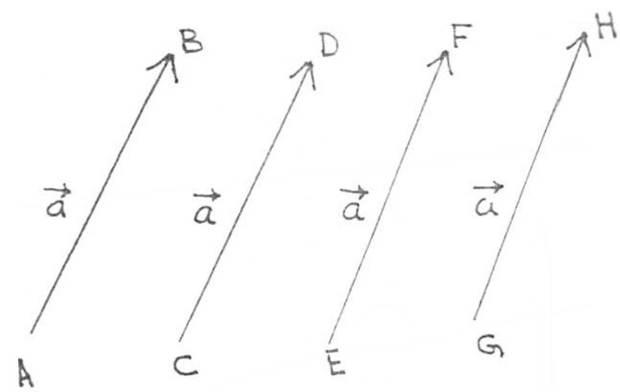
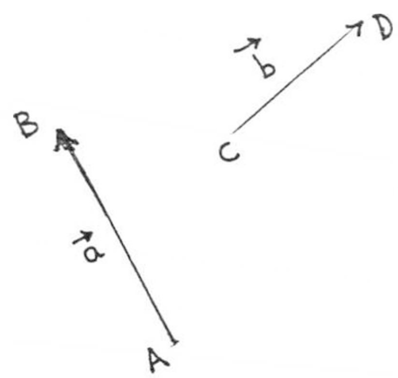


Fig. 3

3. Any two vectors can be represented by the directed line segments having common initial point.



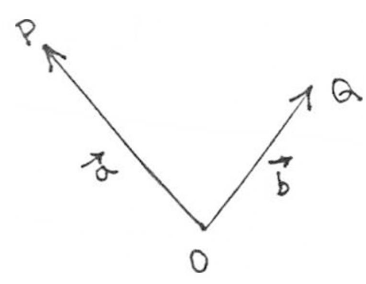


Fig. 4(i) **Fig. 4(ii)**

Here AB = OP, $\overrightarrow{AB}$ and $\overrightarrow{OP}$ have the same direction. Hence $\overrightarrow{AB} = \overrightarrow{OP}$ similarly since CD = OQ and $\overrightarrow{CD}$, $\overrightarrow{OQ}$ have the same direction, $\overrightarrow{CD} = \overrightarrow{OQ}$. In Fig. 4(i) the two given vectors $\vec{a}$ and $\vec{b}$ are represented by directed line segments $\overrightarrow{AB}$ and $\overrightarrow{CD}$. Whereas in Fig. 4(ii) the vectors $\vec{a}$ and $\vec{b}$ have been represented by vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ such that they have a common initial point O.

It is often convenient to take a common initial point for representing given vectors.

3.4 Vector Addition

Concept and Explanations

You know that two scalar quantities can be added and that their sum is a number which we find using rules of number addition. Since vectors are directed line segments, and not just numbers, they cannot be added like ordinary numbers. So, if we wish to add two vectors, we have to define a new kind of addition. Vectors as we have already seen, represent quantities like forces and velocities. From experiments in Physics, it is known that such quantities are added according to a law called 'triangle law of addition'. With this motivation from physical situations, we define addition of vectors in the following way.

Let $\vec{a}$ and $\vec{b}$ be two given vectors. We wish to define a vector $\vec{a} + \vec{b}$ called the sum (resultant) of $\vec{a}$ and $\vec{b}$. To do this, first place the initial point of $\vec{b}$ on the terminal point of $\vec{a}$. Recall that this can be done because we can chose any point as the initial point of a vector. Next join the initial point of $\vec{a}$ to the terminal point of $\vec{b}$.

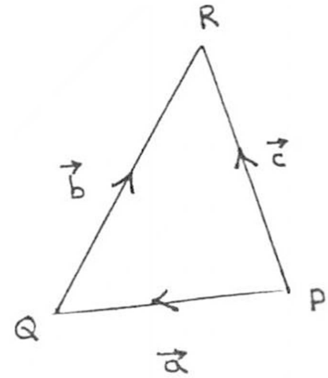


Fig. 5

This procedure gives us the triangle PQR (see Fig. 5). $\vec{a} = \overrightarrow{PQ}$ and $\vec{b} = \overrightarrow{QR}$ in the triangle PQR and $\overrightarrow{PR} = \vec{c}$. We define the vector sum of $\vec{a}$ and $\vec{b}$ as the vector $\overrightarrow{PR} = \vec{c}$ and write $\vec{a} + \vec{b} = \vec{c}$. $\vec{c}$ is also called the resultant of vectors $\vec{a}$ and $\vec{b}$. Therefore, rule of vector addition is

IF TWO VECTORS ARE REPRESENTED BY TWO SIDES OF A TRIANGLE IN ORDER; THEN THEIR SUM (OR RESULTANT) IS REPRESENTED BY THE THIRD SIDE OF THE TRIANGLE BUT IN REVERSE ORDER.

This is called the triangle law of addition of vectors.

We can also define the vector sum $\vec{a} + \vec{b}$ of given vectors $\vec{a}$ and $\vec{b}$ using parallelogram law.

Let any point O be chosen as the origin. Let the initial points of both the vectors $\vec{a}$ and $\vec{b}$ be placed at the origin O. Complete the parallelogram for which $\vec{a}$ and $\vec{b}$ are adjacent sides at the vertex O. From figure 6, $\overrightarrow{OB} = \vec{b}$, $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OC}$ is the diagonal through the common initial point O of vectors $\vec{a}$ and $\vec{b}$.

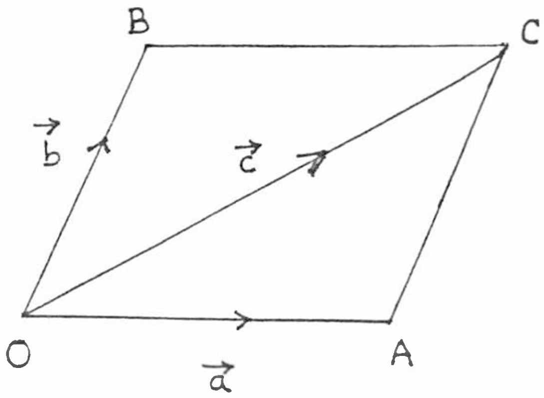


Fig. 6

Let $\vec{OC} = \vec{c}$. We define the vector sum of $\vec{a}$ and $\vec{b}$ as equal to $\vec{OC}$.

i.e. $\vec{a} + \vec{b} = \vec{c}$

or $\vec{OA} + \vec{OB} = \vec{OC}$

We call $\vec{OC}$ as the resultant of $\vec{a}$ and $\vec{b}$. We can state this rule of vector addition as

IF TWO VECTORS ARE REPRESENTED BY THE TWO SIDES OF A PARALLELOGRAM HAVING THE SAME VERTEX AS INITIAL POINT, THEN THEIR SUM IS REPRESENTED BY THE DIAGONAL OF THE PARALLELOGRAM PASSING THROUGH THAT VERTEX, WITH THE VERTEX AS ITS INITIAL POINT.

We discussed two laws of addition of vectors namely, the triangle law and the parallelogram law. But do we get the same sum in both the cases ? We can show that both the laws are equivalent in the sense that the vector sum in both cases is the same.

Let $\vec{a}$ and $\vec{b}$ be two given vectors. Let us represent them as given in Fig. 7. From the parallelogram law,

$$\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OC}$$

$$\text{i.e. } \vec{a} + \vec{b} = \vec{c}$$

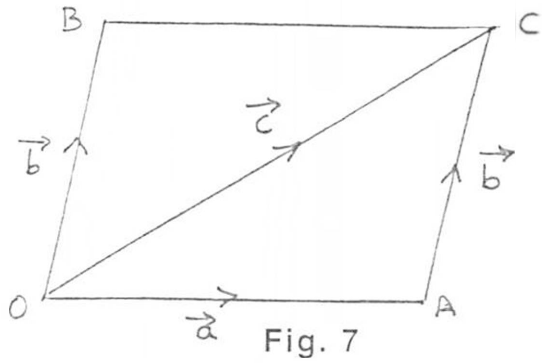


Fig. 7

From the triangle law, $\overrightarrow{OA} + \overrightarrow{AC} = \overrightarrow{OC}$. But since OACB is a parallelogram, OB = AC and OB||AC. Therefore $\overrightarrow{OB}$ and $\overrightarrow{AC}$ have the same magnitude and direction. Hence $\overrightarrow{AC} = \overrightarrow{OB} = \vec{b}$. Therefore $\vec{a} + \vec{b} = \vec{c}$ by triangle law also. Hence by both laws same resultant or vector sum is obtained.

Caution: The plus (+) sign used here to denote addition does not signify number addition. It stands for vector addition which is guided by triangle law of addition. Magnitude of the sum of the vectors is not equal to the sum of the magnitudes except in one case.

Note: Note that triangle law or parallelogram law of addition of vectors can be applied only when the two vectors are not parallel. When they are parallel we have to discuss the cases separately.

1. If $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ have the same direction, their sum (resultant) is $\overrightarrow{PQ} + \overrightarrow{RS} = \overrightarrow{PS}$. Also $|\overrightarrow{PQ}| + |\overrightarrow{RS}| = |\overrightarrow{PS}|$. See Fig. 8(i).

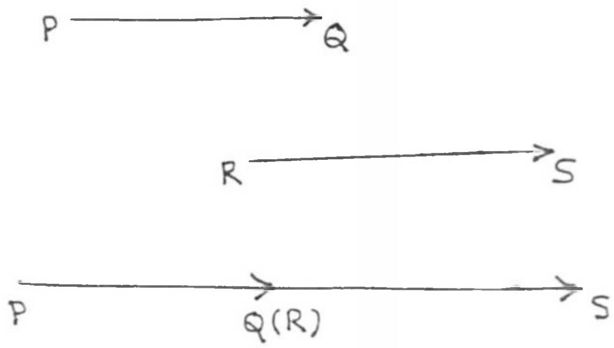


Fig. 8(i)

2. Suppose $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ have opposite directions and length of $\overrightarrow{PQ}$ > length of $\overrightarrow{RS}$.

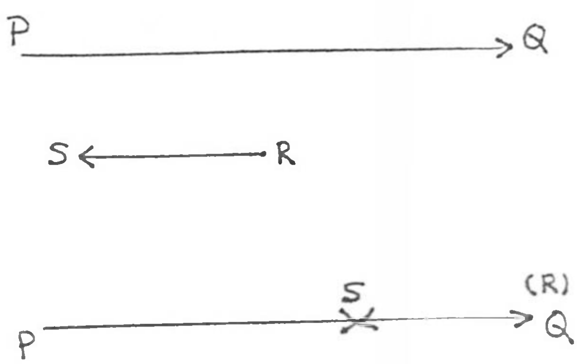


Fig. 8(ii)

If one travels from P to Q and then travel from Q(R) to S (see fig. 8(ii)) in the opposite direction (in the direction of $\overrightarrow{RS}$ by a magnitude $|\overrightarrow{RS}|$) then the resultant displacement is $\overrightarrow{PS}$ shown in Fig. 8(ii). Hence here $\overrightarrow{PQ} + \overrightarrow{RS} = \overrightarrow{PS}$ and $|\overrightarrow{PS}| = |\overrightarrow{PQ}| - |\overrightarrow{RS}|$.

3. Suppose a car travels 30 miles on a straight road from A to B and then travels back 30 miles from B to reach A. These displacements $\overrightarrow{AB}$ and $\overrightarrow{BA}$ are shown in the figure 8(iii).

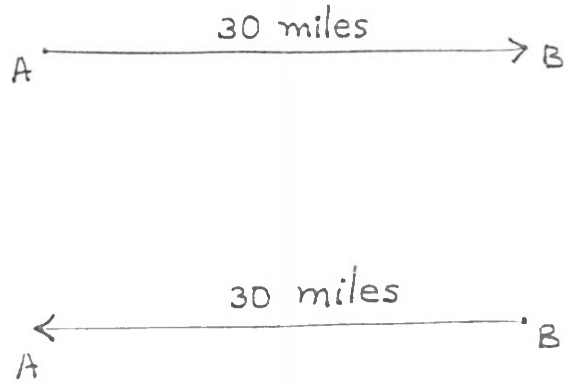


Fig. 8(iii)

The resultant displacement is the vector sum of the displacements $\overrightarrow{AB}$ and $\overrightarrow{BA}$, i.e. $\overrightarrow{AB} + \overrightarrow{BA} = \overrightarrow{AB} + (-\overrightarrow{AB}) = \overrightarrow{0}$ (since $\overrightarrow{BA} = -\overrightarrow{AB}$) which means displacement sum is zero.

We can explain this by saying that travel from A to B and back to A form B is as good as not moving from A at all. $\therefore \overrightarrow{AB}+\overrightarrow{BA}=\overrightarrow{AA}=\overrightarrow{O}$.

Activity

A car is following the route from A to B due east then from B to C, 30°N. Finally from C back to A. Represent these displacements in a diagram. What is the resultant displacement ?

Examples

1. Suppose a car travels 3 miles due north and then 5 miles along north-west. These displacements are shown in the figure 9(i) as $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{AB} = \vec{b}$. Here $|\vec{a}| = 3$ and $|\vec{b}| = 5$.

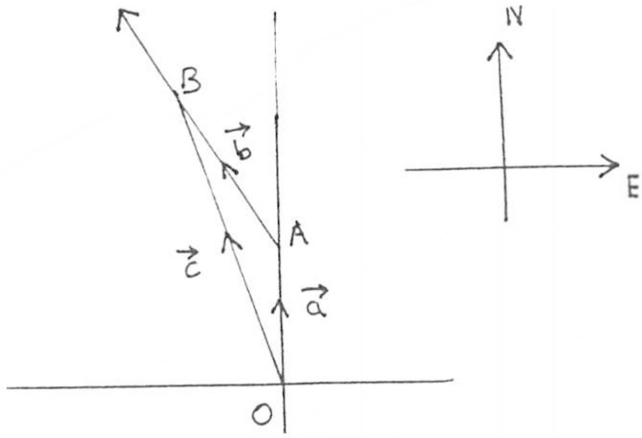


Fig. 9(i)

$\overrightarrow{OB}$ is the direct path from O to B. $\overrightarrow{OB} = \vec{c}$ is the resultant of the displacements $\vec{a}$ and $\vec{b}$. $\therefore \overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$ or $\vec{c} = \vec{a} + \vec{b}$.

Here, we mean that to travel from O to B along $\overrightarrow{OB}$ is in effect the same as to travel first along $\overrightarrow{OA}$ (from O to A) and then along $\overrightarrow{AB}$ (from A to B). In both cases, the effective displacement is $\overrightarrow{OB}$, because the journey starts from O and ends at B.

In the above example, if the car starts from O and reaches B along the path OACB as shown in the figure 9(ii), then the displacements are $\overrightarrow{OA}$, $\overrightarrow{AC}$ and $\overrightarrow{CB}$. Again $\overrightarrow{OB}$ is the resultant or sum of the given three displacements. $\therefore \overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AC} + \overrightarrow{CB}$.

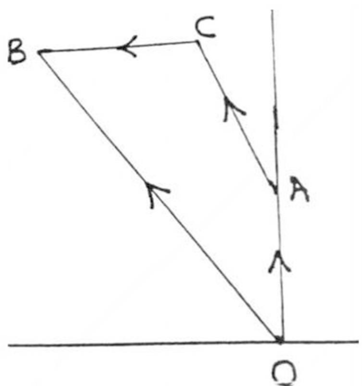


Fig. 9(ii)

In fact, there may be many routes from O to B (routes 1, 2, 3, 4 in fig.), as shown in fig. 9(iii). But all these routes serve the same purpose. That is to start from O and reach B. This is equal to a single displacement $\overrightarrow{OB}$. Hence the sum of displacements (resultant displacement) in every case (from 1 to 4) is equal to the same displacement $\overrightarrow{OB}$.

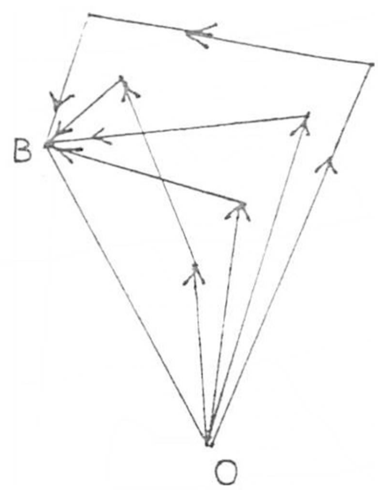


Fig. 9(iii)

2. A man rowing a boat can move at a velocity of 6 ft/sec in still water. He wants to cross a river which is flowing at a velocity of 4 ft/sec. Suppose he starts rowing his boat directly across the river. See what happens in one second. Boat will have travelled 6 ft across the river and the river will have carried the boat downstream by 4 ft.

Representing these velocities in the figure 10. we see that the resultant velocity of the boat is given by $\overrightarrow{OB}$. ($\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$ by triangle law of addition).

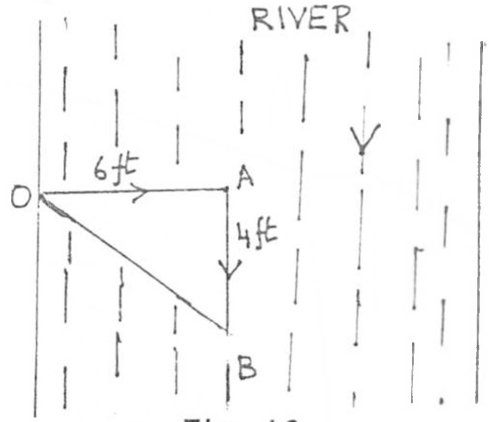


Fig. 10

This means that an observer standing on the river bank sees the boat actually move along $\overrightarrow{OB}$ with a velocity approximately 7.2 ft/sec. ($OB^2 = OA^2 + AB^2 = 62 + 42 = 52$ $\therefore OB \sim 7.2$)

3. Suppose a policeman is taking two police dogs for a walk. He holds both their chains in his right hand. The dogs are both pulling ahead of the policeman. Suppose the first dog is pulling towards north-east with a force of 50 N and second dog is pulling due east with a force of 70 N. These forces are represented in figure.

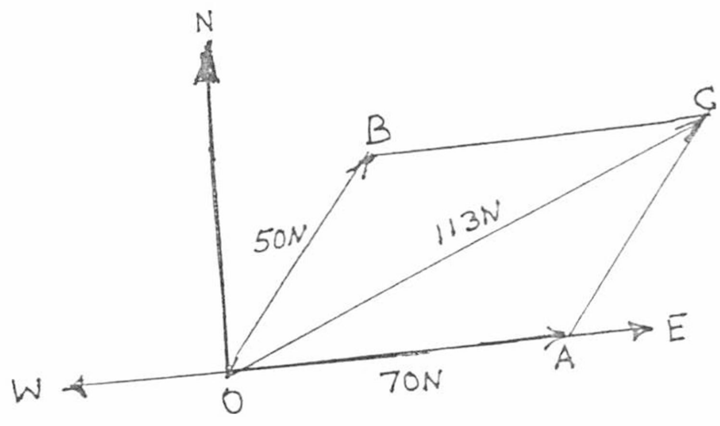


Fig. 11

The resultant of these forces is their vector sum which is given by $\overrightarrow{OC}$.

$\therefore \overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OC}$

(using parallelogram law of forces). Draw to scales and verify that the resultant force is approximately 113 N.

4. An aeroplane is moving with a velocity of 20 m/sec due west. Wind is blowing from north to south with a velocity of 15 m/sec. The situation is represented in Fig. 12.

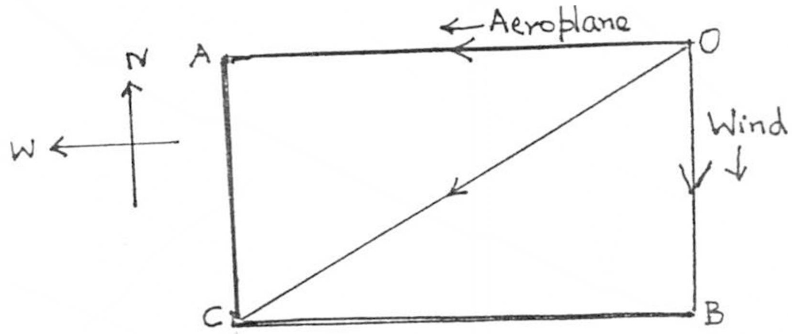


Fig. 12

Here, OA represents velocity of a plane. Therefore, OA = $|\vec{OA}|$ = 20. OB represents velocity of wind. Therefore, OB = $|\vec{OB}|$ = 15.

To find the resultant velocity of the plane, complete the parallelogram OBCA which in this case is a rectangle. By parallelogram law of vector addition, the resultant velocity is given by $\vec{OC}$. $\vec{OA} + \vec{OB} = \vec{OC}$ which is along the diagonal OC.

In rectangle OBCA, OC = $|\vec{OC}| = \sqrt{OA^2+OB^2} = \sqrt{20^2+15^2} = 25$

Therefore, the resultant velocity is 25 m/sec towards south-west making an angle $\tan^{-1}(3/4)$ with west. (Why ?)

5. A river has straight parallel banks. The river flows at a constant velocity of 3 ft/sec eastwards. A boat is heading northward at a velocity of 5 ft/sec. Let us find the resultant velocity.

Figure 13 illustrates the situation described.

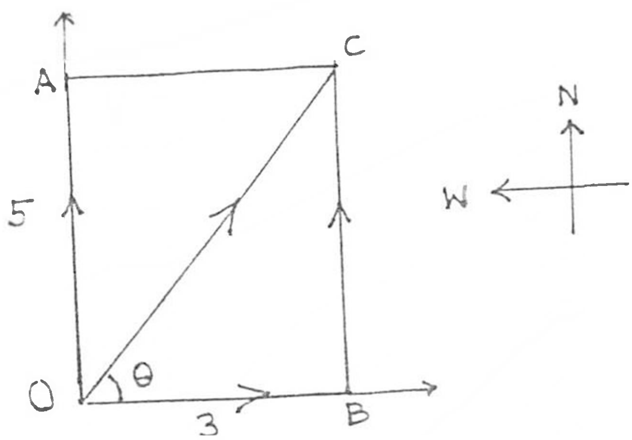


Fig. 13

By completing the rectangle OACB we have $|\overrightarrow{AB}| = |\overrightarrow{OC}|$. The resultant velocity of the boat is given by $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{OB}$.

Magnitude of $\overrightarrow{OC} = |\overrightarrow{OC}| = OC = \sqrt{34}$. If $\angle COB = \theta$, then $\tan\theta = 5/3$. $\therefore\ \theta = \tan^{-1}(5/3)$

Hence the resultant velocity of the boat has magnitude $\sqrt{34}$ ft/sec making an angle $\tan^{-1}(5/3)$ with the bank.

Activity

Draw to scales and find θ approximately in the above example.

3.6 POSITION VECTOR OF A POINT

Concept and Explanations

A vector, as we saw, has magnitude and direction but no fixed position because any point can be taken as its initial point. Hence a given vector may be represented by any one of an unlimited number of directed line segments parallel to one another and with identical magnitude. When we consider a geometrical problem in terms of vectors, it may be convenient to think of all vectors in the diagram with reference to a fixed initial point O. This fixed initial point is called the origin. After fixing a point O as origin, which serves as a reference point, any point P in the plane is uniquely determined by the vector $\overrightarrow{OP}$, i.e. given the vector $\overrightarrow{OP}$ (in magnitude and direction) the position of the point P is fixed and conversely, given the position of P, the vector $\overrightarrow{OP}$ is fixed. $\overrightarrow{OP}$ is called the position vector of the point P. If Q is another point in the plane $\overrightarrow{OQ}$ is its position vector. Suppose we want to represent $\overrightarrow{PQ}$ in the terms of origin O. From Fig. 14, $\overrightarrow{OP} + \overrightarrow{PQ} = \overrightarrow{OQ}$ (triangle law of vector addition).

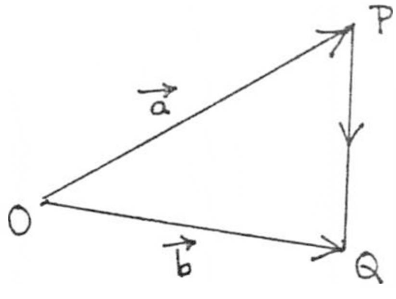


Fig. 14

Hence $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$. If $\overrightarrow{OP} = \vec{a}$, $\overrightarrow{OQ} = \vec{b}$ then $\overrightarrow{PQ} = \vec{b} - \vec{a}$. Hence $\overrightarrow{PQ}$ is got by subtracting from the position vector of Q, the position vector of P.

$\therefore$ $\overrightarrow{PQ}$ = [PV of terminal point of $\overrightarrow{PQ}$ - PV of initial point of $\overrightarrow{PQ}$]

Example

$\vec{a}$, $\vec{b}$, $\vec{c}$ are the position vectors the points A, B and C with respect to the origin O. Let us see how to express the vectors of AC. AB and BC in terms of $\vec{a}$, $\vec{b}$ and $\vec{c}$.

AC = Position vector of the terminal point C - Position vector of initial point A = $\overrightarrow{OC} - \overrightarrow{OA} = \vec{c} - \vec{a}$. Similarly $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \vec{c} - \vec{b}$ and $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \vec{b} - \vec{a}$.

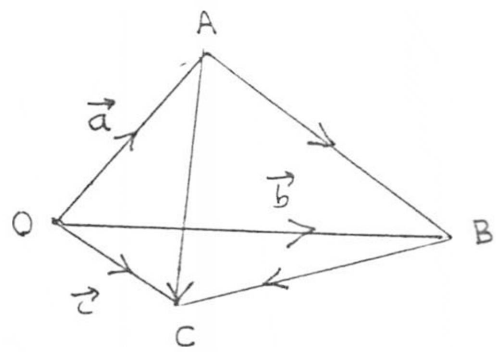


Fig. 15

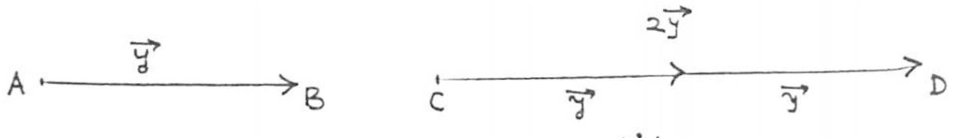
Activity

Try now the problem 2 on p. 68 and problems 5, 10 and 11 on p. 70 of your textbook.

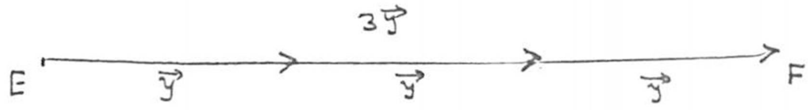
3.7 MULTIPLICATION OF A VECTOR BY A SCALAR (NUMBER)

Concept and Explanations

Let $\overrightarrow{AB}$ be a directed line segment 5 cms long. Let $\overrightarrow{AB} = \vec{y}$. We draw a vector given by $\vec{y} + \vec{y}$ as in Fig. 16(i). Next we draw vector $\vec{y} + \vec{y} + \vec{y}$ (see Fig. 16(ii)). In first case, length of vector $\vec{y} + \vec{y}$ is 10 cms or twice the length of $\vec{y}$. Direction is unchanged. Hence, we write $\vec{y} + \vec{y} = 2\vec{y}$. $2\vec{y}$ here means that 2 and $\vec{y}$ are multiplied together.



(i)



(ii)

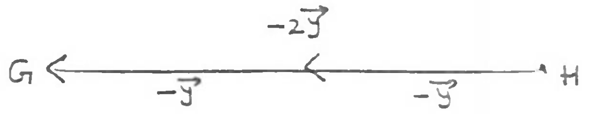


Fig. 16(iii)

Similarly $\vec{y} + \vec{y} + \vec{y}$ is 15 cms long or 3 times as long as $\vec{y}$ and has same directions $\vec{y}$. Hence we write $\vec{y} + \vec{y} + \vec{y} = 3\vec{y}$ which is 3 multiplied with $\vec{y}$. If we draw $-2\vec{y}$. It can be written as sum $-\vec{y} + (-\vec{y})$ which has twice the magnitude of $\vec{y}$ and in opposite direction. Hence $-2\vec{y}$ is -2 multiplied with $\vec{y}$.

In what we discussed here, a new kind of multiplication is defined. We call it multiplication of a vector by a number (or scalar). This is not ordinary multiplication. When we multiplied numbers, product was also a number. Here when a vector is multiplied by a number, we obtain a vector, not a number.

If a vector $\vec{a}$ is multiplied by a scalar (number) k we get a vector $k\vec{a}$, which is k times as long as $\vec{a}$ and has the same direction as $\vec{a}$. The vector $\vec{b} = k.\vec{a}$ is called the scalar multiple of $\vec{a}$. The numerical multiplier k is always written on the left. Note that $|k\vec{a}| =$

$|k||\vec{a}|$. By taking k = 0, we get $|0.\vec{a}| = |0||\vec{a}| = 0|\vec{a}| = 0$. Hence the magnitude of the vector $0.\vec{a}$ is 0. Hence $0\vec{a} = \vec{0}$.

Examples

1. If $\vec{a}$ and $\vec{b}$ are given vectors, let us see what will be $3\vec{a}+2\vec{b}$? (see Fig. 17). It is given by $\overrightarrow{A'C'}$ in $\Delta A'B'C'$.

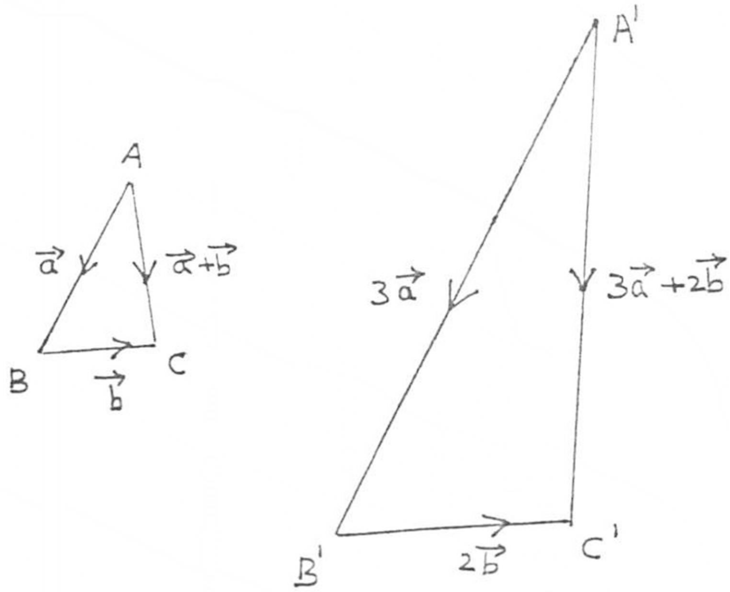


Fig. 17

2. If $|\vec{a}| = 7$, $|5\vec{a}| = |5|.|\vec{a}| = 5x7 = 35$

   $|-4\vec{a}| = |-4|.|\vec{a}| = 4x7 = 28$

   $|0\vec{a}| = |\vec{0}| = 0$

3. ABCD is a quadrilateral in which BC is parallel to AD, ratio of lengths BC:AD is 3:5. Let AB = $\vec{b}$ and AD = $5\vec{a}$. Let us now find $\overrightarrow{BC}$, $\overrightarrow{AC}$, $\overrightarrow{BD}$ and $\overrightarrow{DC}$. BC:AD = 3:5 and $\overrightarrow{AD} = 5\vec{a}$. Hence BC/AD = 3/5. ∴BC = (3/5)AD. Further BC and AD are parallel and hence have the same direction. Therefore $\overrightarrow{BC} = (3/5).\overrightarrow{AD} = (3/5)x5\vec{a} = 3\vec{a}$.

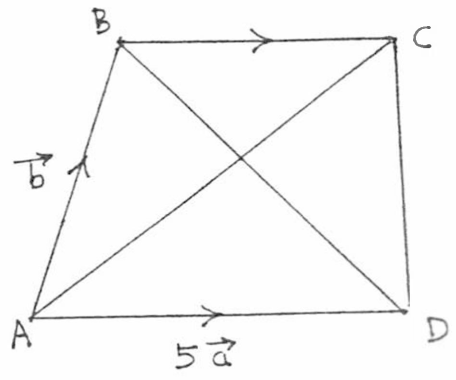


Fig. 18

$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \vec{b} + 3\vec{a}$

$\overrightarrow{BD} = \overrightarrow{BA} + \overrightarrow{AD} = -\vec{b} + 5\vec{a} = 5\vec{a} - \vec{b}$

$\overrightarrow{DC} = \overrightarrow{DA} + \overrightarrow{AC} = -\overrightarrow{AD} + \overrightarrow{AC} = -5\vec{a} + (\vec{b}+3\vec{a}) = \vec{b} - 2\vec{a}$

Problem Solving

1. If D is the mid point of the side BC of a ΔABC, then prove that

   $\overrightarrow{AB} + \overrightarrow{AC} = 2\overrightarrow{AD}$.

   $\overrightarrow{AB} + \overrightarrow{BD} = \overrightarrow{AD}$ (triangle law of addition).

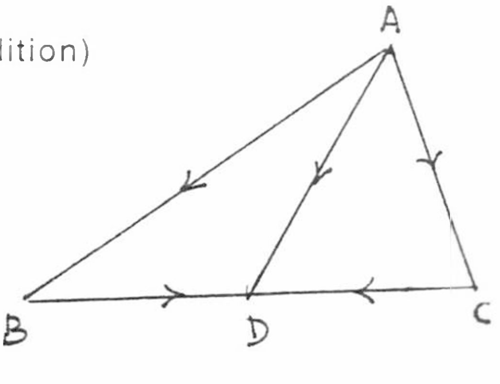
   Also, $\overrightarrow{AC} + \overrightarrow{CD} = \overrightarrow{AD}$ (triangle law of addition)

   $\therefore \overrightarrow{AB} + \overrightarrow{BD} + \overrightarrow{AC} + \overrightarrow{CD} = 2\overrightarrow{AD}$

   $\overrightarrow{AB} + \overrightarrow{AC} + (\overrightarrow{BD} + \overrightarrow{CD}) = 2\overrightarrow{AD}$

   $\therefore \overrightarrow{AB} + \overrightarrow{AC} + \vec{0} = 2\overrightarrow{AD}$ ($\because \overrightarrow{BD} = -\overrightarrow{CD}$)

   Therefore, $\overrightarrow{AB} + \overrightarrow{AC} = 2\overrightarrow{AD}$



2. Find the resultant of three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$.

   We know that the resultant of $\vec{a}$ and $\vec{b}$ is $\vec{a} + \vec{b}$. The vector $\vec{a} + \vec{b}$ is represented by the diagonal of the parallelogram with $\vec{a}$, $\vec{b}$ as sides. Let $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$. Completing the parallelogram OARB and using parallelogram law, $\overrightarrow{OR} = \vec{a} + \vec{b}$ is the resultant of $\vec{a}$ and $\vec{b}$

see Fig. 20). Next we form the resultant of the two vectors $\vec{a} + \vec{b}$ and $\vec{c}$. We have $\overrightarrow{OR} = \vec{a} + \vec{b}$ and $\overrightarrow{OC} = \vec{c}$.

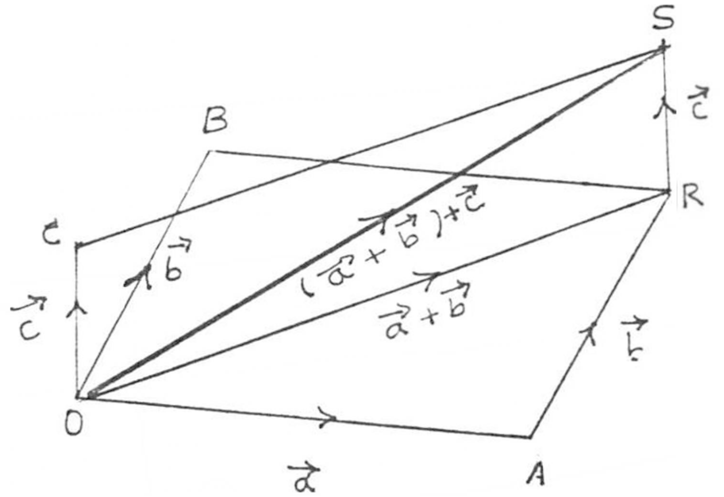


Fig. 20

Since this resultant is $(\vec{a}+\vec{b}) + \vec{c} = \overrightarrow{OR} + \overrightarrow{OC}$, we must construct a parallelogram with $\overrightarrow{OR}$ and $\overrightarrow{OC}$ as its sides. ORSC is the required parallelogram..

$\overrightarrow{OS} = \overrightarrow{OR} + \overrightarrow{OC}$ (by parallelogram law of addition) or $\overrightarrow{OS} = (\vec{a} + \vec{b}) + \vec{c}$. ....(1)

Also since ORSC is a parallelogram, $\overrightarrow{RS} = \overrightarrow{OC} = \vec{c}$.

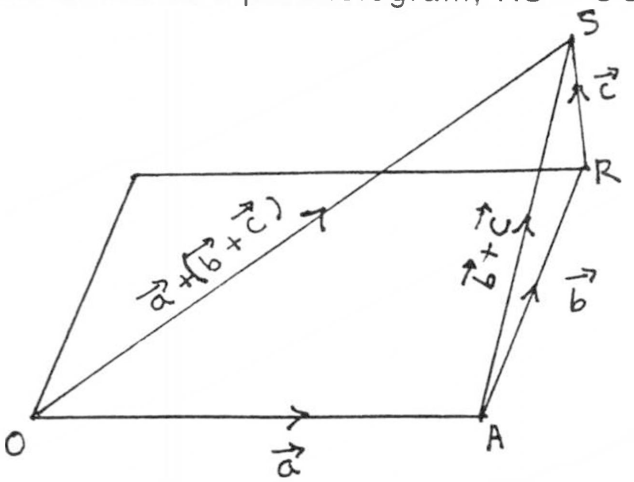


Fig. 21

In parallelogram OARB, AR||OB. $\therefore \overrightarrow{AR} = \vec{b}$

Hence, $\vec{b} + \vec{c} = \overrightarrow{AR} + \overrightarrow{RS} = \overrightarrow{AS}$

Now by triangle law of addition of vectors,

$\vec{a} + (\vec{b} + \vec{c}) = \overrightarrow{OA} + \overrightarrow{AS} = \overrightarrow{OS}$ .... (2)

From (1) and (2) it follows that

$\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$

Hence associative law for addition is true for vectors. Hence we can talk of the sum of three vectors $\vec{a}, \vec{b}, \vec{c}$ and write it as $\vec{a} + \vec{b} + \vec{c}$. In Figs. 20 and 21, the vector OS represents $\vec{a} + \vec{b} + \vec{c}$.

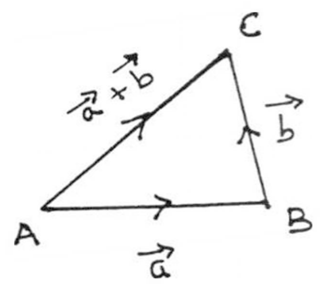
3. Prove that $|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|$

Suppose a and b are not parallel.

Let $\vec{a} = \overrightarrow{AB}$, $\vec{b} = \overrightarrow{BC}$.

Then by triangle law of addition

$\vec{a} + \vec{b} = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$



From elementary geometry, in ΔABC, AC < AB + BC. Therefore, length of $\overrightarrow{AC}$ < length of $\overrightarrow{AB}$ + length of $\overrightarrow{BC}$, i.e. $|\overrightarrow{AC}| < |\overrightarrow{AB}| + |\overrightarrow{BC}|$

or $|\vec{a} + \vec{b}| < |\vec{a}| + |\vec{b}|$. When $\vec{a}$ and $\vec{b}$ have the same direction we have seen that $|\vec{a} + \vec{b}| = |\vec{a}| + |\vec{b}|$.

When $\vec{a}$ and $\vec{b}$ have opposite directions we have seen that $|\vec{a} + \vec{b}| = |\vec{a}| - |\vec{b}|$ or $= |\vec{b}| - |\vec{a}|$. Hence in this case $|\vec{a} + \vec{b}| < |\vec{a}| + |\vec{b}|$.

Hence, $|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|$.

4. If $\vec{a} + 5\vec{b} = \vec{c}$ and $\vec{a} - 7\vec{b} = 2\vec{c}$, then show that $\vec{a}$ has the same direction as $\vec{c}$ and opposite direction to $\vec{b}$.

We have $\vec{a} + 5\vec{b} = \vec{c}$, $\vec{a} - 7\vec{b} = 2\vec{c}$.

$\therefore \vec{a} - 7\vec{b} = 2(\vec{a} + 5\vec{b})$

$\Rightarrow \vec{a} - 7\vec{b} = 2\vec{a} + 10\vec{b}$

$\Rightarrow \vec{a} + 17\vec{b} = \vec{0}$

or $\vec{a} = -17\vec{b}$

$\therefore \vec{a}$ and $\vec{b}$ have opposite direction.

Again, $\vec{a} + 5\vec{b} = \vec{c} \Rightarrow \vec{a} + 5\ ((-\ 1/17)\vec{a}) = \vec{c}$

$\Rightarrow 12\vec{a} = 17\vec{c}$

$\vec{a} = (17/12)\vec{c} \Rightarrow \vec{a}$ and $\vec{c}$ have same directions.

We will now solve some problems from your textbook.

5. $\vec{a}$ and $\vec{b}$ are represented by $\overrightarrow{AB}$ and $\overrightarrow{BC}$ of a ΔABC, show that triangle is equilateral if $|\vec{a}| = |\vec{b}| = |\vec{a} + \vec{b}|$.

Since $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{BC} = \vec{b}$, by the triangle law of vector addition, $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \vec{a} + \vec{b}$.

Now, $AB = |\overrightarrow{AB}| = |\vec{a}|$

$BC = |\overrightarrow{BC}| = |\vec{b}|$

$AC = |\overrightarrow{AC}| = |\vec{a} + \vec{b}|$

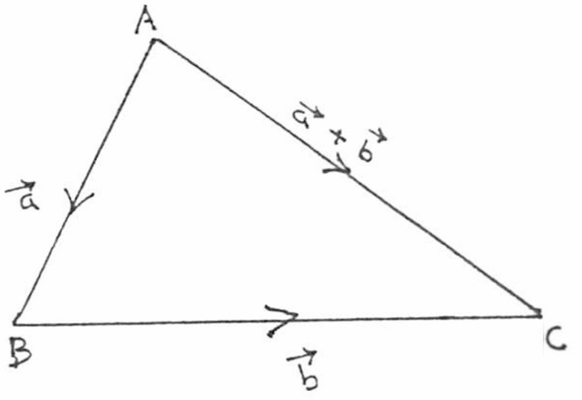


Fig. 23

Since $|\vec{a}| = |\vec{b}| = |\vec{a} + \vec{b}|$, hypothesis, AB = BC = AC. Therefore ΔABC is equilateral.

6. The diagonals of a parallelogram ABCD intersect at O. If $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$, express the sides of the parallelogram in terms of $\vec{a}$ and $\vec{b}$.

$\overrightarrow{AO} = -\overrightarrow{OA} = -\vec{a}$

$\overrightarrow{AO} + \overrightarrow{OB} = \overrightarrow{AB}$ (triangle law of addition) or $-\vec{a} + \vec{b} = \overrightarrow{AB}$. $\therefore \overrightarrow{AB} = \vec{b} - \vec{a}$

Now, $\overrightarrow{OC} = -\overrightarrow{OA} = -\vec{a}$ (diagonals bisect at O)

$\overrightarrow{BC} = \overrightarrow{BO} + \overrightarrow{OC} = -\vec{b} + (-\vec{a}) = -(\vec{b}+\vec{a})$ ($\because \overrightarrow{BO} = -\overrightarrow{OB} = -\vec{b}$)

$\therefore \overrightarrow{BC} = -(\vec{b} + \vec{a})$

$\overrightarrow{CD} = -\overrightarrow{AB} = -(\vec{b} - \vec{a}) = \vec{a} - \vec{b}$

$\overrightarrow{DA} = -\overrightarrow{BC} = -(-(\vec{b}+\vec{a})) = \vec{b} + \vec{a}$

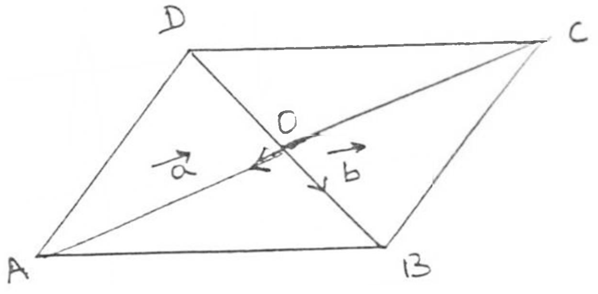


Fig. 24

7. In a parallelogram OACB, $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$. Show that OACB is a rectangle if $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$.

Since OACB is a parallelogram, OB||AC, OB = AC and hence $\overrightarrow{AC} = \vec{b}$. Therefore $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC} = \vec{a} + \vec{b}$. Also OA||BC, OA = BC. $\therefore$ $\overrightarrow{BC} = \overrightarrow{OA} = \vec{a}$. But $\overrightarrow{CA} = -\overrightarrow{AC} = -\vec{b}$. It is given that $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$. So the magnitudes or lengths of diagonals are equal. In parallelogram OACB, the diagonals are equal. Therefore, by elementary geometry, we know that such a parallelogram must be a rectangle.

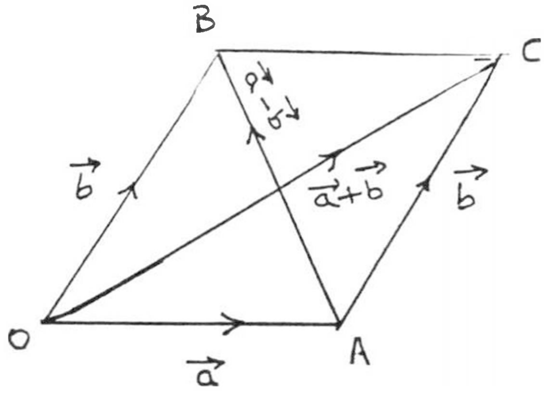


Fig. 25

8. Prove by vector method that the line segment joining the mid points of any two sides of a triangle is equal to half the third side and parallel to the third side.

Let D and E be mid points of sides AB and AC of triangle ABC.

Now $\overrightarrow{DA} + \overrightarrow{AE} = \overrightarrow{DE}$ ... (by triangle law of addition) .... (1)

Again, $\overrightarrow{BA} + \overrightarrow{AC} = \overrightarrow{BC}$ ...(by triangle law of addition) .... (2)

It is given that $\overrightarrow{DA} = \frac{1}{2}\overrightarrow{BA}$, $\overrightarrow{AE} = \frac{1}{2}\overrightarrow{AC}$

Using these in (1), $\overrightarrow{DA} + \overrightarrow{AE} = \frac{1}{2}(\overrightarrow{BA} + \overrightarrow{AC})$

$\therefore \overrightarrow{DE} = \frac{1}{2}\overrightarrow{BC}$ ....(3)

$\therefore \overrightarrow{DE}$ and $\overrightarrow{BC}$ have the same direction.

$\therefore$ DE||BC

Also, $|\overrightarrow{DE}| = \frac{1}{2}|\overrightarrow{BC}|$

$\therefore$ DE = ½ BC

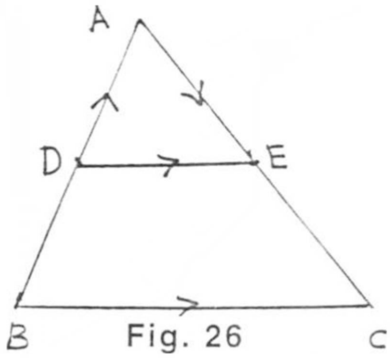


Fig. 26

9. Prove by vector method that the mid points of the sides of any quadrilateral are the vertices of a parallelogram.

Let ABCD be the given quadrilateral and P, Q, R and S, be the mid points of its sides.

Let $\overrightarrow{DA} = \vec{a}$, $\overrightarrow{AB} = \vec{b}$, $\overrightarrow{BC} = \vec{c}$, CD = $\vec{d}$. Then $\overrightarrow{DB} = \vec{a} + \vec{b}$, $\overrightarrow{AC} = \vec{b} + \vec{c}$, $\overrightarrow{BD} = \vec{c} + \vec{d}$, $\overrightarrow{CA} = \vec{d} + \vec{a}$. Hence by previous problem,

$\overrightarrow{PQ} = \frac{1}{2}(\vec{a} + \vec{b})$

$\overrightarrow{QR} = \frac{1}{2}(\vec{b} + \vec{c})$

$\overrightarrow{RS} = \frac{1}{2}(\vec{c} + \vec{d})$

$\overrightarrow{SP} = \frac{1}{2}(\vec{d} + \vec{a})$

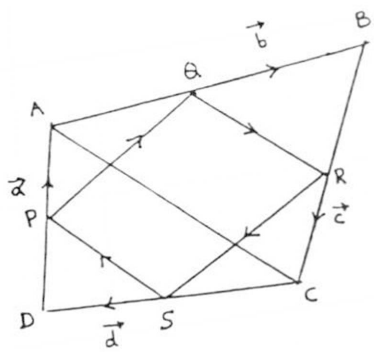


Fig. 27

But $\vec{a} + \vec{b} + \vec{c} + \vec{d} = \vec{0}$ (why ?)

$$\therefore \overrightarrow{PQ} = \frac{\vec{a} + \vec{b}}{2} = -\frac{\vec{c} + \vec{d}}{2} = \overrightarrow{SR}$$

$$\text{Again, } \overrightarrow{QR} = \frac{\vec{b} + \vec{c}}{2} = -\frac{\vec{d} + \vec{a}}{2} = \overrightarrow{PS}$$

Hence $\overrightarrow{PQ} = \overrightarrow{SR}$ and $\overrightarrow{QR} = \overrightarrow{PS}$. $\therefore$ PQ||SR and QR||PS.

We have shown that both pairs of opposite sides are equal and parallel. Hence PQRS is a parallelogram.

10. Prove by vector method, if a pair of opposite sides of a quadrilateral are equal and parallel, then the quadrilateral is a parallelogram.

Let OABC be a quadrilateral such that AB = OC and AB||OC.

$\therefore \overrightarrow{AB} = \overrightarrow{OC}$ .... (1)

By triangle law

$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$ .... (2)

$\overrightarrow{OC} + \overrightarrow{CB} = \overrightarrow{OB}$ .... (3)

$\therefore \overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OC} + \overrightarrow{CB}$

$\therefore \overrightarrow{OA} = \overrightarrow{CB}$ using (1)

$\therefore$ OA||CB and OA = CB

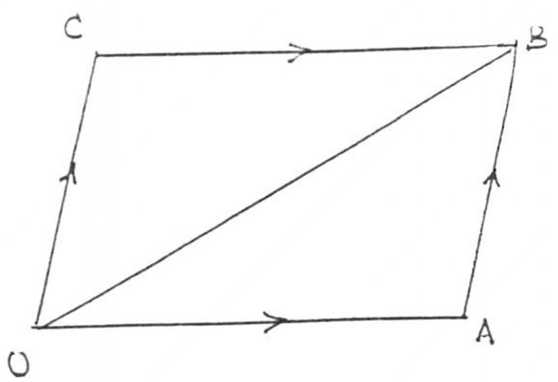


Fig. 28

In quadrilateral OABC

AB||OC and OA||CB

$\therefore$ It is a parallelogram.

11. The position vectors of points A, B, C, D are $\vec{a}$, $\vec{b}$, $2\vec{a}+3\vec{b}$ and $\vec{a}+2\vec{b}$ respectively. Express $\overrightarrow{AC}$ and $\overrightarrow{BD}$ in terms of $\vec{a}$ and $\vec{b}$.

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = (2\vec{a} + 3\vec{b}) - (\vec{a}) = \vec{a} + 3\vec{b}$.

Similarly, $\overrightarrow{BD} = \overrightarrow{OD} - \overrightarrow{OB} = \vec{a} + 2\vec{b} - \vec{b} = \vec{a} + \vec{b}$.

12. $\vec{a}$ and $\vec{b}$ are position vectors of A and B with respect to O, C and D are the points of trisection of AB. Find the position vector of C and D in terms of $\vec{a}$ and $\vec{b}$;.

Now, $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \vec{b} - \vec{a}$

$$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC} = \vec{a} + \frac{\vec{b} - \vec{a}}{3}$$

$$\therefore \overrightarrow{OC} = \frac{2\vec{a} + \vec{b}}{3}$$

$$\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD} = \vec{a} + 2/3(\vec{b} - \vec{a})$$

$$\therefore \overrightarrow{OD} = \frac{\vec{a} + 2\vec{b}}{3}$$

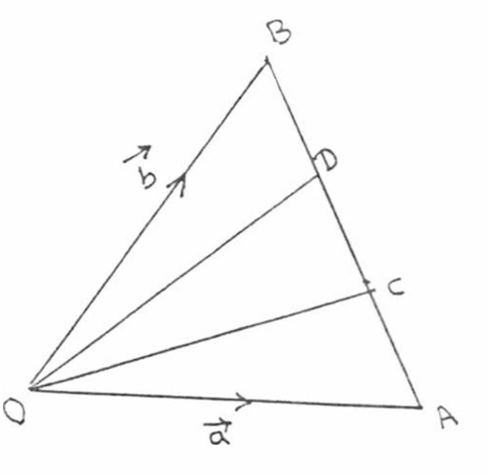


Fig. 28a

13. If $\vec{a}$, $\vec{b}$, $\vec{c}$ are the position vectors of A, B and C show that the position vector of the centroid of ΔABC is $\dfrac{\vec{a} + \vec{b} + \vec{c}}{3}$

First we prove the Section formula:

If the position vectors of two points A and B are $\vec{a}$ and $\vec{b}$ respectively, find the position vector $\vec{r}$ of the point P which divides the segment AB in the ratio m:n.

Solution: Let $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$

Join $\overrightarrow{AB}$

Let P be a point on AB such that AP:PB = m:n

Join $\overrightarrow{OP}$. Now, $\overrightarrow{OP} = \vec{r}$.

Hence $\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \vec{r} - \vec{a}$

and $\overrightarrow{PB} = \overrightarrow{OB} - \overrightarrow{OP} = \vec{b} - \vec{r}$.

Since $\overrightarrow{AP}$ and $\overrightarrow{PB}$ are collinear,

$m\overrightarrow{PB} = n\overrightarrow{AP}$

$\therefore\ m(\vec{b} - \vec{r}) = n(\vec{r} - \vec{a})$

Solving for $\vec{r}$, we get

$$\vec{r} = \frac{m\vec{b} + n\vec{a}}{m + n}$$

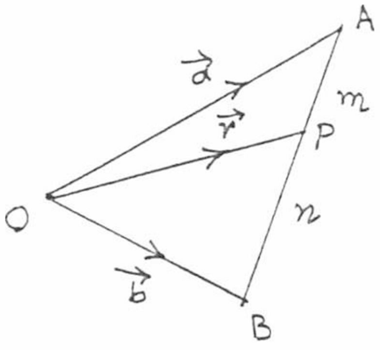


Fig. 29

This formula is called the Section Formula. We use this formula to solve the given problem. Let the position vectors of the vertices A, B, C of ΔABC be $\vec{a}$, $\vec{b}$ and $\vec{c}$.

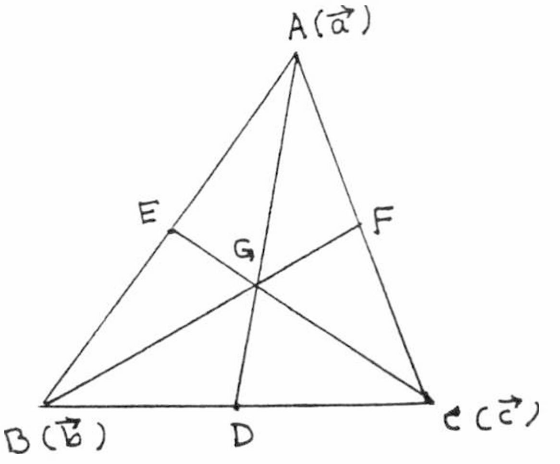


Fig. 30

Let E and D be the mid points of AB and BC.

Then $\overrightarrow{OE} = \dfrac{1.\vec{a} + 1.\vec{b}}{1 + 1} = \dfrac{\vec{a} + \vec{b}}{2}$ (by section formula).

G be the centroid of ΔABC. Then we know that EG:GC = 1:2.

$$\text{Position vector of the centroid } G = \overrightarrow{OG} = \frac{1.(\vec{c}) + 2\,\dfrac{(\vec{a}+\vec{b})}{2}}{1 + 2}$$

$$= \frac{\vec{a} + \vec{b} + \vec{c}}{3} \text{ by section formula}$$

HINTS FOR TEACHING

The prerequisite knowledge that the 10^{th} class children are supposed to have for a reasonable understanding of vectors and their properties are:

(1) Properties of number addition and multiplication,(2) Construction of triangles, parallelograms and quadrilaterals. (3) Preliminary ideas/ notions of displacement, velocity, force, etc. (which they learn in their physics classes) and how to graphically represent them, (4) Statements of theorems from elementary geometry which can be proved by vector methods, (5) Pythagoras theorem.

Pupils of tenth class are generally familiar with the above stated prerequisites. However, teacher will do well to revise them and test the students to ensure better or proper learning.

School children generally find vectors hard to understand. Hence proper and adequate motivation must be provided through concrete examples.

Practical situations in Mechanics (Physics) provide a rich source of motivation for vectors. Problems related to moving cars, ships, aeroplanes, running, swimming, cycling, flowing rivers, blowing of wind, etc. which involve velocity and displacements can be discussed. Diagrams to (using scale drawing) represent these problems can be drawn. Magnitudes and directions of line segments in the diagrams can be measured and recorded. In particular, use of displacements as vectors can be of great help.

It may be desirable to tell the students to find out how the navigators, pilots and people in deserts keep track of directions when they are moving.

When an ordinary line segment is measured, we get its length which is a number. This number (a scalar) is represented by just a point on the number line. But a vector is a line segment with an arrow head. The notion of a directed line segment clashes with the pupils' earlier notion of line segments in elementary geometry. Many line segments of same length say 3" can be drawn in all possible directions. All these line segments are said to be equal but directed line segments are equal only if their directions are also equal, that is, when they are parallel. Sufficient practice of drawing vectors will help to clear this confusion.

The difference in scalar and vector addition needs to be emphasised through concrete examples. Vector sum is to be presented as resultant of two physical quantities like forces, rather than as an abstract algebraic operation. Of course, abstract symbolic treatment can also come later. Same thing can be said to multiplication of a vector by a scalar.

Distinctions between $\overrightarrow{AB}$ and magnitude of $\overrightarrow{AB}$ must be made clear. It is difficult for some students to appreciate how

1. $\overrightarrow{AB} + (-\overrightarrow{AB}) = 0$, but $|\overrightarrow{AB}| + |-\overrightarrow{AB}| \neq 0$.
2. In a triangle ABC, AB + BC > AC but $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$.

Special care must be taken to clarify these distinctions. Need of position vectors, choice of the origin in representing vectors, etc. must be explained through diagrams and examples.

While discussing zero or null vector, following points may be borne in mind.

Although zero vector is indeed a vector, it cannot be represented as a directed line segment because its initial and terminal points coincide. The vectors $\overrightarrow{AA}$, $\overrightarrow{BB}$, $\overrightarrow{CC}$, etc. are zero vectors. Their magnitudes $|\overrightarrow{AA}| = |\overrightarrow{BB}| = |\overrightarrow{CC}| = 0$ (scalar). Clearly zero vector has no definite direction. In fact, it can have any direction. The idea of zero vector can be better clarified while teaching vector addition. Physical illustration leading to the conclusions $\vec{a} + (-\vec{a}) = \vec{0}$ may be helpful. Teachers may discuss the role of zero vectors (as compared to number zero) and its properties. While introducing unit vector, one may emphasise that there are many unit vectors. For any vector $\vec{a}$ we can find a unit vector in the direction of $\vec{a}$.

Magnitude of a unit vector is always unity. If $\vec{a}$ is the given vector, then the unit vector $\hat{a}$ along $\vec{a}$ is equal to $\dfrac{\vec{a}}{|\vec{a}|}$.

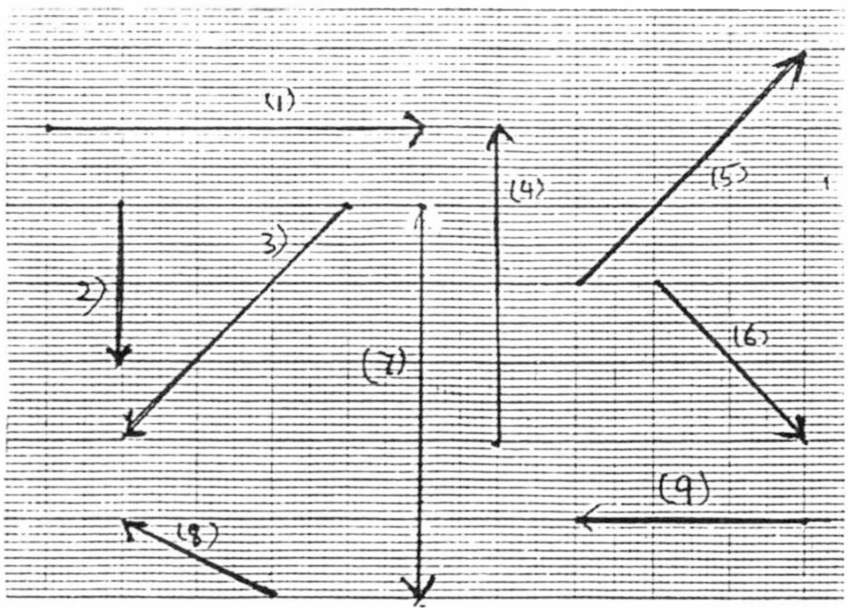
or $\hat{a} = \dfrac{\vec{a}}{|\vec{a}|}$

$\therefore$ Any vector $\vec{a}$ can be expressed as $\vec{a} = |\vec{a}|$ times $\hat{a}$

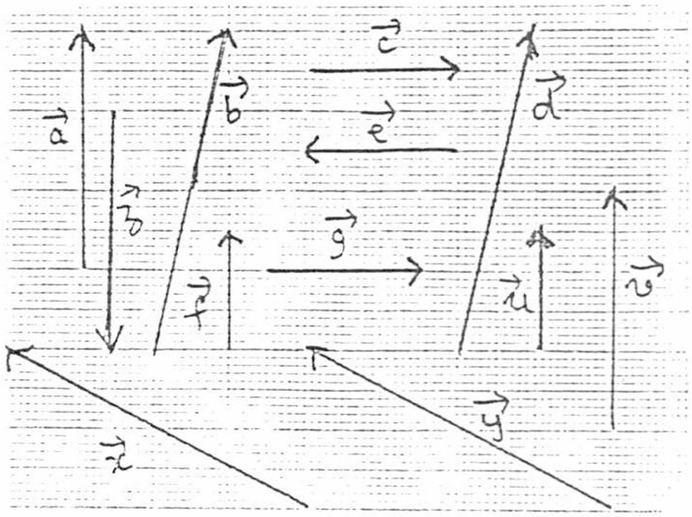
New ideas require children to reorient their thinking. Pupils need time and practice to get used to them. We, as teachers, must keep our patience.

Exercises (for self evaluation)

1. Give the magnitudes and directions of the vectors shown below.



2. Which of the vectors given below are equal ? Find their magnitudes and directions.



3. Given the vectors $\vec{a}$ = 6 kms, 30° North of East, $\vec{b}$ = 4 kms 60° North of West (scale 1 cm = 1 km), $\vec{c}$ = 8 km South West.

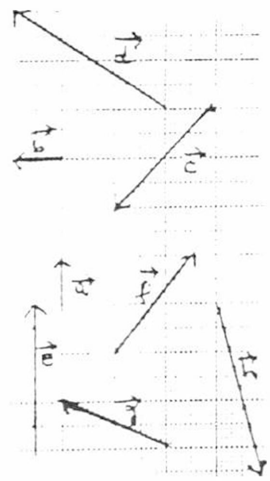
(a) Draw diagrams in which (i) $\vec{a}$ is followed by $\vec{b}$ followed by $\vec{c}$, (ii) $\vec{c}$ is followed by $\vec{a}$ followed by $\vec{b}$, (iii) $\vec{b}$ is followed by $\vec{c}$ ($\vec{a}$ followed by $\vec{b}$ means end point of $\vec{b}$ is placed on initial point of $\vec{a}$).

(b) Construct vectors equal to $2\vec{a}$, $-\vec{c}$, $(½)\vec{b}$, $-5\vec{a}$.

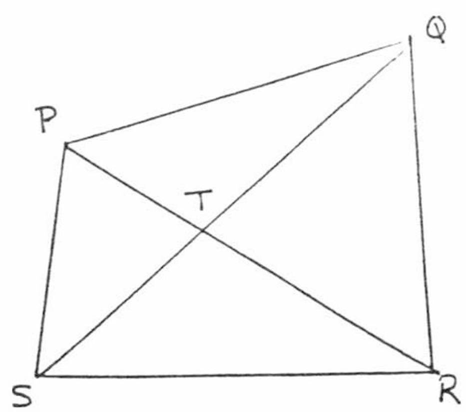
4. If $\vec{a}$ = 1 km north and $\vec{b}$ = 2 km north west, describe in words the following displacements. (i) $2\vec{a}$, (ii) $3\vec{b}$, (iii) $-\vec{b}$, (iv) $-\vec{a}$, (v) $6\vec{b}$, (vi) $4\vec{a}$. Draw these vectors to a scale of 1 cm = 1 km.

5. $\vec{u}$ = 2 km N and $\vec{v}$ = 3 km East, then draw (i) $\vec{a} = \vec{u}+\vec{v}$, (ii) $\vec{b} = -2\vec{u}-\vec{v}$, (iii) $\vec{c} = 4\vec{u} + (-1/2\ \vec{v})$. (Use graph sheet).

6. If $\vec{a}$ = 1 metre west, $\vec{b}$ = 1 metre south, describe the following displacements in terms of $\vec{a}$ and $\vec{b}$.



7. In the given quadrilateral find, vector (i) $\overrightarrow{PT}+\overrightarrow{TR}$, (ii) $\overrightarrow{PS}+\overrightarrow{SQ}$, (iii) $\overrightarrow{PS}+\overrightarrow{SQ}+\overrightarrow{QR}$, (iv) $\overrightarrow{SQ}+\overrightarrow{QT}$.

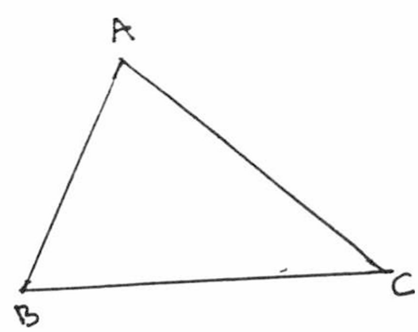


8. Given the figure of ΔABC, simplify

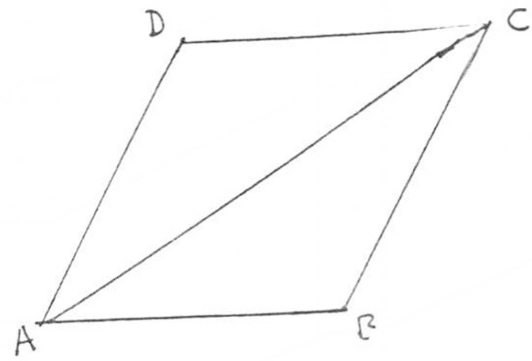
(i) $\overrightarrow{AC} - \overrightarrow{AB}$

(ii) $\overrightarrow{BC} - \overrightarrow{BA}$

(iii) $\overrightarrow{CB} - \overrightarrow{CA}$



9. In the parallelogram ABCD, find $\overrightarrow{DB}$ in terms of the sides in two ways.

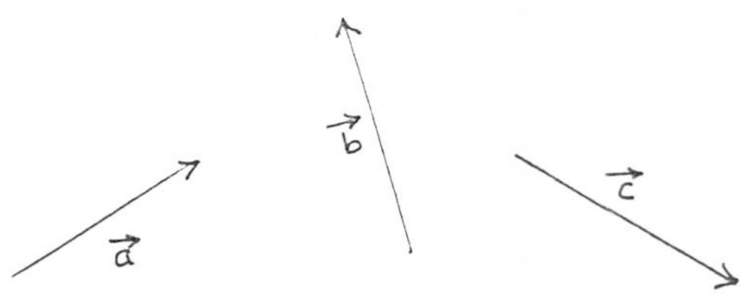


10. Given triangle ABC is equilateral $\overrightarrow{AB} = \vec{c}, \overrightarrow{BC} = \vec{a}, \overrightarrow{CA} = \vec{b}, \overrightarrow{MA} = \vec{d}$ (M is the mid point of BC), say which of the following are true.

(i) $\vec{a} + \vec{b} = \vec{c}$, (ii) $|\vec{a} + \vec{b}| = |\vec{c}|$, (iii) $|\vec{a}| + |\vec{b}| + |\vec{c}| = 3|\vec{a}|$, (iv) $\vec{a} = \vec{b} + \vec{c}$.

(v) If $\vec{x} + \vec{d} = -\vec{c}$ then $\vec{x} = \vec{a}/2$.

11. Given vectors $\vec{a}, \vec{b}, \vec{c}$ (see Fig.)



(a) (Construct) Find the vectors $\vec{a} + \vec{b}$, $\vec{b} + \vec{d}$, $\vec{c} + \vec{a}$, $\vec{a} + \vec{c}$, $\vec{b} + \vec{c}$, $\vec{c} + \vec{b}$ by drawing suitable diagrams.

(b) Find by drawing the resultant of two forces of 10 N and 16 N acting at (i) 70° east of north, (ii) 140° to each other.

12. A monkey is hanging on two ropes which make angles of 23° and 60° with the vertical respectively. If the force in the first rope is

66 N and in the second is 30 N. Find by scale drawing the force supporting the monkey.

13. One ship A is sailing with a velocity of 10 miles per hour in south west direction. A second ship B is sailing with a velocity of 15 miles/hour in the direction 30°E of N. Find the velocity of A in relation to B (Take $\vec{V}_A$ and $\vec{V}_B$ to be the velocities of A and B, then $\vec{V}_A$ - $\vec{V}_B$ is the velocity of A in relation to B).

14. A car is moving with a velocity of 40 kmh, N. A bullet is fired from that car with a velocity of 30 kmph due west. Find the resultant velocity of the bullet.

UNIT NO. 4: ALGEBRAIC STRUCTURES

Unit Introduction

Algebraic structures give the first taste of abstract nature of Mathematics. Because of its abstract nature, most of the children find the chapter on Algebraic structures difficult to understand unless motivated through concrete situations. Children come across algebraic structures in an intuitive manner from the primary classes itself through the study of Number System and the four fundamental operations of addition, subtraction, multiplication and division. Then in their higher primary classes, they come across the algebraic properties of the number system. However, it is only in the tenth standard that we try to make these properties of the number systems completely abstract and build algebraic structures. So the teacher should be very careful in introducing the concepts of algebraic structures. The approach must be from concrete to abstract. The teacher should recall the relevant properties of numbers and then generalise them into abstract algebraic structures.

SUBUNIT NO. 4.1: BINARY OPERATIONS

Concepts and Explanations

In the set of integers, if we add to an integer a an integer b we again get an integer a + b. That means the operation of addition associates with an ordered pair of integers (a,b) a unique integer namely their sum a+b. Similarly if we subtract from an integer a an integer b, then we get again an integer a-b. That means the operation of subtraction associates with every ordered pair (a,b) of integers a unique integer namely a-b. Again in the set of rational numbers, if we multiply a rational number a by a rational number b, we again get a rational number a.b. That means in the set of rational numbers, the operation of multiplication associates with every ordered pair (a,b) of rationals, a unique rational a.b. In all the above cases, the operation considered, associates with every ordered pair of elements in the given set a unique element of the set. Such operations are called binary operations.

Definition : A binary operation on a set S is a rule which associates with every ordered pair of elements of S, a unique element of S.

Therefore a binary operation on a set S is a map from SxS to S (why ?).

In the definition of a binary operation it is important that the pair of elements which we consider are ordered. Subtraction is a binary operation on the set of integers. But whereas with (3,2) it associates 3-2 = 1, with (2,3) it associates 2-3 = -1. Hence the order in which the pair of elements is taken is very important.

We normally denote a binary operation on a set S by a symbol $*$ and denote the unique element associated with the ordered pair (a,b) by $a*b$. If $a,b \in S$ then $a * b \in S$.

Examples

1. On the set of natural numbers N, usual addition, multiplication are binary operations. But subtraction is not a binary operation on N since,

   $2,3 \in N$. But $2-3 = -1 \notin N$.

2. On the set of integers, the set of rational numbers and the set of real numbers, usual addition, subtraction and multiplication are binary operations.

3. On the set of integers Z, division is not a binary operation since

   $1,2 \in Z$, but $\frac{1}{2} \notin Z$ (Think of another reason)

4. On the set of rational numbers Q, division is not a binary operation since $1,0 \in Q$, but $1 \div 0$ is not defined. For the same reason division is not a binary operation as the set of real numbers R also.

5. On the set of non-zero rational numbers Q^*, division is a binary operation. On the set of non-zero real numbers R^* division is a binary operation. Note that on both Q^* and R^* the operation of multiplication is a binary operation.

6. On the set of 2x2 matrices, with integer entries (rational or real entries) the addition of matrices, subtraction of matrices, multiplication of matrices are all binary operations.

7. On the set $A = \{a,b,c\}$, the operation $*$ given by the following table is a binary operation. Here the unique element $x*y$ associated with the ordered pair (x,y) is the entry lying on both the row labelled x and column labelled y in the table given below. For example, the encircled letter b is $b*c$.

*	a	b	c
a	b	c	a
b	c	a	b
c	a	b	a

Since all the entries in the table are from the set A, it follows that if $x,y \in A$ then $x*y \in A$. Hence $*$ is a binary operation.

8. Multiplication is a binary operation on the set $\{-1,0,1\}$. The multiplication on the set $\{-1,0,1\}$ can be written in the tabular form as follows.

*	0	1	-1
0	0	0	0
1	0	1	-1
-1	0	-1	1

All the entries in the table belong to the set {-1,0,1}. This verifies that multiplication is a binary operation on {-1,0,1}.

9. Let S = {a,b,c,d,e}. Define an operation * on S by the following table.

*	a	b	c	d	e
a	a	b	c	d	e
b	b	c	d	e	a
c	c	d	e	a	b
d	d	e	a	f	c
e	e	a	b	c	d

Then * is not a binary operation on S. Why ?

Problem Solving

1. Which of the following is a binary operation on the set ?

(a) a * b = H.C.F. of a and b on N.

If a and b are natural numbers, then H.C.F. of a and b is also a natural number.

$\therefore$ a,b $\in$ N $\Rightarrow$ a*b $\in$ N

Hence * is a binary operation on N.

(b) $a*b = \dfrac{a+b}{a-b}$ on Z.

If a and b are integers, then $\dfrac{a+b}{a-b}$ need not be an integer. For example, let a = 2, b = -1. Then a+b = 2+(-1)=1, a-b = 2-(-1)=2+1=3.

$$\therefore\ a*b = \frac{a+b}{a-b} = \frac{1}{3}$$ which is not an integer.

Thus, $a,b \in Z \not\Rightarrow a * b \in Z$.

Hence $*$ is not a binary operation on Z.

In fact $*$ is not a binary operation on Q also. By taking $a = b$,

we see that $a*b = \frac{2a}{a-a} = \frac{2a}{0}$ which is not defined.

(c) $a*b = \frac{a+b}{2}$ on Q.

If a and b are rational numbers, then $a + b$ is a rational number and hence $\frac{a+b}{2}$ is also a rational number. Hence, $a,b \in Q \Rightarrow a*b \in Q$

Hence $*$ is a binary operation on Q.

2. Find the true statements in the following:

(a) Subtraction is a binary operation on the set of odd integers.

False because if a and b are odd integers, then a-b is not an odd integer. In fact a-b is an even integer.

(b) Multiplication is a binary operation on the set {0,1}.

True. 0x0 = 0, 0x1 = 0, 1x0 = 0, 1x1 = 1

Hence $a,b \in \{0,1\} \Rightarrow axb \in \{0,1\}$

$\therefore$ Multiplication is a binary operation on {0,1}.

3. On Z, $a*b = 2a+3b \ \forall \ a,b \in Z$. Find $2*3$ and $3*2$. Is $2*3 = 3*2$?

$2*3 = 2 \times 2 + 3 \times 3 = 4+9 = 13$

$3*2 = 2 \times 3 + 3 \times 2 = 6+6 = 12$

$\therefore \ 2*3 \neq 3*2$.

4. On N, $a*b$ = L.C.M. of a and b $\forall \ a,b \in N$, find (i) $4*6$, (ii) $1*5$, (iii) $(2*4)*6$.

(i) $4*6$ = L.C.M. of 4 and 6 = 12

(ii) $1*5$ = L.C.M. of 1 and 5 = 5

(iii) $(2*4)*6$ = (L.C.M. of 2 and 4)$*6$

= $4*6$ = L.C.M. of 4 and 6 = 12

5. On N, $a*b$ = H.C.F. of a and b $\forall \ a,b \in N$, find (i) $2*3$, (ii) $8*12$, (iii) $(2*3)*6$.

(i) $2*3$ = H.C.F. of 2 and 3 = 1

(ii) $8*12$ = H.C.F. of 8 and 12 = 4

(iii) $(2*3)*6$ = (H.C.F. of 2 and 3)$*6$

= $1*6$ = H.C.F. of 1 and 6 = 1.

6. Which of the following is a binary operation ?

(a) $a*b = \dfrac{a}{b} + \dfrac{b}{a}$ on R.

When $a = 0$, $\dfrac{b}{a}$ is not defined for any value of b.

$\therefore$ $a*b$ is not defined in this case.

$\therefore$ $a,b \in R \not\Rightarrow a*b \in R$,

(Similarly when $b = 0$, $\dfrac{a}{b}$ is not defined for any value of b and hence in this case also $a*b$ is not defined.)

Therefore $*$ is not a binary operation.

(b) $\quad a*b = \dfrac{ab}{7}$ on Q^+.

If a and b are positive rationals, then their product ab is also a positive rational number and hence $\dfrac{ab}{7}$ is also a positive rational.

Hence $a,b \in Q^+ \Rightarrow \dfrac{ab}{7} \in Q^+$

$\therefore$ $a,b \in Q^+ \Rightarrow a*b \in Q^+$

$\therefore$ $*$ is a binary operation on Q^+.

(c) $\quad a*b = \dfrac{ab}{a+b}$ on R^+.

$a,b \in R^+ \Rightarrow a+b \in R^+$ and $a.b \in R^+$. Hence $\dfrac{ab}{a+b} \in R^+$.

Hence, $a,b \in R^+ \Rightarrow a*b \in R^+$

$\therefore$ $*$ is a binary operation on R^+.

(d) $a \in Q^+ \Rightarrow 1/a \in Q^+$ $b \in Q^+ \Rightarrow 1/b \in Q^+$

$$\therefore \frac{1}{a} + \frac{1}{b} \in Q^+$$

Thus $a,b \in Q^+ \Rightarrow a*b \in Q^+$

$\therefore$ $*$ is a binary operation on Q^+.

SUBUNIT NO. 4.2: ASSOCIATIVE BINARY OPERATIONS

Concept and Explanations

Consider the set of integers Z and the binary operation of addition +. Now if we consider three integers say, 3, -4 and 7, then we can first add 3 to -4 and then add the sum to 7 or we can add to 3 the sum of -4 and 7. Do we get the same integer in both the cases ?

$[3 + (-4)] + 7 = -1 + 7 = 6$

$3 + [(-4) + 7] = 3 + 3 = 6$

$\therefore [3 + (-4)] + 7 = 3 + [(-4) + 7]$

In fact, if we consider any three integers a, b and c, we always get the two sums to be equal, i.e. $(a+b) + c = a + (b+c)$ for every a,b,c in Z.

However, this property is not true for all binary operations. For example, consider the binary operation of subtraction on the set of integers. For the same integers 3, -4 and 7

$3 - [(-4) - 7] = 3 - [-11] = 3 + 11 = 14$

whereas

$[3 - (-4)] - 7 = [3 + 4] - 7 = 7 - 7 = 0$

Thus, $3 - [(-4) - 7] \neq [3 - (-4)] - 7$

Thus for the binary operation of subtraction on the set of integers Z, $a - (b - c) \neq (a - b) - c$ in general.

Definition:A binary operation $*$ on the set S is said to be associative if $a*(b*c) = (a*b)*c$ for every a,b,c in S.

Examples

1. Addition and multiplication on the set of integers Z, on the set of rationals Q, on the set of reals R are associative binary operations.
2. Subtraction is not an associative binary operation on Z, Q or R.
3. Multiplication is an associative binary operation on the set of non-zero rationals Q^* and on the set of non-zero reals R^*.
4. Division is not an associative binary operation on Q^* and R^*.

$$\frac{1}{2} \div \frac{2}{3} \quad \div \quad \frac{3}{4} \quad \neq \quad \frac{1}{2} \quad \div \quad \frac{2}{3} \div \frac{3}{4}$$

4. Let $*$ be a binary operation on the set of reals R defined by $a*b = 2a + b$ for all a,b in R.

Then, $(a*b)*c = (2a+b)*c = 2(2a+b)+c = 4a+2b+c$, whereas

$a*(b*c) = 2a+(b*c) = 2a+(2b+c) = 2a+2b+c$

Hence, $a*(b*c) \neq (a*b)*c$

Hence $*$ is not an associative binary operation on R.

5. The union and intersection of sets are associative binary operations on the set of all sets.

   For, $A \cup (B \cup C) = (A \cup B) \cup C$

   and $A \cap (B \cap C) = (A \cap B) \cap C$ for any sets A, B and C, which is proved in earlier classes.

6. Addition of matrices and multiplication of matrices are associative binary operations on the set of all 2x2 matrices with real entries. Because,

(i,j)th element of $(A+B)+C = (A+B)_{ij} + C_{ij} = (A_{ij}+B_{ij})+C_{ij} = A_{ij}+(B_{ij}+C_{ij})$

(by associativity of addition of real numbers)

$= A_{ij}+(B+C)_{ij} =$ (i,j)th element of $A+(B+C)$.

addition of matrices is associative.

Show the associativity of multiplication on the set of 2x2 matrices (!)

SUBUNIT NO. 4.3: COMMUTATIVE BINARY OPERATIONS

Concepts and Explanations

If a and b are any two real numbers, we know that $a+b = b+a$. Hence the binary operation + satisfies $a+b = b+a$ for every a and b in the set of real numbers R. For the binary operation of multiplication also, we know that $a.b = b.a$ for every a and b in the set of real numbers R. Such binary operations are called commutative binary operations.

Definition: A binary operation $*$ on a set S is said to be commutative if $a*b = b*a$ for every a, b in S

Examples

1. Addition and multiplication are commutative binary operations on the set of natural numbers, on the set of integers, on the set of rational numbers and on the set of real numbers.
2. As $7 - 5 = 2$ and $5 - 7 = -2$, $7 - 5 \neq 5 - 7$. Hence in general $a-b \neq b-a$ in the sets Z, Q and R. Hence subtraction is not a binary operation in Z, Q or R.
3. On the set of non-zero rational numbers Q^* and non-zero real numbers R^*, multiplication is a commutative binary operation.
4. Multiplication of matrices is not a commutative binary operation, since

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & 6 \end{bmatrix}$$

whereas $\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -2 & 5 \end{bmatrix}$

Hence $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$

5. The binary operations * on the set A = {a,b,c} given by the following table.

*	a	b	c
a	b	c	a
b	c	c	b
c	a	b	a

is a commutative binary operation as a*b = c = b*a, a*c = a = c*a, b*c = c = c*b.

SUBUNIT NO. 4.4: BINARY OPERATIONS WITH IDENTITY ELEMENTS

Concepts and Explanations

In the set of real numbers R, we know that for any real number a,

$a + 0 = a$ and $0 + a = a$

For no other real number b, $a + b = a$ and $b + a = a$

since, once $a + b = a$, we have $b = a - a = 0$

Hence 0 is the unique element in the set of real numbers R with the property

$a + 0 = a$ and $0 + a = a$ for every a in R.

Similarly 1 is the unique element in the set of real numbers with the property

$a.1 = a$ and $1.a = a$ for every a in R.

The elements such as 0 and 1 are called identity elements for the binary operations of addition and multiplication respectively in R.

Definition: Let $*$ be a binary operation on a set S. An element e in S is said to be the identity element for the binary operation $*$ if

$a * e = a$ and $e * a = a$ for every a in S.

One can show that if the identity element exists, it is unique.

Examples

1. On the set of integers Z, the set of rationals Q, and the set of real numbers R, zero is the identity element for the binary operation of addition.

2. On the set of integers Z, the set of rationals Q, and the set of real numbers R, 1 is the identity element for the binary operation of multiplication.

3. On the set of natural numbers N, 1 is the identity element for the binary operation of multiplication. But there is no identity element in N for the binary operation of addition.

4. For the binary operation of intersection of sets, the universal set U is the identity element since

   $A \cap U = A$ and $U \cap A = A$ for every set A.

5. For the binary operation of union of sets, the null set ϕ is the identity element since $A \cup \phi = A$ and $\phi \cup A = A$ for every set A.

6. For the binary operation of multiplication on the set {-1, 0, 1}, 1 is the identity element.

7. For the binary operation of addition of matrices on the set of 2x2 matrices, the 2x2 zero matrix $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is the identity element, whereas for the multiplication of matrices $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the identity element since

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

SUBUNIT NO. 4.5: INVERSE ELEMENT WITH RESPECT TO A BINARY OPERATION

Concepts and Explanations

Consider the set of Real numbers R and the binary operation of addition. Then, given any real number a in R, the real number -a in R is such that

$$a + (-a) = 0 \text{ and } (-a) + a = 0$$

Note here that 0 is the identity element in R with respect to the binary operation of addition. -a is called the inverse of a with respect to '+'.

In the set of non-zero real numbers R* and with respect to the binary operation of multiplication, given any non-zero real number a in R*, the real number 1/a is a non-zero real number and hence belongs to R* and a.(1/a) = 1 and (1/a).a = 1.

Note here that 1 is the identity element in R*, with respect to the operation of multiplication. (1/a) is called the inverse of a with respect to '.'.

Definition: Let $*$ be a binary operation on a set S with the identity element e. An element b in S is said to be the inverse of the element a in S with respect to the binary operation $*$ if

$$a*b = e \text{ and } b*a = e.$$

It is possible to show that inverse, if it exists, is unique. We write the inverse of an element a by a^{-1}.

Examples

1. In the set of rational numbers Q, the rational number - p/q is the inverse element of the rational number p/q with respect to the binary operation of addition.
2. In the set of non-zero rational numbers Q^*, rational number q/p is the inverse element of the rational number p/q with respect to the binary operation of multiplication.
3. In the set of real numbers R with respect to the binary operation of multiplication, every non-zero real number has an inverse 1/a in R. But the real number 0 has no inverse with respect to multiplication. (why ?)
4. In the set of natural numbers N, 1 is the identity element with respect to the binary operation of multiplication in N. But except the element 1, no other natural number has an inverse with respect to multiplication.
5. In the set {-1,0,1}, with respect to the binary operation of multiplication, the element 1 and -1 have inverses, viz. 1 and -1 respectively. But the element 0 has no inverse.
6. In the set of all 2x2 matrices, every element A has an inverse - A with respect to the binary operation of addition of matrices. Here,

if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $-A = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$

Problem Solving on subunits 4.2, 4.3, 4.4 and 4.5

1. Find whether $*$ is a Binary operation? Associative? Commutative?

(a) $a*b = a+b+2ab$ on N

$a,b \in N \Rightarrow a+b+2ab \in N$

$\therefore a*b \in N$

$\therefore *$ is a binary operation.

$(a*b)*c = (a+b+2ab)*c$

$= (a+b+2ab)+c+2.(a+b+2ab).c$

$= a+b+2ab+c+2ac+2bc+4abc$

$= a+b+c+2ab+2ac+2bc+4abc$

$a*(b*c) = a*(b+c+2bc)$

$= a+(b+c+2bc)+2a(b+c+2bc)$

$= a+b+c+2bc+2ab+2ac+4abc$

$\therefore (a*b)*c = a*(b*c) \ \forall \ a, c \in N$

$\therefore$ * is associative.

$a*b = a+b+2ab$

$b*a = b+a+2ba$

$= a+b+2ab$

$\therefore a*b = b*a \ \forall \ a,b \in N$

$\therefore *$ is commutative

(b) $a*b = 2a+3b$ on Z.

$a,b \in Z \Rightarrow 2a \in Z, 3b \in Z \Rightarrow 2a+3b \in Z \Rightarrow a*b \in Z$

$\therefore$ $*$ is a binary operation.

$(a*b)*c = (2a+3b)*c = 2(2a+3b)+3c$

$= 4a + 6b + 3c$

$a*(b*c) = a*(2b*3c) = 2a+3(2b+3c)$

$= 2a + 6b + 9c$

$\therefore$ $(a*b)*c \neq a*(b*c)$

$\therefore$ $*$ is not associative.

$a*b = 2a+3b$, $b*a = 2b+3a$

$\therefore$ $a*b \neq b*a$

$\therefore$ $*$ is not commutative.

(c) $a*b = a^b$ on N

$a,b \in N \Rightarrow a^b \in N \Rightarrow a*b \in N$

$\therefore$ $*$ is a binary operation.

$(2*2)*3 = (2^2)^3 = 4^3 = 64$

$2*(2*3) = 2*(2^3) = 2*8 = 2^8 = 256$

$\therefore$ $(2*2)*3 \neq 2*(2*3)$

$\therefore$ $*$ is not associative.

$2*3 = 2^3 = 8$

$3*2 = 3^2 = 9$

$\therefore$ $2*3 \neq 3*2$

$\therefore$ $*$ is not commutative.

(d) $a*b = \sqrt{ab}$ on N.

Let a = 2, b = 3

Then $a*b = 2*3 = \sqrt{2x3} = \sqrt{6} \notin N$

$\therefore$ is not a binary operation on N.

(e) $a*b = \sqrt{a+b}$ on R.

Let a = 1, b = -2

Then $a*b = \sqrt{1-2} = \sqrt{-1} \notin R$

$\therefore$ is not a binary operation on R.

2.

(a) If $a*b = a+b-2 \;\forall\; a,b \in Z$, find the identity element and a^{-1} for any $a \in Z$. Also find 3^{-1} and 5^{-1}.

$a*e = a+e-2 = a \;\forall\; a \in Z$

$\therefore$ e-2 = 0, e = 2

$\therefore$ 2 is the identity element in Z.

The element a^{-1} is such that $a*a^{-1} = e$ and $a^{-1}*a = e$. But $a*a^{-1} = a+a^{-1}-2$ and e = 2.

$\therefore a+a^{-1}-2 = 2$

$\therefore a+a^{-1} = 4$ $\qquad\qquad \therefore a^{-1} = 4-a$

$\therefore 3^{-1} = 4-3 = 1$ and $5^{-1} = 4-5 = -1$

(b) If $a*b = \frac{3}{4} ab \; \forall \; a,b \in R'$, find the identity element and a-1 for any element a with respect to $*$. Hence find 2^{-1} and $\left(\frac{3}{2}\right)^{-1}$

Let e be the identity element. Then $a*e = a \; \forall \; a \in R'$, $\therefore \frac{3}{4} ae = a \; \forall \; a \in R'$

$\therefore (3/4)e = 1 \qquad \therefore e = 4/3$

$\therefore a*a^{-1} = 4/3 \qquad \therefore (3/4)\, a\, a^{-1} = 4/3$

$\therefore a^{-1} = (16/9)\, a$

$$\therefore 2^{-1} = \frac{16}{9 \times 2} = \frac{8}{9} \text{ and } \left(\frac{3}{2}\right)^{-1} = \frac{16}{9 \times (3/2)} = \frac{32}{27}$$

3. $a*b = a+b-ab \; \forall \; a,b \in R$

(In the book, this problem has been printed as $a,b \in Q^+$. But on Q^+, $*$ is not a binary operation, since $2*3 = 2+3-(2\times3) = 5-6 = -1 \in Q^+$.)

(a) Find the identity element with respect to $*$.

(b) Find the element having no inverse.

Let e be the identity element. Then $a*e = a \; \forall \; a \in R$

But $a*e = a+e-ae$

$\therefore a+e-ae = a \; \forall \; a \in R$

$\therefore e(1-a) = 0 \; \forall \; a \in R$

Put a = 2. We get $-e = 0 \qquad \therefore e = 0$

$\therefore$ 0 is the identity element.

Now, $a*a^{-1} = e$

$\therefore\ a*a^{-1} = 0, \qquad \therefore\ a+a^{-1}-a.a^{-1} = 0$

$\therefore\ a+a^{-1}(1-a) = 0 \qquad \therefore\ a^{-1} = \dfrac{a}{a-1}$ if $a \neq 1$.

But if we put $a = 1$, a^{-1} does not exist.

$\therefore$ 1 has no inverse.

5. Show that there is no identity element with respect to $*$ in Q^+ if

$$a*b = \frac{ab}{a+b}.$$

If there exists an identity element let it be e.

Then $a*e = a\ \forall\ a \in Q^+$

$$\therefore\ \frac{ae}{a+e} = a\ \forall\ a \in Q^+$$

$\therefore\ ae = a^2 + ae\ \forall\ a \in Q^+ \qquad \therefore\ a^2 = 0\ \forall\ a \in Q^+$

But in particular put $a = 1$. $\therefore\ 1 = 0$ which is absurd.

$\therefore$ There exists no identity element with respect to $*$.

6. Given $a*b = a+b+3\ \forall\ a,b \in Z$. Find

(a) $3*2$, (b) $(2*3)*4$, (c) $3^{-1}*3^{-1}$

$3*2 = 3+2+3 = 8$

$(2*3)*4 = (2+3+3)*4 = 8*4 = 8+4+3 = 15$

To find inverse of any element, first of all we should know the identity element. If e is the identity element, then

$a*e = a\ \forall\ a \in Z \qquad \therefore\ a+e+3 = a\ \forall\ a \in Z$

$\therefore$ e + 3 = 0

$\therefore$ e = -3

$\therefore$ $3*3^{-1} = -3$ $\qquad$ $\therefore$ $3+3^{-1}+3 = -3$ $\qquad$ $\therefore$ $3^{-1} = -9$

$\therefore$ $3^{-1}*3^{-1} = (-9)*(-9) = (-9)+(-9)+3 = -15$

7. Given $a*b = \dfrac{ab}{3}$ in Q, find (a) $3*1$, (b) 3^{-1}, (c) $2*3^{-1}$

(a) $3*1 = \dfrac{3x1}{3} = 1$

(b) To find 3^{-1} first we have to find the identity element in Q^+.

If e is the identity element, then $a*e = a \ \forall \ a \in Q^+$.

$\therefore \dfrac{a \times e}{3} = a \ \forall \ a \in Q^+$

$\therefore$ $ae = 3a \ \forall \ a \in Q^+$

Put a = 1

We get e = 3 $\qquad$ $\therefore$ 3 is the identity element.

Now $3*3^{-1} = 3$ $\qquad$ $\therefore \dfrac{3 \times 3^{-1}}{3} = 3$ $\qquad$ $3 \times 3^{-1} = 9$

$\therefore$ $3^{-1} = 3$

(c) $2*3^{-1} = 2*3 = 2$ (3 is the identity element)

Exercises (for self Evaluation)

1. Show that multiplication is a binary operation on the set {1,-1,i,-i} where $i^2 = -1$.

2. Find whether the following operations given along with the sets are binary or not. In case they are binary, find whether they have identity elements ? Also find which elements have inverses ?

(i) $\{0,2,4,6,8,10,...\}$, +

(ii) $\{0,2,4,6,8,10,...\}$, x

(iii) $\{0,\pm2,\pm4,\pm6,\pm8,\pm10,...\}$, +

(iv) $\{0,\pm2,\pm4,\pm6,\pm8,\pm10,...\}$, x

(v) $\{0,\pm2,\pm4,\pm6,\pm8,\pm10,...\}$, -

(vi) $\{0,\pm2,\pm4,\pm6,\pm8,\pm10,...\}$, ÷

(vii) $\{-1,0,1\}$, -

(viii) $\{-1,0,1\}$, ÷

(ix) $\left\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \middle| \; a,b \in R \right\}$; Matrix addition

(x) $\left\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \middle| \; a,b \in R \right\}$; Matrix multiplication

(xi) $\left\{\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \middle| \; a,b \in R \right\}$; Matrix addition

(xii) $\left\{\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \middle| \; a,b \in R \right\}$; Matrix multiplication

3. Find whether the binary operation $*$ on $A = \{a,b,c,d\}$ given by

*	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

is commutative ? Does there exist identity element ? If identity exists which elements have inverses ? If they have inverses, find them.

4. Find whether the binary operation $*$ as $A = \{a,b,c,d\}$ given by

*	a	b	c	d
a	d	a	c	b
b	a	c	b	d
c	c	d	a	b
d	c	b	d	a

is commutative ? Does there exist identity element ? If the identity element exists, find the elements which have inverses.

5. Let $S = \{A,B,C,D\}$ where $A = \phi$, $B = \{a,b\}$, $C = \{a,c\}$ and $D=\{a,b,c\}$. Show that the set union $\cup$ is a binary operation on S, whereas the set intersection $\cap$ is not a binary operation. Does S have an identity element with respect to the union operation ? Find the inverses of A, B, C, D if they exist.

SUBUNIT NO. 4.6: GROUPS

Concepts and Explanations

We have seen that addition is a binary operation on the set of integers Z. Moreover, addition is associative. 0 is the identity element in Z with respect to addition and every integer a in Z has inverse -a in Z with respect to addition. These properties can be listed as follows:

1. $a, b \in Z \Rightarrow a+b \in Z$
2. $(a+b)+c = a+(b+c)\ \forall\ a, b, c \in Z$
3. $0 \in Z$ such that $a+0 = a$, $0+a = a\ \forall\ a \in Z$
4. Given $a \in Z$, there exists $-a \in Z$ such that $a+(-a) = 0$, $(-a)+a = 0$

The set Q of rational numbers as well as the set R of real numbers satisfy the above 4 conditions with respect to the operation of addition. Such sets are called groups. To be more precise, we define a group as follows:

Definition: A set G with a binary operation $*$ is said to be a group if following conditions are satisfied:

1. $a, b \in G \Rightarrow a*b \in G$ (Closure)
2. $(a*b)*c = a*(b*c)\ \forall\ a, b, c \in G$ (Associativity)
3. There exists $e \in G$ such that $a*e = a$, $e*a = a\ \forall\ a \in G$ (Existence of identity)
4. Given $a \in G$, there exists $b \in G$ such that $a*b = e$ and $b*a = e$ (Existence of inverse).

Examples

1. The set of integers is a group with respect to addition.
2. The set of rational numbers and set of real numbers are groups with respect to addition.
3. The set of positive rationals Q^+ is a group with respect to multiplication. If a and b are positive rationals, then a, b is a positive rational. $\therefore$ $a,b \in Q^+ \Rightarrow a.b \in Q^+$

   We know that multiplication of rationals is associative and in particular multiplication is associative in Q^+.

Further 1 is a positive rational number.

So $1 \in Q^+$ and $a.1 = a$ and $1.a = a \ \forall \ a \in Q^+$.

Also, given a positive rational $a \in Q^+$, $a = \dfrac{p}{q}$ and we can take $p>0, q>0$

Hence $\dfrac{1}{a} = \dfrac{q}{p}$ is a positive rational and $a.\dfrac{1}{a} = \dfrac{p}{q} . \dfrac{q}{p} = 1$,

$\dfrac{1}{a}.a = \dfrac{q}{p}.\dfrac{p}{q} = 1.$

$\therefore$ Q^+ is a group with respect to multiplication.

4. Consider the set of all non-zero rationals Q^*. Given any two non-zero rationals p and q, their product p.q is again a non-zero rational. Hence $p,q \in Q^* \Rightarrow p.q \in Q^*$. We know that multiplication of rational numbers is associative. Hence multiplication of non-zero rational numbers is also associative.

$\therefore$ p.(q.r) = (p.q).r for all rationals p,q,r $\in Q^*$

1 is a non-zero rational and

p.1 = p   1.p = p $\forall$ p $\in Q^*$

Also given any zero rational $p = \frac{a}{b}$, we have a $\neq$ 0. Hence $q = \frac{b}{a}$ is a non-zero rational number and $p.q = \frac{a}{b}.\frac{b}{a} = 1$ $q.p = \frac{b}{a}.\frac{a}{b} = 1$

Hence given p $\in Q^*$,there exists q $\in Q^*$ such that p.q = 1 and q.p = 1

Hence Q^* is a group with respect to multiplication.

5. On similar lines R^*, the set of non-zero real numbers is a group with respect to multiplication.

6. R^+, the set of positive real numbers is a group with respect to multiplication.

If a and b are positive real numbers, then their product a.b is again a positive real. Hence a,b $\in R^+ \Rightarrow$ a.b $\in R^+$. We know that multiplication of real numbers is associative. Hence, multiplication of positive real numbers is also associative.

$\therefore$ (a.b).c = a.(b.c) $\forall$ a,b,c $\in R^+$

Now, 1 is a positive real number and hence 1 $\in R^+$. Further

a.1 = a and 1.a = a $\forall$ a $\in R^+$

Given any positive real number a, $b = \frac{1}{a}$ is a positive real number and $a.b = a.\frac{1}{a} = 1$ and $b.a = \frac{1}{a}.a = 1$. Hence given a $\in R^+$, there exists $b \in R^+$ such that $a.b = 1$ and $b.a = 1$. Thus R^+ is a group with respect to multiplication.

7. Let M be the set of all 2x2 matrices with integer entries. Then if A and B are two 2x2 matrices with integer entries, then we know that A+B is also a 2x2 matrix with integer entries. Hence, $A,B \in M \Rightarrow A+B \in M$. We know that the addition of matrices is associative. Hence $(A+B)+C = A+(B+C) \ \forall A,B,C \in M$.

The zero 2x2 matrices $\mathbf{O} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in M$ and $A + \mathbf{O} = A$ and $\mathbf{O} + A = A \ \forall A \in M$.

Given any matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M$, let $B = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$

Then $A + B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbf{0}$

Similarly $B + A = \mathbf{0}$.

Thus given any matrix, $A \in M$, there exists a matrix $B \in M$ such that $A + B = 0$ and $B + A = 0$. Hence M is a group.

8. Let V be the set of all vectors in the plane. If $\vec{a}$ and $\vec{b}$ are any two vectors in the plane, then we know that $\vec{a} + \vec{b}$ is also a vector in the plane. Hence,

$$\vec{a}, \vec{b} \in V \Rightarrow \vec{a} + \vec{b} \in V$$

We know that the vector addition is associative. Hence,

$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c}) \quad \forall \ \vec{a}, \vec{b}, \vec{c} \in V$$

The zero vector $\vec{0}$ is a vector in the plane. Hence

$\vec{0} \in V$ and $\vec{0} + \vec{a} = \vec{a}$ and $\vec{a} + \vec{0} = \vec{a} \quad \forall \ \vec{a} \in V$

Given any vector $\vec{a}$ in the plane, consider the vector $\vec{b} = -\vec{a}$. Then

$$\vec{a} + \vec{b} = \vec{a} + (-\vec{a}) = \vec{0}$$

$$\vec{b} + \vec{a} = (-\vec{a}) + \vec{a} = \vec{0}$$

Thus the set of all vectors in a plane is a group.

Problem Solving

1. Which of the following is a group ?

(i) [N,+]

If $a,b \in N$, then $a+b \in N$.

$\therefore$ Closure property holds good in N.

$(a+b)+c = a+(b+c) \quad \forall \ a,b,c \in N$

$\therefore$ Associativity law holds good in N.

But for any natural number a, there exists no natural number e such that $a+e = a$.

$\therefore$ Identity element does not exist.

$\therefore$ [N,+] is not a group.

(ii) [Z,-]

If $a,b \in Z$, then $a - b \in Z$

$\therefore$ Closure property holds good in Z

But $a-(b-c) \neq (a-b)-c$ in general in Z

For example let $a = 1$, $b = 2$, $c = 2$. Then $a-(b-c) = 1-(2-2) = 1 - 0 = 1$

But $(a-b)-c = (1-2)-2 = -1-2 = -3$

Thus [Z,-] is not a group.

(iii) $[Q^+,+]$

As in problem (i), here also closure and associativity properties holds good. But given $a \in Q^+$, there exists no $e \in Q^+$ such that $a+e = a$ $\therefore$ $[Q^+,+]$ is not a group.

(iv) $[R^+,x]$

$a,b \in R^+ \Rightarrow axb \in R^+$

$\therefore$ Closure property holds good in R^+

$(axb)xc = ax(bxc)$ $\forall$ $a,b,c \in R^+$

$\therefore$ Associativity holds good in R^+

$1 \in R^+$ and $ax1 = a$, $1xa = a$ $\forall$ $a \in R^+$

$\therefore$ Identity element exists with respect to x in R^+

Given $a \in R^+$, $1/a \in R^+$ and $ax(1/a) = 1$, $(1/a)xa = 1$

$\therefore$ $a^{-1} = 1/a$ $\quad\quad$ $\therefore$ a^{-1} exists for every $a \in R^+$ with respect to x

$\therefore$ $[R^+,x]$ is a group.

2. Why are the following not groups ?

(a) $S = \{1,2,3\}$, +

$2 \in S$, $3 \in S$, but $2+3 = 5 \notin S$

$\therefore$ Closure property does not hold good in S.

$\therefore$ [S,+] is not a group.

(b) $[R^+,+]$ Though R^+ is closed with respect to +, and associative law holds good, but identify element with respect to + does not belong to R^+. Hence not a group.

(c) Set of all odd integers with respect to +

If a and b are odd integers, then a+b is an even integer and hence not an odd integer. Hence closure property does not hold good.

Hence set of all odd integers with respect to + is not a group.

3. Verify that {1,-1} is a group with respect to multiplication.

$1\text{x}1 = 1$, $1\text{x}(-1) = -1$, $(-1)\text{x}1 = -1$, $(-1)\text{x}(-1) = 1$

Hence {1,-1} is closed under multiplication.

Multiplication of integers is associative. Hence multiplication is associative in {1,-1} also.

Clearly 1 is the identity element with respect to x.

Also since $1\text{x}1 = 1$, $(-1)\text{x}(-1) = 1$, inverse of 1 is 1 and inverse of -1 is -1 with respect to multiplication. Hence every element has inverse in {1,-1}.

Hence {1,-1} is a group with respect to multiplication.

4. Show that Z is a group with respect to $*$ if $a*b = a+b-1 \quad \forall\ a,b \in Z$. What is the identity element in the group ? Also find the inverse of 5.

$a,b \in Z \Rightarrow a+b-1 \in Z$

$\therefore a,b \in Z \Rightarrow a*b \in Z$ (closure)

$(a*b)*c = (a+b-1)*c = a+b-1+c-1 = a+b+c-2$

$a*(b*c) = a*(b+c-1) = a+b+c-1-1 = a+b+c-2$

$\therefore$ $(a*b)*c = a*(b*c)$ $\forall$ $a,b,c \in Z$ (Associativity)

$1 \in Z$ and $a*1 = a+1-1 = a$, $1*a = 1+a-1 = a$ $\forall$ $a \in Z$

$\therefore$ 1 is the identity element in Z (Existence of identity).

Given $a \in Z$, $2-a \in Z$ and

$a*(2-a) = a+2-a-1 = 1$

$(2-a)*a = 2-a+a-1 = 1$

$\therefore$ 2-a is the inverse of a in Z (Existence of inverse)

$\therefore$ Z is a group with respect to $*$.

5. Verify that R^+ is a group with respect to $*$, if $a*b = \dfrac{ab}{11}$ $\forall$ $a,b \in R^+$

$a,b \in R^+ \Rightarrow a\,b \in R^+ \Rightarrow \dfrac{ab}{11} \in R^+ \Rightarrow a*b \in R^+$ (Closure)

$$(a*b)*c = \frac{ab}{11} *c = \frac{abc}{11x11} = \frac{abc}{121}$$

$$a*(b*c) = a* \frac{ab}{11} = \frac{abc}{11x11} = \frac{abc}{121}$$

$\therefore$ $a*(b*c) = (a*b)*c$ $\forall$ $a,b,c \in R^+$ (Associativity)

$11 \in R^+$ and

$$a*11 = \frac{ax11}{11} = a, \qquad 11*a = \frac{11xa}{11} = a \quad \forall \; a \in R^+$$

$\therefore$ 11 is the identity element with respect to $*$ (Existence of identity).

$$a * \frac{121}{a} = \frac{a \times 121}{a \times 11} = 11$$

$$\frac{121}{a} * a = \frac{121}{a} \times \frac{a}{11} = 11$$

Hence $\frac{121}{a}$ is the inverse of a with respect to $*$ (Existence of inverse)

$\therefore$ R^+ is a group with respect to $*$.

Inverse of 5 $= \frac{121}{5}$

8. Let $G = \{3^n \mid n \in Z\}$ Verify that [G,x] is a group.

$a, b \in G \Rightarrow a = 3^m$ and $b = 3^n$ for some $n, m \in Z$

$\Rightarrow a \times b = 3^m \times 3^n = 3^{m+n} \in G$ ($m+n \in Z$). (Closure)

Multiplication is associative in $\mathbb{Q}$ and hence in G also.

$1 = 3^0 \in G$ and $a \times 1 = a$, $1 \times a = a \quad \forall \quad a \in G$

$\therefore$ 1 is the identity element in G.

Given any $a \in G$, $a = 3^n$ for some $n \in Z$

Then $3^{-n} \in G$ and $a \times 3^{-n} = 3^n \times 3^{-n} = 3^{n-n} = 3^0 = 1$ and

$3^{-n} \times a = 3^{-n} \times 3^n = 3^{-n+n} = 3^0 = 1$

Thus every element has an inverse in G.

$\therefore$ G is a group.

SUBUNIT NO. 4.7: CONGRUENCE MODULO m

Concepts and Explanations

In our usual clock, there are only 12 hours and the time indicated on the clock 1 hour after 12 hours is 1 hour, 2 hours after 12 hours is 2 hours, 3 hours after 12 hours is 3 hours, etc. If the time on the clock shows 7 hours now, after 8 hours, it will show 3 hours though it should have been 7+8 = 15 hours. In other words, in a 12 hour clock, 7 hours + 8 hours = 3 hours or briefly 7+8 = 3 or 15 = 3. Similarly, on a 12 hour clock, 16 = 4, 17 = 5, 18 = 6, etc. If the time shown is 9 hours now, after exactly 12 hours, the time shown will be again 9 hours, after 24 hours, the time shown will be again 9 hours, after 36 hours the time shown will be again 9 hours, etc. Thus 12 and its multiples will be as good as 0, as far as a 12 hour clock is concerned. Let us observe carefully what we have seen above. In a 12 hour clock.

15 = 3., 18 = 6, 21 = 9, 24 = 12, 28 = 4, 37 = 1, 47 = 11

Note that 15-3 = 12, 18-6 = 12, 21-9 = 12, 24-12 = 12, 28-4 = 24, 37-1 = 36, 47-11 = 36.

On the right hand side of each of the above equations, we have a multiple of 12. So we can generalise the rule observed above as follows: In a 12 hour clock 15 hours = 3 hours because 15-3 is a multiple of 12, 28 hours = 4 hours because 28-4 is a multiple of 12. 37 hours = 1 hour because 37-1 is a multiple of 12.

If we apply this rule, 12 hours = 0 hours, since 12-0 = 12, is a multiple of 12. Since in a 12 hour clock, 12 is the starting point we can as well consider it as 0 and every multiple of 12 hours will again bring back the clock to the starting point and hence to 0.

We use the above analogy to define modular arithmetic.

Let m be any positive integer. Then given any two positive integers a and b we say that

a is congruent to b modulo m and write it as $a \equiv b \pmod{m}$ if a-b is multiple of b, or equivalently if m divides a-b. Sometime we simply write it as a=b (mod m). We know by division algorithm that $a = qm + r$ where $0 \le r \le m-1$ where q is the quotient on division of a by m and r is the remainder. Then $a-r = qm$, a multiple of m. Therefore $a = r \pmod{m}$. So any integer is congruent to one of 0,1,2,...,m-1 since on division by m, any integer gives of 0, 1, 2, 3, ... m-1 as remainders. These integers are called residues of integers modulo m.

Also if $a = qm+r$, $0 \le r \le m-1$ and $b = q_1m+r_1$ $0 \le r_1 \le m-1$, then $a-b = (q-q_1)m + (r-r_1)$.

Hence a-b will be a multiple of m if and only if $r_1-r = 0$ i.e. if $r = r_1$. Thus any two integers a and b are congruent modulo m if and only if both have the same remainder on division by m.

Examples

1. $14 \equiv 0 \pmod{7}$ since $14-0 = 14$ is a multiple of 7
2. $29 \equiv 4 \pmod{5}$ since $29-4 = 25$ is a multiple of 5

3. $293 \equiv 23 \pmod{30}$ since $293-23 = 270$ is a multiple of 30

4. Any integer is congruent to 0,1,2,3,4,5 modulo 6:

$0 \equiv 0 \pmod 6$, $1 \equiv 1 \pmod 6$, $2 \equiv 2 \pmod 6$, $3 \equiv 3 \pmod 6$, $4 \equiv 4 \pmod 6$,

$5 \equiv 5 \pmod 6$, $6 \equiv 0 \pmod 6$, $7 \equiv 1 \pmod 6$, $8 \equiv 2 \pmod 6$, $9 \equiv 3 \pmod 6$,

$10 \equiv 4 \pmod 6$, $11 \equiv 5 \pmod 6$, $12 \equiv 0 \pmod 6$ ...

Hence 0,1,2,3,4,5 are residues of integers modulo 6.

SUBUNIT NO. 4.8: ADDITION AND MULTIPLICATION MODULO m

Concepts and Explanations

Consider residues of integers module 6. They are 0,1,2,3,4,5.

If we consider usual addition, we get

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	6
2	2	3	4	5	6	7
3	3	4	5	6	7	8
4	4	5	6	7	8	9
5	5	6	7	8	9	10

But $6 \equiv 0 \pmod 6$, $7 \equiv 1 \pmod 6$, $8 \equiv 2 \pmod 6$, $9 \equiv 3 \pmod 6$, $10 \equiv 4 \pmod 6$.

Hence we get

$\oplus_6$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

This is known as addition modulo 6. Thus,

$$2 + 4 = 0 \pmod 6$$

We write this as $2 \oplus_6 4 = 0$

Similarly,

$3 \oplus_6 5 = 2$, $4 \oplus_6 4 = 2$, $5 \oplus_6 4 = 3$, etc.

In general, when we consider residues of integers modulo a positive integer m, if $a+b \equiv r \pmod m$ where r is one of $0,1,2,\ldots,m-1$ then we write $a \oplus_m b = r$. This is called addition modulo m. r is in fact the remainder on division of a+b by m.

On similar lines, we can define multiplication modulo m. If $a \times b \equiv r \pmod m$ where r is one of $0,1,2,\ldots,m-1$, then we write

$$a \otimes_m b = r$$

r in fact is the remainder on division of axb by m.

For example modulo 5, since $3 \times 3 = 9 \equiv 4 \pmod 5$ and $3 \times 2 = 6 \equiv 1 \pmod 5$

$3 \otimes_5 3 = 4$ $3 \otimes_5 2 = 1$

The multiplication modulo 6 table will be as follows:

$\otimes_6$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

With respect to addition modulo m, the set of residues of integers modulo m, namely {0,1,2,3,...,m-1} form a group. This has been explained well in the textbook with examples.

Examples: Refer to your textbook.

Problem Solving

1. Find the sums:

(a) $13 \oplus_4 5$

$13 + 5 = 18 = 4x4 + 2 \equiv 2 \pmod 4$

$\therefore 13 \oplus_4 5 = 2$

(b) $14 \oplus_3 6$

$14 + 6 = 20 = 6x3 + 2 \equiv 2 \pmod 3$

$\therefore 14 \oplus_3 6 = 2$

(c) $(11 \oplus_9 3) \oplus_9 4$

$11 + 3 = 14 = 9+5 \equiv 5 \pmod 9$

$\therefore 11 \oplus_9 3 = 5$

$\therefore (11 \oplus_9 3) \oplus_9 4 = 5 \oplus_9 4$

$5 + 4 = 9 \equiv 0 \pmod 9$

$\therefore (11 \oplus_9 3) \oplus_9 4 = 0$

2. Find the products:

(a) $13 \otimes_5 3$

$13 \otimes 3 = 39 = 7x5 + 4 \equiv 4 \pmod 5$

$\therefore 13 \otimes_5 3 = 4$

(b) $10 \otimes_8 5$

$10 \times 5 = 50 = 6 \times 8 + 2 \equiv 2 \pmod 8$

$\therefore\ 10 \otimes_8 5 = 2$

(c) $5 \otimes_7 (6 \otimes_7 8)$

$6 \times 8 = 48 = 6 \times 7 + 6 \equiv 6 \pmod 7$

$\therefore\ 6 \otimes_7 8 = 6$

$\therefore\ 5 \otimes_7 (6 \otimes_7 8) = 5 \otimes_7 6 = 30 \pmod 7 \equiv 2 \pmod 7$

$\therefore\ 5 \otimes_7 (6 \otimes_7 8) = 2$

3. Verify:

(a) $(5 \oplus_6 7) \oplus_6 11 = 5 \oplus_6 (7 +_6 11)$

$5 \oplus_6 7 = 12 \pmod 6 \equiv 0 \pmod 6$ $\qquad \therefore\ 5 \oplus_6 7 = 0$

$\therefore\ (5 \oplus_6 7) \oplus_6 11 = 0 \oplus_6 11 = 11 \pmod 6 \equiv 5 \pmod 6$

$\therefore\ (5 \oplus_6 7) \oplus_6 11 = 5$

$(7 \oplus_6 11) = 18 \pmod 6 \equiv 0 \pmod 6$ $\qquad \therefore\ 7 \oplus_6 11 = 0$

$\therefore\ 5 \oplus_6 (7 \oplus_6 11) = 5 \oplus_6 0 \equiv 5 \pmod 6$

$\therefore\ 5 \oplus_6 (7 \oplus_6 11) = 5$

4. Find x such that

(a) $3 \oplus_5 4 = x$

$3 \oplus_5 4 = 3 + 4 \pmod 5 = 7 \pmod 5 \equiv 2 \pmod 5$

$\therefore\ 3 \oplus_5 4 = 2$ $\qquad \therefore\ x = 2$

(b) $5 \oplus_6 x = 5$

$5 \oplus_6 x = 5 + x \pmod 6$ $\qquad \therefore\ 5 + x \equiv 5 \pmod 6$ $\qquad \therefore\ x = 0$

(c) $2 \oplus_3 x = 0$

$2 \oplus_3 x = 2 + x \pmod 3$ $\quad \therefore\ 2 + x \equiv 0 \pmod 3$ $\quad \therefore\ x = 1$

(d) $3 \oplus_4 x = 1$

$3 \oplus_4 x = 3 + x \pmod 4$ $\quad \therefore\ 3 + x \equiv 1 \pmod 4$ $\quad \therefore\ x = 2$

(e) $x \otimes_6 5 = 5$

$x \otimes_6 5 = 5x \pmod 6$ $\quad \therefore\ 5x \equiv 5 \pmod 6$ $\quad \therefore\ x = 1$

(f) $5 \otimes_6 x = 1$

$5 \otimes_6 x = 5x \pmod 6$ $\quad \therefore\ 5x \equiv 1 \pmod 6$ $\quad \therefore\ x = 5$

6. Construct Cayley's table and verify that each of the following is a group.

(a) {0,1,2,3} under $\oplus_4$. This is worked Example 2 of the textbook.

(b) {1,2,3,4} under $\otimes_5$.

$\otimes_5$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

Since all the entries in the table belong to the set {1,2,3,4}, the closure law holds good.

We know that associativity holds good for multiplication modulo any positive integer m. We can see from the table that 1 is the identity element with respect to $\otimes_5$.

Also from the table $1^{-1} = 1$, $2^{-1} = 3$, $3^{-1} = 2$, $4^{-1} = 4$ with respect to $\otimes_5$. Hence every element has inverse with respect to $\otimes_5$.

$\therefore$ {1,2,3,4} is a group with respect to $\otimes_5$.

(c) {1,5} under $\otimes_6$

$\otimes_6$	1	5
1	1	5
5	5	1

Clearly {1,5} is closed under $\otimes_6$. Associative law holds good for $\otimes_6$.

1 is the identity element for $\otimes_6$.

$1^{-1} = 1$, $5^{-1} = 5$ with respect to $\otimes_6$

$\therefore$ {1,5} is a group under $\otimes_6$

(d) {1,3,7} under $\otimes_{10}$

$\otimes_{10}$	1	3	7
1	1	3	7
3	3	9	1
7	7	1	9

Clearly {1,3,7} is closed under $\otimes_{10}$

Associative law holds good for $\otimes_{10}$

1 is the identity element with respect to $\otimes_{10}$

Inverse of 3 is 7 and Inverse of 7 is 3.

$\therefore$ {1,3,7} is a group with respect to $\otimes_{10}$

So, $1^{-1} = 1$, $3^{-1} = 7$, $7^{-1} = 3$ with respect to $\otimes_{10}$

Exercises for Self Evaluation

1. Is the set {1,-1} a group with respect to addition ? Justify.
2. Is the set {1,0,-1} a group with respect to multiplication ? Justify.
3. Is the set of all 2x2 matrices with determinant $\pm$ 1 a group with respect to multiplication ? Why ?
4. Is the set of all 2x2 matrices with determinant 0 a group with respect to multiplication ? Justify.
5. Is the set of all irrational numbers a group with respect to multiplication ? Justify.
6. Is the set of all negative rational numbers a group with respect to addition ? Justify.
7. Is the set of all real numbers a group with respect to multiplication ? Justify.
8. Is the set of all integer multiples of 5 a group with respect to addition ? Multiplication ? Justify.
9. Is the set of all odd integers a group with respect to multiplication? Justify.
10. If $2x = 3 \oplus_6 5$, what is x ?
11. If $5 \oplus_5 y = 2 \otimes_5 4$, what is y ?

12. If $3 \otimes_3 y = 2 \oplus_3 x$, what is x ?

13. If $z = (4 \oplus_7 6) \otimes_7 3$, what is z ?

14. If $x \otimes_5 2 = 1$, what is x ?

15. If $y \oplus_6 4 = 0$, what is y ?

16. Write addition modulo 4 and multiplication modulo 4 tables for the set {1,2,3}.

UNIT NO. 5: SIMILAR TRIANGLES

Introduction to Similarity

Figures having the same shape but not necessarily the same size are called similar figures. However if the sizes are also the same, then the figures are described as congruent figures. When a figure's dimensions are increased or decreased in the same ratio, the figure so got is similar to the original figure. Dissimilarity is the opposite of similarity. To cite some examples of similar figures - Any two - line segments, circles, squares, regular polygons with the same number of sides - are similar.

The concept of similarity comes to our experience in many life situations. An object and its photograph are similar to each other. The idea of similarity is an extremely important and useful one for design engineers and finds ample applications in subjects like graphics. Spider webs found in old and delapidated buildings remind us of the idea of similarity.

Thales (Greek mathematician 600 B.C.) who is said to have introduced geometry in Greece, is believed to have found the heights of pyramids in Egypt using the lengths of shadows and the principles of similar triangles.

Proportionality is the single basic concept at the heart of the idea - similarity.

Tips to Teach

This chapter can be made interesting by motivating the students through examples drawn from our surroundings and life-situations. Thus

(i) a person and his photograph look alike

(ii) an object and its shadow (projection on a plane) are similar, in general

(iii) at any time of a day, the lengths of objects are proportional to the lengths of their shadows

Give a number of examples before generating any formal statements - definitions - theorems, etc.

When it comes to theorems, give a feel of it through a (numerical) example. This will help to develop a belief in the truth of the theorem to be proved. Caution, at the same time, that a verification cannot replace a mathematical proof.

Coming to the proof of a theorem - an analysis - What is given ? What is to be proved ? Do we require any additional input to help the proof ? (-like constructions) is a must. Then follows the proof, to be sequential (i.e. order of the steps), logical (i.e. each step to be supported by some known result(s)). At each step (or turn) let the students know the why ? of it.

Geometry is and must be made interesting and teaching it well is an art.

I. Concepts/Notations/Terminology/Theorems

1. Two rectilinear plane figures with the same number of sides are similar figures if the angles of one are congruent to the angles of the other and the sides of one are in the same ratio with the sides of the other figure.
2. Two triangles are said to be similar to each other if the sides of one triangle are proportional to the sides of the other. Notation: |||
3. In the statement of proportionality of the sides of similar triangles, the sides in the same ratio (i.e. the proportional sides) are called the corresponding sides. The vertices of the triangles opposite to corresponding sides are called the corresponding vertices. Notation for correspondence (of sides/vertices): $\longleftrightarrow$
4. **Important Theorems**

a. Thale's Basic proportionality theorem and its converse corollary

b. Theorems on similar triangles

(i) Equiangular triangles are similar and conversely - AAA - correspondence.

(ii) The SAS similarity theorem* : If two sides of a triangle are proportional to two sides of another triangle and the included angles are congruent then the triangles are similar - SAS correspondence.

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* This theorem is not included in the syllabus/textbook. However, it is an important and useful theorem.

(iii) Theorem on a right angled triangle: In a right angled triangle, the perpendicular from the right angled vertex to the hypoteneuse divides the triangle into two similar triangles each of which is similar to the given triangle.

(iv) Areas of similar triangles: The areas of similar triangles are proportional to the squares on the corresponding sides of the triangles.

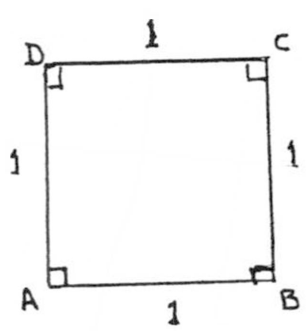
II. Explanations

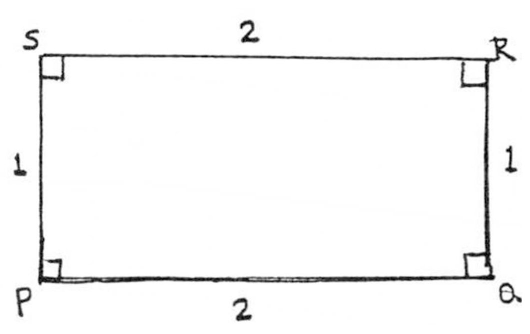
(i) Suppose two quadrilaterals are given. They will be similar if and only if (a) the angles of the two quadrilaterals are equal pairwise (i.e. the figures are equi angular) and (b) the sides are proportional.

If one or the other of these conditions only hold, then the figures are not similar.

Example

1. A square and a rectangle with unequal adjacent sides are not similar even though they are equi angular.



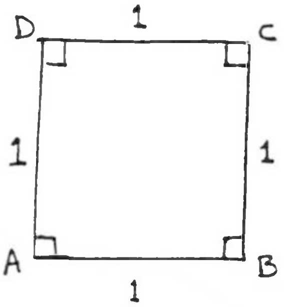


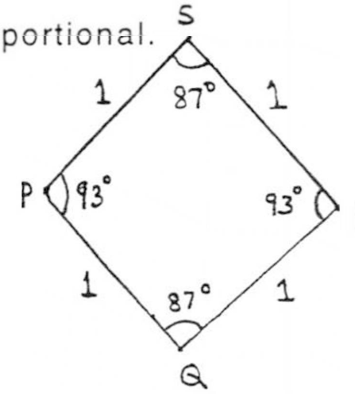
$$\frac{AD}{PS} = \frac{1}{1} = 1$$

$$\frac{AB}{PQ} = \frac{1}{2}$$

$$\text{and } 1 \neq \frac{1}{2}$$

2. A square and a rhombus are not similar even though the sides of the two are proportional.





$$\frac{AB}{PQ} = \frac{BC}{QR} = \frac{CD}{RS} = \frac{AD}{PS} = \frac{1}{1}$$

But $\hat{A} \neq \hat{P}$, $\hat{B} \neq \hat{Q}$

$\hat{C} \neq \hat{R}$ and $\hat{D} \neq \hat{S}$

(ii) Consider two triangles ABC and DEF.

$$\text{Let } \frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF} \qquad \text{...............(1)}$$

Then the triangles ABC and DEF are called similar triangles. (1) is called the proportionality condition or statement for sides of the triangles.

Side (of Δ ABC) Corresponding side (of Δ DEF)

$AB \longleftrightarrow DE$

$BC \longleftrightarrow EF$ ....(2)

$AC \longleftrightarrow DF$

(2) is called a correspondence between the sides of the Δles ABC and DEF.

Note: (1) means (2) and vice versa.

Accordingly, if two Δles are similar, then there is a correspondence between the sides.

Given a correspondence between the sides of two Δles, the opposite vertices of the corresponding sides are called corresponding vertices of the Δles. When (2) is given we get

Vertex (of Δ ABC) Corresponding vertex (of Δ DEF)

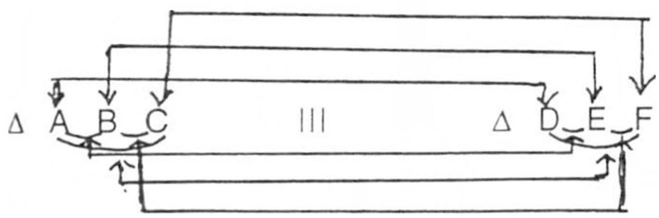
$A \longleftrightarrow D$

$B \longleftrightarrow E$ ....(3)

$C \longleftrightarrow F$

A Teaching Strategy

A way to write the statement that Δles ABC and DEF are similar in which (1) is true (or (2) holds).



⇒ (3)

⇒ (2)

The above diagram helps to write the

(i) correspondence between the sides (i.e. (2))

(ii) correspondence between the vertices (i.e. (3)) and

(iii) proportionality statement (or condition) (i.e. (1)) Correctly and mechanically.

Note: In the above, the corresponding vertices of the two Δles are written in the same order.

In writing, this way

(i) write Δ ABC

(ii) then write ||| (i.e. similar to) and

(iii) write the vertices of Δ DEF in the same order of the vertices corresponding to A, B and C respectively.

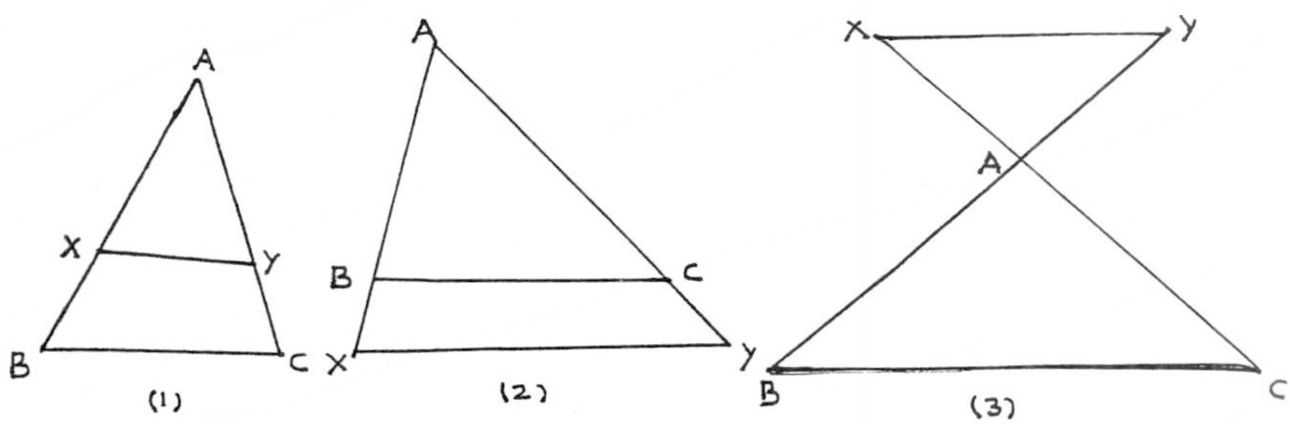
eg: If Δles ABC and DEF are similar and the vertices corresponding to A, B and C are F, E and D respectively. Then we write ΔABC ||| ΔFED (not Δ DEF). Then the proportionality condition (for sides) is

$$\frac{AB}{FE} = \frac{BC}{ED} = \frac{AC}{FD}$$

Theorems on similar Δles

1(a) Thale's Basic Proportionality Theorem

A straight line drawn parallel to a side of a triangle divides the other two sides proportionally. This means, if XY||BC in a triangle ABC, as in the figures (1), (2) or (3) -



then $\dfrac{AX}{XB} = \dfrac{AY}{YC}$ or briefly XY||BC $\Rightarrow \dfrac{AX}{XB} = \dfrac{AY}{YC}$

(b) Converse of the above theorem

If a line divides the two sides of a triangle proportionally, then the line is parallel to the third sides of the triangle.

This means in reference to the above figures if $\frac{AX}{XB} = \frac{AY}{YC}$ then XY||BC or briefly, $\frac{AX}{XB} = \frac{AY}{YC} \Rightarrow XY \parallel BC$

(c) A corollary: If XY||BC in Δ ABC, then

$$\frac{AX}{AB} = \frac{AY}{AC} = \frac{XY}{BC}$$

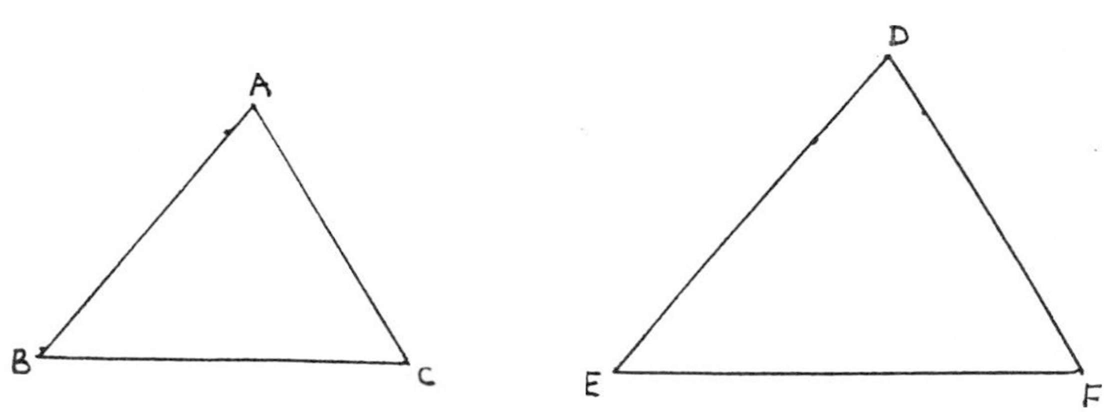
Connected equivalent results

$$XY \parallel BC \Leftrightarrow \frac{AX}{XB} = \frac{AY}{YB}\ ;\ \frac{AX}{AB} = \frac{AY}{AC}\ ;\ \frac{AB}{XB} = \frac{AC}{YC}\ .$$

These are important and useful in a number of problems.

2. Two triangles are said to be equi angular triangles if the angles of one are congruent to the angles of the other.

(a) Two equiangular triangles are similar



In triangles ABC and DEF, $\hat{A} = \hat{D}$; $\hat{B} = \hat{E}$ and $\hat{C} = \hat{F}$ (say). Then ΔABC III ΔDEF.

(b) Converse of the above theorem: Two similar triangles are equi angular.

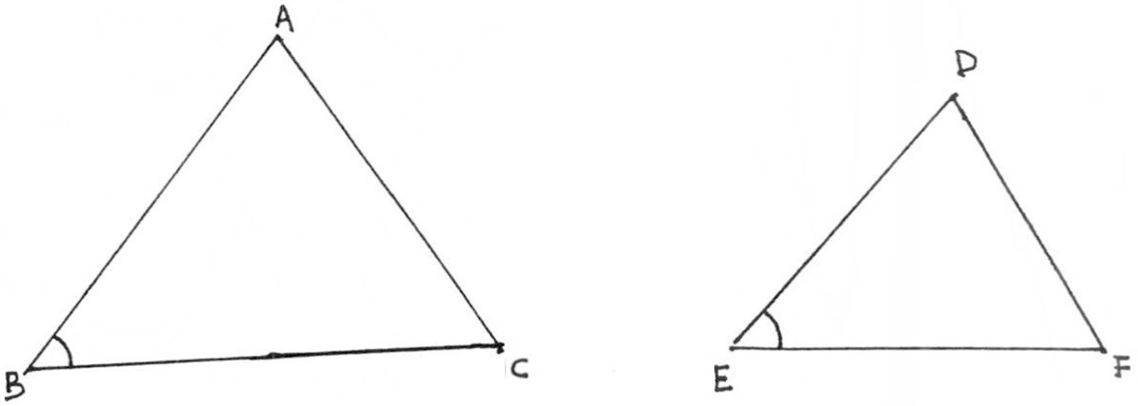
In triangles ABC and DEF, if ΔABC and ΔDEF are similar, then the triangles are equi angular.

This means, if $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$

(because ΔABC III ΔDEF) then $\hat{A} = \hat{D}$; $\hat{C} = \hat{E}$ and $\hat{C} = \hat{F}$ (i.e. the angles opposite to corresponding sides of the triangles are congruent).

Note: The condition of equi angularity of triangles for similarity of triangles is called AAA - similarity condition (A means Angle).

(c) SAS Similarity Condition

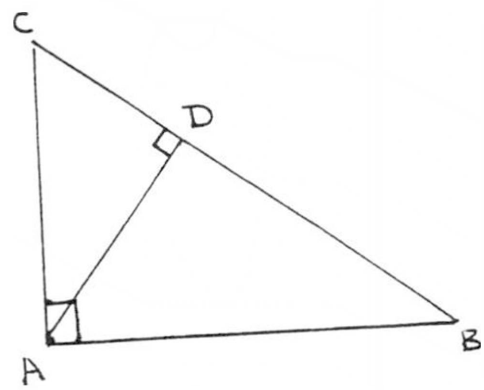


In two triangles ABC, DEF, if two sides of one triangle are proportional to two sides of the other triangle, and the included angles are equal, then the triangles are similar and conversely.

i.e. suppose $\dfrac{AB}{DE} = \dfrac{BC}{EF}$ (say)

and $\angle ABC = \angle DEF$, then $\Delta ABC \parallel\!| \Delta DEF$.

(d) A theorem on a right angled triangle



In ΔABC, $\hat{A} = 90°$ and $AD \perp BC$. Then the triangles ABD, ADC and ABC are all similar triangles to each other.

This theorem leads to some very useful results -

$$BC.BD = AB^2 \qquad (1)$$

$$BC.CD = AC^2 \qquad (2)$$

$$BD.CD = AD^2 \qquad (3)$$

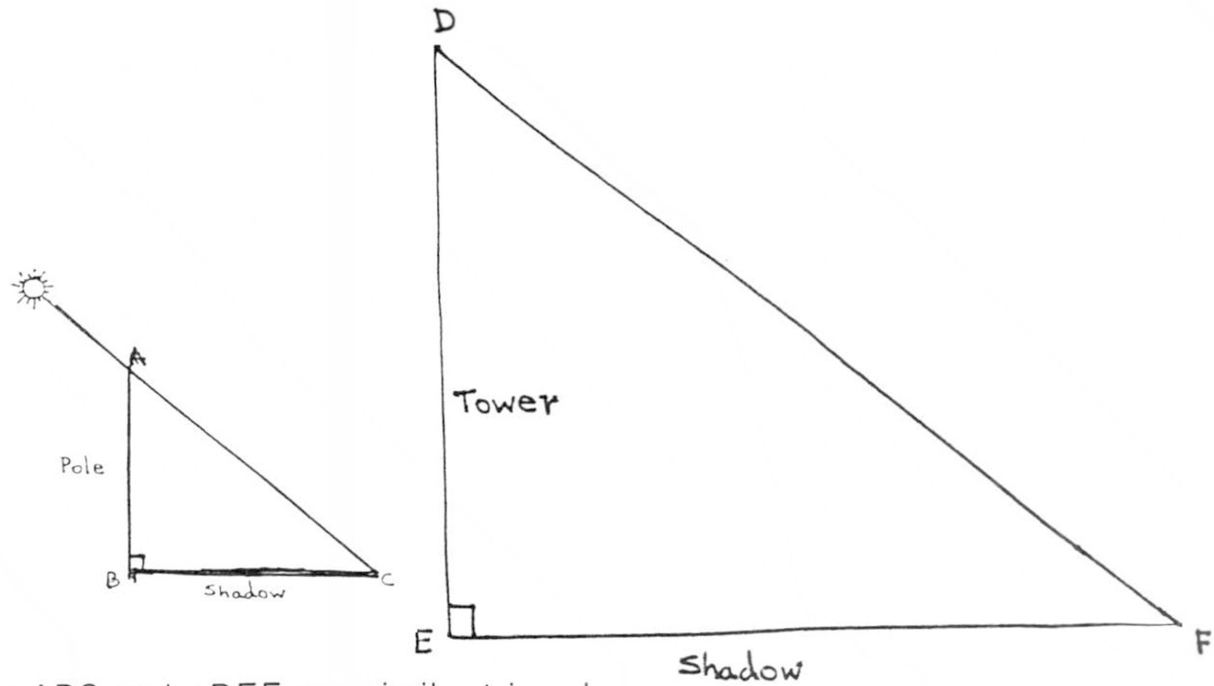
(e) Theorem on areas of similar triangles

If ΔABC and ΔDEF are similar triangles and

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}, \text{ then } \frac{\text{The areas of } \Delta ABC}{\text{The areas of } \Delta DEF} = \frac{AB^2}{DE^2}$$

III. Examples

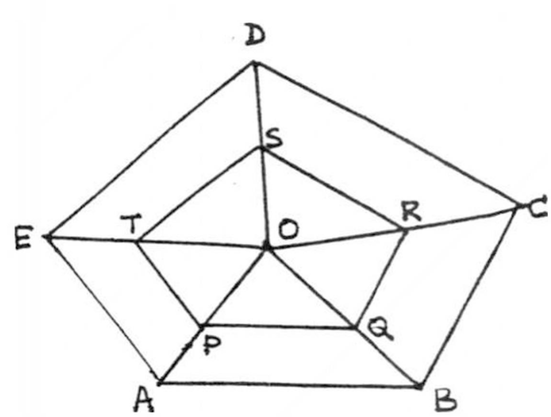
1. In Fig. (1) AB is a pole at right angles to a level ground and BC is its shadow in the morning. In fig. (2) DE is a tower and EF is its shadow at the same moment. Then



ΔABC and ΔDEF are similar triangles.

2. A metal sheet ABC (in the shape of a triangle) is uniformly heated. After some time the metal sheet is DEF. Then ΔABC and ΔDEF are similar, because, when it is heated uniformly, the sides of the triangle increase proportionally.

3. The diagram is that of a spider's web. In this, the figures ABCDE and PQRST are similar with corresponding sides parallel (i.e. $AB \parallel PQ$, etc.) Also the pairs of triangles - OAB, OPQ; OBC, OQR; OCD, ORS; ODE, OST and OAE, OPT are similar.



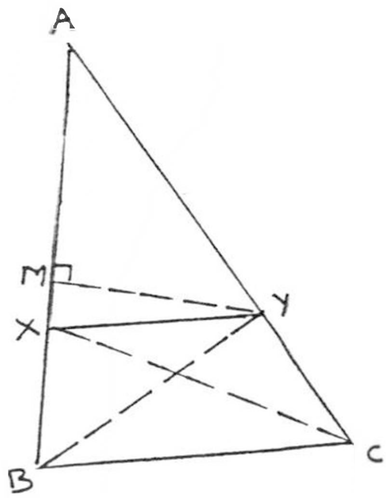
Suggested activity: Find some more life situations involving the concept of similarity.

IV. Proofs (of the theorems on similar triangles)

(a) Thale's Basic Proportionality Theorem

If XY is || to BC in ΔABC, then $\dfrac{AX}{XB} = \dfrac{AY}{YC}$

Proof of the theorem: Let $YM \perp AB$



Area $\Delta AXY = \frac{1}{2}$ AX.YM and

Area $\Delta XBY = \frac{1}{2}$ XB.YM

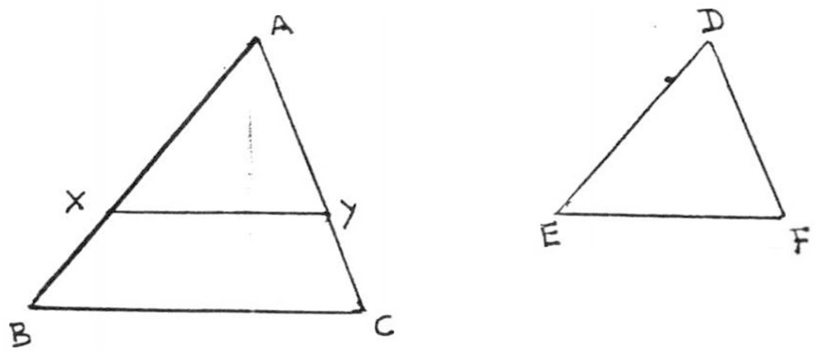
$\therefore \dfrac{\text{Area } \Delta AXY}{\text{Area } \Delta XBY} = \dfrac{AX}{XB}$ (i)

Similarly $\dfrac{\text{Area } \Delta AXY}{\text{Area } \Delta XCY} = \dfrac{AY}{YC}$ (ii)

Since XY || BC, Area ΔXBY = Area ΔXCY (iii)

$\therefore$ From (i), (ii) and (iii), $\dfrac{AX}{XB} = \dfrac{AY}{YC}$

(b) Equiangular triangles are similar



Proof of the theorem

In triangles ABC, DEF, $\hat{A} = \hat{D}$, $\hat{B} = \hat{E}$ and $\hat{C} = \hat{F}$ (Data). Mark X on AB, Y on AC such that AX = DE, AY = DF (construction).

$\Delta AXY \equiv \Delta DEF$, because $\hat{A} = \hat{D}$ (data) AX = DE and AY = DF (construction)

$\therefore XY = EF$ .... (1)

$\widehat{AXY} = \widehat{DEF} = \widehat{ABC}$
$\widehat{AYX} = \widehat{DFE} = \widehat{ACB}$ (which theorem is used ?) .... (2)

$\therefore XY \parallel BC$ (Why ?)

$\therefore \frac{AB}{AX} = \frac{AC}{AY} = \frac{BC}{XY}$ (because of the converse of the basic proportionality theorem and its corollary.)

$\therefore \frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$

Hence $\Delta ABC \parallel\!\mid \Delta DEF$

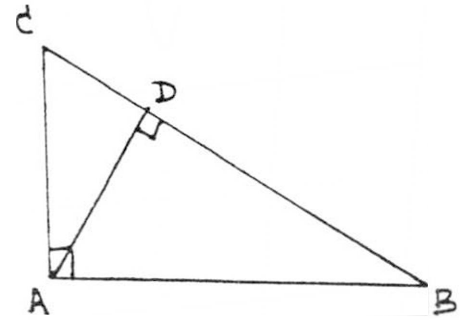
Note: The converse of this theorem is also true. Therefore, similarity of triangles can be defined by either of the properties. Two triangles are similar if (a) the sides of the triangles are proportional or if (b) the triangles are equiangular.

(c) Theorem on a right angled triangle

Proof of the theorem

ABC is a right angled triangle at A

and AD $\perp$ BC (the hypotenuse) (data)



(i) Comparing triangles ABD and ABC

$A\hat{B}D = A\hat{B}C$ (common angle)

$A\hat{D}B = 90° = B\hat{A}C$

$\therefore$ $B\hat{A}D = A\hat{C}B$ (why ?)

$\therefore$ ΔABD and ΔABC are similar

i.e. ΔABD III ΔCBA ....(1) (why do we write like this ?)

(ii) Comparing triangles ACD and ABC

$A\hat{C}D = A\hat{C}B$ (common angle)

$A\hat{D}C = 90° = B\hat{A}C$

$\therefore C\hat{A}D = A\hat{B}C$

$\therefore \Delta ACD$ and ΔABC are similar

i.e. ΔACD III ΔBCA ....(2) (why do we write like this ?)

From (1) and (2) the triangles ABD, ACD and ABC are mutually similar.

Since ΔABD and ΔACD are similar

ΔABD III ΔCAD .... (3) (why do we write like this ?)

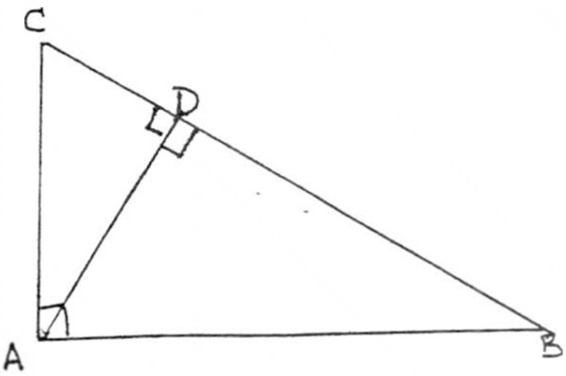
Hence ΔABC III ΔDEF

(d) Proof of

(i) $AB^2 = BD.BC$

(ii) $AC^2 = CD.BC$

(iii) $AD^2 = BD.CD$



We have proved

$\Delta ABD \ ||| \ \Delta CBA$ .... (1)

$\therefore \dfrac{AB}{CB} = \dfrac{BD}{BA} = \dfrac{AD}{CA}$

$\therefore AB^2 = BD.BC$ .... (i)

$\Delta ACD \ ||| \ \Delta BCA$ .... (2)

$\therefore \dfrac{AC}{BC} = \dfrac{CD}{CA} = \dfrac{AD}{BA}$

$\therefore AC^2 = CD.BC$ .... (ii)

$\Delta ABD \ ||| \ \Delta CAD$ .... (3)

$\therefore \dfrac{AB}{CA} = \dfrac{BD}{AD} = \dfrac{AD}{CD}$

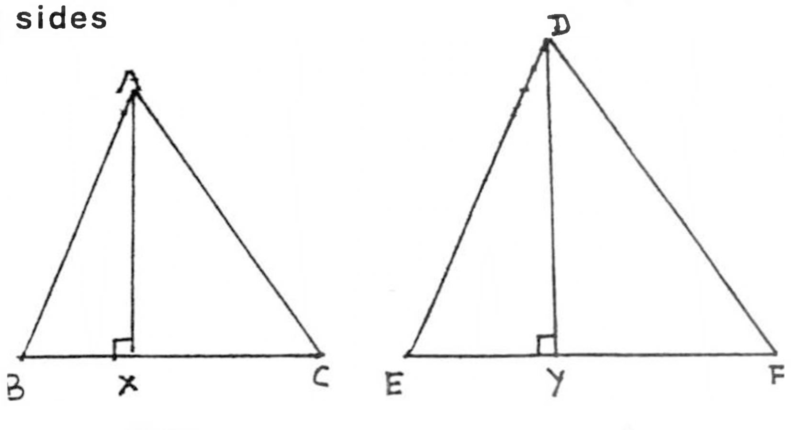
$\therefore AD^2 = BD.CD$ .... (iii)

The results (i), (ii) and (iii) are very important and useful in solving some problems.

(e) Areas of similar triangles are proportional to the squares on the corresponding sides

Proof of the theorem

$\Delta ABC \ ||| \ \Delta DEF$



and $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$ ....(i) (Data)

Let $AX \perp BC$ and $DY \perp EF$.

$$\therefore \frac{\Delta ABC}{\Delta DEF} = \frac{\frac{1}{2} BC.AX}{\frac{1}{2} EF.DY} = \frac{BC}{EF} \cdot \frac{AX}{DY} \quad \text{...............(ii)}$$

$\Delta ABX \;|||\; \Delta DEY$ because $ABX = DEY$ (why ?)

$AXB = 90° = DYE$

$\therefore BAX = CDY$

(i.e. the triangles are equiangular)

$$\therefore \frac{AB}{DE} = \frac{BX}{EY} = \frac{AX}{DY} \quad \text{............... (iii)}$$

From (ii) and (iii)

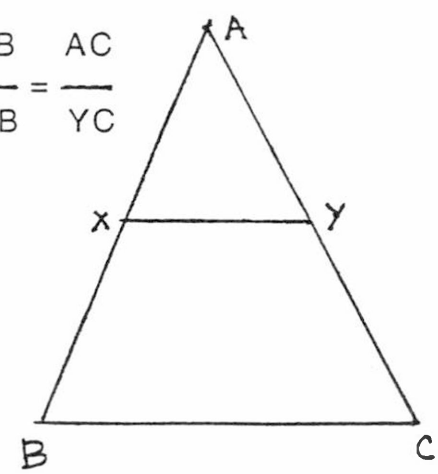
$$\therefore \frac{\Delta ABC}{\Delta DEF} = \frac{BC}{EF} \cdot \frac{AB}{DE} = \frac{AB^2}{DE^2} \text{ or } \frac{BC^2}{EF^2}$$

or $\boxed{\frac{\text{Area } \Delta ABC}{\text{Area } \Delta DEF} = \frac{AB^2}{DE^2}}$

V. Problem Solving

1. In a ΔABC, $XY \parallel BC$

show that $\frac{AX}{XB} = \frac{AY}{YC} \Rightarrow$ (a) $\frac{AX}{AB} = \frac{AY}{AC}$ (b) $\frac{AB}{XB} = \frac{AC}{YC}$



Solution

$\frac{AX}{XB} = \frac{AY}{YC}$ (given)

$\therefore \frac{XB}{AX} = \frac{YC}{AY}$ $\quad \therefore 1 + \frac{XB}{AX} = 1 + \frac{YC}{AY}$ (by adding 1 both sides)

$$\therefore \frac{AX + XB}{AX} = \frac{AY + YC}{AY} \text{ or } \frac{AB}{AX} = \frac{AC}{AY}$$

$$\therefore \frac{AX}{AB} = \frac{AY}{AC} \qquad \ldots\ldots\ldots\ldots\ldots \text{(a)}$$

From (a), $-\frac{AX}{AB} = -\frac{AY}{AC}$

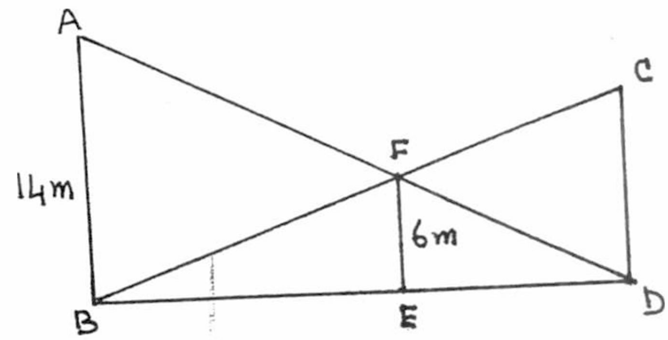
$$\therefore 1 - \frac{AX}{AB} = 1 - \frac{AY}{AC}$$

$$\therefore \frac{AB - AX}{AB} = \frac{AC - AY}{AC} \qquad \ldots\ldots\ldots\ldots\ldots \text{(b)}$$

$$\therefore \frac{XB}{AB} = \frac{YC}{AC} \text{ or } \frac{AB}{XB} = \frac{AC}{YC}$$

2. AB = 14 m and CD are two vertical poles on a horizontal ground. The top of each is tied to the foot of the other pole by ropes which intersect at a point F, 6 m above the ground. Find the length of CD.

Solution



Let CD = x

AB, CD and EF are parallel to each other

$$\therefore \frac{BE}{BD} = \frac{EF}{CD} = \frac{6}{x} \quad \ldots\ldots\ldots \text{(1)} \quad (\text{In } \Delta BCD) \text{ (How ?)}$$

$$\therefore \frac{DE}{DB} = \frac{EF}{AB} = \frac{6}{14} \quad \ldots\ldots\ldots (2) \quad (\text{In } \triangle ABD) \ (\text{How ?})$$

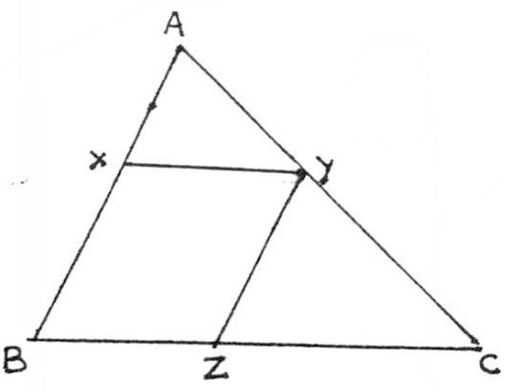
$$\therefore \frac{BE}{BD} + \frac{DE}{BD} = \frac{6}{x} + \frac{6}{14} \quad \text{or} \quad \frac{BE + DE}{BD} = \frac{6}{x} + \frac{6}{14}$$

$$\therefore \frac{6}{x} + \frac{6}{14} = 1 \qquad \therefore \frac{6}{x} = 1 - \frac{6}{14} = \frac{8}{14}$$

$$\therefore 8x = 14 \times 6 \qquad \therefore x = \frac{42}{4} = 10.5$$

$\therefore CD = 10.5$ m

3. If a line is drawn parallel to a side of a triangle then show that the sides of the intercepted triangles are proportional to the sides of the given triangle.



i.e. In $\triangle ABC$, if $XY \parallel BC$, then

$$\therefore \frac{AX}{AB} = \frac{AY}{AC} = \frac{XY}{BC}$$

Solution

Proof: Drawing $YZ \parallel AB$,

XBZY is a parallelogram $\quad \therefore XY = BZ \quad \ldots\ldots\ldots\ldots\ldots (1)$

$$\frac{AX}{AB} = \frac{AY}{AC} \text{ (by the basic proportionality theorem)}$$

Also since $YZ \parallel AB$, $\frac{AY}{AC} = \frac{BZ}{BC}$ (by the basic proportionality theorem)

$$\therefore \frac{AX}{AB} = \frac{AY}{AC} = \frac{BZ}{BC}$$

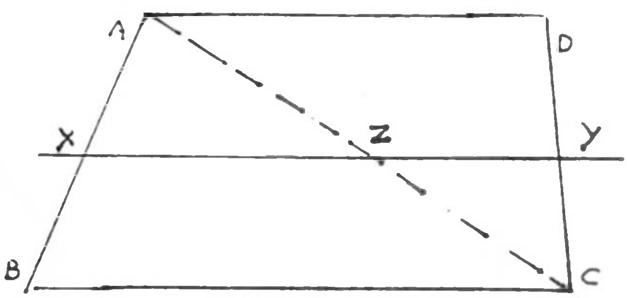
But from (1), this becomes

$$\frac{AX}{AB} = \frac{AY}{AC} = \frac{XY}{BC}$$

4. Show that in a trapezium, a line drawn parallel to either of the parallel sides divides the non-parallel sides proportionally.

Solution

ABCD is a trapezium in which AD || BC and XY || BC (or AD).



Then $\frac{AX}{XB} = \frac{DY}{YC}$

Let AC intersect XY at Z.

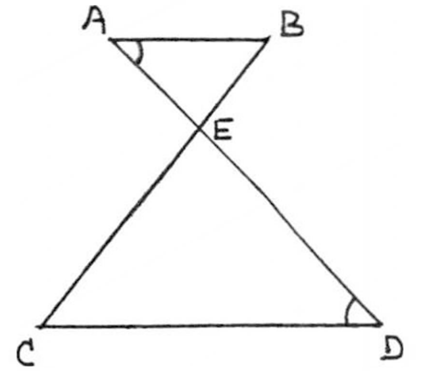
In ΔABC, $\frac{AX}{XB} = \frac{AZ}{ZC}$ (1) [by the basic proportionality theorem]

In ΔACD, $\frac{AZ}{ZC} = \frac{DY}{YC}$ (2) [Why ?]

Hence, from (1) and (2) we get $\frac{AX}{XB} = \frac{DY}{YC}$

5. In the figure BAD = ADC and CD = 4AB, show that BC = 5BE.

Solution



BAD = ADC

$\therefore$ AB || CD

$$\therefore \frac{BE}{CE} = \frac{AB}{CD} = \frac{AE}{DE} \text{ (by the basic proportionality theorem)}$$

$$\therefore \frac{BE}{CE} = \frac{AB}{4AB} = \frac{1}{4} \qquad \therefore CE = 4BE$$

$\therefore BC = BE + CE = BE + 4BE = 5BE$

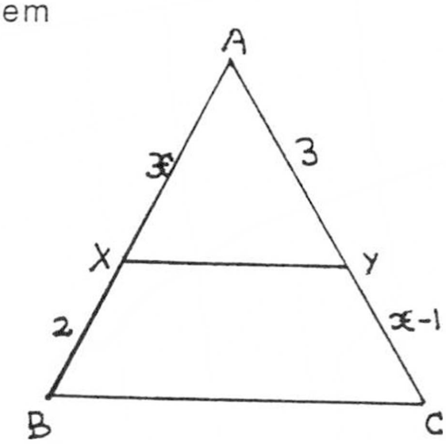
$\therefore BC = 5BE$

6. In the figure, find x.

Solution

By the basic proportionality theorem

since XY || BC,



$$\frac{AX}{XB} = \frac{AY}{YC}$$

$$\Rightarrow \frac{x}{2} = \frac{3}{x-1}$$

$\Rightarrow x^2 - x = 6 \qquad \therefore x^2 - x - 6 = 0$

$\therefore (x - 3)(x + 2) = 0 \qquad \therefore x = 3, x = -2$

Since $x > 0$, $\mathbf{x = 3}$

7. Find the lengths of DE and AE in the figure.

Solution

Using the proportionality theorem,

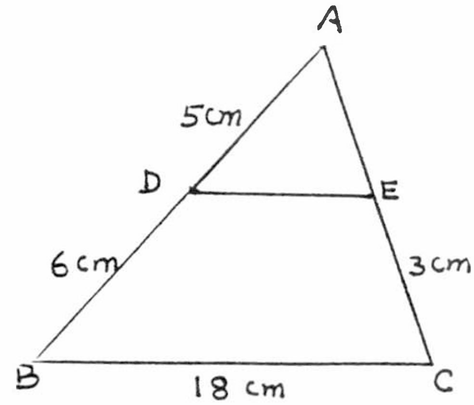
$$\frac{AD}{BD} = \frac{AE}{EC} = \frac{DE}{BC} \Rightarrow \frac{5}{6} = \frac{AE}{3} = \frac{DE}{18}$$

$\therefore DE = \frac{5}{6} \times 18 = 15$

$\therefore$ DE = 15 cm

and $AE = \frac{5 \times 3}{6} = \frac{5}{2}$

$\therefore$ AE = 2.5 cm



8. In $\triangle ABC$, $BE \perp AC$ and $CF \perp AB$. Show that $\frac{BE}{CF} = \frac{AB}{AC}$

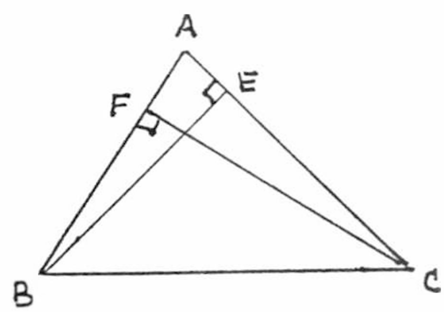
Solution

In triangles ABE and ACF,

$\widehat{BAE} = \widehat{CAF}$ (common angle)

$\widehat{AEB} = 90° = \widehat{AFC}$

$\therefore \widehat{ABE} = \widehat{ACF}$



Hence $\triangle ABE \;|||\; \triangle ACF$ (by which theorem ?)

$\therefore \frac{AB}{AC} = \frac{BE}{CF} = \frac{AE}{AF}$ Hence the result.

9. XY is parallel to BC in a $\triangle ABC$. Show that $\triangle AXY \;|||\; \triangle ABC$.

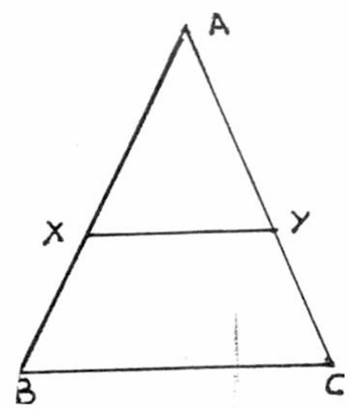
Solution

$\therefore$ XY II BC (data)

$\therefore \widehat{AXY} = \widehat{ABC}$

and $\widehat{AYX} = \widehat{ACB}$

Also $\widehat{XAY} = \widehat{BAC}$ (common angle)



Hence triangles AXY and ABC are equiangular so that $\triangle AXY \;|||\; \triangle ABC$.

10. ABCD is a trapezium and its diagonals AC, BD intersect at E.

Show that AE.ED = BE.EC.

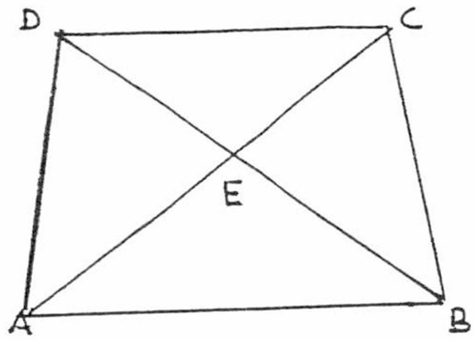
Solution

In triangles AEB and CED,

$\widehat{AEB} = \widehat{CED}$

$\widehat{EAB} = \widehat{ECD}$

and $\widehat{EBA} = \widehat{EDC}$ $\quad$ ($\because$ AB || CD)



$\therefore$ $\Delta AEB \ ||| \ \Delta CED$

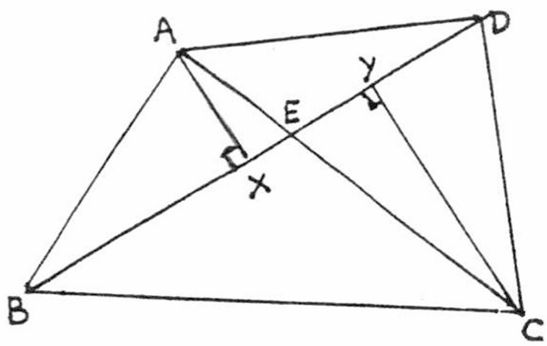
$$\therefore \frac{AE}{CE} = \frac{EB}{ED} = \frac{AB}{CD}$$

$\therefore AE.ED = BE.EC$

11. ABCD is a quadrilateral whose diagonals AC and BD intersect at E. Show that

(a) $$\frac{\text{Area of } \Delta ABD}{\text{Area of } \Delta CBD} = \frac{AE}{EC}$$

and (b) $$\frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta ADC} = \frac{BE}{ED}$$



Solution

Proof of (a)

Drawing AX, CY $\perp$ BD,

in triangles AEX and CEY,

$\widehat{AXE} = 90° = \widehat{CYE}$

$\widehat{AEX} = \widehat{CEY}$

$\therefore \ \widehat{EAX} = \widehat{ECY}$

$\therefore \Delta AEX \;|||\; \Delta CEY$ (by which theorem ?)

$$\therefore \frac{AE}{CE} = \frac{AX}{CY} \qquad \ldots\ldots\ldots\ldots\ldots(1)$$

$$\frac{\text{Area } \Delta ABD}{\text{Area } \Delta CBD} = \frac{\tfrac{1}{2}\, BD.AX}{\tfrac{1}{2}\, BD.CY} = \frac{AX}{CY} = \frac{AE}{CE} \quad \text{(from (1))}$$

Hence $\dfrac{\text{Area } \Delta ABD}{\text{Area } \Delta CBD} = \dfrac{AE}{CE}$

Exercise: Prove (b) (hint: on the same lines)

12. In ΔABC, DE is drawn such that $\widehat{ADE} = \widehat{ACB}$ as shown in the figure. Show that AD.AB = AE.AC.

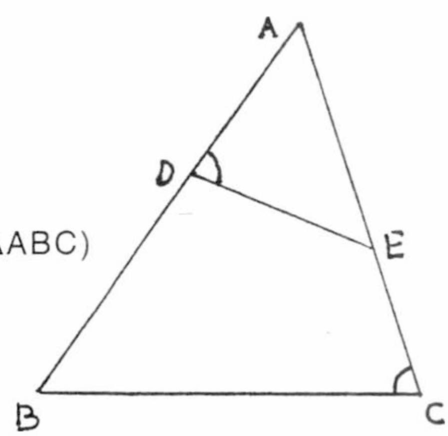
Solution: In triangles ADE and ABC,

$\widehat{DAE} = \widehat{BAC}$ (common angle)

$\widehat{ADE} = \widehat{ACB}$ (data)

$\therefore \ \widehat{AED} = \widehat{ABC}$

$\therefore \ \Delta ADE \;|||\; \Delta ACB$ (why not write $\Delta ADE \;|||\; \Delta ABC$)



$$\therefore \frac{AD}{AC} = \frac{AE}{AB} \qquad \therefore AD.AB = AE.AC$$

(The segment DE is called an antiparallel to BC)

13. ΔDEF is the middle point triangle of ΔABC. Prove that $\Delta DEF \;|||\; \Delta ABC$.

Solution

D, E, F are the mid points of BC, CA and AB respectively.

Hence DE || AB, EF || BC and DF || AC.

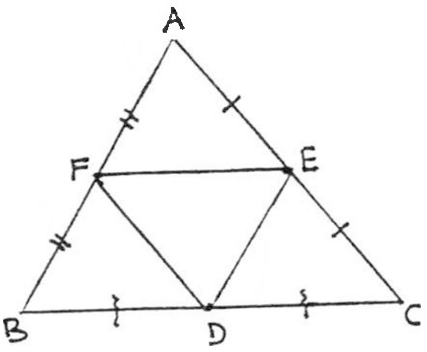
$\therefore \widehat{EDF} = \widehat{BFD}$ (DE $\parallel$ AB)

and $\widehat{BFD} = \widehat{BAC}$ (DF $\parallel$ AC)

$\therefore\ \widehat{EDF} = \widehat{BAC}$

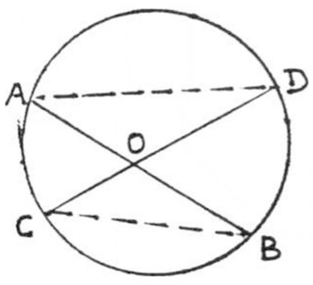
Similarly $\widehat{DEF} = \widehat{ABC}$

and $\widehat{DFE} = \widehat{ACB}$ (Prove)

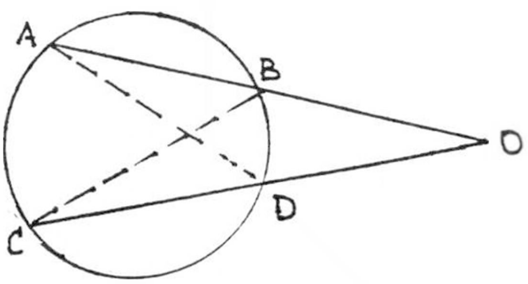


$\therefore$ Triangles DEF and ABC are equiangular and hence similar, i.e., $\triangle DEF \ |||\ \triangle ABC$.

14. AB and CD are two chords of a circle and they intersect at 0. Prove that AO.OB = CO.OD. (The chords intersect inside or outside the circle).



(1)



(2)

Solution

In triangles AOD and BOC,

$\widehat{AOD} = \widehat{BOC}$ (how ?)

$\widehat{OAD} = \widehat{OCB}$ (how ?)

Hence $\widehat{ADO} = \widehat{CBO}$

$\therefore$ Triangles AOD and BOC are equiangular and hence similar.

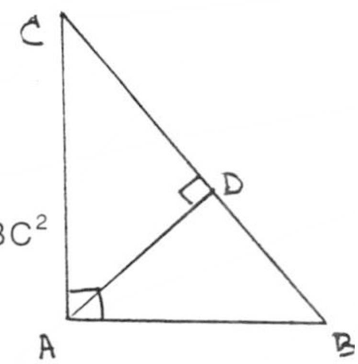
i.e. $\Delta AOD \;|||\; \Delta COB$

$\therefore \dfrac{AO}{CO} = \dfrac{OD}{OB}$ $\qquad \therefore AO.OB = CO.OD$

15. Deduce pythagoras theorem for a right angled traingle, from the similarity theorem on a right angled triangle.

Solution: In triangle ABC right angled at A, AD $\perp$ BC.

Hence, triangles ABD, ACD and ABC are similar triangles. (by the theorem on a right angled triangle).



$\therefore AB^2 = BD.BC$

$AC^2 = DC.BC$

Adding: $AB^2 + AC^2 = (BD + DC).BC = BC.BC = BC^2$

Hence $AB^2 + AC^2 = BC^2$

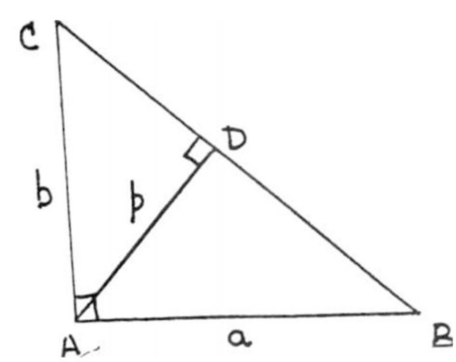
16. In ΔABC, $\hat{A} = 90°$ and AD = p, AB = a, AC = b, prove that

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{p^2}$$

Solution: Since $\hat{A} = 90°$ in ΔABC

and AD $\perp$ BC,

we have



(i) $AB^2 = BD.BC$

(ii) $AC^2 = CD.BC$

(iii) $AD^2 = BD.CD$

$$\therefore \frac{1}{AB^2} + \frac{1}{AC^2} = \frac{1}{BD.BC} + \frac{1}{CD.BC}$$

$$= \frac{CD+BD}{BD.CD.BC} = \frac{BC}{BD.CD.BC}$$

$$\therefore \frac{1}{AB^2} + \frac{1}{AC^2} = \frac{1}{BD.CD}$$

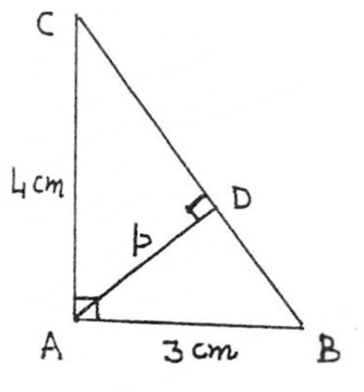
or $\frac{1}{AB^2} + \frac{1}{AC^2} = \frac{1}{AD^2}$ [from (iii)]

$$\therefore \frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{p^2}$$

17. In a right angled triangle ABC, $\hat{A} = 90°$, AB = 3 cm, AC = 4 cm, find the distance of BC from A.

Solution: Drawing AD $\perp$ BC,

let AD = p



Then using the result $\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$

a = 3, b = 4 $\quad \therefore \frac{1}{p^2} = \frac{1}{9} + \frac{1}{16} = \frac{25}{9 \times 16}$

$$\therefore p = \frac{3x4}{5} = \frac{12}{5} = 2.4 \text{ cm}$$

$\therefore$ The distance of A from BC = 2.4 cms

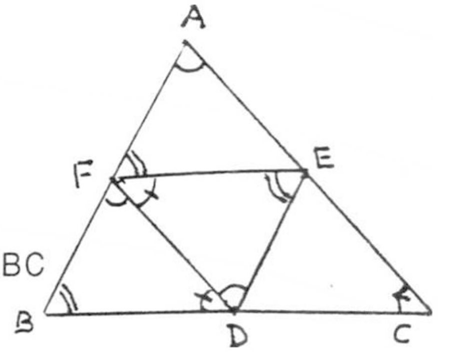
18. Show that the area of the middle point triangle of a triangle ABC is one-fourth the area of the ΔABC.

Solution: D, E, F are mid points of the sides of the ΔABC, as shown.

$\therefore$ EF || BC, FD || CA and DE || AB

Hence ΔDEF ||| ΔABC (how ?)

$$\frac{\text{Area } \Delta DEF}{\text{Area } \Delta ABC} = \frac{EF^2}{BC^2}$$ (by theorem areas of similar triangles)



Since EF = ½ BC $\Rightarrow \dfrac{EF}{BC} = \dfrac{1}{2}$

$$\frac{\text{Area } \Delta DEF}{\text{Area } \Delta ABC} = \frac{1}{4}$$ $\therefore$ Area ΔDEF = ¼ Area ΔABC

19. Two similar triangles ABC and DEF have areas 392 sq cm and 200 sq cm. One side of ΔABC is 14 cm, find the length of the corresponding side of ΔDEF.

Solution: ΔABC ||| ΔDEF (Data)

Let AB $\leftrightarrow$ DE and AB = 14 cm

Then

$$\frac{\text{Area } \Delta ABC}{\text{Area } \Delta DEF} = \frac{AB^2}{DE^2}$$ (by which theorem ?)

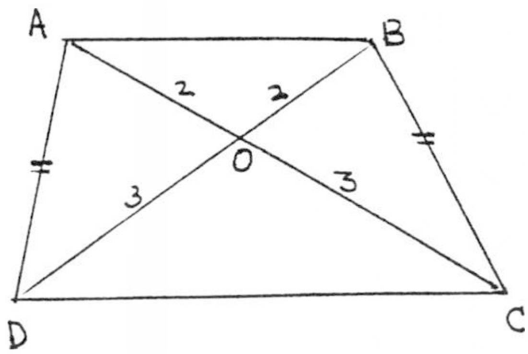
$$\Rightarrow \frac{392}{200} = \frac{14^2}{DE^2}$$

$$\Rightarrow \frac{196}{100} = \frac{196}{DE^2} \Rightarrow DE^2 = 100$$

or DE = 10 cm

20. ABCD is an isosceles trapezium of area 100 sq cms in which AB||CD. Its diagonals AC and BD divide each other in the ratio 2:3 at O. Show that

$$\frac{\text{Area } \Delta AOB}{\text{Area } \Delta COD} = \frac{4}{9}$$



Solution

In trapezium ABCD,

AB||CD and AD = BC

($\because$ it is an isosceles trapezium)

$$\frac{AO}{OC} = \frac{BO}{OD} = \frac{2}{3} \text{ (Data)}$$

In triangles AOB and COD,

$$\left.\begin{aligned} \widehat{AOB} &= \widehat{COD} \\ \widehat{ABO} &= \widehat{CDO} \\ \widehat{BAO} &= \widehat{DCO} \end{aligned}\right] \quad \therefore \Delta AOB \,|||\, \Delta COD$$

$$\therefore \frac{\text{Area } \Delta AOB}{\text{Area } \Delta COD} = \frac{AO^2}{OC^2} = \frac{4}{9}$$

VI. Activities

(i) List some life situations involving similarity and find the various fields where the idea of similarity is applied.

(ii) Explain how use of similar triangles is used in the geometrical constructions of

(a) dividing a line segment into a number of equal parts

(b) dividing a line segment in a given ratio internally/externally.

EXERCISES (FOR ENRICHMENT)

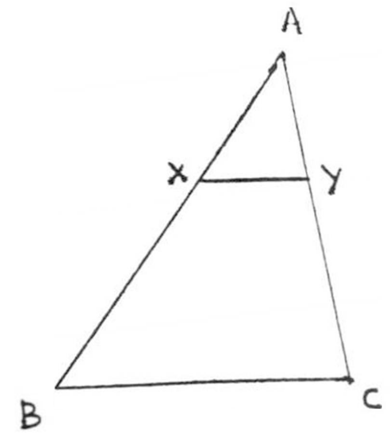
1. In the fig. XY || BC.

   Find x in each case below.

   (a) AB = 5, AC = 3

   AX = 1+x, AY = 1-x

   (b) AX = x; XB = 1+x, $\frac{AY}{YB} = \frac{2}{3}$



2. ABC is an isosceles triangle with AB = AC. BY and CX are $\perp$ to AC and AB respectively. Show that XY || BC.

3. Prove that the line segments joining the mid points of non-parallel sides in a trapezium is parallel to the parallel sides.

4. ABC is a right angled triangle at A and AD $\perp$ BC. If BD = 16 cms, DC = 4 cms, find AD, AB and AC.

5. In a right angled triangle ABC, with $\hat{A} = 90°$, AD = 2.4 cms, AB = 3 cms, find AC and BC, if AD $\perp$ BC.

6. If the sides of two triangles are respectively parallel then show that the triangles are similar.

7. If the sides of two triangles are respectively perpendicular, then show that the triangles are similar.

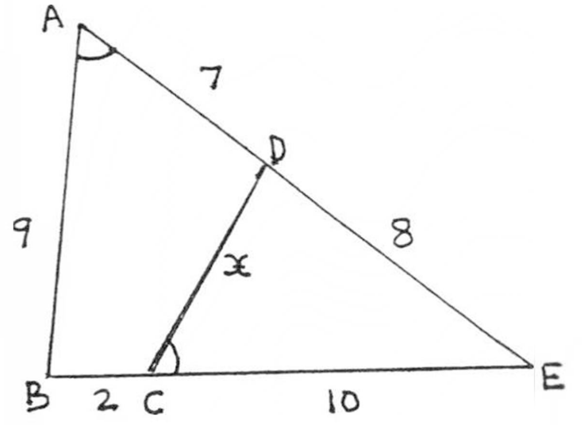
8. In similar triangles, show that the following are in the ratio of the corresponding sides -

   (a) Medians, (b) Altitudes, (c) Internal bisectors and (d) Circum radii.

9. The length of the shadow of a person 160 cm tall standing vertically on a horizontal ground is 240 cm at some time on a day and the length of the shadow of a tower at the same time on the day is 75 m. What is the height of the tower.

10. Two polls 8 m and 6 m tall stand vertically on a horizontal ground, separated by a distance 16 m. Find the position of a point between the sections of the polls where the segment joining the tops of the polls subtends a right angle.*

11. In the figure, ABCD is a cyclic quadrilateral. Find x.



12. ABCD is a parallelogram and E is the point of intersection of BD and AM, M being the mid point of CD. Prove that

$$\frac{DE}{BE} = \frac{DF}{DC} = \frac{1}{2}$$

13. Show that the internal bisector of an angle of a triangle divides the base of the triangle in the ratio of the other sides (This theorem is called the angle bisector theorem or Apollonian Theorem).*

14. Using the above theorem prove - ABCD is a quadrilateral in which AB = AD, E and F are the points of BC and CD such that AE and AF are the bisectors of $\widehat{BAC}$ and $\widehat{CAD}$ respectively. Then show that EF || BD and EF = ½ BD.*

15. ABC is a right angled triangle at A and AD $\perp$ BC. AB = 5 cm and AC = 12 cm, find the areas of ΔABD and ΔACD.

16. ABCD is a trapezium with AB and CD parallel. The diagonals of the trapezium divide each other in the ratio 2:1 at O. Find the ratio of the areas of ΔAOB and ΔCOD.

17. In ΔABC, D and E are on AB and AC such that AD:BD::3:5 and DE||BC. If area of ΔABC = 64 sq cms, find the area of the trapezium BCED.

18. The ratio between the areas of two similar triangles is 25:16. If a side of the bigger triangle is 10 cm, find the length of the corresponding side of the smaller triangle.

19. In a ΔABC, the altitudes through B and C intersect at O and meet the opposite sides at E and F respectively. Show that

$$\frac{\Delta BOF}{\Delta COE} = \frac{BF^2}{CE^2}$$

20. The parallel sides AB and CD of a trapezium ABCD are as 2:3 and the diagonals intersect at O. Find the ratio of the areas of triangles AOB and COD.

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* These problems are a bit more challenging.

UNIT NO. 6: GRAPHS

Introduction to Graphs

A classical problem of crossing all the seven bridges across the river Bregel of Königsberg in Germany one after the other without recrossing any, led to more general problems of traversability of paths. Graph theory (or briefly graphs - not to be mistaken with graphs of functions) is the part of mathematics of relatively recent times dealing with traversability of paths. A graph, in this context, consists of two or more points and lines (straight or curved) joining them. Given a graph, the question is whether the graph can be traced in one sweep without retracing any part of the graph. Electronics, electrical engineering, network analysis, traffic control, route-map design and graphics are the fields of knowledge where graph theory is used. A graph is also called a network.

I. Concepts - Terminology

(i) A graph consists of two or more *points* and *lines* connecting the points.

(ii) The points of the graph are called *nodes* and the lines are called the *arcs* of the graph.

(iii) The *order of a node* - even and odd nodes. Given a graph, a node is said to be of order n (or it is called an n-node) if n arcs pass through the node. If n is an even number, the node is called an *even node* and if n is an odd number, the node is called an *odd node*.

(iv) **Loop at a node in a graph**: The presence of a circle from a node to the node indicates a loop at the node. It increases the order of the node by 2.

(v) **Traversibility of a graph**: Given a graph, it is said to be ***traversible*** or ***non-traversible*** according as whether it can be traced in *one sweep* or not. This means, you should not retrace any arc though you can visit any node more than once.

(vi) **Euler's solution**

(a) A graph with no odd node or at the most two odd nodes is traversible.

(b) If a graph has more than two odd nodes then it is non-traversible.

(vii) **Euler's formula for graphs**: In a graph, let N = the number of nodes, A = the number of arcs and R = the number of regions into which the graph divides the plane.

Then $N + R = A + 2$

(viii) **Euler's formula for polyhedra**: If a polyhedron (a solid in which each face is a polygon) has V vertices, F faces and E edges.

Then $V + F = E + 2$

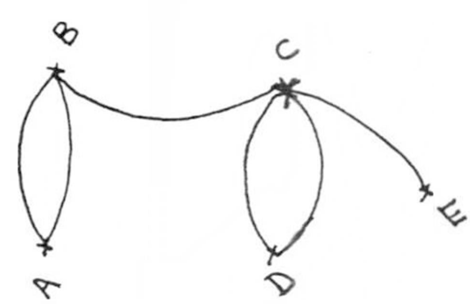
(ix) **Regular polyhedron**: A polyhedron whose faces are identical regular polygons is called a regular polyhedron (also called a platonic solid - named after Plato, 380 B.C.).

(x) **Matrix of a graph**: The matrix which gives the number of arcs in the graph is called the matrix of the graph.

II. Explanations: Illustrations

(a) Look at the diagram. There are five points which are connected by arc (lines). The points are nodes and lines connecting the nodes are arcs.

Nodes	Order
A	2
B	3
C	4
D	2
E	1



Total of orders = 12 = 2×(The number of arcs)

This graph has two odd nodes B and E. Hence it is traversible. However one has to start from an odd node B or E; after traversing the graph one will reach the other odd node. You cannot return to the starting point. The conclusion follows because of Euler's solution.

The matrix of the graph is

$$
\begin{array}{c|ccccc|c}
 & A & B & C & D & E & \text{Row totals} & & \text{Order of the nodes} \\
A & 0 & 2 & 0 & 0 & 0 & 2 & = & A \\
B & 2 & 0 & 1 & 0 & 0 & 3 & = & B \\
C & 0 & 1 & 0 & 2 & 1 & 4 & = & C \\
D & 0 & 0 & 2 & 0 & 0 & 2 & = & D \\
E & 0 & 0 & 1 & 0 & 0 & 1 & = & E \\
\text{Col. total} & 2 & 3 & 4 & 2 & 1 & 12 & = & 2\text{x(No. of arcs)}
\end{array}
$$

Observe

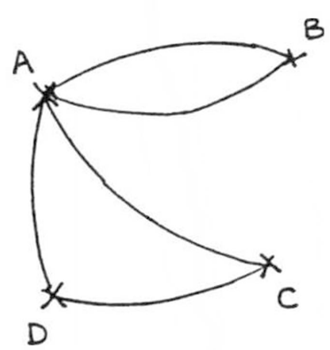
1. The number of rows (or columns) = The number of nodes.
2. The sum of the numbers in a row (or column) = The order of the corresponding node.
3. The sum of the numbers in the matrix = The sum of the orders.

   = 2 x The number of arcs in the graph.

Another Illustration

The adjacent graph has four nodes - A, B, C, D and 5 arcs.

Nodes	Order
A	4
B	2
C	2
D	2



Total = 10 = 2 x Number of arcs

This graph is traversible since it has no odd nodes (all are even node). Further you can start from any node and return to the starting point. Again this is because of Euler's solution.

The matrix of the graph

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 2 & 1 & 1 \\ 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Row totals	Order of the nodes
4	= Order of A
2	= Order of B
2	= Order of C
2	= Order of D

Total 10 = 2 x Number of arcs in the graph

In the illustration 1

The graph divides the plane of the graph into three regions. Accordingly, N = Number of nodes = 5, A = The number of arcs = 6 and R = The number of regions = 3.

$\boxed{N + R = A + 2}$ (=8)

In the illustration 2

N = 4, A = 5, R = 3 and $\boxed{N + R = A + 2}$ (= 7)

The formula N + R = A + 2 connecting

N = the number of nodes in the graph

R = the number of regions into which the plane is divided by the graph

A = the number of arcs in the graph

is called *Euler's formula for graphs*.

b. A polyhedron is a solid whose faces are polygons. Basically, it is a three dimensional object occupying space (like a ball or a box).

It is called a regular polyhedron if the faces are congruent polygons.

There are only Five Regular Polyhedra.

They are

Sl.No.	Name	Shape of the faces
1	Tetrahedron	Equilateral triangle
2	Hexahedron (or cube)	Square
3	Octahedron	Equilateral triangle
4	Dodecahedron	Regular pentagon
5	Icosahedron	Equilateral triangle

These regular solids are also called *Platonic solids* (Named after Greek mathematician Plato who lives in 380 B.C.).

$$\text{Let}\quad \left.\begin{aligned} V &= \text{Number of vertices} \\ E &= \text{Number of edges} \\ F &= \text{Number of faces} \end{aligned}\right] \text{— of a polyhedron}$$

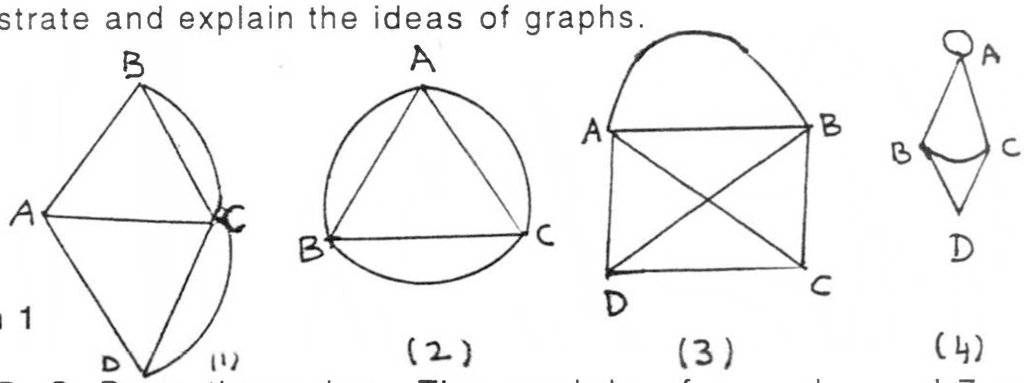
Then these are connected by the formula $V + F = E + 2$ called Euler's formula for polyhedron. We can verify these for regular polyhedra.

Polyhedron	V	F	E	V + F	E + 2
Tetrahedron	4	4	6	8	8
Hexahedron	8	6	12	14	14
Octahedron	6	8	12	14	14
Dodecahedron	20	12	30	32	32
Icosahedron	12	20	30	32	32

III. Examples - Problem solving

In this chapter, you have ample choice to illustrate the ideas of graphs.

(a) Here are some graphs and each one of them may be used to illustrate and explain the ideas of graphs.



Graph 1

(i) A, B, C, D are the nodes - The graph has four nodes and 7 arcs (i.e. N = 4, A = 7).

Node	Order of the node	Nature of the node
A	3	Odd
B	3	Odd
C	5	Odd
D	3	Odd

Total: 14 = 2 x The total number of arcs in the graph.

By Euler's solution for traversibility, this graph is **not traversible** because it has **more than two odd nodes**.

(ii) We construct the matrix for the graph

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 2 & 0 \\ 1 & 2 & 0 & 2 \\ 1 & 0 & 2 & 0 \end{bmatrix} \end{array} \qquad \begin{array}{cl} \text{Row totals} & \\ 3 & \text{(order of A)} \\ 3 & \text{(order of B)} \\ 5 & \text{(order of C)} \\ 3 & \text{(order of D)} \end{array}$$

Total of numbers in the matrix = 14

(iii) The graph divides the plane on which it is drawn into five regions (i.e. R = 5).

N = 4, A = 7 and $\boxed{N + R = A + 2}$ (=9)

This verifies Euler's formula for the graph.

Graph 2

(i) The graph has 3 nodes A, B and C (i.e. N=3) and 6 arcs (i.e. A=6)

Node	Order of the node	Nature of the node
A	4	Even
B	4	Even
C	4	Even

Total 12 = 2 x A

This graph has no odd node. Hence it is a traversible graph.

(ii) The matrix of the graph

$$\begin{array}{c|ccc|cl} & A & B & C & \text{Row totals} & \\ A & 0 & 2 & 2 & 4 & = \text{order of A} \\ B & 2 & 0 & 2 & 4 & = \text{order of B} \\ C & 2 & 2 & 0 & 4 & = \text{order of C} \end{array}$$

Total = $\boxed{12 = 2 \times \text{ The number of arcs}}$

(iii) The graph divides the plane of the paper into five regions (i.e. R = 5). Then N + R = A + 2 (=8)

Graph 3

(i) For this graph, N = 5, A = 9.

Node	Order of the node	Nature of the node
A	4	Even
B	4	Even
C	3	Odd
D	3	Odd
E	4	Even

Total 18 = 2 x A

The graph is traversible since the number of odd nodes is only two (not more than two). However to traverse it start from an odd node and you will reach the other odd node after tracing the graph.

(ii) The matrix of the graph

$$\begin{array}{c|ccccc|c|l}
 & A & B & C & D & E & \text{Row total} & \text{Order of the nodes} \\
A & 0 & 2 & 0 & 1 & 1 & 4 & = \text{order of } A \\
B & 2 & 0 & 1 & 0 & 1 & 4 & = \text{order of } B \\
C & 0 & 1 & 0 & 1 & 1 & 3 & = \text{order of } C \\
D & 1 & 0 & 1 & 0 & 1 & 3 & = \text{order of } D \\
E & 1 & 1 & 1 & 1 & 0 & 4 & = \text{order of } E \\
\end{array}$$

$$\text{Total} \quad = \quad 18 \quad = 2 \times A$$

Note: When two nodes are connected, it means it is connected by a ***single arc***. That is why the number of arcs connecting A and C (or B and D) is taken as zero.

(iii) The graph divides the plane of the paper into six regions (i.e. R = 6).

N + R = A + 2 (=11), which verifies Euler's formula.

Graph 4

In this graph, there is a loop at the node A. A loop at a node is **a single arc** connecting a node to itself. Since a loop can be traced in the clockwise as well as the anticlockwise directions, the order of the node containing a loop is increased by 2. However, while counting the number of arcs, a loop is taken as a single arc.

For the graph

(i) N = 4, A = 6

Node	Order of the node	Nature of the node
A	4	Even
B	3	Odd
C	3	Odd
D	2	Even

Total = 12 = 2 x A

This graph is traversible, since it has just two odd nodes. You have to start from one odd node (B or C) and you will end up at the other odd node.

(ii) The matrix of the graph

$$\begin{array}{c|c|cc} & \begin{matrix} A & B & C & D \end{matrix} & \text{Row totals} & \\ \begin{matrix} A \\ B \\ C \\ D \end{matrix} & \begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} & \begin{matrix} 4 \\ 3 \\ 3 \\ \underline{2} \end{matrix} & \begin{matrix} = \text{order of } A \\ = \text{order of } B \\ = \text{order of } C \\ = \text{order of } D \end{matrix} \\ & \text{Total} = & 12 & = 2 \times A \end{array}$$

(iii) N = 4, A = 6, R = Number of regions formed by the graph = 4.

N + R = A + 2 (=8), which verifies Euler's formula.

(b) **Given the matrix of the graph, to construct (draw) the graph — Illustrations.**

Draw the graph using the matrix, in each case

(1)

$$\begin{array}{c} \\ A \\ B \end{array}\begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array}$$

(2)

$$\begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} \end{array}$$

(3)

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 3 & 2 & 1 \\ 3 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{bmatrix} \end{array}$$

To draw the graph

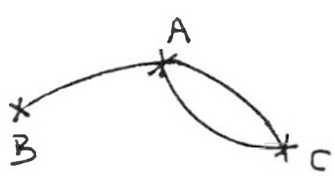
(a) First we mark the nodes - A and B.

(b) Looking at the number of arcs connecting each node to the other nodes, we draw the graph, by connecting the nodes.

Then, the graph for the matrix (1) is

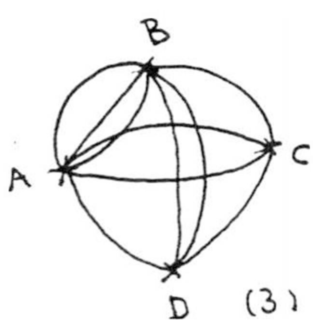
(1)

The graph for the matrix (2) is



(2)

The graph for the matrix (3) is

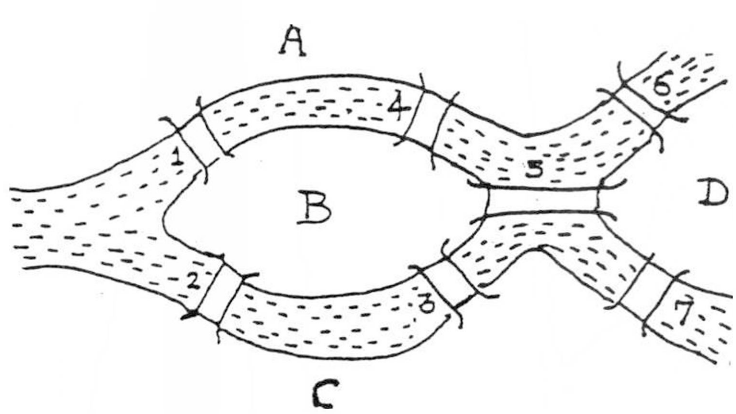


(3)

(c) The Königsberg Problem (of seven bridges)

An ancient problem

In Euler's* time the river Pregel flowing through the University town of Königsberg in Germany had seven bridges across it (see the Figure). People tried and tried to cross all the seven bridges in one walk without recrossing any bridge but could not. They were wondering whether it can be done (traversing all the seven bridges, each bridge only once) at all. It was to the genius of Euler who could come out with a convincing solution to the problem and the likes of it. As we know it today, it is impossible to cross all the seven bridges one after the other, each bridge only once.



KÖNIGSBERG 7-BRIDGES

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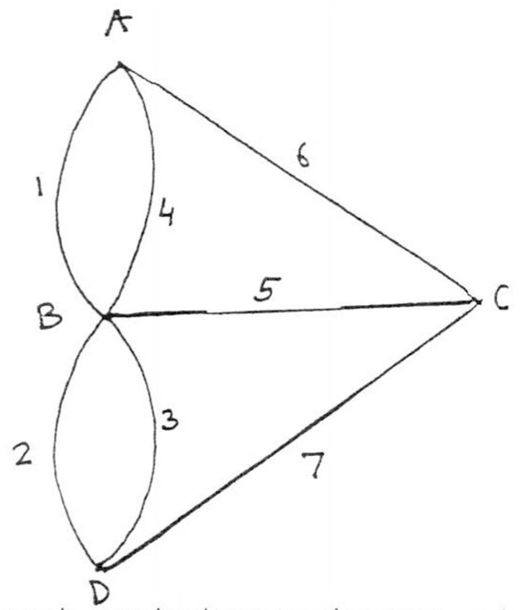
* Euler was a Swiss mathematician of 18th century (1707-1783) whose contribution to mathematics is abundant.



The graph equivalent to the seven-bridges

For this graph, N = 4, A = 7

Node	Order of the node	Type of the node
A	3	Odd
B	5	Odd
C	3	Odd
D	3	Odd

It is evident that the graph has more than two odd nodes. Hence it is not traversible according to Euler's formula for traversibility of a graph.

(d) When we count the number of vertices (V), the number of faces (F) and the number of edges (E) for **any polyhedron**, it can be verified that

$V + F - E = 2$ (always). Equivalently

$V + F = E + 2$. This relation between V, F and E is called Euler's formula for polyhedra.

Consider a few polyhedra (not necessary the regular polyhedra which have already been discussed earlier).

(a) A pentagonal based pyramid

(b) A hexagonal based prism

(c) A square based pyramid

(d) An octagonal based prism

(e) A truncated square based prism

got by removing from square based pyramid such a pyramid (with square base) by a section parallel to the base.

Sl. No.	Name of the solid	V	F	E	V+F-E	
a	Pentagonal based pyramid	6	6	10	2	Hence V+F=E+2
b	A hexagonal based prism	12	8	18	2	
c	A square based pyramid	5	5	8	2	
d	An octagonal based prism	9	9	16	2	
e	A truncated square based prism	8	6	12	2	

IV. Suggested Activities

1. List examples of graphs from daily life experiences like - road maps of a town, networks, etc.
2. Examine the validity of Euler's formula for polyhedra, by discussing (i) different polyhedra, (ii) combinations of polyhedra - like a cube which carries a square based prism on each face.

3. Construct models of polyhedra (both regular ones and others).

4. Cut across some of the edges of a polyhedra so that the faces are on a flat surface. This is a graph. Verify Euler's formula N+R = A+2.

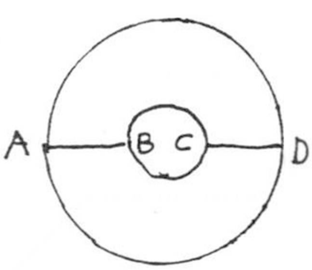
V. Exercises

1. Explain the terms in reference to a graph.

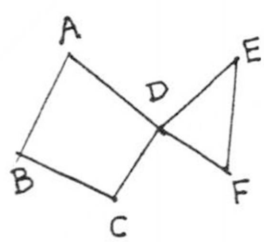
   (i) Graph, (ii) Node, (iii) Arc, (iv) Order of a node, (v) Even node, (vi) Odd node, (vii) Matrix of a graph, (viii) Loop at a node.

2. Examine the traversibility of the following graphs, stating the reason for the conclusion and verify Euler's formula: N+R = A+2.

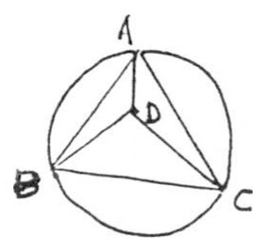
(i)



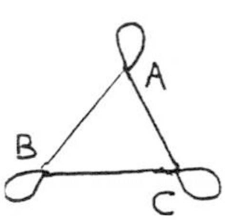
(ii)



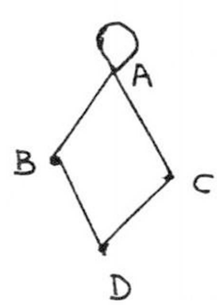
(iii)



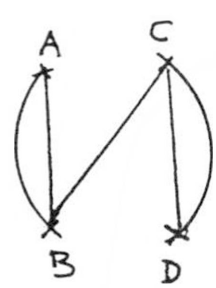
(iv)



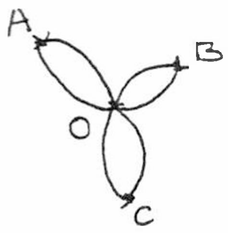
(v)

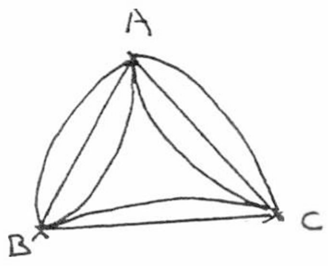


(vi)



(vii) (viii)





3. For the graphs in 2. write the matrices.

4. Write the graphs for the following matrices -

(i) $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 2 & 3 & 0 \end{bmatrix}$ $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

(ii) $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

5. Verify Euler's formula for the following polyhedra -

(i) A ten-sided polygon based prism

(ii) A pentagonal based pyramid.

(iii) A cube with prisms on one pair of opposite faces.

(iv) A truncated prism (a solid got by removing a prism from another prism by a section parallel to the base) with octagonal base.

(v) A solid obtained by joining the bases of two identical pentagonal based prisms.