RESOURCE MATERIAL
OF

THREE-WEEK REFRESHER COURSE FOR PGTs IN MATHEMATICS

OF NAVODAYA VIDYALAYA SAMITHI

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REGIONAL INSTITUTE OF EDUCATION
(NATIONAL COUNCIL OF EDUCATIONAL RESEARCH AND TRAINING)
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PREFACE

With the request of Navodaya Vidyalaya Samithi, three week Refresher Course was conducted for PGTs in Mathematics of NVS by Regional Institute of Education Mysore from 28-04-2008 to 16-05-2008. Totally 28 participants from all over the country attended this program.

This program was conducted to address their needs in content as well as pedagogical aspects. As the achievement of higher secondary students particularly in Mathematics was not satisfactory, it was felt that there is a need to enrich the competence of teachers in teaching Mathematics. Also in view of new text books, there is a need to address the changes, selection of new topics, presentation and evaluation based on the recommendations of National Curriculum Frame-Work 2005.

Based on the interactions with participants and Resource persons, this material was developed. We expect that the teachers can study on their own whenever they find difficulty in teaching.

The academic coordinator is indeed grateful to Prof. G.T. Bhandage, Principal, Regional Institute of Education, Mysore for their encouragement and help in all respect. He is also thankful to the Head, DEE and the staff, RIE, Mysore, for their guidance and encouragement. He is thankful to Navodaya Vidyalaya Samithi for sending their teachers to this program. He is thankful to all

internal and external resource persons for their efforts and cooperation. Thanks to our library staff, ICT staff and all supporting staff, for their support throughout this program. He is very much thankful to Mr. B.K. Venkatesh, Kowshik DTP Centre, Mysore, for his neat and timely completion of the materials required.

Last, but not the least, he is thankful to all the participants who have come from all parts of country to attend this programme and congratulate them for taking keen interest in participating in this programme for the benefit of our teacher and student committee.

Mathematics teaching can and should be an intellectually stimulating and ever challenge. It should also be an endless source of satisfaction.

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PART - I

RELATIONS

Let A be a set. A **relation** R on A is a subset of the Cartesian product A x A.

A relation R on A is called,

1) **reflexive** if (a , a) is in R for every a in A.
2) **symmetric** if (a , b) is in R implies (b , a) is in R for all a,b in A.
3) **transitive** if (a , b) is in R and (b , c) is in R implies (a , c) is in R for all a,b,c in A.

A relation R is said to be an **equivalence relation** if R is reflexive, symmetric and transitive.

Examples :

1) The relation R in the set {1,2,3 given by, R ={(1,1),(2,2),(3,3),(1,2)} is reflexive only.

2) On the set L of all straight lines in a plane the ' Perpendicular ' relation is symmetric only.

3) On the set Q of all rational numbers , the 'less then' relation is transitive only.

4) On the set A = {1,2,3 } the relation R given by, R = {(1,1),(2,2),(3,3),(1,2), (2,1)} is reflexive and symmetric but not transitive.

5) On the set Q of all rational numbers the 'less then or equal' relation is reflexive and transitive but not symmetric.

6) On the set A = {1,2,3} the empty relation is symmetric and transitive but not reflexive.

7) On the set A = {1,2,3} the relation R given by R = {(1,1),(2,2),(3,3)} is an equivalence relation.

8) On the set of all integers the relation R given by R={(a,b) /(a–b) is divisible by 3} is an equivalence relation.

9) On the set of all human beings in a town at a particular time, the relation R given by R = { (a,b) / a is a father of b } is neither reflexive nor symmetric nor transitive.

Problems :

1) Here is a 'proof' that, " every relation R that is both symmetric and transitive is also reflexive"; " Since R is symmetric (a,b) is in R implies (b,a) is in R. Since R is transitive (a,b) is in R and (b,a) is in R implies (a,a) is in R as desired". Find the flow in this statement.

2) Let A be a nonempty set and let R be a relation on A. Suppose the following properties hold;
   a) For every x in A , (x,x) is in R.
   b) For every x,y,z, in A , (x,y) is in R and (y,z) is in R implies (z,x) is in R (it is called **circular property**)

   Then show that R is an equivalence relation.

3) Let A be a nonempty set and let R be a relation on A. Suppose the following properties hold;
   a) For every x in A , (x,x) is in R.
   b) For every x,y,z in a , (x,y) is in R and (x,z) is in R implies (y,z) is in R (it is called **triangular property**)

   Then show that R is an equivalence relation.

BINARY OPERATION

A **binary operation** * on a set A is a function * : A x A →A . That is * is an operation which operates between any two elements of a such that the resultant unique element should belongs to A. That for a ,b, in A , a*b is in A.

The following chart gives a brief idea about the basic operations + , – , × and ÷ on each number sets:

	N	W	Z	Q	Q'	R	R^+	C
+	YES	YES	YES	YES	NO	YES	YES	YES
–	NO	NO	YES	YES	NO	YES	NO	YES
×	YES	YES	YES	YES	NO	YES	YES	YES
÷	NO	NO	NO	NO	NO	NO	YES	NO

A binary operation * is said to be **commutative** if for all a,b,in A

a*b = b*a.

A binary operation * is said to be **associative** if for all a,b,c in A

a*(b*c) = (a*b)*c

EXAMPLES:

1) On the set Q of all rational numbers,

a) $a*b = (a - b)^2$ is commutative but not associative.
b) $a*b = a^2 + b^2$ commutative but not associative.
c) $a*b = ab^2$ is neither associative nor commutative.

Functions

Introduction

Functions play a very important role in Calculus. Without functions, Calculus cannot exist. The fundamental processes of calculus called differentiation and integration are the processes applied on functions. To understand these processes and to be able to carry them out, you have to be thorough with the concept of functions. Here we discuss some of the basic notions concerning functions.

Definition of a function/domain and range of a function

Let D be a subset of the set of real numbers. By a function f on D, we mean a rule which assigns to each number x in D a unique real number denoted by f(x).

The set D is called the *domain* of the function f. The set of all real numbers f(x) as x varies over D is called the *range* of the function f.

Range f = $\{ f(x) \in \Re \mid x \in D \}$

The real number f(x) is called the image of x under f.

Examples 1 : Suppose to each real number x, we assign its square x^2, we get the function f, where $f(x) = x^2$ for every real number x.

Since we have assigned to every real number x, its square viz., x^2, the domain of f is the whole of $\Re$.

But the assigned value is always a square of a real number and hence is non negative. Hence the elements in the range of f are non-negative real numbers. Also given any non negative real number y has a real square root $\sqrt{y}$ and the real number $\sqrt{y}$ is assigned by f its square y. Here every non-negative real number is in the range of f. Thus the range of f is precisely the set of all non negative real numbers.

In short,

Domain f = $(-\infty, \infty)$, Range of f = $[0, \infty)$

Example 2 : Now consider the function which assigns to a real number x, the positive square root of the real number x + 4

i.e. $g(x) = \sqrt{x+4}$

But in order that the square root of x + 4 exists, x + 4 must be non negative. For this $x \geq -4$. Hence g(x) is defined only for $x \geq -4$. Hence the domain of this function g is $[-4, \infty)$.

As we are taking the positive square root of x + 4, g(x) is non negative. Hence the elements in the range of g are non negative real numbers. Further given any non negative real number say y, it is assigned to the real number $y^2 - 4$ since

$$g(y^2 - 4) = \sqrt{y^2 - 4 + 4} = \sqrt{y^2} = y$$

Hence every non negative real number is in the range of g. Hence range g is the set of all non negative real numbers. In short,

Domain g $= [-4, \infty)$ Range g $= [0, \infty)$

In the above example, we can restrict the domain of g to any subset of $[-4, \infty)$ also. For example, let us define

$$g(x) = \sqrt{x+4} \quad \forall x \in [0,5]$$

Then the domain is clearly given as [0, 5]. However, to find the range of g, observe that g(0) = 2, and g(5) = 3. For $0 < x \leq 5$, $0 + 4 \leq x + 4 \leq 5 + 4$ and hence $\sqrt{4} \leq \sqrt{x+4} \leq \sqrt{9}$.

$$\therefore \quad 2 \leq \sqrt{x+4} \leq 3$$

Here Range g = [2, 3].

Many a times, as in example 2 above, the domain of the function is not explicitly given. We have to find it out.

For example, let us take the functions

$$f(x) = x^3, \; g(x) = \sqrt{x}, \; h(x) = \frac{1}{\sqrt{2-x}} + 5.$$

Here clearly f(x) is defined for all real numbers x.

Hence domain of f is the whole set of real numbers. Also every real number has a real cube root and hence every real number is the cube of a real number. Hence range of f is also the whole of $\Re$.

For g(x) = $\sqrt{x}$, $\sqrt{x}$ is real only when $x \geq 0$. Hence domain of g is the set of non negative real numbers i.e. $[0, \infty)$. Similarly, the range is also $[0, \infty)$.

However, in case of h, for h(x) to be real $\sqrt{2 - x}$ should be a non-zero real number. For this to happen $2 - x > 0$ i.e., $x < 2$. Hence the domain of h is $(-\infty, 2)$. Sine for $x < 2$, $\sqrt{2 - x}$ ranges over all positive real numbers, $\frac{1}{\sqrt{2 - x}}$ also ranges over all positive real numbers. Hence $\frac{1}{\sqrt{2 - x}} + 5$ ranges over all real numbers greater than 5. Hence Range h = $(5, \infty)$.

Algebraic Operations on functions :

Functions which have the same domain D can be added, subtracted, multiplied, divided (with a restricted domain) and multiplied by a constant real number as follows :

$(f + g)(x) = f(x) + g(x) \ \forall x \in D$

$(f - g)(x) = f(x) - g(x) \ \forall x \in D$

$(f . g)(x) = f(x) . g(x) \ \forall x \in D$

$(\frac{f}{g})(x) = \frac{f(x)}{g(x)} \ \forall x \in D$ for which $g(x) \neq 0$.

$(kf)(x) = kf(x) \ \forall \ x \in D$ where k is a constant real number.

Note: If $f(x) = x + 2$ and $g(x) = x - 2$

then the domains of both f and g are $\Re$. However for $\frac{f}{g}$, since $g(x) = 0$ when $x = 2$, $\frac{f}{g}(x)$ is not defined when $x = 2$. Here the domain of $\frac{f}{g}$ is $\Re \rightarrow \{2\}$ i.e. $(-\infty, 2) \cup (2, \infty)$.

Polynomial and Rational functions :

Starting with the function $f_1(x) = x$ which assigns to every real number x, the real number x itself called the identity function, we can get

$f_1(x) \bullet f_2(x) = x \bullet x = x^2 \qquad \forall\ x \in \Re$

$f_3(x) = f_2(x) \bullet f(x) = x^2 \bullet x = x^3,\ \forall\ x \in \Re$

$f_4(x) = f_3(x) \bullet f(x) = x^3 \bullet\ x = x^4\ \forall\ x \in \Re$

Proceeding this way, for every positive integer n,

$f_n(x) = x^n,\ \forall\ x \in \Re$ is a function.

Also, $g_0(x) = a_0,\ \forall\ x \in \Re$ where a_0 is a constant is a function called constant function.

$g_k(x) = a_k x^k,\ \forall\ x \in \Re$ where a_k is a constant for $1 \leq k \leq n$ is a function.

Hence

$a_0 + a_1x + a_2x^2 + a_nx^n$ is a function $\forall\ x \in \Re$. This function is called polynomial function.

If $f(x) = a_0 + a_1x + \ldots.. + a_nx^n$

$g(x) = b_0 + b_1\ x + \ldots.. + b_mx^m$ with $b_0, \ldots.. b_m$ not all zero.

Then $h(x) = \dfrac{f(x)}{g(x)},\ \forall\ x \in \Re$ is a function called a rational function.

Trigonometric Functions

To every real number x, if we assign the real number $\sin(x^c)$, the trigonometric ratio for general angles sine of x radians, we get the function $f(x) = \sin x,\ \forall\ x \in \Re$. This function is called sine function. Here the domain is the set of real numbers $\Re$ and since the trigonometric ratio *sin x* takes precisely all values between –1 and 1, its range is [-1, 1].

Similar is the case for the function g(x) = cos x. Since $\tan x = \dfrac{\sin x}{\cos x}$, the domain of definition of tan x is

$\Re$ - { the set of zeros of cos x } = $\Re$ - $\{ n\pi + \pi/2 \mid n \in \mathbf{Z} \}$.

As x varies from $-\pi/2$ to 0, sin x varies from –1 to 0 whereas cos x varies from 0 to 1. Hence tan x varies from $-\infty$ to 0. Further as x varies from 0 to $\pi/2$, sin x varies from 0 to 1 whereas cos x varies from 1 to 0. Hence tan x varies from 0 to ∞. Thus the range of tan x is $(-\infty, \infty)$.

Also cosec x = $\frac{1}{\sin x}$ and hence its domain of definition is

$\Re$ - { the set of zeros of sin x } = $\Re - \{ n\pi \mid n \in Z\}$

and its Range is $(-\infty\ \ 0) \cup (0\ \ \infty)$.

Since sec x = $\frac{1}{\cos x}$ its domain of definition is the same as that of tan x. It is clear that sec x is never 0. Further as x varies from $-\pi$ to π, cos x varies from -1 to 1. Hence sec x = $\frac{1}{\cos x}$ varies from $-\infty$ to ∞ except 0, the

Range sec x = $(-\infty,\ 0) \cup (0,\ \infty)$. Also cot x = $\frac{\cos x}{\sin x}$ and hence its domain of definition is the same as that of cosec x and its Range is $(-\infty,\ \infty)$.

You might have already come across the number e which is the sum of the series $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots$.. In fact e is an irrational number and it lies between 2 and 3. Now for any real number x, we consider the real number e^x. The function $f(x) = e^x$ is called the exponential function. The domain of the exponential function is the set of real numbers of $\Re$. If x = 0, $e^x = e^0 = 1$. If x > 0, $e^x > 1$ and as x increases, e^x also goes on increasing. We can see that e^x takes all values between 1 and ∞ (as x varies from 0 to ∞). Further when x < 0, we can put x = -y where y > 0. Hence $e^x = e^{-y}\ (e^y)^{-1} = \frac{1}{e^y}$. Since y > 0, e^y takes all values between 1 and ∞. Hence $\frac{1}{e^y}$ takes all values between 1 and 0, except 0. Thus the range of e^x is $(0, \infty)$.

Problem solving :

1. Find the domain and range of the following functions.

   i) $f(x) = x^2 - 1$

Since x^2 is a real number for all real number x, so is $x^2 - 1$. Hence f(x) is defined for all real numbers x. Here Domain f = $\Re$.

For all real numbers x, $x^2 \geq 0$. Further as x varies from 0 to ∞, x^2 also increases by taking all values from 0 to ∞. Here $x^2 - 1$ takes all values between -1 and ∞. Here Range f = $[-1, \infty)$.

ii) $f(x) = 3x - 2$

3x is a real number for all real numbers x. Here 3x-2 is a real number for all numbers x. Hence domain of f is $\Re$.

As x increases from $-\infty$ to ∞, 3x also increases taking all values between $-\infty$ and ∞ and hence 3x-2 takes all values between $-\infty$ and ∞.

Hence Range f(x) = $\Re$.

(iii) $g(x) = \sqrt{x} - 3$

$\sqrt{x}$ is a real number only for non negative real numbers x, so is $\sqrt{x} - 3$. Hence domain of g(x) is the set of non negative real numbers i.e. $[0, \infty)$.

As x increases from 0 to ∞, $\sqrt{x}$ takes all values between 0 and ∞. Hence $\sqrt{x} - 3$ takes all values between -3 and ∞. Here Range $g = [-3, \infty)$.

(iv) $f(x) = \sqrt{1-x} - 1$

$\sqrt{1-x}$ is real for all values of x for which $1 - x \geq 0$ i.e. $1 \geq x$. Hence $\sqrt{1-x}$ is real for all real numbers $x \leq 1$. Hence $f(x) = \sqrt{1-x} - 1$ is real for all $x \leq 1$. Therefore, Domain f = $[-\infty, 1]$. As x decreases from 1 to $-\infty$, $1 - x$ takes all values between 0 and ∞, and hence $\sqrt{1-x}$ also takes all values between 0 and ∞. Hence $\sqrt{1-x} - 1$ takes all values between -1 and ∞. $\therefore$ Range f = $[-1, \infty)$.

(iv) $f(x) = \begin{cases} 1-x^2 & x \leq 0 \\ x & 0 < x < 1 \\ x^2 & x \geq 1 \end{cases}$

Clearly domain of f is the whole set of real numbers $\Re$. As x varies from $-\infty$ to 0, x^2 varies from ∞ to 0 and hence $-x^2$ varies from 0 to $-\infty$. Hence $1-x^2$ varies from 1 to $-\infty$. As x varies from 0 to 1, f(x) = x varies from 0 to 1. Further as x varies from 1 to ∞, x^2 varies from 1 to ∞. Hence Range f = $\Re$.

Exercises for Self Evaluation

Find the domain and range of

i) $f(x) = \frac{1}{x^2}$

ii) $g(x) = \sqrt{x - 1} - 1$

iii) $f(x) = \frac{1}{\sqrt{4 - x^2}}$

iv) $g(x) = -\frac{x}{2} - 3$

v) $f(x) = \sqrt{x} - \frac{1}{\sqrt{x}}$

vi) $g(x) = \begin{cases} 1+x & 0 \leq x \leq 1 \\ x & 1 < x < 2 \\ \frac{1}{2}x + 1 & x \geq 2 \end{cases}$

Limits

1.1 Introduction

We live in a world of change – our values, ideals, hopes and intuitions are undergoing constant change. It is interesting to note that certain changes are happening too rapidly, while other changes are not occurring fast enough. This illustrates that, although the topic of change is important, often the concept of rate of change is more relevant. For example, in the study of population growth, it is not sufficient to know that the population is changing by doubling. We need to know the rate at which this doubling is taking place. It is significant that at one time the doubling of the world population took a thousand years, but now the doubling takes only a few decades time. The mathematical tool for measuring rates of change is the concept of limits. The concept of limit is needed to pass from the average rate of change to the more useful concept of an instantaneous rate of change. Indeed it is this concept of the limit, that resulted in the invention of Calculus. It may be surprising to discover that Newton did not have a complete understanding of the limit. Many years later Cauchy put the concept of limit on a sound mathematical basis. In this section, the approach to the concept of limit is initially intuitive and later the mathematically elegant Cauchy's epsilon-delta approach is given.

There are many topics in school mathematics through which limits can be illustrated. For instance, consider the problem of finding circumference of a circle. The circumference of a circle can be taken as the limit of perimeter of inscribed regular polygon as the number of sides goes on increasing and tends to infinity. Teachers can also use the action of a bouncing ball. If $\{ h_n \}$, n = 1,2,3,..... is the sequence of heights of a bouncing ball, then 0 is the limit of such a sequence.

1. Concept and Explanations :

1.2 Limit of a Function :

Consider the function f(x) = $\frac{x^2-4}{x-2}$ for $x \neq 2$.

f(x) is not defined at x = 2 because the direct substitution 2 for x results in 0/0 which is an indeterminate. Let us calculate the values of f(x) for some values x that are very close to but not equal to 2.

From the table it appears that if x is very close to 2, then f(x) = $\frac{x^2-4}{x-2}$ is very near to 4.

x	f(x) = $\frac{x^2-4}{x-2}$
1.98	3.98
1.99	3.99
2.0	4.01
2.02	4.02

We represent this statement mathematically as,

Limit of f(x) = $\frac{x^2-4}{x-2}$ as x approaches 2 is 4 or $\lim_{x \to 2} f(x) = 4$.

Using the same argument as above you can easily see that

i) $$\lim_{x \to 3} \frac{x^2 - 5x + 6}{x - 3} = 1$$

ii) $\lim_{x \to 3} \frac{x^2 - 9}{x+3} = 0$

Now we provide intuitive definition of limit of a function.

Definition: Let f be a real valued function defined on a subset D of real numbers and let a ∈ D. We say that limit of f(x) as x → a is a real number l if f(x) is very close to l, whenever x is very close to a.

We write this as $\lim_{x \to a} f(x) = l.$

If such a l does not exist then we say that $\lim_{x \to a} f(x)$ does not exist. For instance $\lim_{x \to 0} \frac{1}{x}$ does not exist.

x	$f(x) = \frac{1}{x}$
.1	10
.01	100
.001	1000
.0001	10000
.00001	100000

We see that as x comes nearer and nearer to 0, f(x) does not come very close to any real number l.

Next we shall introduce the idea of left hand limit and right hand limit of a function at a point. Let f(x) be a function defined as follows :

$$f(x) = \begin{cases} \frac{1}{2} x + 2 & \text{if } x < 2 \\ x + 4 & \text{if } x \geq 2 \end{cases}$$

We shall examine whether $\lim_{x \to 2} f(x)$ exists.

First suppose x → 2 from the right side of 2 (or x → 2 and x > 2).

Symbolically it is written as x → 2+.

Then $\lim_{x \to 2+} f(x) = \lim_{x \to 2+} (x + 4) = 2 + 4 = 6$

This limit is called as right hand limit of f(x) at 2.

Next suppose x → 2 from the left side of 2 (or x → 2 and x < 2).

Symbolically it is written as x → 2-.

Then $\lim_{x \to 2-} f(x) = \lim_{x \to 2-} \left(\frac{1}{2}x + 2\right) = 1 + 2 = 3.$

$\lim_{x \to 2-} f(x)$ is called as left hand limit of f(x) at 2.

Thus $\lim_{x \to 2+} f(x) \neq \lim_{x \to 2-} f(x)$.

Here as x comes very close to 2 from right, f(x) is very close to 4 whereas as x comes very close to 2 from left, f(x) is very close to 3. So $\lim_{x \to a} f(x)$ does not exist. $\lim_{x \to a} f(x)$ exists if and only if $\lim_{x \to 2+} f(x) = \lim_{x \to 2-} f(x)$.

Earlier we got $\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = 4$.

In this case, notice that $\lim_{x \to 2+} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2-} \frac{x^2 - 4}{x - 2} = 4.$

The definition of limit given above is intuitive and suffers from shortcomings. In the first instance, it lacks mathematical rigour and further it is hardly useful in the development of theory of limits. What is meant by very close ?

1.999999....99 to 100 digits is very close to 2 . If a quantity is very close to 1.999999 (100 places can we say that it is very close to 2? So if the limit is 1.9999999 can we say that limit is also 2 ? We can examine more closely the idea of limit so as to arrive at Cauchy's mathematical definition.

Let us begin with $\lim_{x \to 3} (2x + 1) = 7$. This means that when x is very close to 3, 2x + 1 is very close to 7. Since "close to" is not mathematically defined so far, we have trouble in understanding what we mean by these words. Therefore, our first attempt to explain $\lim_{x \to 3} (2x + 1) = 7$ is unsatisfactory. In our second attempt to explain $\lim_{x \to 3} (2x + 1) = 7$ we mean that the value of 2x + 1 can be made as near to 7 as we wish to have it by making x near enough to 3. This leads us to the 'Cauchy's definition' for limit of a function.

Definition : $\lim_{x \to a} f(x) = L$ if for every $\varepsilon > 0$ however small, there exists $\delta > 0$ such that $| f(x) - L | < \varepsilon$ whenever x is such that $0 < | x - a | < \delta$.

Examples : Use the above Cauchy definition of limit and show that $\lim_{x \to 3} (2x + 1) = 7$

Solution : Let $\varepsilon > 0$ be any given number. Then we have to find δ such that

$|(2x + 1) - 7| < \varepsilon$ whenever $0 < |x - 3| < \delta$.

Now $|(2x + 1) - 7| = 2|x - 3| < \varepsilon$ if and only if $0 < |x - 3| < \varepsilon/2$

Hence choose $\delta = \varepsilon/2$, so that $|(2x + 1) - 7| < \varepsilon_o$.

for $0 < |x - 3| < \delta = \varepsilon/2$

$\therefore \quad \lim_{x \to 3} (2x + 1) = 7$

Example 2 : Use the Cauchy definition of limit and show that $\lim_{x \to 2} \left[\frac{1}{2} x - 4\right] = -3$.

Solution : Let $\varepsilon > 0$ be any given number.

Then

$$\left|\left(\frac{1}{2} x - 4\right) - (-3)\right| < \varepsilon \text{ iff } \left|\frac{1}{2} x - 1\right| < \delta$$

$$\left|\left(\frac{1}{2} x - 4\right) - (-3)\right| < \varepsilon \text{ iff } \frac{1}{2}|x - 2| < \delta$$

Choose $\delta = 2\varepsilon$, so that $|(1/2\, x - 4) - (-3)| < \varepsilon$ whenever $0 < |x - 2| < \delta$.

Hence, $\lim_{x \to 2} \left[\frac{1}{2} x - 4\right] = -3$.

Now we shall illustrate the use of this definition of limit in proving some of the important properties of limits.

Properties of limits and their proofs.

Property 1 : $\lim_{x \to a} c = c$ (c is any constant).

(i.e. limit of a constant function is constant itself.)

Proof : Let $\varepsilon > 0$ be given.

Because $|c - c| = 0$ is always true for any x and in particular for x such that $0 < |x - a| < \delta$ whenever $\delta > 0$. In particular we can take $\delta = 1$. Hence we have

$0 < | x - a | < 1 \Rightarrow | c - c | < \varepsilon$

$\therefore \lim_{x \to c} = c$

Property 2 : If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$

Then, $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = L + M$

(i.e. limit of a sum is sum of its limits).

Proof : Let $\varepsilon > 0$ be given. Then $\varepsilon/2 > 0$.

Since $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$,

by definition of limit, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

$| f(x) - L | < \varepsilon/2$ for $0 < | x - a | < \delta_1$ and

$| g(x) - M | < \varepsilon/2$ for $0 < | x - a | < \delta_2$

Let δ be the smaller of δ_1, δ_2. Then

$| f(x) - L | < \varepsilon/2$ and $| g(x) - M | < \varepsilon/2$ for $0 < | x - a | < \delta$

Now $| f(x) + g(x) - (L + M) | = | (f(x) - L) + (g(x) - M)|$

$\leq f(x) - L | + | g(x) - M) | < \varepsilon/2 + \varepsilon/2 = \varepsilon$

such that $0 < | x - a | < \delta$

$\therefore \quad \lim_{x \to a} \{ f(x) + g(x) \} = L + M = \lim_{x \to a} f(x) + \lim_{x \to a} g(x).$

On the same lines as above, some more properties on the limits may be proved.

Property 3 :

$$\lim_{x \to a} [f(x) - g(x)] = l - m, \text{ where } \lim_{x \to a} f(x) = l, \ \lim_{x \to a} g(x) = m.$$

Let $\varepsilon > 0$ be given. Let $\varepsilon_1 = \dfrac{\varepsilon}{2}$. Then $\varepsilon_1 > 0$.

Since $\lim_{x \to a} f(x) = l$, there exists $\delta_1 > 0$ such that

$| f(x) - l | < \varepsilon_1 \ \forall \ x$ such that $0 < | x - a | < \delta_1$

and since $\lim_{x \to a} g(x) = m$, there exists $\delta_2 > 0$ such that

$| g(x) - m | < \varepsilon_1 \quad \forall$ x such that $0 < | x - a | < \delta_2$

Let $\delta = \min \{ \delta_1, \delta_2 \}$. Then

$| f(x) - g(x) - (l - m) | = | [f(x) - l] + [-g(x) + m] |$

$\leq | f(x) - l | + |-g(x) + m |$

$= | f(x) - l | + | g(x) - m |$

$< \varepsilon_1 + \varepsilon_1 = 2\varepsilon_1 = \varepsilon, \ \forall$ x such that $0 < | x - a | < \delta$.

$\therefore \quad \lim_{x \to a} [f(x) - g(x)] = l - m.$

Property 4 : $\lim_{x \to a} f(x) .g(x) = l . m.$

Since $\lim_{x \to a} g(x) = m$, given $\varepsilon = 1 > 0$, there exists $\delta_1 > 0$ such that

$| g(x) - m | < 1 \quad \forall \ 0 < | x - a | < \delta_1$

$\therefore m - 1 < g(x) < m + 1 \ \forall x, \quad 0 < | x - a | < \delta_1.$

$\therefore | g(x) | < \max. \{ | m - 1 |, | m + 1 | \} = k > 0 \ \forall \ 0 < | x - a | < \delta_1$

Let $\varepsilon > 0$ be given.

Let $\varepsilon_1 = \dfrac{\varepsilon}{2k}, \quad \varepsilon_2 = \dfrac{\varepsilon}{2 (| l | + 1)}$. Then $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$.

Then $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$.

Since $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} g(x) = m$, there exists $\delta_2 > 0$ and $\delta_3 > 0$

such that

$| f(x) - l | < \varepsilon_1 \ \forall \ 0 < | x - a | < \delta_2$

$| g(x) - m | < \varepsilon_2 \ \forall \ 0 < | x - a | \leq \delta_3$

Let $\delta = \min \{ \delta_1, \delta_2, \delta_3 \}$. Then

$| f(x) . g(x) - lm |$

$= | f(x) . g(x) - l g(x) + l g(x) - lm |$

$\leq | f(x) . g(x) - l g(x) | + | l g(x) - l m |$

$= | g(x) | . | f(x) - l | + | l | \ | g(x) - m |$

$< k . | f(x) - l | + | l | . | g(x) - m | \ \forall \ x, \quad 0 < | x - a | < \delta$

$< k.\,\varepsilon_1 + |l|\,.\,\varepsilon_2 = k\,.\,\dfrac{\varepsilon}{2k} + |l|\,.\,\dfrac{\varepsilon}{2(|l|+1)} \quad \forall\ x,\ \ 0 < |x-a| < \delta$

$< \dfrac{\varepsilon}{2} + \dfrac{\varepsilon}{2} = \varepsilon,\ \forall_x,\ 0 < |x-a| < \delta,$ since $\dfrac{|l|}{|l|+1} < 1.$

$\therefore \quad \lim\limits_{x \to a} f(x)\,.\,g(x) = l\,.\,m$

Property 5 :

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{l}{m} \text{ where } m \neq 0.$$

Since $m \neq 0$, either $m > 0$ or $m < 0$.

Case (i) : Let $m > 0$. Therefore, there exists k such that $m > k > 0$.

Since $\lim\limits_{x \to a} g(x) = m$, given $k > 0$ there exists $\delta_1 > 0$ such that

$|g(x) - m| < k$ for all x, $0 < |x-a| < \delta_1$

i.e. $m - k < g(x) < m + k$ for all $0 < |x-a| < \delta_1$

Since $m - k > 0$, it follows that $g(x) > 0$ for all $0 < |x-a| < \delta_1$.

$\therefore \quad |g(x)| > |m-k| = k_1$ for all $0 < |x-a| < \delta_1$ where $k_1 > 0$.

Case (ii) : Let $m < 0$. Then there exits $k < 0$ such that $m < k < 0$. $\therefore\ -k > 0$.

Since $\lim\limits_{x \to a} g(x) = m$, given $-k > 0$ there exists $\delta_2 > 0$ such that

$|g(x) - m| < -k$ for all $0 < |x-a| < \delta_2$.

$\therefore \quad m + k < g(x) < m - k$ for all $0 < |x-a| < \delta_2$.

Since $m - k < 0$, therefore, $g(x) < 0\ \forall\ 0 < |x-a| < \delta_2$.

Therefore, $|g(x)| > |m-k| = k_2$ for all $0 < |x-a| < \delta_2$, where $k_2 > 0$.

Thus in both the cases, there exist $k_3 > 0$ and some $\delta_3 > 0$ such that

$|g(x)| > k_3$ for all $0 < |x-a| < \delta_3$

Let $\varepsilon > 0$.be given. Let $\varepsilon_1 = \dfrac{\varepsilon k_3}{2}$ and $\varepsilon_2 = \dfrac{\varepsilon\, k_3\, |m|}{2\,(|l| + 1)}$. Then $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$.

Since $\lim\limits_{x \to a} f(x = l$ *and* $\lim\limits_{x \to a} g(x) = m$.

There exists $\delta_4 > 0$ and $\delta_5 > 0$ such that

$|f(x) - l| < \varepsilon_1$ for all $0 < |x-a| < \delta_4$

$| g(x) - m | < \varepsilon_2$ for all $0 < | x - a | < \delta_5$

Let $\delta = \min \{\delta_3, \delta_4, \delta_5\}$. Then $\delta > 0$.

Now,

$$\left| \frac{f(x)}{g(x)} - \frac{l}{m} \right| = \left| \frac{f(x)\, m - g(x)\, l}{g(x)\,.\, m} \right|$$

$$= \frac{| f(x)m - g(x)\, l |}{| g(x)\, m |}$$

$$= \frac{| f(x).\, m - lm + lm - g(x)\, l|}{| g(x) | . |m|}$$

$$\leq \frac{| f(x).\, m - lm| + | lm - g(x)\, l|}{| g(x) | . |m|}$$

$$= \frac{| m| \,. | f(x) - l |}{| g(x) | . |m|} + \frac{| l | \,. |g(x) - m |}{| g(x) | . |m|} \qquad \because | m - g(x) | = | g(x) - x |$$

$$< \frac{| f(x) - l|}{k_3} + \frac{|l| . | g(x) - m |}{k_3 \,. |m|}$$

$$< \frac{\varepsilon_1}{k_3} + \frac{| l |}{| m| \,.\, k_3} \,.\, \varepsilon_2 \quad \forall\, 0 | x - a | < \delta.$$

$$= \frac{\varepsilon\, k_3}{2k_3} + \frac{|l|}{| m | \,.\, k_3} \,.\, \frac{\varepsilon\, k_3 \,|m|}{2\,(| l | + 1)}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \forall\, x,\ 0 < | x - a | < \delta, \text{ since } \frac{| l |}{1 + | l |} < 1.$$

$$\therefore \quad \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{l}{m}. \qquad\qquad k \,.\, \lim_{x \to a} f(x)$$

Property 6 : $\lim\limits_{x \to a} k\, f(x) = k\, l$

Note that if $k = 0$, then the proof is clear. Let us assume $k \neq 0$.

Let $\varepsilon > 0$ be given. Let $\varepsilon_1 = \dfrac{\varepsilon}{| k|}$. Then since $\lim\limits_{x \to a} f(x) = l$, there exists $\delta > 0$ such that

$| f(x) - l | < \varepsilon_1 \ \forall\, x,\ 0 < | x - a | < \delta$

Now,

$$| k \,.\, f(x) - kl | = | k | \,.\, | f(x) - l |$$

$$= | k | \,.\, \varepsilon_1 , \ \forall\, x,\ 0 < | x - a | < \delta$$

$$= |k| \cdot \frac{\varepsilon}{|k|} = \varepsilon \ \forall x, \ 0 < |x - a| < \delta$$

$$\therefore \quad \lim_{x \to a} k\, f(x) = k \lim_{x \to a} f(x).$$

Property 7 :

Let $f(x) \le h(x) \le g(x) \ \forall x$. If $\lim_{x \to a} f(x) = l = \lim_{x \to a} g(x)$, then $\lim_{x \to a} h(x) = l$.

We have, $0 \le h(x) - f(x) \le g(x) - f(x)$

Let $\in > 0$ be given. Let $\varepsilon_1 = \frac{\varepsilon}{3}$. Then $\varepsilon_1 > 0$. Since $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} g(x) = l$,

there exists $\delta_1 > 0$ and $\delta_2 > 0$ such that

$| f(x) - l | < \varepsilon_1 \ \forall x, \ < | x - a | < \delta_1$

$| g(x) - l | < \varepsilon_1 \ \forall x, \ 0 < | x - a | < \delta_2$

Let $\delta = \min \{ \delta_1, \delta_2 \}$. Then $\delta > 0$ and

$| f(x) - l | < \varepsilon_1$, $| g(x) - l | < \varepsilon_1 \ \forall x, \ 0 < | x - a | < \delta$.

$$\begin{aligned} | h(x) - l | &= | h(x) - f(x) + f(x) - l | \\ &\le | h(x) - f(x) | + | f(x) - l | \\ &\le | g(x) - f(x) | + | f(x) - l | \quad \because | h(x) - f(x) | \le g(x) - f(x) | \\ &\le | g(x) - l + l - f(x) | + | f(x) - l | \\ &\le | g(x) - l | + | l - f(x) | + | f(x) - l | \\ &< \varepsilon_1 + \varepsilon_1 + \varepsilon_1 = 3 \cdot \frac{\varepsilon}{3} = \varepsilon \ \forall \ 0 < | x - a | < \delta. \end{aligned}$$

$$\therefore \lim_{x \to a} h(x) = l.$$

Next we shall explain limits at infinity and infinite limits.

Let $f(x) = 1/x$

Let us examine the behaviour of f(x) as x approaches zero from right side. The closer x is to zero, the larger f(x) is. In other words, as $x \to 0+$, f(x) goes on increasing without bound. In this case, we write $\lim_{x \to 0+} \frac{1}{x} = +\infty$ read '∞' as "plus infinity").

Similarly, as $x \to 0-$, f(x) goes on decreasing without bound and we write

$$\lim_{x \to 0-} f(x) = \lim_{x \to 0-} \frac{1}{x} = -\infty$$

(Read '$-\infty$' as "minus infinity").

Here, ∞ is a symbol showing the phenomenon of growing larger and larger without bound. Similarly, $-\infty$ is a symbol showing the phenomenon of decreasing without bound. Thus ∞ and $-\infty$ are not numbers.

Next let us consider $\lim\limits_{x \to \infty} \frac{1}{x}$. As x grows larger and larger, the values of 1/x are closer to zero. Therefore, we write $\lim\limits_{x \to \infty} \frac{1}{x} = 0$.

Also as $x \to -\infty$, $\frac{1}{x} \to 0$ and so we write $\lim\limits_{x \to \infty} \frac{1}{x} = 0$.

However, we shall not attempt formal definitions of the above type of limits.

Illustrative examples on properties of limits

Let us assume the following property of ∞ and $-\infty$.

$$a \pm \infty = \pm \infty$$
$$\infty + a = \infty$$
$$-\infty \pm a = -\infty$$
$$a \cdot \infty = (-a)(-\infty) = \begin{cases} \infty & \text{if } a > 0 \\ -\infty & \text{if } a < 0 \end{cases}$$

$$\frac{a}{\infty} = 0$$

$$\text{For } a \neq 0,\ \frac{\infty}{a} = \begin{cases} \infty & \text{if } a > 0 \\ -\infty & \text{if } a < 0 \end{cases}$$

1. Let $f(x) = \frac{x^2 - 2x}{x}$, $g(x) = \frac{x^2 + 2x}{x}$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{x^2 - 2x}{x} = \lim_{x \to 0} \frac{x(x-2)}{x}$$

$$= \lim_{x \to 0} x - 2 \quad (\because \text{ for } x \neq 0\ \frac{x(x-2)}{x} = x - 2).$$

$$= -2$$

$$\lim_{x \to 0} g(x) = \lim_{x \to 0} \frac{x^2 + 2x}{x} = \lim_{x \to 0} \frac{x(x+2)}{x}$$

$$= \lim_{x \to 0} (x + 2) \quad (\because \text{ for } x \neq 0,\ \frac{x(x+2)}{x} = x + 2).$$

$= 2$

Now,

i) $f(x) + g(x) = \dfrac{x^2 - 2x}{x} + \dfrac{x^2 + 2x}{x} = \dfrac{2x^2}{x}$

$\therefore \lim_{x \to 0} (f(x) + g(x)) = \lim_{x \to 0} \dfrac{2x^2}{x}$

$= \lim_{x \to 0} \dfrac{2x}{1}$ ($\because$ for $x \neq 0$, $\dfrac{2x^2}{x} = 2x$).

$= 2 \times 0 = 0$

Also, $\lim_{x \to 0} f(x) + \lim_{x \to 0} g(x) = -2 + 2 = 0$.

$\therefore \lim_{x \to 0} f(x) + \lim_{x \to 0} g(x) = \lim_{x \to 0} ((x) + g(x))$.

is verified.

ii) $f(x) - g(x) = \dfrac{x^2 - 2x}{x} - \dfrac{x^2 + 2x}{x} = \dfrac{-4x}{x}$

$\therefore \lim_{x \to 0} f(x) - g(x) = \lim_{x \to 0} \dfrac{-4x}{x}$.

$= \lim_{x \to 0} \dfrac{-4}{1}$ ($\because$ for $x \neq 0$, $-\dfrac{4x}{x} = -4$)

$= -4$

But $\lim_{x \to 0} f(x) - \lim_{x \to 0} g(x) = -2 - (2) = -4$.

$\therefore \lim_{x \to 0} (f(x) - g(x)) = \lim_{x \to 0} f(x) - \lim_{x \to 0} g(x)$ is verified.

(iii) $f(x)\, g(x) = \dfrac{x^2 - 2x}{x} \cdot \dfrac{x^2 + 2x}{x}$

$= \dfrac{x^4 - 4x^2}{x^2}$

$\therefore \lim_{x \to 0} (f(x)\, g(x)) = \lim_{x \to 0} \dfrac{x^4 - 4x^2}{x^2}$

$= \lim_{x \to 0} \dfrac{x^2 (x^2 - 4)}{x^2}$

$$= \lim_{x \to 0} \frac{x^2 - 4}{1} \qquad (\because \text{ for } x \neq 0, \ \frac{x^2\ (x^2 - 4)}{x^2} = x^2 - 4$$

$$= 0 - 4 = -4.$$

But $(\lim_{x \to 0} f(x))\ (\lim_{x \to 0} g(x)) = (-2) \times (2) = -4.$

$\therefore \quad \lim_{x \to 0} (f(x) \,.\, g(x)) = \lim_{x \to 0} f(x) \,.\, \lim_{x \to 0} g(x)$ is verified.

(iv) $$\frac{f(x)}{g(x)} = \frac{\dfrac{x^2 - 2x}{x}}{\dfrac{x^2 + 2x}{x}} = \frac{x(x^2 - 2x)}{x\,(x^2 + 2x)} = \frac{x^2\ (x - 2)}{x^2\,(x + 2)}$$

$$\therefore \lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{x^2\ (x - 2)}{x^2\,(x + 2)}$$

$$= \lim_{x \to 0} \frac{x - 2}{x + 2} \qquad (\because \text{ for} \neq 0, \ \frac{x^2\ (x - 2)}{x^2\,(x + 2)} = \frac{x - 2}{x + 2})$$

But $\left| \frac{x - 2}{x + 2} + 1 \right|$

$$= \left| \frac{x - 2 + x + 2}{x + 2} \right| = \frac{2\,|x|}{|x + 2|}$$

If $|x - 0| < 1$, $|x| < 1$ $\qquad \therefore -1 < x < 1$

$\therefore -1 + 2 < x + 2 < 1 + 2 + 1 < x + 2 < 3$

$\therefore \ |x + 2| > 1$ for $0 < |x - 0| < 1$

$$\therefore \left| \frac{x - 2}{x + 2} - (-1) \right| = \left| \frac{2x}{x + 2} \right| = \frac{2\,|x|}{|x + 2|} < 2\,|x| \ \forall\, x, \ 0 < |x - 0| < 1$$

$$= \left| \frac{x - 2 + x + 2}{x + 2} \right|$$

Hence given $\varepsilon > 0$,

let $\delta = \min\{1, \varepsilon/2\}$

Then $0 < | x - 0 | < \delta$ implies $\left| \frac{x-2}{x+2} - (-1) \right| < 2 \cdot \frac{\varepsilon}{2} = \varepsilon$

$\therefore \lim_{x \to 0} \frac{x-2}{x+2} = -1.$

But $\frac{\lim_{x \to 0} f(x)}{\lim_{x \to 0} g(x)} = \frac{-2}{2} = -1$

$\therefore \quad \lim_{x \to 0} \frac{f(x)}{g(x)} = \frac{\lim_{x \to 0} f(x)}{\lim_{x \to 0} g(x)}$ is verified.

Note: We cannot substitute x = 0 in ** because that will mean that we are assuming the quotient property of the limits.

2. Let $f(x) = 3x^2 + 1$, $h(x) = 4x^2 + 1$, $g(x) = 5x^2 + 1$.

Then $f(x) < h(x) < g(x)\ \forall\ x$.

And $\lim_{x \to 0} f(x) = \lim_{x \to 0} (3x^2 + 1) = 3 \times 0 + 1 = 1.$

$\lim_{x \to 0} g(x) = \lim_{x \to 0} (5x^2 + 1) = 5 \times 0 + 1 = 1$

and $\lim_{x \to 0} h(x) = 4 \times 0 + 1 = 1.$

$\therefore \lim_{x \to 0} h(x) = \lim_{x \to 0} f(x) = \lim_{x \to 0} g(x).$

Problem Solving :

1. Find $\lim_{x \to 1} (\sqrt{2}\, x - 6x + 1) = \lim_{x \to 1} \sqrt{2}\, x - \lim_{x \to 1} 6x + \lim_{x \to 1} 1.$

$= \sqrt{2} \cdot \lim_{x \to 1} x - 6 \lim_{x \to 1} 1 + \lim_{x \to 1} 1$

$= \sqrt{2} \times 1 - 6 \times 1 + 1 = \sqrt{2} - 5.$

2. Find $\lim_{x \to 2} \frac{x-2}{x^2 - 4}$

$\lim_{x \to 2} \frac{x-2}{x^2 - 4} = \lim_{x \to 2} \frac{x-2}{(x-2)(x+2)}$

$= \lim_{x \to 2} \frac{1}{x+2}$ $\quad \left(\because \text{ for } x \neq 2, \ \frac{x-2}{(x-2)(x+2)} = \frac{1}{x+2} \right)$

$$= \frac{\lim_{x \to 2} 1}{\lim_{x \to 2} x + 2}$$

$$= \lim_{x \to 2} \frac{1}{x+2}$$

$$= \frac{1}{2+2} = \frac{1}{4}$$

3. Find $\lim_{x \to 2} \frac{x^2 + x + 1}{x^2 + 2x}$

Since $\lim_{x \to 2} (x^2 + 2x) = 4 + 4 = 8 \neq 0$

$$\lim_{x \to 2} \frac{x^2 + x + 1}{x^2 + 2x} = \frac{\lim_{x \to 2} (x^2 + x + 1)}{\lim_{x \to 2} (x^2 + 2x)} = \frac{4 + 2 + 1}{4 + 4} = \frac{7}{8}.$$

4. $\lim_{x \to 0} \left(x - \frac{4}{x}\right) = \lim_{x \to 0} x - \lim_{x \to 0} \frac{4}{x} = 0 - \infty = -\infty.$

5. $\lim_{x \to -4} \left(\frac{2x}{x+4} + \frac{8}{x+4}\right) = \lim_{x \to -4} \frac{2x+8}{x+4}$

$$= \lim_{x \to -4} \frac{2(x+4)}{x+4}$$

$= \lim_{x \to -4} 2$ $\quad$ ($\because$ $\frac{2(x+4)}{x+4} = 2$ for all $x \neq -4$).

$= 2$

6. $\lim_{t \to 1} \frac{t^2 - 2t + 1}{t^3 - 3t^2 + 3t - 1} = \lim_{t \to 1} \frac{(t-1)^2}{(t-1)^3}$

$= \lim_{t \to 1} \frac{1}{t-1}$ $\quad$ ($\because$ for $t \neq 1$, $\frac{(t-1)^2}{(t-1)^3} = \frac{1}{t-1}$)

$= \frac{1}{0} = \infty$, which is not defined. Hence the limit does not exist.

7. $\lim_{t \to 0} \frac{t + \frac{1}{t}}{t - \frac{1}{t}} = \lim_{t \to 0} \frac{(t^2 + 1)/t}{(t^2 - 1)/t}$

$$= \lim_{t \to 0} \frac{(t^2 + 1)/t}{(t^2 - 1)/t}$$

$$= \lim_{t \to 0} \frac{t^2 + 1}{t^2 - 1} \qquad \left(\because \text{ for } t \neq 0, \ \frac{(t^2+1)t}{(t^2-1)t} = \frac{t^2+1}{t^2-1}\right)$$

$$= \frac{\lim_{t \to 0} (t^2 + 1)}{\lim_{t \to 0} (t^2 - 1)}$$

$$= \frac{0+1}{0-1} = -1$$

8. $$\lim_{t \to -1} \frac{t^2 + 6t + 5}{t^2 + 3t + 2} = \lim_{t \to -1} \frac{(t+1)(t+5)}{(t+1)(t+2)}$$

$$= \lim_{t \to -1} \frac{t+5}{t+2} \qquad \left(\because \text{ for } t \neq -1, \ \frac{(t+1)(t+5)}{(t+1)(t+2)} = \frac{t+5}{t+2}\right).$$

9. $$\lim_{x \to 3} \frac{\frac{1}{x^3} - \frac{1}{3^3}}{x - 3}$$

$$= \lim_{x \to 3} \frac{3^3 - x^3}{x^3\, 3^3\, (x-3)} = \lim_{x \to 3} \frac{(3-x)(3^2 + 3x + x^2)}{x^3\, 3^3\, (x-3)}$$

$$= \lim_{x \to 3} - \frac{3^2 + 3x + x^2}{x^3\, 3^3}$$

$$\left(\because \text{ for } x \neq 3, \ \frac{(3-x)(3^2 + 3x + x^2)}{x^3\, 3^3\, (x-3)} = -\frac{3^2 + 3x + x^2}{x^3\, 3^3}\right)$$

$$= -\frac{9+9+9}{3^3 . 3^3} = -\frac{27}{3^3 . 3^3} = -\frac{1}{27}.$$

10. $$\lim_{x \to 0} \frac{\sqrt{a+x} - \sqrt{a-x}}{x} \qquad (a > 0)$$

$$= \lim_{x \to 0} \frac{(\sqrt{a+x} - \sqrt{a-x})(\sqrt{a+x} + \sqrt{a-x})}{x(\sqrt{a+x} + \sqrt{a-x})}$$

$$= \lim_{x \to 0} \frac{(a+x) - (a-x)}{x(\sqrt{a+x} + \sqrt{a-x})}$$

$$= \lim_{x \to 0} \frac{2x}{x(\sqrt{a+x} + \sqrt{a-x})}$$

$$= \lim_{x \to 0} \frac{2}{\sqrt{a+x} + \sqrt{a-x}}$$

$$(\because \text{ for } x \neq 0, \frac{2x}{x(\sqrt{a+x} + \sqrt{a-x})} = \frac{2}{\sqrt{a+x} + \sqrt{a-x}})$$

$$= \frac{2}{\sqrt{a} + \sqrt{a}} = \frac{1}{\sqrt{a}}$$

Exercises for Self Evaluation

Evaluate the following limits.

1. $\lim_{x \to 4} \frac{x^2 - 3x + 4}{x^2 - 2x - 8}$

2. $\lim_{x \to 1} \frac{x - x^2}{(x^2 - 1)(3x + 4)}$

3. $\lim_{x \to -1} \frac{x^3 + 1}{4 - 4x^2}$

4. $\lim_{x \to -2} \frac{(x^2 - x - 6)^2}{x + 2}$

5. $\lim_{t \to 1} \frac{t^2 - 2t + 1}{t^3 - 1}$

6. $\lim_{t \to -1} \frac{1 - |t|}{1 + |t|}$

7. $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$

8. $\lim_{n \leftarrow 0} \frac{e^h - 1}{n} = 1.$

2. CONTINUINTY AND DISCONTINUITY OF FUNCTIONS

2.1. Closely dependent on the limit concept is the concept of continuity. We begin with the assumption that you have some idea of continuity. Our purpose is to lead you from an intuitively concept to an appropriate mathematical definition through a discussion that primarily follows the historical development of continuity in mathematics.

Consider first the functions f(x)=x, and

g(x) $= \frac{|x|}{x}$ for $x \neq 0$. We observe that the graph of f(x) can be drawn without lifting the pencil from the paper, whereas the graph of g(x) cannot be drawn that way. Pencil to be lifted from the paper at x=0 and as such there is an interruption of gap in the drawing.

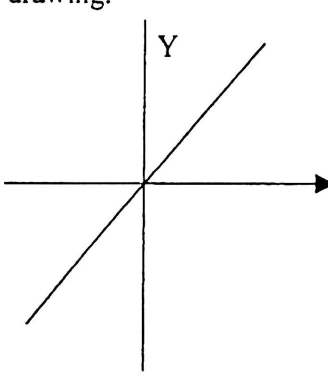


FIG 1f(x)=x

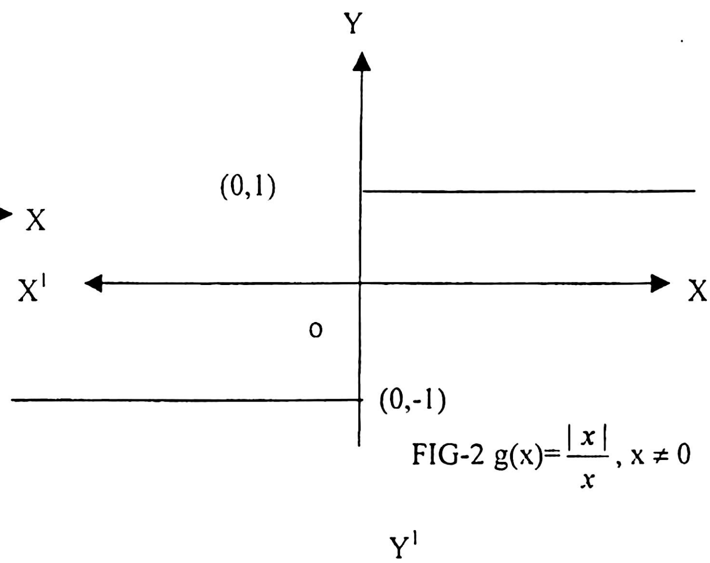


FIG-2 g(x)=$\frac{|x|}{x}$, $x \neq 0$

Intuitively we feel that the graph of f(x) is continuous while the graph of g(x) is not continuous as there is a gap in the graph at x = 0. In fact g(0) is not defined. Even if we define g(0) =0 still the graph of g(x) is not continuous. The reason is that where as $\lim_{x \to 0-} g(x) = -1$, but $\lim_{x \to 0+} g(x) = 1$ and hence $\lim_{x \to 0} g(x)$ does not exist. Hence one requirement for continuity of a function say h(x) at a point 'a' is that $\lim_{x \to a} h(x)$ must exist.

Now consider another function defined as follows:

$$F(x) = \begin{cases} x \text{ if } x \neq 0 \\ 2 \text{ if } x = 0 \end{cases}$$

Here $\lim_{x\to 0-} f(x) = \lim_{x\to 0-} x = 0$ $\quad \lim_{x\to 0+} f(x) = \lim_{x\to 0-} x = 0$ $\quad \therefore \lim_{x\to 0} f(x) = 0$

Hence $\lim_{x\to 0} f(x)$ exists. Now even though $\lim_{x\to 0} f(x)$ exists the graph of f(x) is not continuous at 0. Try to draw the graph of f(x) without lifting the pencil from the paper , you cannot. The reason is that lim $f(x)=0 \neq 2=f(0)$. Only if we alter the definition of f at 0 and define f(0)=0, then f(x) becomes continuous at 0. From these illustrations we conclude that a function f(x) is continuous at a point c if

i) $\lim_{x\to 0} f(x)$ exists, ii) f(c) is defined and

iii) $\lim_{x\to 0} f(x)=f(0)$

Now we are in a position to give the mathematical definition of continuity of function at a point.

Continuity of function at a point:

Definition: Let f(x) be a function defined in an interval containing the point x_1. Then f is said to be continuous at x_1 if I) $f(x_1)$ exists, 2) $\lim_{x\to x_1} f(x)$ exists

3) $\lim_{x\to x_1} f(x)=f(x_1)$

If any one of these three criteria is not met, then f is said to be discontinuous at x_1. Earlier we gave Cauchy definition for limit of a function. Now we shall use this to give another definition of (usually called epsilon delta definition) of continuity.

Condition for continuity of the function:

Definition

Let f(x) be a function defined in an interval containing 'a'. Then f is said to be continuous at a if given $\in >0$ $\exists \delta >0$ such that

* If $(x) - f(a) | < \in \forall$ x with $|x-a|<\delta$.

Note here that in the condition (*) we have not used the expression with $0<|x-a|<\delta$ as in the case of definition of limits. This is obvious since when x=a, f(x)=f(a)

$\therefore | f(x) - f(a) | = | f(a) - f(a) | = |0| = 0 < \epsilon$

2.2 Continuity of a function

Definition: Let $f: A \to R$ (R being set of all real numbers) be a function defined on a subset A of real numbers. Then f is said to be continuous on A if f is continuous at every point of A. Thus f is not continuous on A , if there exists $x_0 \in A$ such that f is not continuous of x_0.

Fr instance consider the identity function f(x)=x defined on any subset of real numbers , then f is continuous on A . Because if a is any point of A then f(a)=a and so f(a) exists. Also

$$\lim_{x \to a} f(x) = \lim_{x \to a} x = a$$

$$\therefore \lim_{x \to a} f(x) = a = f(a)$$

Now, f is continuous at a. But a is an arbitrary point of A. Hence f is continuous at every point of A and so f is continuous on A.

Now we shall prove an important results on limits which is quite useful in deciding whether or not a given function is continuous at a point.

Properties of continuous functions:

By the properties of limits it immediately follows that If f(x) and g(x) are defined on a domain D and if they are continuous at $a \in D$, then

1. $f(x) + g(x)$ is continuous at $a \in D$
2. $f(x) - g(x)$ is continuous at $a \in D$
3. $K.f(x)$ is continuous at $a \in D$
   where k is a constant
4. $f(x) . g(x)$ is continuous at $a \in D$
5. $\frac{f(x)}{g(x)}$ is continuous at $a \in D$ provided $g(a) \neq 0$.

2.3 Discontinuous functions

Definition: A function y=f(x) is said to be discontinuous at x=a if f(x) is not continuous at a.
The discontinuity of f(x) at x=a can occur in any one of the following ways.

1. $\lim_{x \to a} f(x)$ does not exist.

2. $\lim_{x \to a} f(x)$ exists but is not equal to f(a).

Now we shall illustrate these possibilities by means of some examples.

Illustration 1: Let f(x) be a function defined on [0,2] as follows:

$$f(x) = \begin{cases} x, \ \forall x \in [0,1] \\ x+1, \ \forall x \in [1\ 2] \\ \frac{3}{2}, \quad x = 1 \end{cases}$$

As x approaches 1 from the left side

(i.e. $x \to 1$) we have

$\lim_{x \to 1} f(x) = \lim_{x \to 1} x = 1$

As x approaches 1 from right side, (ie $x \to 1+$) we have,

$\lim_{x \to 1+} f(x) = \lim_{x \to 1} x+1 = 2$

Thus $\lim_{x \to 1-} \neq \lim_{x \to 1+} (x)$

$\lim_{x \to 1} f(x)$ does not exist

Such a discontinuity of f(x) at x = 1 is called as ordinary discontinuity or discontinuity of first kind.

f(x) at x=1.

Illustration 2

Let f(x) x= $\begin{cases} x & \forall^2 \in [0,2] \text{ and } x \neq 1 \\ 2 & \text{if } x = 1. \end{cases}$

Then $\lim_{x \to 1+} f(x) = \lim_{x \to 1-} f(x) = \lim_{x \to 1} f(x) = 1$

But f(1) =2.

Hence $\lim_{x \to 1} f(x) \neq f(1)$

Hence f is discontinuous at x=a.

But this discontinuity of f at x=1 can be removed by altering the value of f(1).

Instead of defining f(1)=2, if we define f(1)=1, then f becomes continuous at x=1.

Hence this type of discontinuity of f is called as removable discontinuity at x=1

Illustration 3

Define a function f on [0,1] by,

F(x) = $\begin{cases} +1 & \text{if x is rational} \\ -1 & \text{if x is irrational.} \end{cases}$

Then neither $\lim_{x \to \frac{1}{2}+} f(x)$ nor $\lim_{x \to \frac{1}{2}-} f(x)$ exist.

If neither $\lim_{x \to a+} f(x)$ nor $\lim_{x \to a-} f(x)$ exist then

f(x) is said to have a discontinuity of second kind at x=a.

Hence f has second kind discontinuity at x = $\frac{1}{2}$.

Illustration 4

For instance define a function f(x) on [0,2] as follows:

$$f(x) = \begin{cases} x & \text{if } 0 \le x < 1 \\ 1 & \text{if } x \text{ is irrational and } x \in [1,2] \\ 0 & \text{if } x \text{ is rational and } x \in [1, 2] \end{cases}$$

Then $\lim_{x \to 1-} f(x) = 1$ but $\lim_{x \to 1+} f(x)$ does not exist.

If one of the two limits $\lim_{x \to a+} f(x)$, $\lim_{x \to a-} f(x)$ exists while the other does not exist then the point x=a is called a point of mixed discontinuity for f(x).

Hence f has mixed discontinuity at x = 1 in the above.

x =a.

Illustration 5

Consider $f(x) = \begin{cases} x & \forall\, x < 0 \\ 0 & \text{if } x = 0 \\ \dfrac{1}{x} & \text{if } x > 0 \end{cases}$

Then lim f(x)=∞. If either of the limits $\lim_{x \to a+} f(x)$, $\lim_{x \to a-} f(x)$ is infinite then f(x) is said to have an infinite discontinuity at x=a. Therefore, f has an infinite discontinuity at x=0.

Problem solving:

1. Let $f(x) = \begin{cases} \dfrac{2x^4 - 6x^3 + x^2 + 3}{x-1} & x \neq 1 \\ 0 & \text{if } x=1 \end{cases}$

Is, f continuous at x= 1? If not what type of discontinuity is there at x=1?

$$\lim_{x \to 1} \frac{2x^4 - 6x^3 + x^2 + 3}{x - 1}$$

$$= \lim_{x \to 1} \frac{2x^4 - 2x^3 - 4x^3 + 4x^2 - 3x^2 + 3x - 3x + 3}{x - 1}$$

$$= \lim_{x \to 1} \frac{2x^3(x-1) - 4x^2(x-1) - 3x(x-1) - 3(x-1)}{x - 1}$$

$$= \lim_{x \to 1} \frac{(x-1)(2x^3 - 4x^2 - 3x - 3)}{x - 1}$$

$$= \lim_{x \to 1} \quad 2x^3 - 4x^2 - 3x - 3 \quad (\because x \neq 1))$$

$= 2 - 4 - 3 - 3$

$= -8$

$\therefore \lim_{x \to 1} f(x)$ exists and it is equal to –8.

But $\lim_{x \to 1} f(x) \neq f(1)$, because $f(1) = 0$

$\therefore$ f(x) is not continuous at x=1

Discontinuity is removable discontinuity at x=1

2. Let $f(x) = \dfrac{x}{x^2 - 1}$

Find the values of x at which f(x) is continuous.

f(x) is not defined at $x = \pm 1$. Then f(x) is discontinuous at x=1 and x=-1.

At x=a, $a \neq \pm 1$,

$$\lim_{x \to a} f(x) = \lim_{x \to a} \frac{x}{x^2 - 1} = \frac{a}{a^2 - 1} = f(a)$$

$\therefore$ f(x) is continuous at all real values a with $a \neq \pm 1$.

3. Let $f(x) = \begin{cases} \dfrac{x - |x|}{x} & for\ x \neq 0 \\ 1 & \text{for } x=0 \end{cases}$

Is f(x) continuous at x=0?

$$\lim_{x \to 0-} f(x) = \lim_{x \to 0-} \frac{x - (-x)}{x}$$

$$= \lim_{x \to 0-} \frac{2x}{x} = \lim_{x \to 0} 2 = 2$$

$$\lim_{x \to 0+} f(x) = \lim_{x \to 0+} \frac{x - x}{x}$$

$$= \lim_{x \to 0+} \frac{0}{x} = \lim_{x \to 0+} 0 = 0$$

$$\therefore \lim_{x \to 0+} f(x) \neq \lim_{x \to 0-} f(x) \qquad \left(\therefore for\ x \neq 0, \frac{0}{x} = 0\right)$$

$\therefore$ Function is discontinuous at x=0 Discontinuity is of first kind.

4. Find the points of discontinuity of the function.

$$F(x) = \begin{cases} \dfrac{x}{(x-2)(x-4)} & x \neq 2,4 \\ 1 & \text{If } x=2 \text{ or} 4 \end{cases}$$

$$\lim_{x \to 2+} \frac{x}{(x-2)(x-4)} = -\infty \qquad \lim_{x \to 2-} \frac{x}{(x-2)(x-4)} = \infty$$

$$\lim_{x \to 4+} \frac{x}{(x-2)(x-4)} = \infty \qquad \lim_{x \to 4^-} \frac{x}{(x-2)(x-4)} = -\infty$$

$\therefore$ f(x) has infinite discontinuity at x=2 and x=4.

5. Examine the continuity of f(x) at x=1 where

$$f(x) = \begin{cases} \dfrac{(x-1)^2}{|x-1|}, x \neq 0 \\ 0 \quad \text{if } x=0 \end{cases}$$

Now that $|x-1| = \begin{cases} x-1 & \text{is } x>1 \\ -(x-1) & \text{is } x<1 \end{cases}$

$$\lim_{x \to 1-} f(x) = \lim_{x \to 1-} \frac{(x-1)^2}{|x-1|} = \lim_{x \to 1-} \frac{(x-1)^2}{-(x-1)} = \lim_{x \to 1-} -(x-1)$$

$(\because x - 1 \neq 0)$

$$\lim_{x \to 1+} f(x) = \lim_{x \to 1+} \frac{(x-1)^2}{|x-1|} = \lim_{x \to 1+} \frac{(x-1)^2}{(x-1)} = \lim_{x \to 1+} (x-1) = 0$$

$(\because x - 1 \neq 0)$

$$\therefore \lim_{x \to 1+} f(x) = \lim_{x \to 1-} f(x)$$

$$\therefore \lim_{x \to 1+} f(x) = 0 = f(0)$$

$\therefore$ f(x) is continuous at x=1.

6. Examine the continuity of $f(x)=|x^2-1|$ at x=1

Note that

$$|x^2-1| = \begin{cases} x^2-1 & \text{if } x \geq 1 \\ 1-x^2 & \text{if } x<1 \end{cases}$$

$$\lim_{x \to 1-} f(x) = \lim_{x \to 1-} (1-x^2) = 0$$

$$\lim_{x \to 1+} f(x) = \lim_{x \to 1+} x^2-1=0$$

$$\therefore \lim_{x \to 1+} f(x) = \lim_{x \to 1-} f(x)$$

$\therefore \lim_{x \to 1} f(x) = 0 = f(1)$

$\therefore$ f(x) is continuous at x=1.

7. Examine the continuity of $f(x) = \dfrac{x-1}{|x-1|}$ at x=1

Note that $|x-1| = \begin{cases} x-1 & \text{is } x \geq 1 \\ 1-x & \text{is } x<1 \end{cases}$

$\therefore \lim_{x \to 1-} f(x) = \lim_{x \to 1-} \dfrac{x-1}{1-x} = -1 (\because x-1 \neq 0)$

$\lim_{x \to 1+} f(x) = \lim_{x \to 1+} \dfrac{x-1}{x-1} = 1 \quad (\because x-1 \neq 0)$

$\because \lim_{x \to 1+} f(x) \neq \lim_{x \to 1-} f(x)$

$\because$ The function is not continuous at x=1.
The discontinuity is of first kind.

8. Discuss the continuity of $f(x) = \begin{cases} \dfrac{\sqrt{x-3}}{|x-3|} & \text{if } x \neq 3 \\ 1 & \text{if } x=3 \end{cases}$

Note that

$|x-3| = \begin{cases} x-3 & \text{if } x \geq 3 \\ 3-x & \text{if } x<3 \end{cases}$

$\therefore \lim_{x \to 3-} f(x) = \lim_{x \to 3-} \dfrac{\sqrt{x-3}}{3-x} = \lim_{x \to 3-} \dfrac{-1}{\sqrt{3-x}} = -\infty$

$\lim_{x \to 3+} f(x) = \lim_{x \to 3+} \dfrac{\sqrt{x-3}}{x-3} = \lim_{x \to 3+} \dfrac{1}{\sqrt{x-3}} = +\infty$

$\therefore \lim_{x \to 3-} f(x) \neq \lim_{x \to 3+} f(x)$

$\therefore$ f(x) if not continuous at x=3 and discontinuity is infinite discontinuity of first kind.

9. If f (x) is continuous at x=c, then show that their exist δ >0 such that f is bounded on (c-δ $c+\delta$)

Solution:

Since f(x) is continuous at x=c, given $\varepsilon = 1 > 0$,
$\exists\ \delta > 0$ such that
|f(x)-f©<1 for all |x-c|<δ

| f(x) - f(c) | < 1
i.e., f(c)-1 < f(x) <f(c)+1 for all x $\in$ (c-δ $c+\delta$)

$\therefore$ f(x) is bounded on (c-δ $c+\delta$)

10. Let f(x) = $\begin{cases} 2x+1 & \forall\ x<1 \\ 3 & \text{if } x=1 \\ x+2 & \forall\ x>1 \end{cases}$

Examine the continuity of f(x) at x=1

$$\lim_{x \to 1-} f(x) = \lim_{x \to 1-} 2x+1 = 3$$

$$\lim_{x \to 1+} f(x) = \lim_{x \to 1+} x+2 = 3$$

$$\therefore \lim_{x \to 1-} f(x) = \lim_{x \to 1-} f(x)$$

$\therefore \lim_{x \to 1}$ f(x) exists and $\lim_{x \to 1} f(1) = 3 = f(1)$
$\therefore$ f(x) is continuous at x=1

Exercises for self Evaluation

Determine whether or not the given function is continuous at the indicated point. If not continuous name the type of discontinuity.

(i) f(x) = x^3-5x+1 at x =2

(ii) g(x) = $\sqrt{(x-1)^2+5}$ at x=1

(iii) $f(x) = \begin{cases} x^2+4 & x<2 \\ x^3 & x \geq 2 \end{cases}$ at x=2

(iv) $f(x) = \begin{cases} x^2+4 & , x<2 \\ 5 & x=2 \\ x^3 & x>2 \end{cases}$ at x=2

(v) $g(x) = \begin{cases} \dfrac{x^2-1}{x+1}, & x \neq -1 \\ -2 & x=-1 \end{cases}$

(vi) $f(x) = \begin{cases} \dfrac{1}{x+1} & x \neq -1 \\ 0 & x = -1 \end{cases}$ at x=-1

(vii $g(x) = \begin{cases} -x^2 & x<0 \\ 1-\sqrt{x} & x \geq 0 \end{cases}$ at x=0

(viii) $f(x) = \begin{cases} 1 & x \leq -2 \\ \dfrac{1}{2}x & -2<x<4 \\ \sqrt{x} & x \geq 4 \end{cases}$

Differentiation

Two main motivating factors for the invention of Calculus were the problems of (i) finding the tangent line to a curve at any given point and (ii) finding the area under a given curve. The tools developed in attempting to find a solution to these two problems led to the invention of derivatives and integrals by Newton and Leibniz independently almost at the same time. Though invented for finding solution to the above problems, both differentiation and integration have found applications to an enormous number of different types of problems in diverse academic fields. In the present day context, calculus has found applications in building abstract models for the study of the ecology of populations, management practices, economics and various other fields.

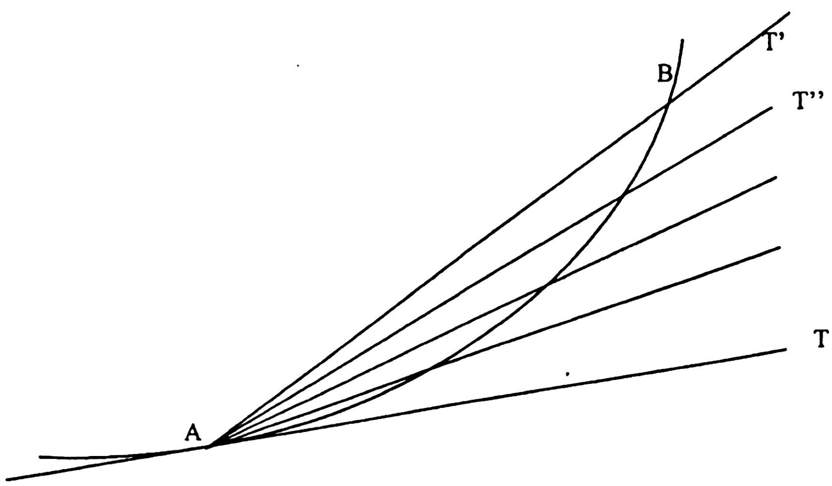
Gradient of a curve :

The gradient of a curve, which is a measure of its slope, changes continually as the point varies along the curve.

The gradient of a curve at any point is defined as the slope of the tangent to the curve at that point.

As drawing of tangent to a curve at a point is not always an easy task (nor will it be accurate) so is finding its slope and hence the gradient of the curve at the given point. Hence a method has to be found to find the gradient of the curve at a given point. Once the gradient is found, tangent can be drawn.

So, consider first the problem of finding the gradient of a curve at a given point A. If B is another point on the curve (not too far from A), then the slope of the chord AB gives us an approximate value for the slope of the tangent at A. The closer B is to A, the better is the approximation. In other words, as $B \to A$, slope of chord $AB \to$ slope of the tangent at A.



Let us now consider an example where we can use this definition to find the gradient of a curve at a particular point of the curve.

For this purpose, we introduce the following symbols. A variable quantity, prefixed by δ, means a small increase in that quantity,

δx is a small increase in x,

δy is a small increase in y

Here δ is only a prefix and it cannot be treated as a factor.

Now consider the curve $y = x(2x - 1)$ and the problem of finding gradient at the point (1,1) on the curve. Let A be the point (1,1). Let B be a point on the curve very close to A. Then x coordinate of B is $1 + \delta x$, where δx is very small.

y coordinate of B $= (1 + \delta x)\,[\,2\,(1 + \delta x) - 1\,]$

$$= (1 + \delta x)\,(2\delta\,x + 1)$$

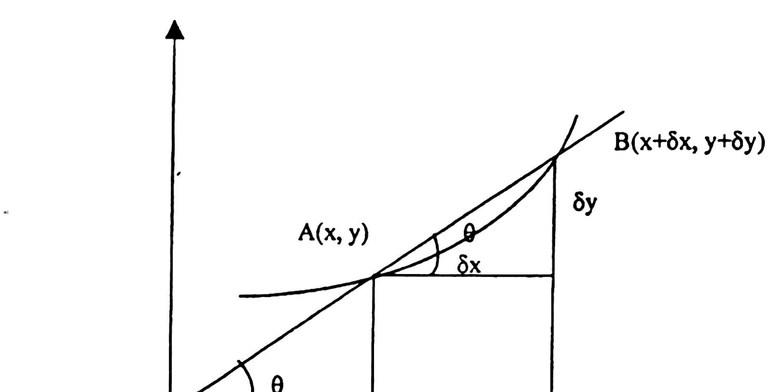


Fig.2

Slope of AB = $\tan\theta = \frac{BM}{AM} = \frac{\delta y}{\delta x} = \frac{change\ in\ y}{change\ in\ x}$

$= \frac{(1 + \delta x)(2\delta x+1) - 1}{(1 + \delta x) - 1}$

$= \frac{2(\delta x)^2 + 3\delta x}{\delta x}$

$= 2\delta x + 3$

As B approaches A, $\delta x \to 0$.

Hence gradient of the curve at A = $\lim_{B \to A}$ [slope of AB]

$= \lim_{\delta x \to 0} [2\delta x + 3]$

$= 3$

Now we found that the gradient of the curve y = x (2x – 1) is 3 at the point (1,1) on the curve. We will now derive a function for the gradient at any point on the curve. Then we can find the gradient at a particular point by substitution into this derived function. Instead of taking a fixed point on the curve, we shall take A as any point (x,y) on the curve. Let B be another point on the curve whose x coordinate is

$x + \delta x$.

Then B is the point $(x + \delta x,\ [x + \delta x] \bullet [2x + 2\delta x - 1]$

The slope of chord AB = $\frac{(x + \delta x)(2x+2\delta x-1) - x(2x-1)}{x}$.

$= \frac{2x^2 + 4x\delta x + 2(\delta x)^2 - \delta x - x - 2x^2 + x}{\delta x}$

$= \frac{4x\delta x - \delta x + 2(-\delta x)^2}{dx}$

$= [4x - 1 + 2\delta x]$

Then the gradient at any point A on the curve = $\lim_{B \to A} \{ slope\ of\ AB \}$

$\lim_{\delta x \to 0} 4x - 1 + 2\delta x$

$= 4x - 1$

So the function 4x – 1 gives the gradient at any point on the curve y = x(2x – 1).

We can now find the gradient of the curve at a particular point on $y = x(2x - 1)$ by substituting the x coordinate of that point into the function $4x - 1$. Thus the gradient of the curve at $x = 1$ is $4.1 - 1 = 3$ which we obtained earlier.

The function $4x - 1$ is called the gradient function of $y = x(2x - 1)$ and the process of deriving is called differentiation with respect to x. Since $4x - 1$ was derived from the function $x(2x - 1)$ it is called the derivative of $x(2x - 1)$. Symbolically, we write, $\frac{d}{dx}[x(2x-1)] = 4x - 1$ where $\frac{d}{dx}$ stands for "derivative w.r.t. x of". We also write $\frac{dy}{dx} = 4x - 1$. Sometimes, we call $\frac{dy}{dx}$ as "differential coefficient of y w.r.t. x". The above method of finding derivatives is called as "finding derivatives from first principles".

In general, for any curve $y = f(x)$,

Gradient of $y = f(x)$ at a point (x,y)

$= \lim_{B \to A}$ slope of AB

$= \lim_{\delta x \to 0} \frac{BM}{AM}$ (since as $B \to A$, $x + \delta x \to x$, hence $\delta x \to 0$ and conversely).

$= \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$

Thus limiting value of $\frac{\delta y}{\delta x}$ is called the derivative of y w.r.t. x and is denoted by $\frac{dy}{dx}$ or $f'(x)$.

Examples :

1. Let $y = x^2$. Hence $f(x) = x^2$.

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{(x+\delta x)^2 - x^2}{\delta x} = \lim_{\delta x \to 0} \frac{x^2 + 2x.\delta x + \delta x^2 - x^2}{\delta x}.$$

$$= \lim_{\delta x \to 0} (2x + \delta x) \qquad (\because \delta x \neq 0)$$

$= 2x$

2. Let $y = \frac{1}{x^2}$

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\frac{1}{(x+\delta x)^2} - \frac{1}{x^2}}{\delta x} = \lim_{\delta x \to 0} \frac{x^2 - (x + \delta x)^2}{x^2 (x+\delta x)^2 . \delta x}$$

$$= \lim_{\delta x \to 0} \frac{x^2 - x^2 - 2x . \delta x - (\delta x)^2}{x^2 (x+\delta x)^2 \delta x} = \lim_{\delta x \to 0} \frac{-2x - (\delta x)}{x^2 (x + 8x)^2} \quad (\because \delta x \neq 0)$$

$$= \frac{-2x}{x^4} = \frac{-2}{x^3}.$$

3. Let $y = x^3 - 2x + 1$

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{(x+\delta x)^3 - 2(x+\delta x) + 1 - x^3 + 2x - 1}{\delta x}$$

$=$

$$\lim_{\delta x \to 0} \frac{x^3 + 3x^2 . \delta x + 3x(\delta x)^2 + (\delta x)^3 - 2x - 2\delta x + 1 - x^3 + 2x - 1}{\delta x}$$

$$= \lim_{\delta x \to 0} 3x^2 + 3x\,\delta x - 2 + (\delta x)^2 \quad (\because \delta x \neq 0)$$

$$= 3x^2 - 2$$

4. Let $y = \sin 2x$

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\sin(2(x+\delta x)) - \sin 2x}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\sin(2x + 2\delta x) - \sin 2x}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{2\cos(2x+\delta x)\sin \delta x}{\delta x}$$

$$= 2\cos 2x \qquad \because \lim_{\delta x \to 0} \frac{\sin \delta x}{\delta x} = 1$$

Equations of Tangents and Normals :

Now that we know how to find the gradient of a curve at a given point on the curve, we can find the equation of the tangent or normal to the curve at that point.

Illustration 1 :

Find the equation of the tangent to the curve

$y = x^2 - 3x + 2$ at the point where it cuts the y-axis

$y = x^2 - 3x + 2$ cuts the y-axis where x = 0 and y = 2.

The slope of the tangent at (0,2) = the value $\frac{dy}{dx}$ when x = 0.

$$= \left[\frac{d}{dx} \left[x^2 - 3x+2\right]\right]_{x=0} = [2x - 3]_{x=0} = -3$$

Thus the tangent is a line with slope –3 and passing through (0,2). So its equation is y-2 = -3 (x – 0).

Hence the desired equation is y = -3x + 2.

Illustration 2 :

Find the equation of the normal to the curve $y = x^2 + 3x - 2$ at the point where the curve cuts the y-axis.

As shown in the illustration 1, the slope of the tangent to the curve at (0,2) is –3.

Hence the slope of normal to the curve at (0.2) is 1/3. Hence the equation of normal to the curve at (0,2) is given by $y - 2 = \frac{1}{3}x$ or $3y = x + 6$.

Properties of Differentiable Functions :

By the properties of limits it follows that :

If f(x) and g(x) are defined in a domain D and if f(x) and g(x) are differentiable at $a \in D$, then

1. f(x) + g(x) is differentiable at $a \in D$.
2. f(x) – g(x) is differentiable at $a \in D$.
3. k . f(x) is differentiable at $a \in D$ where k is a constant.
4. f(x). g(x) is differentiable at $a \in D$.

5. $\dfrac{f(x)}{g(x)}$ is differentiable at a $\in$ D provided g(a) $\neq$ 0.

Problem Solving :

1. Find the derivatives by first principle :

i) $y = x^2 + 3\cos x$

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{(x+\delta x)^2 + 3\cos(x+\delta x) - x^2 - 3\cos x}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{x^2 + 2x\,\delta x + (\delta x)^2 + 3(\cos(x+\delta x) - \cos x) - x^2}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{2x\,\delta x + (\delta x)^2 + 3\left[2\sin\frac{-\delta x}{2} \cdot \sin\left(x + \frac{\delta x}{2}\right)\right]}{\delta x}$$

$$= \lim_{\delta x \to 0} 2x + \delta x - 3\left(\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}\right)\sin\left(x + \frac{\delta x}{2}\right)$$

$$= 2x - 3\sin x \;\left(\because \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1\right)$$

ii) $y = \sqrt{x}$

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\sqrt{x+\delta x} - \sqrt{x}}{\delta x} = \lim_{\delta x \to 0} \frac{x + \delta x - x}{\sqrt{x+\delta x} + \sqrt{x}} \cdot \frac{1}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\delta x}{\sqrt{x+\delta x} + \sqrt{x}} \cdot \frac{1}{\delta x} \;(\because \delta x \neq 0)$$

$$= \frac{1}{2\sqrt{x}}$$

iii) $y = \sin\sqrt{x}$

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\sin\sqrt{x+\delta x} - \sin\sqrt{x}}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\sin\sqrt{x+\delta x} - \sin\sqrt{x}}{\sqrt{x+\delta x} - \sqrt{x}} \cdot \frac{\sqrt{x+\delta x} - \sqrt{x}}{\delta x}$$

$$= \lim_{\delta x \to 0} 2\cos\left(\frac{\sqrt{x+\delta x} + \sqrt{x}}{2}\right) \cdot \left[\frac{\sin\left(\sqrt{x+\frac{\delta x}{2}} - \sqrt{x}\right)}{2 \times \frac{\sqrt{x+\delta x} - \sqrt{x}}{2}}\right] \bullet \frac{x+\delta x - x}{\sqrt{x+\delta x}+\sqrt{x}} \bullet \frac{1}{\delta x}$$

$$= \cos\sqrt{x} \times \frac{1}{2\sqrt{x}}.$$

iv) $y = e^{\sqrt{x}}$

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{e^{\sqrt{x+\delta x}} - e^{\sqrt{x}}}{\delta x}.$$

$$0 = \lim_{\delta x \to 0} e^{\sqrt{x}} \left[\frac{e^{\sqrt{x+\delta x} - \sqrt{x}} - 1}{\delta x}\right]$$

$$= \lim_{\delta x \to 0} e^{\sqrt{x}} \left[\frac{e^{\sqrt{x+\delta x} - \sqrt{x}} - 1}{\sqrt{x+\delta x} - \sqrt{x}} \times \frac{\sqrt{x+\delta x} - \sqrt{x}}{\delta x}\right]$$

$$= e^{\sqrt{x}} \lim_{\delta x \to 0} \frac{e^{\sqrt{x+\delta x} - \sqrt{x}} - 1}{\sqrt{x+\delta x} - \sqrt{x}} \times \frac{x+\delta x - x}{\sqrt{x+\delta x} + \sqrt{x}} \times \frac{1}{\delta x}$$

$$= e^{\sqrt{x}} \lim_{\delta x \to 0} \frac{e^{\sqrt{x+\delta x} - \sqrt{x}} - 1}{\sqrt{x+\delta x} - \sqrt{x}} \times \frac{\delta x}{\sqrt{x+\delta x} + \sqrt{x}} \times \frac{1}{\delta x} \quad (\because \delta x \neq 0)$$

$$= e^{\sqrt{x}} \cdot 1 \cdot \frac{1}{2\sqrt{x}} \quad (\because \lim_{\theta \to 0} \frac{e^{\theta} - 1}{\theta} = 1$$

$$= \frac{1}{2\sqrt{x}} \cdot e^{\sqrt{x}}$$

2. Find the equation of tangent to the curve $y = x^2 + 5x - 2$ at the point where this curve cuts the line $x = 4$.

To know the equation of the tangent to the curve, we should know the slope of the tangent and the point through which it passes. It is given that the tangent is at the point where $x = 4$ cuts the given curve. But when $x = 4$, $y = 16 + 20 - 2 = 34$. Hence the line $x = 4$ cuts the given curve at (4, 34). Also, slope of the tangent at (4, 34) to the given curve $= \frac{dy}{dx}\Big|_{(4,\,34)} = (2x+5)\Big|_{(4,\,34)}$

$$= 8 + 5 = 13$$

$\therefore$ the equation of the tangent line to the given curve at (4, 34) is

$$y - 34 = 13\,(x - 4)$$

i.e. $y - 34 = 13x - 52$

or $y - 13x + 18 = 0$

3. Find the equation of the normal to the curve $y = x^2 - 5x + 6$ at the points where this curve cuts the x-axis.

As in the previous problem to know the equation of any line, it is sufficient if we know its slope and a point through which it passes. X-axis cuts the given curve $y = x^2 - 5x + 3$ at two points viz.,

When $y = 0$ (on the x-axis, $y = 0$), $x^2 - 5x + 6 = 0$

$\therefore$ $(x - 2)\ (x - 3) = 0$ $\therefore$ $x = 2$ or 3

$\therefore$ The two points are (2, 0), (3, 0).

Since the tangent is perpendicular to the normal at a given point to a curve

Slope of the normal $= \dfrac{-1}{\frac{dy}{dx}}$ at that point.

Now for the given curve $\frac{dy}{dx} = 2x - 5$

$$\therefore \left(\frac{dy}{dx}\right)_{(2,\,0)} = 4 - 5 = -1$$

$$\left(\frac{dy}{dx}\right)_{(3,\,0)} = 6 - 5 = 1$$

$\therefore$ The equation of normals at the two points are

$y - 0 = -\frac{1}{-1} \cdot (x - 2)$ i.e. $y = x - 2$ i.e. $x - y - 2 = 0.$

and $y - 0 = -\frac{1}{1} \cdot (x - 3)$ i.e. $y = —\ (x - 3)$ i.e. $x + y - 3 = 0$

4. Find the coordinates of the point on $y = x^2$ at which the gradient is 2. Hence find the equation of the tangent to $y = x^2$ whose slope is 2.

We know that the gradient of a curve at a point is the slope of the tangent to it at that point which in turn is $\frac{dy}{dx}$ at that point P.

Given gradient is 2.

$\therefore \left(\frac{dy}{dx}\right)_P = 2$

$\therefore\ 2x = 2 \qquad \therefore\ x = 1$

Since the equation of the curve is $y = x^2$, $y = 1$.

Here the point is (1,1).

$\therefore$ The equation to the tangent is

$y - 1 = 2(x - 1)$

i.e. $y - 2x + 1 = 0$

5. Find the value of x for which $y = 2x + k$ is a normal to $y = 2x^2 - 3$.

We know that the slope of the normal to a curve $y = f(x)$ is $-\frac{1}{\frac{dy}{dx}}$.

Here slope of the normal is 2.

$\therefore \quad -\frac{1}{\frac{dy}{dx}} = 2 \qquad \therefore \frac{dy}{dx} = -\frac{1}{2}$

Let $\frac{dy}{dx} = 4x \qquad \therefore\ 4x = -\frac{1}{2}.$

$\therefore\ x = -\frac{1}{8}$

$\therefore\ y = 2 \times \left(-\frac{1}{8}\right)^2 - 3$

$= +\frac{1}{32} - 3 = -\frac{95}{32}$

$\therefore$ The point is $\left(-\frac{1}{8}, -\frac{95}{32}\right)$.

$\therefore$ The equation of normal is

$$\left(y + \frac{95}{32}\right) = 2\left(x + \frac{1}{8}\right)$$

i.e. $y + \frac{95}{32} = 2x + \frac{1}{4}$

$\therefore$ $y = 2x + \left(\frac{1}{4} - \frac{95}{32}\right) = 2x + k$, where $k = \frac{-87}{32}$

$\therefore$ $k = \frac{-87}{32}$.

6. Find the equation of the tangent to y = (x - 5) (2x + 1) which is parallel to x-axis.

If the tangent is parallel to x-axis, its slope must be zero. Hence we have to find the point on the curve where gradient is zero.

i.e. $\frac{dy}{dx} = 0$

But $y = (x - 5)(2x + 1)$

$= 2x^2 - 9x - 5$

$\therefore$ $\frac{dy}{dx} = 4x - 9$

$\therefore$ $\frac{dy}{dx} = 0 \Rightarrow 4x - 9 = 0 \quad \therefore x = \frac{9}{4}$

$\therefore$ $y = 2 \cdot \frac{81}{16} - 9 \times \frac{9}{4} - 5$

$= \frac{81}{8} - \frac{81}{4} - 5 = \frac{-81}{8} - 5$

$= \frac{-121}{8}$

The point is y $\left(\frac{9}{4}, \frac{-121}{8}\right)$.

$\therefore$ Equation of the tangent is

$$y + \frac{121}{8} = 0 \times \left(x - \frac{9}{4}\right)$$

$$\therefore\ y + \frac{121}{8} = 0$$

or $8y + 121 = 0$

7. Differentiate cos (x + 2) by first principle.

$$y = \cos(x+2)$$

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\cos(x + 2 + \delta x) - \cos(x+2)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{2.\sin\left(-\frac{\delta x}{2}\right).\sin\left(x+2+\frac{\delta x}{2}\right)}{\delta x}$$

$$= -\lim_{\delta x \to 0} \frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}} \ .\ \sin\left(x+2+\frac{\delta x}{2}\right)$$

$$= -\sin(x+2) \quad \left(\because \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1\right).$$

8. Differentiate e^{mx+n} by first principle.

$$y = e^{mx+n}$$

$$\therefore \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{e^{m(x+\delta x)+n} - e^{mx+n}}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{e^{mx+n}.e^{m\delta x} - e^{mx+n}}{\delta x}$$

$$= e^{mx+n} \lim_{\delta x \to 0} \frac{e^{m\delta x} - 1}{\delta x}$$

$$= e^{mx+n} \lim_{\delta x \to 0} .\ m\ .\ \frac{e^{m\delta x} - 1}{m\,\delta x}$$

$$= e^{mx+n}.\ m \lim_{\delta x \to 0} \frac{e^{m\delta x} - 1}{m\,\delta x}$$

$$= e^{mx+n}.\ m\ .\ 1 \qquad \because \lim_{\theta \to 0} \frac{e^{\theta} - 1}{\theta} = 1$$

$$= m\ .\ e^{mx+n}$$

Continuity and Differentiability of Hyperbolic Functions

By definition

$$\text{Sinh } x = \frac{e^x - e^{-x}}{2} \text{ and } \cosh x = \frac{e^x + e^{-x}}{2}$$

Since e^x and e^{-x} are continuous and differentiable at all $x \in \Re$, by the properties of continuity and differentiability, it follows that $e^x - e^{-x}$ and $e^x + e^{-x}$ are continuous and differentiable at all $a \in \Re$ and hence $\frac{1}{2}(e^x - e^{-x})$ and $\frac{1}{2}(e^x + e^{-x})$ and hence sinh x and cosh x are continuous and differentiable at all $a \in \Re$.

Again by the properties of continuity and differentiability it follows that

$$\text{Tanh } x = \frac{\sinh x}{\cosh x},\ \text{cosech } x = \frac{1}{\sinh x},\ \text{sech } x = \frac{1}{\cosh x}$$

$\coth x = \dfrac{\cosh x}{\sinh x}$ are also continuous and derivable at all points $a \in \Re$ except for the points where they are not defined.

Differentiability and Continuity

If a function $y = f(x)$ is differentiable (i.e. its derivative exists at a point) $a \in D$, then it is also continuous at a.

For, since f(x) is differentiable at x = a

$$\lim_{\delta x \to 0} \frac{f(a+\delta x) - f(a)}{\delta x} \text{ exists. It is denoted by } f'(a).$$

$$\therefore \lim_{\delta x \to 0} \left[\frac{f(a+\delta x) - f(a)}{\delta x} - f'(a)\right] = 0$$

Now, $\lim_{\delta x \to 0} [\, f(a + \delta x) - f(a)\,]$

$$= \lim_{\delta x \to 0} \left[\frac{f(a+\delta x) - f(a)}{\delta x} . \delta x\right]$$

$$= \lim_{\delta x \to 0} \left[\frac{f(a+\delta x) - f(a)}{\delta x}\right] . \lim_{\delta x \to 0} \delta x$$

$$= f'(a) . 0 = 0$$

$$\therefore \lim_{\delta x \to 0} f(a + \delta x) = f(a)$$

$\therefore \lim_{\delta x \to 0} f(x) = f(a)$

$\therefore$ f(x) is continuous at x = a.

However, the converse is not true.

i.e. A continuous function need not be differentiable.

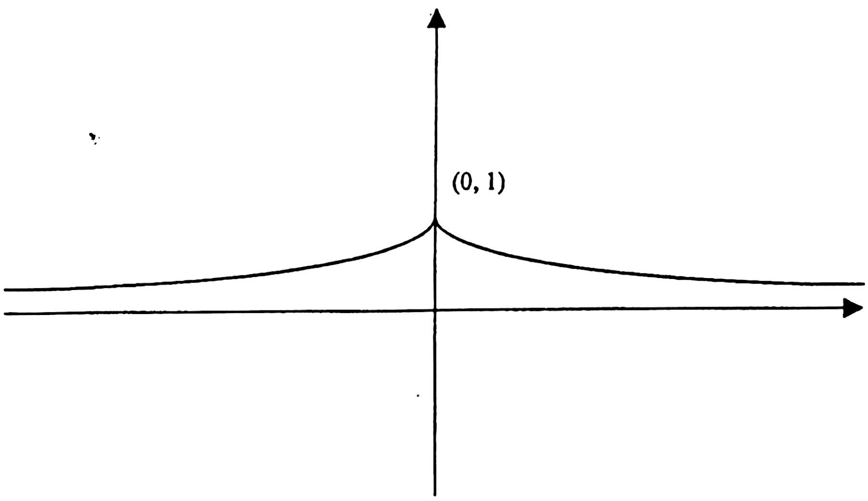
i.e. There exists functions which are continuous but not differentiable.

Usual example given in the textbooks for this is the modulus functions | x |.

We can also give the following examples.

1. Let $f(x) = \begin{cases} e^{x} & if\ x<0 \\ e^{-x} & if\ x \geq 0 \end{cases}$

Then the graph of f(x) looks like



Then $\lim_{\delta x \to 0-} \frac{f(0+\delta x)-f(0)}{\delta x} = \lim_{\delta x \to 0-} \frac{e^{0+\delta x} - e^{0}}{\delta x}$

$= \lim_{\delta x \to 0} \frac{e^{\delta x} - 1}{\delta x} = 1$

and $\lim_{\delta x \to 0+} \frac{f(0+\delta x)-f(0)}{\delta x} = \lim_{\delta x \to 0+} \frac{e^{-\delta x} - e^{0}}{\delta x}$

$= \lim_{\delta x \to 0} \frac{e^{-\delta x} - 1}{\delta x}$

$= \lim_{\delta x \to 0} \frac{1-e^{\delta x}}{e^{\delta x} . \delta x}$

$= - \lim_{\delta x \to 0} \frac{e^{\delta x} - 1}{\delta x} \times \frac{1}{e^{\delta x}}$

$$= -1 \times \frac{1}{1} = -1$$

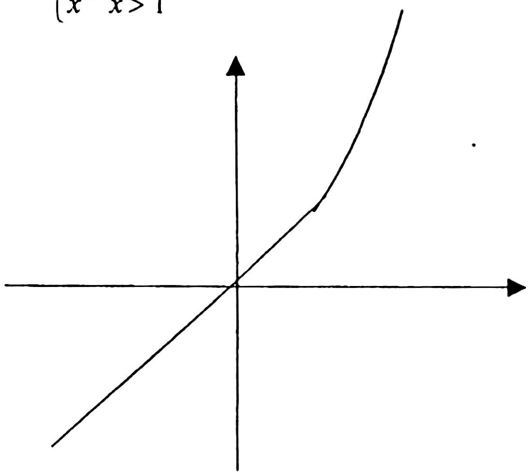
$\therefore$ The f(x) has no derivative at x = 0. But f(x) is continuous at x = 0, since

$$\lim_{x \to 0+} f(x) = \lim_{x \to 0+} e^{-x} = e^{-0} = 1$$

$$\lim_{\delta x \to 0-} f(x) = \lim_{x \to 0-} e^{x} = e^{0} = 1$$

$$\therefore \quad \lim_{x \to 0} f(x) = 1 = f(0)$$

2. Let $y = \begin{cases} x & x \leq 1 \\ x^2 & x > 1 \end{cases}$



$$\lim_{x \to 1-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1-} \frac{x - 1}{x - 1} = 1 \quad (\because x - 1 \neq 0)$$

$$\lim_{x \to 1+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1+} \frac{x^2 - 1}{x - 1}$$

$$= \lim_{x \to 1+} \frac{(x+1)(x-1)}{x-1}$$

$$= \lim_{x \to 1+} x + 1 \ (\because x - 1 \neq 0)$$

$$= 2$$

$$\therefore \lim_{x \to 1+} \frac{f(x) - f(1)}{x - 1} \text{ does not exist.}$$

$\therefore$ The given function does not have a derivative at x = 1. However, f(x) is continuous at x = 1 as $\lim_{x \to 1-} f(x) = \lim_{x \to 1-} x = 1$ and

$$\lim_{x \to 1+} f(x) = \lim_{x \to 1+} x^2 = 1.$$

3. Let

$$y = \begin{cases} \sin x & x \le \dfrac{\pi}{2} \\ \dfrac{2x}{\pi} & x > \dfrac{\pi}{2} \end{cases}$$

Then, $\displaystyle\lim_{x \to \frac{\pi}{2}-} \frac{f(x) - f(\pi/2)}{x - \pi/2} = \lim_{x \to \frac{\pi}{2}-} \frac{\sin x - 1}{x - \pi/2}$

$$= \lim_{x \to \frac{\pi}{2}-} \frac{\sin x - \sin \pi/2}{x - \pi/2}$$

$$= \lim_{x \to \frac{\pi}{2}-} \frac{2 \cos\left(\frac{\pi/2}{2} + x\right) \sin\left(\frac{x - \pi/2}{2}\right)}{x - \pi/2}$$

$$= \lim_{x \to \frac{\pi}{2}-} \cos\left(\frac{(\pi/2)+x}{2}\right) \frac{\sin \frac{x - \pi/2}{2}}{\left(\frac{x - \pi/2}{2}\right)} = 0 \times 1 = 0$$

But, $\displaystyle\lim_{x \to \pi/2+} \frac{f(x) - f(\pi/2)}{x - \pi/2} = \lim_{x \to \pi/2+} \frac{\frac{2x}{\pi} - 1}{x - \frac{\pi}{2}}$

$$= \lim_{x \to \pi/2+} \frac{2x - \pi}{\pi\left(x - \frac{\pi}{2}\right)}$$

$$= \lim_{x \to \pi/2+} \frac{2\left(x - \frac{\pi}{2}\right)}{\pi\left(x - \frac{\pi}{2}\right)} = \frac{2}{\pi} \qquad \because x - \frac{\pi}{2} \ne 0$$

$$\therefore \quad \lim_{x \to \pi/2-} \frac{f(x) - f\left(\frac{\pi}{2}\right)}{x - \frac{\pi}{2}} \ne \lim_{x \to \pi/2+} \frac{f(x) - f\left(\frac{\pi}{2}\right)}{x - \frac{\pi}{2}}$$

$\therefore$ f(x) is not differentiable at x = $\frac{\pi}{2}$.

However, f(x) is continuous at x = $\frac{\pi}{2}$ as

$$\lim_{x \to \pi/2-} f(x) = \lim_{x \to \pi/2-} \sin x = \sin \frac{\pi}{2} = 1$$

and $$\lim_{x \to \pi/2+} f(x) = \lim_{x \to \pi/2+} \frac{2x}{\pi} = \frac{2}{\pi} \times \frac{\pi}{2} = 1.$$

Exercises for Self Evaluation

I. Show that the following functions are continuous but not differentiable at the given ponts.

1. $f(x) = \begin{cases} x^3 & x < 1 \\ x^2 & x \geq 1 \end{cases}$

2. $f(x) = \begin{cases} \sin x & x < 1 \\ 2x & x \geq 1 \end{cases}$

3. $f(x) = \begin{cases} x^2 + 1 & x < 1 \\ x + 1 & x \geq 1 \end{cases}$

II. Differentiate by first principles

1. a) $4x^2 - 3x$ b) $\frac{1}{x^3}$ c) $x^3 + 3$

   d) $\sin(x + 3)$ e) e^{x-1}

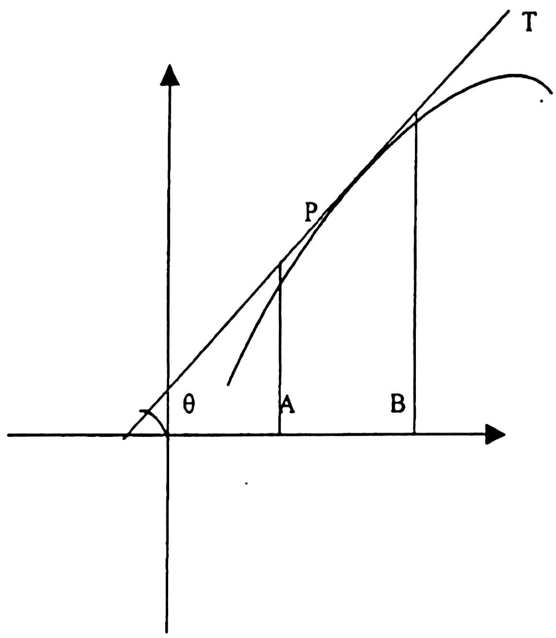
2. Find the equation of the tangent to $y = 2x^2 - 3x$ whose slope is 1.

3. Find the equation of the normal to the curve $y = x^2 - 3x + 2$ whose slope is 2.

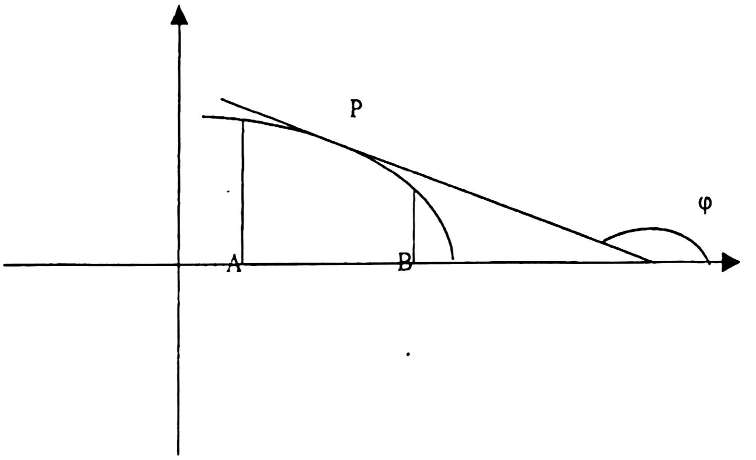
Applications of Derivatives

I. Increasing and Decreasing functions

To study the nature of a curve, it is often useful to know on what intervals a given function is increasing and on what intervals a given function is decreasing. If the function is differentiable, we can answer such questions easily by looking at the sign of the derivative since the derivative at a point is a slope of the tangent at that point to the given curve. Intuitively, if a function is increasing throughout an interval the shape of the curve will be approximately as follows:



Then at any point in the interval the tangent will make an acute angle with the +ve x-axis. Hence its slope is positive. Hence the derivative at that point is +ve Conversely, if derivative is positive,t hen the slope of the tangent is +ve and hence tangent should make an acute angle with +ve x-axis. Therefore, function is increasing. Similarly, if a function is decreasing in an interval, the shape of the curve will be seen approximately as follows :



Then at any point in the interval, the tangent will make an obtuse angle ϕ with +ve x-axis. Hence its slope is negative. Hence the derivative at that point is negative and conversely.

Thus intuitively we see that a function is increasing in an interval if and only if its derivative is positive. A function is decreasing in an interval if and only if its derivative is negative. Now we will try to prove it analytically. First we define.

Definition: A function f is said to be

i) increasing on the interval I iff for every two numbers x_1, x_2 in I

$x_1 < x_2$ implies $f(x_1) < f(x_2)$.

ii) Decreasing on the interval I iff for every two numbers x_1, x_2 in I

$x_1 < x_2$ implies $f(x_1) > f(x_2)$.

Now we will state what we had observed earlier intuitively.

Theorem : Let f be differentiable on the open interval I.

i) If $f'(x) > 0$ for all x in I, then f is increasing on I.

ii) If $f'(x) < 0$ for all x in I, then f is decreasing on I.

iii) If $f'(x) = 0$ for all x in I, then f is constant on I.

The proof needs Mean Value Theorem which is out of scope of the PUC syllabus.

Examples :

1. $f(x) = \sqrt{1 - x^2}$, $-1 \leq x \leq 1$.

Then $f'(x) = \dfrac{\frac{1}{2}}{\sqrt{1-x^2}} \; X \; -2x \; = \dfrac{-x}{\sqrt{1-x^2}}$

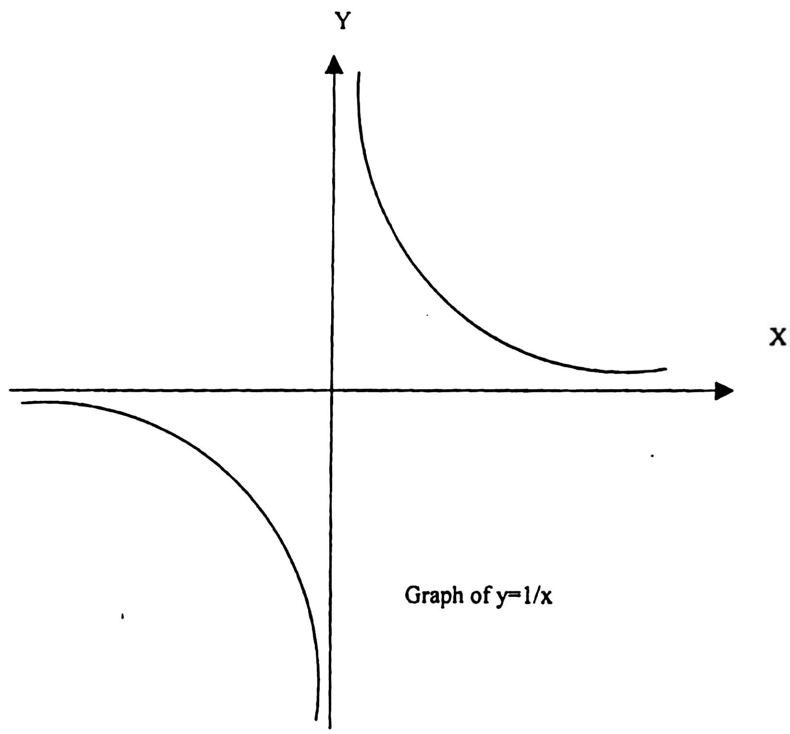
Then, $f'(x) > 0$ if $-1 < x < 0$.

$f'(x) < 0$ if $0 < x < 1$

Therefore, the function is increasing in (-1 0) and decreasing in (0 1).

2. $f(x) = \dfrac{1}{x}$, $x \neq 0$.

Then $f'(x) = -\dfrac{1}{x^2} < 0$ for all $x \neq 0$. Hence the function is decreasing for all $x \neq 0$.



Graph of y=1/x

which can easily be seen from the graph of $f(x) = \dfrac{1}{x}$.

3. $g(x) = 4x^5 - 15x^4 - 20x^3 + 110x^2 - 120x + 40$

Then g(x) is differentiable and

$g'(x) = 20x^4 - 60x^3 - 60x^2 + 220x - 120$

$= 20(x+2)(x-1)^2(x-3)$

Then $g'(x) = 0$ at $x = -2$, $x = 1$, $x = 3$.

Now in $(-\infty, -2)$, $x+2 < 0$, $(x-1)^2 > 0$, $x-3 < 0$ and so $g'(x) > 0$. Hence $g(x)$ is increasing in $(-\infty, -2)$.

In $(-2\ \ 1)$, $x+2 > 0$, $(x-1)^2 > 0$, $(x-3) < 0$ and so $g'(x) < 0$.

Hence $g(x)$ is decreasing on $(-2, 1)$.

Again in $(1,3)$, $x+2 > 0$, $(x-1)^2 > 0$, , $x-3 < 0$ and so $g'(x) < 0$. Hence $g(x)$ is decreasing on $(1,3)$.

However in $(3, \infty)$

$x+2 > 0$, $(x-1)^2 > 0$, $x-3 > 0$ and so

$g'(x) > 0$. Hence $g(x)$ is increasing on $(3,\infty)$. Then $g(x)$ is increasing in $(-\infty, -2)$ and $(3, \infty)$ and it is decreasing in $(-2, 1)$ and $(1,3)$.

Problem Solving

Find the intervals in which f is increasing and those in which it is decreasing.

1. $f(x) = x^3 - 3x + 2$

Here $f'(x) = 3x^2 - 3 = 3(x^2 - 1)$

Hence if $x^2 > 1$, $f'(x) > 0$

$x^2 < 1$, $f'(x) < 0$

But $x^2 > 1$ if either $x < -1$ or $x > 1$.

Hence $x^2 > 1$ in the intervals $(-\infty, -1)$ and $(1\ \infty)$.

Also $x^2 < 1$ if $-1 < x < 1$.

Hence $x^2 < 1$ in the interval $(-1, 1)$.

Therefore,

$f(x)$ is increasing in $(-\infty, -1)$ and $(1, \infty)$ and $f(x)$ is decreasing in $(-1, 1)$.

2. $f(x) = x + \frac{1}{x}$

Then $f'(x) = 1 - \frac{1}{x^2}$

If $x^2 > 1$ then $\frac{1}{x^2} < 1$ and hence $f'(x) = 1 - \frac{1}{x^2} > 0$.

But if $x^2 < 1$, $\frac{1}{x^2} > 1$ and hence $f'(x) = 1 - \frac{1}{x^2} < 0$.

Therefore in $(-\infty, -1)$ and $(1, \infty)$, f(x) is increasing and in $(-1, 1)$, f(x) is decreasing.

3. $f(x) = (x + 1)^4$.

Then, $f'(x) = 4(x + 1)^3$

If $x > -1$, $x + 1 > 0$, Hence $f'(x) > 0$ if $x > -1$.

If $x < -1$, $x + 1 < 0$. Hence $f'(x) < 0$ if $x < -1$.

Hence in -1) , f(x) is decreasing and in $(-1, \infty)$, f(x) is increasing.

4. $f(x) = \frac{1}{|x-2|}$

Then, $f(x) = \begin{cases} \frac{1}{x-2} & \text{if } x > 2 \\ \frac{1}{2-x} & \text{if } x < 2 \end{cases}$

$$f'(x) = \begin{cases} -\left(\frac{1}{x-2}\right)^2 & \text{if } x > 2 \\ \left(\frac{1}{2-x}\right)^2 & \text{if } x < 2 \end{cases}$$

Here $f'(x) < 0$ if $x > 2$.

$f'(x) > 0$ if $x < 2$.

Hence f(x) is increasing in $(-\infty, 2)$ and f(x) is decreasing in $(2, \infty)$

5. $f(x) = |x^2 - 5|$

Then $f(x) = \begin{cases} x^2 - 5 & \text{if } x^2 > 5 \\ 5 - x^2 & \text{if } x^2 < 5 \end{cases}$

Hence f'(x) = $\begin{cases} 2x \text{ if } x^2 > 5 \text{ i.e. } x \in (-\infty, -\sqrt{5}) \cup (\sqrt{5}, \infty) \\ -2x \text{ if } x^2 < 5 \text{ i.e. } x \in (-\sqrt{5}, \sqrt{5}) \end{cases}$

Hence $f'(x) > 0$ if $x \in (\sqrt{5}, \infty)$ or $(-\sqrt{5}, 0)$.

$f'(x) < 0$ if $x \in (-\infty, -\sqrt{5})$ or $(0, \sqrt{5})$.

Hence f(x) is increasing in $(-\sqrt{5}, 0)$ and $(\sqrt{5}, \infty)$.

f(x) is decreasing in $(-\infty, -\sqrt{5})$ and $(0, \sqrt{5})$.

6. $f(x) = \frac{x-1}{x+1}$. Then $f'(x) = \frac{(x+1) - (x-1)}{(x+1)^2} = \frac{2}{(x+1)^2} > 0$ for all values of x except at x = -1. Hence the function is increasing for all values of x in

$(-\infty, -1) \cup (-1\ \infty)$.

7. $f(x) = |x+1|\,|x-2|$. Then f(x) = $\begin{cases} (x+1)(x-2) & in\ (-\infty, -1) \\ (x+1)(2-x) & in\ (-1, 2) \\ (x+1)(x-2) & in\ (2, \infty) \end{cases}$

$\therefore f'(x) = \begin{cases} 2x-1 & in\ (-\infty, -1) \\ -2x+1 & in\ (-1, 2) \\ 2x-1 & in\ (2, \infty) \end{cases}$

$\therefore$ **Function is decreasing in $(-\infty, -1)$, increasing in $\left(-1, \frac{1}{2}\right)$, decreasing in $\left(\frac{1}{2}, 2\right)$ and increasing in $(2, \infty)$.**

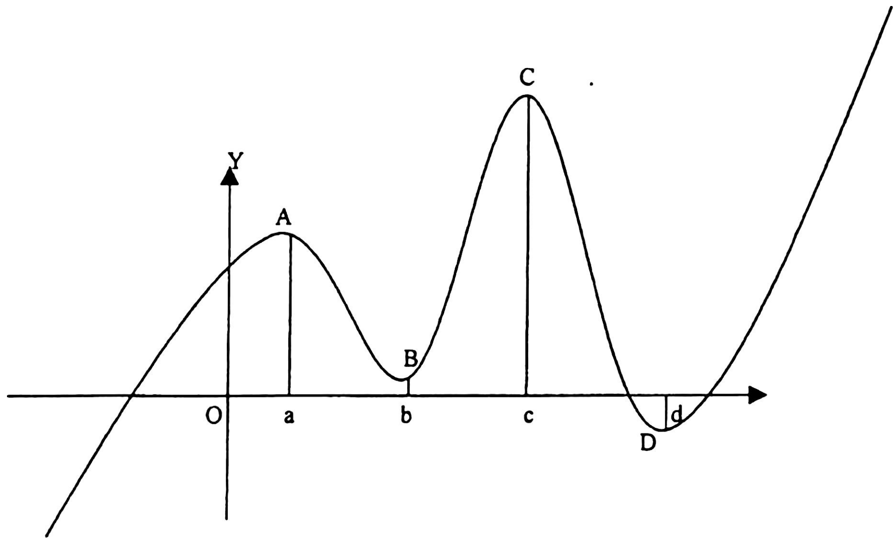
Exercises for Self Evaluation:

Find intervals in which f(x) is increasing and decreasing.

1. $f(x) = \frac{x^2+1}{x^2-1}$ 2. $f(x) = x^2 + \frac{16}{x^2}$ 3. $f(x) = |x+2|\,|x-3|$

Maxima and Minima

Problem of finding out how large or how small a certain quantity may become very important in many a situations in Economics, Engineering and Physics. If the problem admits a mathematical formulation, it is often reducible to the problem of finding the extreme values of some function. First let us observe the following graph of some function.



Note that at x = a i.e. at the point A on the curve, though we do not have the maximum value of the function, if we consider a small neighbourhood around the point A f(x) has a maximum value in this neighbourhood. Similar is the case at C. Also at the point B, though the value of the function is not the minimum, if we consider a small neighbourhood around B, the function has a minimum in this neighbourhood. So is the case at D. So though we do not have the maximum or the minimum values of the function y = f(x) at these point A, B, C, D, in their small neighbourhoods they are maximum or a minimum values of f(x). Hence we call them as local maximum (or relative maximum) and local minimum (or relative minimum) for the function f(x). We define now local maximum and local minimum more mathematically.

7. **Definition** :

A function f is said to have a local (or relative) maximum at c if there exists a $\delta > 0$ such that $f(c) \geq f(x)$ for all $x \in (c - \delta \;\; c + \delta)$.

A function f is said to have a local (or relative) minimum at c if there exists a $\delta > 0$ such that $f(c) \leq f(x)$, for all $x \in (c - \delta \;\; c + \delta)$.

Now we prove a necessary condition for a local maximum or local minimum.

Theorem : If f has a local maximum or a minimum at c, then either

$f'(c) = 0$ or $f'(c)$ does not exist.

Proof: If $f'(c) > 0$ then we know that f(x) will be increasing at c.

Then, $f(x_1) < f(c)$

for all $x_1 < c$, x_1 sufficiently close to c and $f(c) < f(x_2)$ for all $x_2 > c$, sufficiently close to c.

Hence f(x) cannot have either a local maximum or local minimum at x = c.

Similar is the case when $f'(c) < 0$ as f(x) will then be decreasing at x = c.

$\therefore \;\; f'(c) > 0, f'(c) < 0.$

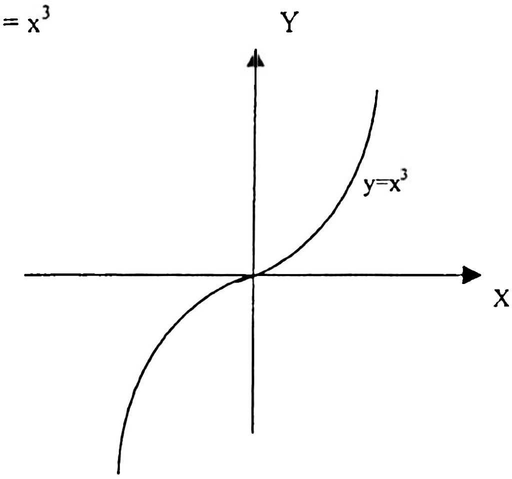
If f(x) has a local maximum or local minimum at x = c, then $f'(c)$ is either zero or $f'(c)$ does not exist.

Critical Points :

Since at points c where f(x) has a local maximum or local minimum $f'(c) = 0$, the points where $f'(x) = 0$ are called the critical points of the function $y = f(x)$.

Note that if f(x) has a local maximum or minimum at c, then $f'(c) = 0$. However converse need not be true i.e. whenever $f'(c) = 0$, there is no guarantee that f(x) has a local maximum or local minimum.

For example, let $f(x) = x^3$



Then $f'(x) = 3x^2$

$\therefore$ $f'(0) = 0$

$\therefore$ (0,0) is a critical point.

However, f(x) does not have a local maximum or local minimum at (0,0) whereas

$f(x) = x^3 < 0 = f(0)$ for all $x < 0$.

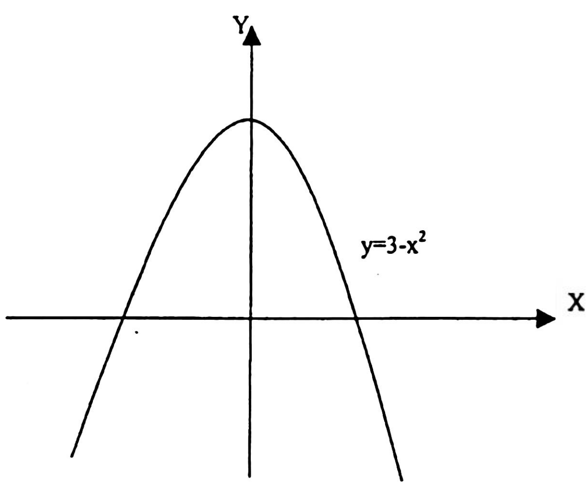
$f(x) = x^3 > 0 = f(0)$ for all $x > 0$.

Thus, if c is a critical point then it is not necessary that f(x) either local maximum or local minimum.

Examples on Relative Maximum and Relative Minimum

1. Let $f(x) = 3 - x^2$

Then $f'(x) = -2x$ exists everywhere. Since $f'(x) = 0$ only at $x = 0$, (0,3) is the only critical point.



For all $x < 0$, $x^2 > 0$ and hence $3 - x^2 < 3$

$\therefore$ $f(x) < f(0)\ \forall\ x < 0$

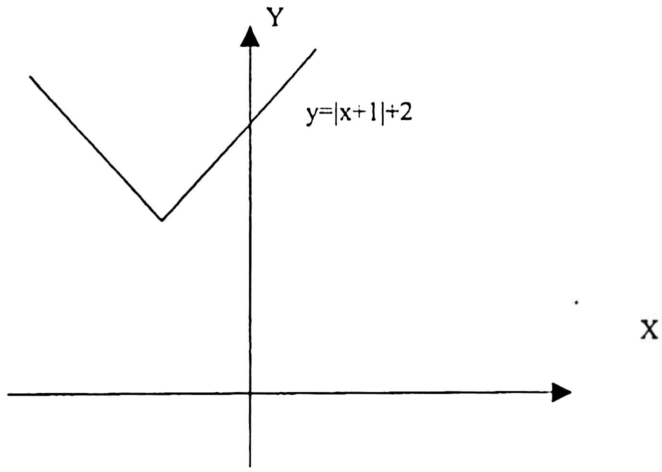
$\therefore$ also for all $x > 0$, $x^2 > 0$ and hence $3 - x^2 < 3$, $f(x) < f(0)\ \forall\ x > 0$.

Here f(x) has a local maximum at (0,3).

2. Let $f(x) = |x + 1| + 2$

Then $f(x) = |x + 1| + 2$

Then, $f(x) = \begin{cases} x+1+2 = x+3 & for\ x > -1 \\ -(x+1)+2 = -x+1 & for\ x < -1 \end{cases}$



i.e. $f'(x) = \begin{cases} 1 \text{ for } x > -1 \\ -1 \text{ for } x < -1 \end{cases}$

$\therefore$ f'(x)is never zero in $(-\infty\ -1) \cup (-1\ \infty)$.

At -1,

$$\lim_{x \to -1+} \frac{f(x) - f(-1)}{x - (-1)} = \lim_{x \to -1+} \frac{x + 3 - 2}{x + 1} = \lim_{x \to -1+} \frac{x+1}{x+1} = 1 \quad (\because \ x+1 \neq 0)$$

and $\lim_{x \to 1-} \frac{f(x) - f(-1)}{x - (-1)} = \lim_{x \to 1-} \frac{-x + 1 - 2}{x + 1} = \lim_{x \to 1-} \frac{-(x+1)}{x+1} = -1$. $(\because \ x+1 \neq 0)$

$\therefore$ f'(x) does not exist at x = -1.

Also f(-1) = 2

For all x < -1, -x > 1 and hence f(x) = -x + 1 > 2 = f(-1)

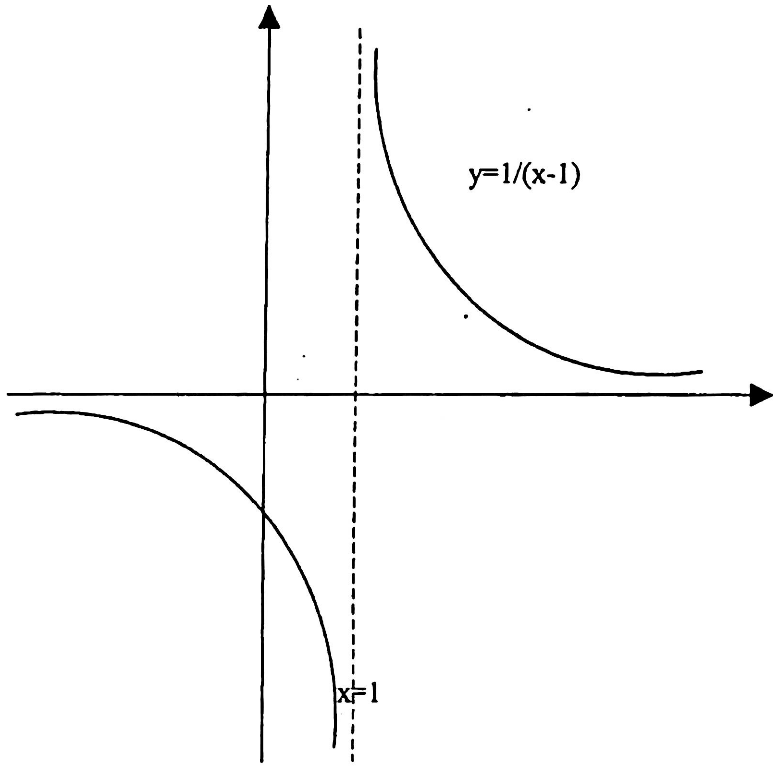
and for all x >-1, and hence f(x) = x + 3 > -1 + 3 = 2 = f(-1)

$\therefore$ f(x) has a local minimum at x = -1.

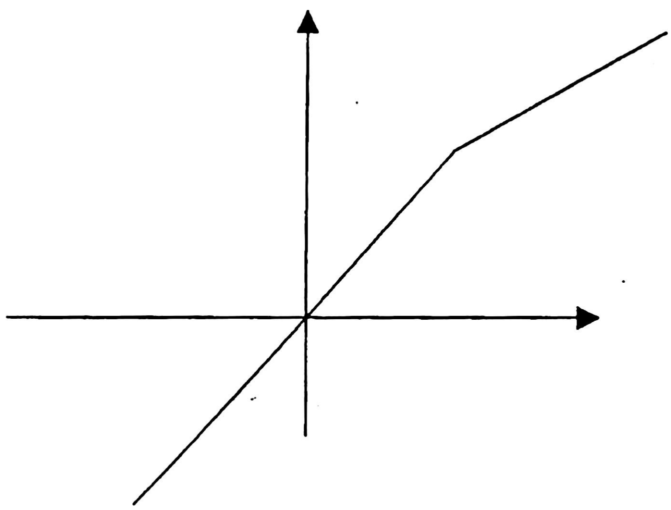
3. Let f(x) = $\frac{1}{x-1}$.

Then f'(x) = $\frac{1}{(x - 1)^2}$ which is never zero.

Hence for no value of x can there be a local maximum or local minimum.



3. $f(x) = \begin{cases} 2x & x < 1 \\ \frac{x}{2} + \frac{3}{2} & x \geq 1 \end{cases}$



Then $f'(x) = \begin{cases} 2 & x < 1 \\ \frac{1}{2} & x > 1 \end{cases}$

But $f'(1)$ does not exist (Prove!).

Hence 1 is a critical point.

But f(x) does not have a local maximum on local minimum at 1 as f(x) is increasing on both sides of 1.

Tests for Local Maxima Local Minima

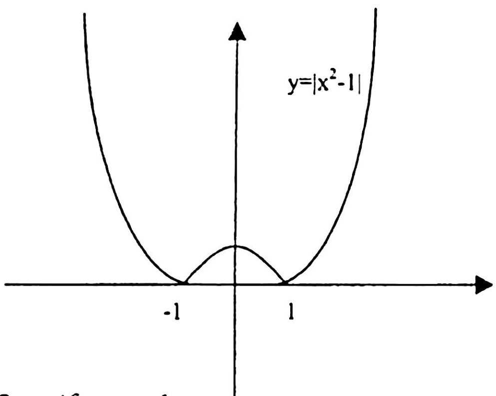
Test 1 : The first derivative test

Suppose c is a critical point for f(x) (i.e. f'x(c) = 0) and let f(x be continuous at c. If there exists $\delta > 0$ such that $f'(x) > 0$ for all x in (c - δ, c) and $f'(x) < 0$ for all x in (c, c + 8) , then f(x has a local maximum at c. If there exists $\delta > 0$, such that $f'(x) < 0$ for all x in (c - δ) and $f'(x) > 0$ for all x in (c, c+δ), then f(x) has a local minimum at c.

Example :

1. Let $f(x) = | x^2 - 1 |$

Then $f(x) = \begin{cases} x^2 - 1 & if\ x \le -1 \\ 1 - x^2 & if\ -1 < x < 1 \\ x^2 - 1 & if\ x \ge 1 \end{cases}$



∴ $f'(x) = \begin{cases} 2x & if\ x < -1 \\ -2x & if\ -1 < x < 1 \\ 2x & if\ x > 1 \end{cases}$ ∴ $f'(x) = 0 \Rightarrow x = 0.$

and $f'(x)$ does not exist at -1 and 1.

$\therefore$ The critical points are -1, 0 and 1.

Also $f'(x)$ is negative for $x < -1$,

$f'(x)$ is positive for $-1 < x < 0$,

$f'(x)$ is negative $0 < x < 1$,

$f'(x)$ is positive for $1 < x < \infty$,

$\therefore$ $f(x)$ has a local minimum at $x = -1$,

$f(x)$ has a local maximum at $x = 0$,

$f(x)$ has a local minimum at $x = 1$.

2. Let $f(x) = (x-2)(x-1)^4$

Then $f'(x) = 4(x-2)(x-1)^3 + (x-1)^4$

$= (x-1)^3 [x - 1 + 4(x-2)]$

$= (x-1)^3 [5x - 9]$

$\therefore$ $f'(x) = 0 \Rightarrow x = 1$ or $x = \frac{9}{5}$

Also for $x < 1$, $(x-1)^3 < 0$, $5x - 9 < 0$

$\therefore$ $f'(x) = (x-1)^3 (5x-9) > 0$

For $1 < x < \frac{9}{5}$, $(x-1)^3 > 0$, $5x - 9 < 0$

$\therefore$ $f'(x) = (x-1)^3 (5x-9) < 0$

For $x > \frac{9}{5}$, $(x-1)^3 > 0$, $5x - 9 > 0$

$\therefore$ $f'(x) = (x-1)^3 (5x-9) > 0$.

Hence $f(x)$ has a local maximum at $x = 1$ and local minimum at $x = \frac{9}{5}$.

Test 2: The second derivative test

Let $f(x)$ be a function such that $f'(c)$ exists and

suppose $f'(c) = 0$, then if $f''(c) > 0$, f has a local minimum at c;

if $f''(c) < 0$, f has a local minimum at c.

Examples :1. Let $f(x) = x^3 - x$.

Then $f'(x) = 3x^2 - 1$

$\therefore \quad f'(x) = 0 \Rightarrow x = \pm \frac{\sqrt{3}}{3}.$

$\therefore \quad -\frac{\sqrt{3}}{3}$ and $\frac{\sqrt{3}}{3}$ are the critical points.

Also $f''(x) = 6x$

$\therefore \quad f''(-\frac{\sqrt{3}}{3}) < 0 \quad f''(\frac{\sqrt{3}}{3}) > 0$

$\therefore$ f(x) has a local maximum at $-\frac{\sqrt{3}}{3}$ and a local minimum at $\frac{\sqrt{3}}{3}$.

Problem Solving

Find the critical points and the local extreme values of the following functions.

1. $f(x) = x^3 + 3x - 2$

$f'(x) = 3x^2 + 3 \ \forall \ x$. Then $f'(x)$ is never zero. Therefore, no critical points.

2. $f(x) = x\ (x + 1)(x + 2) \quad \therefore \ f(x) = x\,(x^2 + 3x + 2) \ = x^3 + 3x^2 + 2x.$

$\therefore \quad f'(x) = 3x^2 + 6x + 2.$

$$f'(x) = 0 \Rightarrow x = \frac{-6 \pm \sqrt{36 - 24}}{6}$$

$$= -1 \pm \frac{2\sqrt{3}}{6}$$

$$= -1 \pm \frac{\sqrt{3}}{3}.$$

$\therefore$ the critical points are $-1 + \frac{\sqrt{3}}{3}$ and $-1 - \frac{\sqrt{3}}{3}$.

Also, $f''(x) = 6x + 6$, $f''(-1 + \frac{\sqrt{3}}{3}) = -6 + 2\sqrt{3} + 6 > 0$

$\therefore$ Local minimum at $-1 + \frac{\sqrt{3}}{3}$.

$f''(-1 - \frac{\sqrt{3}}{3}) = -6 - 2\sqrt{3} + 6 < 0.$

$\therefore$ Local maximum at $-1 - \dfrac{\sqrt{3}}{3}$.

3. $f(x) = x + \dfrac{1}{x}$.

$f'(x) = 1 - \dfrac{1}{x^2}$

$f'(x) = 0 \Rightarrow x = \pm 1$.

$\therefore$ 1, -1 are critical points.

Further,

$f''(x) = \dfrac{2}{x^3}$

$\therefore$ $f''(-1) = -2 < 0$ $\therefore$ Local max at -1.

$f''(1 = \dfrac{2}{1} = 2 > 0$ $\therefore$ Local min at 1.

4. $f(x) = \dfrac{1}{|x-2|}$

Then $f(x) = \begin{cases} \dfrac{1}{x-2} & if \ x > 2 \\ \dfrac{1}{2-x} & if \ x < 2 \end{cases}$

$\therefore$ $f''(x) = \begin{cases} -\left(\dfrac{1}{x-2}\right)^2 & if \ x > 2 \\ +\left(\dfrac{1}{2-x}\right)^2 & if \ x < 2 \end{cases}$

and f″ does not exist at x = 2 and f′(x) is never zero.

So only critical point is at 2.

Also, $f'(x) > 0$ for $x < 2$.

$f'(x) < 0$ for $x > 2$.

$\therefore$ Local max at x = 2.

Exercises for Self Evaluation

Find critical points and extreme values of the following functions.

1. $x^2(1-x)$

2. $\dfrac{1+x}{1-x}$

3. $|x^2-16|$

4. $\dfrac{|x|}{1+|x|}$

5. $\dfrac{1}{x+1}-\dfrac{1}{x-2}$.

6. $|x-3|+|2x+1|$.

7. $\dfrac{2-3x}{2+x}$.

Integral Calculus

If a function f is differentiable in an interval I, then the natural question arises that, given its derivative f' at each point of I, can we determine the function? Such functions are called **antiderivatives**, further the formula that gives all these antiderivatives, is called the '**indefinite integral**' of the function. The process of finding antiderivatives is called '**integration**'.

The development of integral calculus arises out of the efforts of solving the problems of the following types :

a) The problem of finding a function whenever its derivative is given.

b) The problem of finding the area bounded by the graph of a function under certain conditions.

These two problems lead to the two forms of the integrals i.e. indefinite and definite integrals, which together constitute the 'integral calculus'.

In this unit, we illustrate some techniques of solving the indefinite integral problems.

Techniques of Indefinite Integration

A. Change of variable (or substitution) method :

Illustration

1. $I = \int \frac{2\,dx}{\sqrt{1-5x^2}}$

Solution : Put $t = \sqrt{5}\,x$ then $dt = \sqrt{5}\,dx$ i.e. $dx = \frac{dt}{\sqrt{5}}$.

So we have

$$I = \int \frac{2\,dt}{\sqrt{5}\,(\sqrt{1-t^2}} = \frac{2}{\sqrt{5}} \qquad \int \frac{dt}{\sqrt{1-t^2}} = \frac{2}{\sqrt{5}} \sin^{-1} t + C$$

$$I = \int \frac{2}{\sqrt{3}} \sin^{-1} (\sqrt{5}\,x) + c$$

Note : Here the standard form used is $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + c$

2. $I = \frac{x^3 \sin(\tan^{-1} x^4)}{1+x^8}\,dx$

Solution : Put $\tan^{-1}(x^4) = t$ then $dt = \dfrac{4x^3}{1+x^8} . dx$

$$\therefore \quad \frac{dt}{4} = \frac{x^3\, dx}{1+x^8}$$

$$\therefore \quad I = \int \sin t \frac{dt}{4} = \frac{1}{4}\int \sin dt = \frac{1}{4}(-\cos t) + c$$

$$= \frac{1}{4} - \cos(\tan^{-1} x^4) + c$$

Integrals of Trigonometric Functions

Illustration : Most of the integrals containing trigonometric functions are reducible to those containing sines and cosines only. So we reduce generally the trigonometric functions in terms of sine and cosine. Some times expressing sines and cosines in terms of tangent and cotangent also prove useful. Illustrative examples are given below:

$$I = \int \frac{1}{1+\tan x} dx$$

Here $\dfrac{1}{1+\tan x} = \dfrac{1}{1+\dfrac{\sin x}{\cos x}} = \dfrac{\cos x}{\cos x + \sin x} = \dfrac{2\cos x}{2(\cos x + \sin x)}$

$$= \frac{(\cos x + \sin x) + (\cos x + \sin x)}{2(\cos x + \sin x)}$$

$$= \frac{1}{2} + \frac{\cos x - \sin x}{2(\cos x + \sin x)}$$

Put t = cos x + sin x, then dt = (cos x - sin x) dx

So the integral

$$I = \int \frac{dx}{2} + \int \frac{1}{2}\frac{dt}{t} = \frac{x}{2} + \frac{1}{2}\log|t| + c$$

$$= \frac{x}{2} + \frac{1}{2} Log|Cos\ x + \sin x| + c$$

$$I = \int \sin^{-1}(\cos x) dx$$

Put cos x = t sin x dx = dt

$$dx = \frac{dt}{\sin x} = \frac{dt}{\sqrt{1-\cos^2 x}} = \frac{dt}{\sqrt{1-t^2}}$$

So, I = $\int \sin^{-1} t \dfrac{dt}{\sqrt{1-t^2}}$

Put $\sin^{-1}$ t = u, $\dfrac{1}{\sqrt{1-t^2}} dt$ = du

$$\therefore \ I = \int u\, du = \frac{4^2}{2} + c$$

$$= \frac{(\sin^{-1} t)^2}{2} + c'$$

$$= \frac{(\sin^{-1} c \cos x))^2}{2} + c$$

3. $\displaystyle\int \cos^6 x\, dx = \int \left(\frac{1+\cos 2x}{2}\right)^3 dx$

$$= \frac{1}{8}\int (1 + 3\cos 2x + 3\cos^2 2x + \cos^3 2x)\, dx$$

$$= \frac{1}{8}\int (1 + 3\cos 2x + \frac{3(1+\cos 4x)}{2} + (1-\sin^2 2x)(\cos 2x)\, dy$$

$$= \frac{1}{8}\left[x + \frac{3\sin 2x}{2} + \frac{3}{2}\left(x + \frac{\sin 4x}{4}\right) + \frac{\sin 2x}{2} - \frac{\sin^3 2x}{3.2}\right] + c$$

$$= \frac{1}{8}\left(\frac{5}{2}x + 2\sin 2x + \frac{3}{8}\sin 4x - \frac{1}{6}\sin^3 2x\right) + c$$

Integration of Rational Algebraic Functions

Illustration :

1. $\displaystyle I = \int \frac{x+4}{x^3+3x^2-10x}\, dx$

Resolving the integral into partial functions we have

$$\frac{x+4}{x^3+3x^2-10x} = \frac{x+4}{x(x-2)(x+5)} = \frac{-2}{5x} + \frac{3}{7(x-2)} - \frac{1}{35(x+5)}$$

So, using proper substitution we get

$$I = \frac{-2}{5} Log|x| + \frac{3}{7} Log|x-2| - \frac{1}{35} Log|x+5| + c$$

2. $\displaystyle I = \int \frac{Cos\, x}{(1-\sin x)(2-\sin x)}\, dx$

Put sin x = t cos x dy = dt

$$I = \int \frac{dt}{(1-t)(2-t)} = Log \left| \frac{2-\sin x}{1-\sin x} \right| + c$$

Integration by Parts

When we have product of functions to be integrate, this technique is useful.

Rule : $\int f\, g\, dx = f(x) \int g(x)\, dx - (\int g(x)\, dx\, f'(x))\, dx$

Illustrations :

1. $$\int \frac{x\cos^{-1} x}{\sqrt{1-x^2}}\, dx = (\cos^{-1} x)(-\sqrt{1-x^2}) - \int \frac{-1}{\sqrt{1-x^2}} (-\sqrt{1-x^2})\, dx$$

$$= (-\sqrt{1-x^2}) \cos^{-1} x - x + c$$

2. $$\int (\sin^{-1} x)^2\, dx = (\sin^{-1} x)^2 (x) - \int x - \frac{2\sin^{-1} x}{\sqrt{1-x^2}}\, dx$$

$$= [- x(\sin^{-1} x)^2 - 2[-\sqrt{1-x^2} \sin^{-1} x + x + c]$$

$$= x(\sin^{-1} x)^2 + 2\sqrt{1-x^2} \sin^{-1} x - 2x + c$$

II. Definite Integral

A definite integral is denoted by $\int_a^b f(x)\,dx$, where a is called the lower limit of the integral and b is called the upper limit of the integral. The definite integral is introduced either as the limit of a sum or if it has an antiderivative F in the interval [a,b], then its value is the difference between the values of F at the end points i.e. F(b) – F(a).

We shall illustrate these two cases separately by considering some examples.

A. Definite integral as the limit of a sum

Note : The definition of the definite integral can be used with profit to evaluate easily the limits of the sums of certain series, when the number of terms in the series tend to infinity. The method lies in identifying a definite integral equal to series. In fact,

$$\int_a^b f(x)\,dx = \lim_{h \to 0} h \sum f(a + nh) \quad \text{where } nh = b - a$$

$$\text{or } \lim_{n \to \infty} \left(\frac{b-a}{n}\right) \sum \left(f\left(a + \frac{r(b-a)}{n} \right)\right) = \int_a^b f(x)\,dx$$

If a = 0, and b = 1, we have

$$\lim_{n \to \infty} \frac{1}{n} \sum f\left(\frac{r}{n}\right) = \int_0^1 f(x)\,dx$$

Illustrations :

1. Show that $\lim_{n \to \infty} \left(\frac{1}{n+m} + \frac{1}{n+2m} + \ldots + \frac{1}{n+nm} \right) = \frac{1}{m} \log(1+m)$

Solution: Look at

$$\lim_{n \to \infty} \left(\frac{1}{n+m} + \frac{1}{n+2m} + \ldots + \frac{1}{n+nm} \right) = \lim_{n \to \infty} \frac{1}{n} \sum_{r=1}^{n} \frac{1}{1+m/n}$$

$$= \int_0^1 \frac{1}{1+mx}\,dx$$

Put 1 + mx = t, then $dx = \frac{dt}{m}$.

When x = 0, t = 1 and when x = 1, t = 1 + m.

$$\therefore \quad I = \frac{1}{m} \int_1^{1+m} \frac{1}{t}\,dt$$

$$= \frac{1}{m} Log\ t \ \Big]_1^{1+m}$$

$$= \frac{1}{m} [\ Log\ (1+m) - Log\, 1]$$

$$= \frac{1}{m} [\ Log\ (1+m) - 0\ ]$$

$$I = \frac{1}{m} Log\ (1+m)$$

$$\lim_{n \to \infty} \left[\left(1 + \frac{1}{n}\right)\left(1 + \frac{2}{n}\right) \ldots\ldots \left(1 + \frac{n}{n}\right) \right]^{1/n} = \frac{4}{e}.$$

Let us consider

$$\lim_{n \to \infty} \left[\left(1 + \frac{1}{n}\right)\left(1 + \frac{2}{n}\right) \ldots\ldots \left(1 + \frac{n}{n}\right) \right]^{1/n} = \lim_{n \to \infty} x^{1/n}$$

By taking log we get,

$$\text{Log} \lim_{n \to \infty} x^{1/n} = \lim_{n \to \infty} Log\ x^{1/n}$$

$$= \lim_{n \to \infty} \frac{1}{n} Log\ x$$

$$= \lim_{n \to \infty} \left[\frac{1}{n} \sum_{r=1}^{n} Log \left(1 + \frac{r}{n}\right) \right]$$

$$= \int_0^1 Log\ (1+x)\ dx$$

By applying integration by parts, we have

$$\text{I} = (1 + \text{x}) \text{ Log} (1 + \text{x}) \ \ (1 + \text{x}) \ \Big]_0^1$$

$$= (2 \text{ Log } 2 - 2) - (\text{Log } 1 - 1)$$

$$= \text{Log } 4 - 2 + 1$$

$$\text{I} = \text{Log } 4 - 1$$

$$\therefore \quad \text{Log} \lim_{n \to \infty} x^{1/n} = Log\ 4 - 1$$

$$\therefore \quad \lim_{n \to \infty} x^{1/n} = e^{(Log 4 - 1)} = \frac{4}{e}$$

$$\therefore \quad \lim_{n \to \infty} \left[\left(1 + \frac{1}{n}\right)\left(1 + \frac{2}{n}\right) \ldots\ldots \left(1 + \frac{n}{n}\right) \right]^{1/n} = \frac{4}{e}$$

Similarly, try this:

$$\lim_{n \to \infty} \left[\left(1 + \frac{1}{n^2}\right)^{2/n^2} \left(1 + \frac{2^2}{n^2}\right)^{4/n^2} \ldots\ldots \left(1 + \frac{n^2}{n^2}\right)^{\frac{2n}{n^2}} \right] = \frac{4}{e}$$

Exercise :

1. Show that

$$\lim_{n \to \infty} \left[\frac{\sqrt{n+1}\ \sqrt{n+2}\ + \ldots\ldots + \sqrt{2n}}{n\sqrt{n}} \right] = \frac{4}{3}\sqrt{2} - \frac{2}{3}.$$

Illustrates by using properties of definite integrals :

The following examples will illustrate the use of the properties of the definite integrals in solving problems.

1. Show that $\int_0^{\pi/2} Log \sin x \, dx = \frac{\pi}{2} Log\left(\frac{1}{2}\right)$

Solution: Let us consider :

$$\int_0^{\pi/2} Log\ \sin x \, dx \qquad \ldots \qquad (1)$$

By using the property $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

We have,

$$I = \int_0^{\pi/2} Log \sin(\pi/2 - x)\,dx$$

$$I = \int_0^{\pi/2} Log \cos\ x \, dx \qquad \ldots \qquad (2)$$

$\therefore$ By adding (1) and (2) on both sides, we get

$$2I = \int_0^{\pi/2} (Log \sin x + Log \cos x)\,dx$$

$$= \int_0^{\pi/2} (\text{Log}\ (\sin x \,.\, \cos x)\ dx$$

$$= \int_0^{\pi/2} Log\left(\frac{\sin 2x}{2}\right) dx$$

$$= \int_0^{\pi/2} Log \sin 2x \, dx - \int_0^{\pi/2} Log\, 2\, dx$$

Note that

$$\int_0^{\pi/2} Log \sin 2x \, dx = \int_0^{\pi/2} Log \ \sin x \, dx \text{, we have}$$

$$2I = I - Log\ 2 \int_0^{\pi/2} dx$$

$$\therefore \qquad I = (-Log\ 2)\ (x\ ]_0^{\pi/2}$$

$$= \frac{\pi}{2} (- Log\ 2)$$

$$I = \frac{\pi}{2} Log \left(\frac{1}{2}\right)$$

2. $\displaystyle\int_0^1 \frac{Log\ (1+x)}{1+x^2}\, dx$

Let $I = \displaystyle\int_0^1 \frac{Log\ (1+x)}{1+x^2}\, dx$

Put $x = \tan \theta$.

$dx = \sec^2 \theta \, d\theta$

If $x = 0$, then $\theta = 0$ and if $x = 1$ $\tan \theta = \dfrac{\pi}{4}$.

$$\therefore \qquad I = \int_0^{\pi/4} \frac{Log\ (1+\tan\theta) \sec^2 \theta \, d\theta}{1+\tan^2 \theta} = \int_0^{\pi/4} Log (1+\tan\theta)\, d\theta$$

By using the property $\displaystyle\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$, we have

$$I = \int_0^{\pi/4} Log \left(1 + \tan\left(\frac{\pi}{4} - \theta\right)\right) d\theta$$

$$= \int_0^{\pi/4} Log \left(1 + \frac{\tan\frac{\pi}{4} - \tan\theta}{1+\tan\frac{\pi}{4} \cdot \tan\theta}\right) d\theta$$

$$= \int_0^{\pi/4} Log \left(1 + \frac{\tan\theta + 1 - \tan\theta}{1+\tan\theta}\right) d\theta$$

$$= \int_0^{\pi/4} Log\left(\frac{2}{1+\tan\theta}\right) d\theta$$

$$= \int_0^{\pi/4} Log\, 2\, d\theta - \int_0^{\pi/4} Log\ (1+\tan\theta)\, d\theta$$

$$I = \int_0^{\pi/4} Log\ 2 d\theta\ - I$$

$$\therefore \quad 2I = \int_0^{\pi/4} Log\, 2\, d\theta = Log\, 2\ [\theta]_0^{\pi/4} = \frac{\pi}{4} Log\ 2$$

$$\therefore \quad I = \frac{\pi}{8}.Log\, 2$$

$$\therefore \quad \int_0^1 \frac{Log\ (1+x)}{1+x^2}\, dx = \frac{\pi}{8} Log\ 2$$

Exercises :

1. Show that $\int_0^{\pi} \frac{x\, dx}{a^2\cos^2 x + b^2 \sin^2 x} = \frac{\pi^2}{2ab}$

2. $\int_0^{\pi/4} \frac{\sin^{3/2}\theta\, d\theta}{\sin^{3/2}\theta + \cos^{3/2}\theta} = \frac{\pi}{4}$

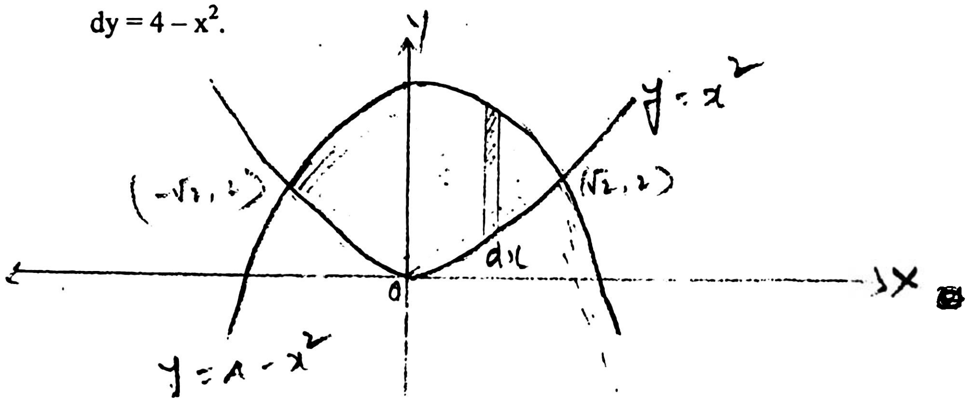
3. $\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}} = \frac{\pi}{12}$

B. As area function

Having known the different techniques of indefinite integration, we are in a better position to use the fundamental theorem of integral calculus in solving problem on area by integration.

Here we will illustrate the techniques of finding the area of a region by integration.

1. Find the area of the region bounded by the two parabolas $y = x^2$ and $dy = 4 - x^2$.



The given curves are

$$y = x^2 \quad .... \quad (1)$$

$$y = 4 - x^2 \quad .... \quad (2)$$

which are as given in the figure.

Now for the points of intersection of (1) and (2),

$$x^2 = 4 - x^2$$

i.e. $2x^2 = 4$

i.e. $x = \pm \sqrt{2}$

and the corresponding values of y are 2 and 2 respectively. So the points are intersection of (1) and (2) are $(-\sqrt{2}, 2)$ and $(\sqrt{2}, 2)$. As is clear from the figure, for a given x in the region, the value of y ranges from x^2 which is the lower limit, to $4 - x^2$ which is the upper limit. So the length of the vertical strip in the figure is (upper limit of y) = (lower limit of y) = $(4 - x^2) - x^2 = 4 - 2x^2$.

Therefore, the area of the region is

$$A = \int_{-\sqrt{2}}^{\sqrt{2}} 4 - 2x^2\, dx$$

$$= 2 \int_{0}^{\sqrt{2}} (4 - 2x^2)\, dx \text{ as the region is symmetric about y-axis.}$$

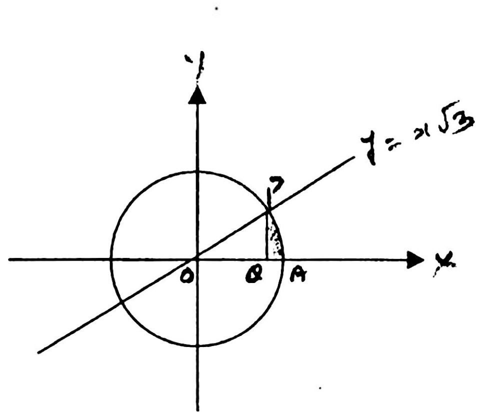
$$= 2\left[4x - \frac{2}{3}x^3\right]_0^{\sqrt{2}} = 2\left[4\sqrt{2} - \frac{2}{3}(\sqrt{2})^3\right] - 0$$

$$= 2\left[4\sqrt{2} - \frac{2}{3}(\sqrt{2})\right]$$

$$= 8\sqrt{2} - \frac{8}{3}\sqrt{2} = \frac{16}{3}\sqrt{2}$$

2. Show that the area in the first quadrant, enclosed by the x-axis, the line $x = y\sqrt{3}$ and the circle $x^2 + y^2 = 4$ is $\pi/3$

Solution : The equation of the circle is $x^2 + y^2 = 4 = 2^2$ (1)



So the center is O(0,0) and radius = 2. The point of intersection of (1) with x-axis curve (2,0) and (-2, 0). So the point A as shown in the figure is in the first quadrant and so is (2,0).

Now the given line $x = y\sqrt{3}$. That is $y = \frac{x}{\sqrt{3}}$ ... (2)

(1) and (2) gives

$3x^2 + x^2 = 12$, i.e. $4x^2 = 12$ i.e. $x = \pm\sqrt{3}$

As P is in the first quadrant, P is $(\sqrt{3}, 1)$ and Q is $(\sqrt{3}, 0)$. Here P is the point of intersection of 1 and 2 in the first quadrant and PQ ⊥ x-axis. So OQ = 3, PQ = 1. Let Area of ΔOQP $= A_1$ and shaded area $= A_2$. Therefore, the required area

$A = A_1 + A_2$

A_1 = area of ΔOQP = ½ . OQ . PQ = ½ $\sqrt{3}$.1 = $\frac{\sqrt{3}}{2}$ sq. units. ... (3)

$A_2 = \int_{\sqrt{3}}^{2} y\,dx$ where $x^2 + y^2 = 4$

Now $x^2 + y^2 = 4 \Rightarrow y^2 = 4 - x^2 \Rightarrow y = \sqrt{4 - x^2}$ in the first quadrant.

So,

$$A_2 = \int_{\sqrt{3}}^{2} \sqrt{2 - x^2}\,dx$$

$$= \left[\frac{x}{2}\sqrt{4 - x^2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_{\sqrt{3}}^{2}$$

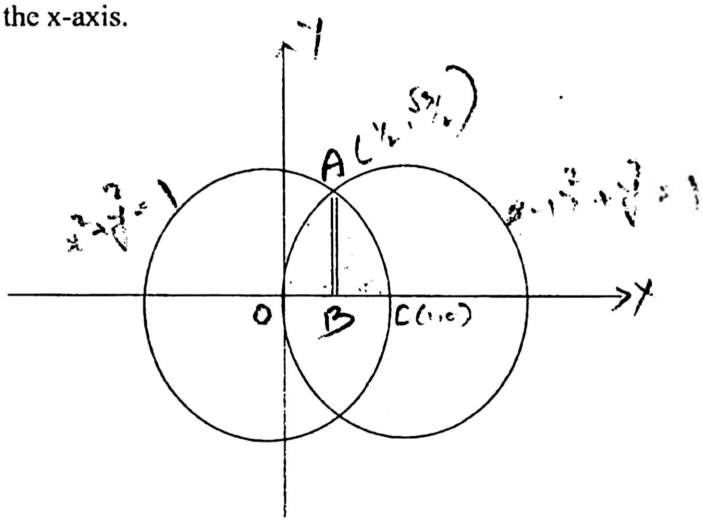
$$= [\,0 + 2\sin^{-1} 1\,] - \left[\frac{\sqrt{3}}{2}\sqrt{4 - 3} - 2\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right]$$

$$= 2\sin^{-1}(1) - \frac{\sqrt{3}}{2} - 2\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

From (3) and (4) we get,

$$A = A_1 + A_2 = \frac{\sqrt{3}}{2} + \frac{\pi}{3} - \frac{\sqrt{3}}{2} = \frac{\pi}{3} \text{ sq. units.}$$

3. Find the area of the region enclosed between two circles $x^2 + y^2 = 1$ and $(x - 1)^2 + y^2 = 1$ and the x-axis.



Solution :

$x^2 + y^2 = 1$ (1)

$(x - 1)^2 + y^2 = 1$ (2)

From the forms of the equations (1) and (2) it is clear that their centers are O(0,0) and C(1,0) and their radius = 1. Solving (1) and (2), their points of intersections are found to be A $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ in the first quadrant and $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ in the second quadrant.

So the required area is

A = Area of OACB

= 2 (area of OAB)

$$=4 \int_0^{1/2} \sqrt{1-(x-1)^2}\, dx \text{ as for circle (2) } y = \sqrt{1-(x-1)^2}$$

$$= \sin^{-1}\left(-\frac{1}{2}\right) - \frac{\sqrt{3}}{4} - \sin^{-1}(-1)$$

$$= -\frac{\pi}{3} - \frac{\sqrt{3}}{4} + \frac{\pi}{2}$$

$$= \left(\frac{\pi}{2} - \frac{\pi}{3}\right) - \frac{\sqrt{3}}{4}$$

$$= \frac{\pi}{6} - \frac{\sqrt{3}}{4}$$

Exercise :

1. Using integration show that the area of Δ ABC where points A, B, C are A(-1, 1), B(0,5) and C(3,2) respectively is eqiual to $\frac{15}{2}$ sq. mts.
2. Find the area bounded by the curve $y^2 = 4ax$ and the lines $y = 2a$ and $x = 0$.
3. Find the area enclosed between the curve $y = \cos x$ and the x-axis between $x = 0$ and $x = \pi$.

Applications of Definite Integral

I. The Theorem of Bliss

The Theorem of Bliss assures that the definite integral is the sum of a sequence of sums, even through the sums, even though the sums involved are obtained in a rather free fashion. In short, we are permitted as unexpected flexibility in forming our sums.

Statement : Let ({I_1, s_1), (I_2, s_2),....(I_n, s_n) } and { (I_1, t_1), (I_2, t_2),.....(I_n, t_n) be sequences defined on [c,d] and let f and g be continous on [c,d].

$$\text{Lim}\left\{f(s_1)\, g(t_1)\, |I_1| + f(s_2)\, g(t_2)\, |I_2| + \ldots + f(s_n)\, g(t_n)\, |I_n|\right\} = \int_c^d f \cdot g$$

II. Length of a Curve

Let f and f' are continuous on [c,d]. Then the length of the graph of f between (c, f(c)) and (d, f(d)) is given by

$$L = \int_c^d \sqrt{1 + (f^1)^2}$$

Ex : Calculate the length of the circumference of a semicircle with radius r.

Solution : $L = \int_{-r}^{r} \frac{r}{\sqrt{r^2 - x^2}}\, dx = \pi\, r\,.$

Note : Let f and f' are continuous on [c,d]. Then t, a real number such that t $\in$ (c,d) such that the length of the graph of f between (c, f(c)) and (d, f(d)) is

$$L = (d - c)\ \sqrt{1 + (f'\,(H)^2}$$

III. Area of a Surface of Revolution

Let f be continuous on [c,d] and f(t) $\geq$ 0 for t $\in$ [c,d]. Then the area of the surface obtained by rotating the graph of f in [c,d] about the x-axis is given by

$$A = \int_c^d 2\pi\, f\, \sqrt{1 + (f')^2}\, dx\ .$$

provided the definite integral exists.

Ex: Determine the area of the surface formed by rotating the graph of $\frac{x^2}{2}$ in [0,1] about x-axis.

Solution : $A = 2\pi \int_0^1 \frac{x^2}{2}\sqrt{1+x^2}\, dx$

$$= \pi\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{8} - \left(\frac{1}{8}\right)\log(1+\sqrt{2})\right) \approx 1.32$$

IV. Volume of solid of Revolution

Let f is continuous on [c,d] and $f(t) \geq 0$ for $t \in [c,d]$. Then the volume of the solid obtained by rotating (about x-axis), the region bounded by the graph of f, ordinate s at c and d and x-axis is given by

$$V = \int_c^d \pi f^2\, dx$$

Ex : Compute volume of the solid of revolution obtained by rotating about x-axis, the region bounded by the graph of x^2, ordinate s at 1 and 2 and x-axis.

Solution : $V = \int_1^2 \pi x^4\, dx = \frac{31\pi}{5}$

V. Average value of a function

The average value of f over [a,b] is

$$A_v = \frac{1}{(b-a)}\int_a^b f\, dx$$

Ex : Compute the average value of sin x over $[0, \pi]$

Solution: $A_v = \frac{1}{\pi}\int_0^\pi \sin x\, dx = \frac{2}{\pi}$

VI. Centre for Gravity : Centroid

Let f be continuous on [c,d] and $f(t) \geq 0$ for $t \in [c,d]$. Then the centroid of the region bounded by the graph of f ordinates at c and d and x-axis is $(\bar{x}, \bar{y})$ where

$$\bar{x} = \frac{\int_c^d x\, f(x)\, dx}{\int_c^d f(x)\, dx} \quad \text{and} \quad \bar{y} = \frac{\frac{1}{2}\int_c^d f^2\, f(x) dx}{\int_c^d f(x)\, dx}$$

Example : Determine the centroid of the region bounded by the graph of x^2, ordinates at 2 and 4 and x-axis.

Solution : $\bar{x} = \dfrac{\int_2^4 x^3\, dx}{\int_2^4 x^2\, dx} = \dfrac{45}{14}, \quad \bar{y} = \dfrac{\frac{1}{2}\int_2^4 x^4\, dx}{\int_2^4 x^2\, dx} = \dfrac{186}{35}$

$\therefore \ (\bar{x}, \bar{y}) = \left(\dfrac{45}{14}, \dfrac{186}{35} \right)$

Differential Equations

Sub Units covered

1. Introduction
2. Definition, Order and Degree of a Differential Equation
3. Formation of Differential Equations
4. Solution of a Differential Equation
5. Solving Differential Equations
6. Applications of Differential Equations

Introduction

We come across differential equations in the context of numerous problems studied in different branches of knowledge – science, engineering and economics. Some of these are

a) the motion of a projectile, rocket, satellite or planet
b) the current in an electric circuit
c) the conduction of heat in a rod or a slab
d) the flow of a fluid
e) the rate of decomposition of radioactive substance or the rate of growth of a population
f) the theory of marginal utility in economics
g) the reaction of chemicals
h) the curves having certain geometrical properties.

The mathematical formulation of the above problems (particularly rate problems) gives rise to differential equations. In each of the situations cited above, the objects involved obey certain laws. These laws involve various rates of change of one or more quantities with respect to other quantities. Such rates can be expressed as derivatives and the laws themselves can be mathematically expressed as equations containing derivatives i.e. differential equations. A differential equation which describes any of the above problem is a mathematical formulation of the problem itself. Therefore, solving the differential equation amounts to solving the problem itself.

Some of the greatest mathematicians of the past three centuries who have contributed to the theory and methods of differential equations are Fermat, Newton, the Bernoullis, Euler, Lagrange, Laplace, Gauss, Abel, Hamilton, Liouville, Chebyshev, Hermite, Riemann and Poincare.

Here in this write up, we aim at familiarize the reader the basic terminology and method of solving the most elementary problem in differential equations of the type $\frac{dy}{dx} = f(x, y)$ by separating the variables and most elementary applications of differential equations.

2. Order and Degree of a Differential Equation

Definition : A relation between a function y of a variable x, the variable x and the successive derivatives of y with respect to x upto order n is called *an ordinary differential equation of order n.* But hereafter in this write up, we will call an ordinary differential equation simply as a differential equation. Symbolically, a differential equation is written as $f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots\ldots, \frac{d^ny}{dx^n}\right) = 0$.

Or $f(x, y, y_1, y_2, \ldots\ldots y_n) = 0$ or $f(x, y, y', y'', \ldots\ldots y^{(n)}) = 0$

The following are examples of differential equations of orders indicated against each of them.

Differential Equations	**Order**
$\left(\frac{dy}{dx}\right)^2 + y = 0$	1
$\frac{d^2y}{dx^2} - y^2 + x = 0$	2
$\sqrt{(y_4)^3 + y_2} + \sin x = x^2$	4
$\sin x\, y''' - a^2 y' = x^2$	3

Definition: The *degree* of a differential equation is the degree of the derivative of the highest order in the differential equation, after the equation is written in a form free from radicals and fractions.

Example 1 : The equation $x\left(\frac{d^3y}{dx^3}\right)^4 + y\left(\frac{dy}{dx}\right)^5 - \sin x = 0$ is an equation of order 3 and degree 4, because of the degree of $\frac{d^3y}{dx^3}$, the derivative of the highest order is 4.

Example 2 : $\frac{d^3y}{dx^3} = \left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}$.

As this equation contains an expression with exponent 3/2, so by squaring both sides, we get,

$$\left(\frac{d^3y}{dx^3}\right)^2 = \left\{1+\left(\frac{dy}{dx}\right)^2\right\}^3$$

i.e. $(y_3)^2 = \{1 + y^2\}^3$

i.e. $(y_3)^2 = 1 + 3y_1^2 + 3y_1^4 + y_1^6$

Here the degree of the order of the highest order derivative i.e. y_3 is 2.

So the equation is of order 3 and degree 2.

Example 3 : Find the order and degree of the equation

$$y = px + \sqrt{2p^2 + a^2} \quad \text{where } p = \frac{dy}{dx}.$$

Solution : $y = px + \sqrt{2p^2 + a^2}$

i.e. $y - px = \sqrt{2p^2 + a^2}$

i.e. $(y - px)^2 = 2p^2 + a^2$

i.e., $y^2 - 2px + p^2x^2 = 2p^2 + a^2$

i.e, $(x^2 - 2)\,p^2 - 2xp + y^2 - a^2 = 0$

i.e., $(x^2 - 2)\left(\frac{dy}{dx}\right)^2 - 2x\left(\frac{dy}{dx}\right) + (y^2 - a^2) = 0$

So the equation is order 1 and degree 2.

Exercises for Self Evaluation

Find the order and the degree of each of the following equations.

1. $x\left(\frac{dy}{dx}\right) + \frac{4}{y\left(\frac{dy}{dx}\right)} = y^3$

2. $\left(\frac{d^3y}{dx^3}\right)^4 + \left(\frac{dy}{dx}\right)^5 = \sin^2 x.$

3. $\left\{\frac{d^2y}{dx^2} + \sin x\right\}^{5/3} = \frac{2dy}{dx}$

4. $y_2 + y_1^2 + 1 = 0$

5. $(p^2 + p - 1)^{1/3} = \cos x$ where $p = \frac{dy}{dx}$.

Answers : 1. 1,2, 2. 3,4, 3. 2, 15, 4. 2,1 5. 1,3

3. Formation of Differential Equations

Suppose an equation, which represents a family of curves contains n arbitrary constants. Differentiate the given equation n times successively. Now we have, along with the given equation, n more equations. We can eliminate n arbitrary constants for those n+1 equations. The equation so obtained is the differential equation of order n representing the family of given curves. Illustrations are given below.

Ex.1. Find the differential equation of the family of all straight lines of a given slope.

Solution: Let the given slope be m. Then any line of the family is represented by the equation y = mx + c where c is arbitrary constant. (Note that m is not an arbitrary constant as m is the given slope).

$$y = mx + c \quad (1)$$

So, y = mx + c (1)

Differentiating (1) w.r.t. x, we get,

y' = m (2)

which is the required differential equation representing all straight line with slope m.

Example 2 : Find the differential equation of the family of all circles passing through the origin.

Solution : Any circle passing through the origin (0,0) is given by

$$x^2 + y^2 + 2gx + 2fy = 0 \qquad (1)$$

Here g and f are arbitrary constants.

Differentiating (1) twice w.r.t. x, we get the following two equations.

$$2x + 2yy' + 2g + 2fy' = 0 \qquad (2)$$

$$2 + 2yy'' + 2y'^2 + 2fy'' = 0 \qquad (3)$$

Eliminating g and f from (1), (2) and (3), we get

$$\begin{vmatrix} x^2 + y^2 & 2x & 2y \\ 2x + 2yy' & 2 & 2y' \\ 2 + 2yy'' + 2y' & 0 & 2y'' \end{vmatrix} = 0$$

i.e. $$\begin{vmatrix} x^2 + y^2 & 2x & 2y \\ x + yy' & 1 & y' \\ 1 + yy'' + y'^2 & 0 & y'' \end{vmatrix} = 0$$

i.e. $(x^2 + y^2)\, y'' - 2x\,(xy'' + yy'\, y'' - y - yy'y''' - y'^3) - 2y(1 + yy'' + y'^2)$

i.e., $-x^2y'' - y^2\, y'' + 2xy' + 2xy'^3 - 2y - 2yy'^2 = 0$

i.e. $(x^2 + y^2)\, y'' - 2xy' - 2xy'^3 + 2yy'^2 + 2y = 0$

which is the required differential equation.

Note : Here in the equation for the family contain two arbitrary constants g and f. So the resulting differential equation is of second degree.

Example 3 : Find the differential equation for the family of curves

$y = A \cos x + B \sin x$.

Solution : Here A and B are arbitrary constants.

Now, $y = A \cos x + B \sin x \qquad (1)$

Differentiating (1) twice w.r.t. x, we get

$y' = - A \sin x + B \cos x \qquad (2)$

$y'' = -A \cos x - B \sin x \qquad (3)$

Now (3) can be written as

$$y'' = -(A \cos x + B \sin x) \qquad (4)$$

So from (1) and (4) we get

$$y'' = -y$$

i.e. $y'' + y = 0$ which is the required differential equation.

Note : It is customary to set the above exercise in a class test as follows :

Eliminate A and B from the equation $y = A \cos x + b \sin x$.

The phrase "formation of a differential equation" has another meaning as well. This meaning is different from the one given above. Here a differential equation is formed to give a mathematical description of a situation in physics, biology, economics or any other discipline. Examples are given below. These examples are similar to those given in the 'Introduction of this write up'.

Example 4 : The population of a country increases at the rate proportional to the population. Form the differential equation to represent this situation.

Solution : Let x be the population in t years. Then $\frac{dx}{dt} = kx$ $(k > 0)$ is the required differential equation.

Example 5 : A savings account pays 5% interest per year. Besides, the income from another investment is credited to the account continuously at the rate of Rs.800/- per year. Form the differential equation to describe this situation.

Solution : Let $x = x(t)$ denote the amount in rupees in the account after t years. Then,

$$\frac{dx}{dt} = \frac{5}{100} x + 800$$

which is the required differential equation.

Example 6 : Radium disappears at a rate proportional to the amount present. Express this model as a differential equation.

Solution : Let x be the amount of radium at time t. Then the required differential equation is

$$\frac{dx}{dt} = -kx \quad (k > 0)$$

Exercises for Self Evaluation

1. Form the differential equation for the family of all circles with centers on the y-axis.
2. Form the differential equations for the family of all hyperbolas with the coordinate as asymptotes.
3. Form the differential equation by eliminating A and B from the equation $(x - A)^2 + (y - B)^2 = 1$
4. A spherical rain drop evaporates at a rate proportional to its surface area. Form a differential equation corresponding to this situation.

Answers :

1. $y' (x^2 + y^2 - 1 - yy'' + y'^2) = 0$
2. $y + xy' = 0$
3. $(y'^3 + y')^2 + (y'^2 + 1)^2 = y''^2$
4. $\frac{dr}{dt} = -k$

4. Solution of a Differential Equation

Definition : A function y = f(x) is called *a solution of a given differential equation* which when substituted in the differential equation reduces the equation into an identity.

A solution which contains a number of arbitrary constants equal to the order of the given differential equation is called *a general solution or a complete integral* of the differential equation. Solutions obtained from the general solution by giving particular values to the constants are known as *particular solutions* or particular integrals.

Example 1 : Verify if the function $y = Ae^{2x} + Be^{-2x}$ is a general solution of the differential equation $y'' = 4y$.

Solution : Consider the function $y = Ae^{2x} + Be^{-2x}$ (1)

Where A and B are constants.

Differentiating (1) successively twice w.r.t. x, we get

$$y' = 2Ae^{2x} - 2Be^{-2x} \qquad (2)$$

$y'' = 4Ae^{2x} + 4Be^{-2x}$

$= 4(Ae^{2x} + Be^{-2x})$ (3)

Substituting the values of y and y″ form (1) and (3) in the given differential equation $y'' = 4y$, we find that the given differential equation becomes an identity for

L.H.S. of the given equation

$y'' = 4(Ae^{2x} + Be^{-2x})$ by (3)

$= 4y$ by (1)

Also (1) contains A and B as *two arbitrary constants* and the *order* of the differential equation is *two*. So the solution (1) is a general solution.

Example 2 : Verify if $y = \sin x$ is a solution of the differential equation $y^2 - y'^2 = 1$

Solution : $y = \sin x$ (1)

$\Rightarrow$ $y' = \cos x$ (2)

Substituting from (1) and (2) in $y^2 - y'^2 = 1$ (3)

We find that (3) *does not reduce to an identity.*

For L.H.S. of (3) $= \sin^2 x - \cos^2 x$

$= -\cos 2x$

$\neq$ R.H.S. of (3)

as $-\cos 2x \neq 1$ for all real values of x.

Hence (1) is *not a solution* of (3).

Example 3 : Verify if $y = \sin x$ is a general solution of $y^2 + y'^2 = 1$.

Solution : Now $y = \sin x$ (1)

So, $y' = \cos x$ (2)

(1) and (2) imply $y^2 + y'^2 = \sin^2 x + \cos^2 x = 1$

So the given differential equation reduces to an identity.

But the function (1) does not contain any arbitrary constant, whereas the given differential equation is of order (1). So (1) is a particular solution, *not a general one.*

Note: In some problems one is asked to "verify that" ..... In such cases that what is to be verified is supposed to hold. So in this sense the phrase "verify that" mean "prove that". But in the phrase "prove that" has a larger connotation than that of "verify that".

Initial Value Problem and its Solution

We have seen earlier that $y'' + y = 0$ has the general solution $y = A \cos x + B \sin x$. Suppose that $y(0) = 4$, $y'(0) = 5$. Then we have $A = 4$, $B = 5$. So $y = 4 \cos x + 5 \sin x$ is a particular solution. This particular solution satisfies the conditions $y(0) = 4$, $y'(0) = 5$.

Definition : A given differential equation with additional condition(s) as in the above paragraph is known as

Initial Value Problem

Thus $y'' + y = 0$ with $y(0) = 4$, $y'(0) = 5$ is an initial value problem.

This initial value problem is written as

$y'' + y = 0$ - the differential equation

$y(0) = 4, y'(0) = 5$ - the initial conditions.

The condition(s) in the initial value problem is called the *initial conditions* of the problem

$y = 4 \cos x + 5 \sin x$ is the solution of the initial value problem :

$y'' + y = 0$ with $y(0) = 4$, $y'(0) = 5$.

Geometrical Meaning of an I.V.P.

A differential equation represents a family of curves which is given by the general solution of the differential equation. As is clear from the above example, an I.V.P. corresponds to a particular solution of the differential equation. So the solution of an *I.V.P. represents a subset of the family of curves represented by the differential equation without any condition whatsoever attached.*

Exercises for Self Evaluation :

Verify if the following functions are solutions of the corresponding differential equations written against them (Prob 1 to 4) :

1. $y^2 = 4ax, \quad y = xy' + a\left(\dfrac{dx}{dy}\right)$
2. $y = a_1 \cos 2x + a_2 \sin 2x, \quad y'' + 4y = 0$
3. $y = ax^3 + bx^2, \quad y''' = 6a$

4. $y = e^{3x}(A + Bx)$, $y'' - 6y' + 8y = 0$

(5), (6), (7), (8) – Verify if the solution (if any) given in (1), (2), (3), (4) are general or particular.

Verify if the function given is a solution of the corresponding I.V.P. (Problems 9 and 10).

9. $x^2 + y^2 = 25$; $yy' + x = 0$, $y(3) = 4$

10. $y = e^{-x}$; $y' + y = 0$, $y(0) = 2$

Assuming the given general solution, find the particular solution satisfying the initial condition (Problems 11 and 12).

11. $xy' = 2y$, $y = ax^2$, $y(1) = 1$

12. $yy' = e^{2x}$, $y^2 = e^{2x} + c$, $y(0) = 1$

Answers :

1. Yes,	2. Yes	3. Yes	4. No	5. General
6. General	7. Particular	8. Not applicable		
9. Yes	10. No	11. $y = x^2$	12. $y^2 = e^{2x} + 1$	

5. Solving Differential Equations

Here we will consider differential equations of the form

$$P(x) \,.\, dx + Q(y) \,.\, dy = 0 \qquad (1)$$

or reducible to the form (1) by some manipulation or by suitable substitution. The solution of (1) is found by integrating both sides of (1). The solution is

$$\int P(x) \,.\, dx + \int Q(y) \,.\, dy = c \qquad (2)$$

where c is an arbitrary constant.

As (1) is a differential equation of first degree, (2) is the general solution of (1).

Example 1 : Solve $\dfrac{dy}{dx} + 2x = e^{3x}$.

Solution : $\dfrac{dy}{dx} + 2x = e^{3x}$

$\Leftrightarrow \quad \dfrac{dy}{dx} + 2x - e^{3x} = 0 \Leftrightarrow dy + (2x - e^{3x})\, dx = 0$

$\Leftrightarrow \quad \int dy + \int (2x - e^{3x} / dx) = C$

$\Leftrightarrow \quad y + \left(x^2 - \dfrac{e^{3x}}{3} \right) = c$

$\Leftrightarrow \quad y = \dfrac{e^{3x}}{3} - x^2 + c$ which is the required solution.

Example 2 : Solve $\dfrac{dy}{dx} = \sin^3 x \,.\, \cos^2 x + xe^x$

Solution :

$$\frac{dy}{dx} = \sin^3 x \,.\, \cos^2 x + xe^x$$

$\Leftrightarrow \quad dy = (\sin^3 x \,.\, \cos^2 x + xe^x)\, dx$

$\Leftrightarrow \quad \int dy = \int (\sin^3 x \,.\, \cos^2 x + xe^x)\, dx$

$\Leftrightarrow \quad y = \int \sin^3 x \,.\, \cos^2 x \,.\, dx + \int xe^x \,.\, dx$ (1)

Now,

$\int \sin^3 x \,.\, \cos^2 x \,.\, dx = - \int (1 - t^2)\, t^2 \,.\, dt$ where t = cos x.

$= - \int (t^2 - t^4) dt = -\left(\dfrac{t^3}{3} + \dfrac{t^5}{5} \right) + a \dfrac{t^5}{5} - \dfrac{t^3}{3} + c_1 = \dfrac{\cos^5 x}{5} - \dfrac{\cos^3 x}{3} + c_1$ (2)

and $\int xe^x\, dx = xe^x - \int e^x \,.\, dx = xe^x - e^x + c_2$ (3)

So from (1), substituting the values of integrals from (2) and (3) in (1), we get

$$y = \frac{1}{5} \cos^5 x - \frac{1}{3} \cos^3 x + xe^x - e^x + (c_1 + c_2)$$

$$\Leftrightarrow y = \frac{1}{5} \cos^5 x - \frac{1}{3} \cos^3 x + xe^x - e^x + c$$

as $c_1 + c_2$ can be treated as single arbitrary constant c.

Solve : $(x+y)^2 \cdot \frac{dy}{dx} = k^2$.

Solution : Put $z = x + y$

Then $\frac{dz}{dx} = 1 + \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{dz}{dx} - 1$.

So the given differential equation reduces to

$$z^2\left(\frac{dz}{dx} - 1\right) = k^2$$

$$\Leftrightarrow \quad z^2 \cdot \frac{dz}{dx} = k^2 + z^2$$

$$\Leftrightarrow \quad \frac{z^2 \cdot dz}{k^2 + z^2} = dx$$

$$\Leftrightarrow \quad \int \frac{z^2}{k^2 + z^2} \cdot dz = \int dx \Leftrightarrow \int\left(1 - \frac{k^2}{k^2 + z^2}\right) \cdot dz = \int dx$$

$$\Leftrightarrow \quad z - k^2 \cdot \frac{1}{k} \tan^{-1} \frac{z}{k} = x + c$$

$$\Leftrightarrow \quad (x+y) - k \tan^{-1}\left(\frac{x+y}{k}\right) = x + c$$

which is the required general solution.

Example 4 : Solve $\frac{dy}{dx} = \frac{x - y + 3}{2(x-y) + 7}$.

Solution : Put $x - y = z$. Then $1 - \frac{dy}{dx} = \frac{dz}{dx}$ i.e. $\frac{dy}{dx} = 1 - \frac{dz}{dx}$.

After making these substitutions for $x - y$ and $\frac{dy}{dz}$, the given differential equation becomes

$$1 - \frac{dz}{dx} = \frac{z+3}{2z+7}$$

$$\Leftrightarrow \quad \frac{dz}{dx} = 1 - \frac{z+3}{2z+7} \Leftrightarrow \frac{dz}{dx} = \frac{3z+4}{2z+7}.$$

$$\Leftrightarrow \quad \left(\frac{2z+7}{3z+4}\right) dz = dx$$

$$\Leftrightarrow \frac{1}{3}\left[\frac{6z+21}{3z+4}\right] dz = dx.$$

$$\Leftrightarrow \frac{1}{3}\left[2 + \frac{13}{3z+4}\right] dz = dx$$

$$\Leftrightarrow \frac{1}{3}\int\left(2 + \frac{13}{3z+4}\right) dz = \int dx.$$

$$\Leftrightarrow \frac{1}{3}\,[2z + (13/3)\log(3z+4)] = x + C$$

$$\Leftrightarrow \frac{2}{3}\,(x-y) + \frac{13}{9}\log(3(x-y)+4) = x + C$$

Example 5 : Solve $(x + y)\,dx - (x - y)\,dy = 0$.

Solution: The given differential equation can be written as

$$\frac{dy}{dx} = \frac{x+y}{x-y}$$

$$\Leftrightarrow \quad \frac{dy}{dx} = \frac{1 + y/x}{1 - y/x} \qquad (1)$$

Put $z = y/x$. then $zx = y \Rightarrow x\,\frac{dz}{dx} + z = \frac{dy}{dx}$.

So (1) can be written as

$$x\,\frac{dz}{dx} + z = \frac{1+z}{1-z}$$

$$\Leftrightarrow \quad x\,\frac{dz}{dx} = \frac{1+z}{1-z} - z \Leftrightarrow x\,\frac{dz}{dx} = \frac{1+z^2}{1-z}.$$

$$\Leftrightarrow \quad \frac{1-z}{1+z^2}\,.\,dz = \frac{dx}{x} \Leftrightarrow \left(\frac{1}{1+z^2} - \frac{z}{1+z^2}\right) dz = \frac{dx}{x}$$

$$\Leftrightarrow \quad \int\left(\frac{1}{1+z^2} - \frac{z}{1+z^2}\right) dz = \int\frac{dx}{x}.$$

$$\Leftrightarrow \quad \tan^{-1} z - ½\log(1 + z^2) = \log x + c$$

$$\Leftrightarrow \quad \tan^{-1}\frac{y}{x} - \frac{1}{2}\log\left(1 + \frac{y^2}{x^2}\right) = \log x + C$$

$\Leftrightarrow \quad \tan^{-1}\frac{y}{x} - \frac{1}{2}\log\left(\frac{x^2+y^2}{x^2}\right) = \log x + C$

$\Leftrightarrow \quad \tan^{-1}\frac{y}{x} - \frac{1}{2}\,[\log(x^2+y^2) - \log x^2] = \log x + C$

$\Leftrightarrow \quad \tan^{-1}\frac{y}{x} - \frac{1}{2}\log(x^2+y^2)\ \frac{1}{2}\ .\ \log x^2 = \log x + C$

$\Leftrightarrow \quad \tan^{-1}\frac{y}{x} - \log\sqrt{x^2+y^2} - \frac{1}{2}\ .\ 2\log x = \log x + C$

$\Leftrightarrow \quad \tan^{-1}\frac{y}{x} - \log\sqrt{x^2+y^2} = 2\log x + C$

which is the required solution.

Exercises for Self Evaluation

Solve the following differential equations.

1. $(1+y^2)\,dx + (1+x^2)\,dy = 0.$
2. $x(y^2+1)\,dx + y(x^2+1)\,dy = 0$
3. $\frac{dy}{dx} = \frac{x^2+x+1}{y^2+y+1}$
4. $x^2\frac{dy}{dx} + y = 1$
5. $\frac{dy}{dx} + \frac{y(y-1)}{x(x-1)} = 0$
6. $y\,dx + (1+x^2)\tan^{-1}x\,dy = 0$
7. $e^{x-y}\,.\,dx + e^{y-x}\,.\,dy = 0$
8. $\frac{dy}{dx} + 1 = e^{x+y}$
9. $\frac{dy}{dx} = \sqrt{y-x}$
10. $\frac{dy}{dx} + \frac{y^2+y+1}{x^2+x+1} = 0$
11. $\sin^{-1}\left(\frac{dy}{dx}\right) = x + y$
12. $x\,.\cos^2 y\,.\,dx - y\,.\cos^2 x\,.\,dy = 0$
13. $(x^2+y^2)\,dy = xy\,dx$

14. $(1 + 3e^{x/y})\,dx + 3e^{x/y}\left(1 - \frac{x}{y}\right).\,dy = 0.$

Answers :

1. $\tan^{-1} x + \tan^{-1} y = C.$
2. $(x^2 + 1)(y^2 + 1) = C.$
3. $\frac{1}{3}(x^3 - y^3) + \frac{1}{2}(x^2 - y^2) + (x - y) = C$.
4. $y = 1 + C\,e^{1/x}$
5. $xy = C(x - 1)(y - 1)$
6. $y \tan^{-1} x = C$
7. $e^{2x} + e^{2y} = C$
8. $e^y = \frac{1}{2} e^x + Ce^{-x}$
9. $\sqrt{y - x} + \log(\sqrt{y - x} - 1) = \frac{1}{2} x + C$
10. $2xy + x + y + C(x + y + 1) = 1$
11. $\tan(x + y) - \sec(x + y) = C$
12. $x \tan x - \log \sec x = y \tan y - \log \sec y + c$
13. $y = C\,e^{(x^2/2y^2)}$
14. $x + 3ye^{x/y} = C$

6. Applications of Differential Equations

As we have seen earlier that differential equations occur in subjects other than mathematics e.g. physics, chemistry, biology and economics. We form differential equations to describe problems of these subjects. Solving these differential equations give the solution to the problems and issues therein. Here we will discuss some most elementary solutions (problems/issues of these subjects) as illustrations.

Example 1 : The temperature T of a cooling object drops at a rate proportional to the difference T – S, where S is the constant temperature of the surrounding medium. Find an expression for the temperature as a function T(t) of the time variable t, provided T(0) = 150.

Solution : Clearly, we have the differential equation describing the situation of the problem as

$$\frac{dT}{dt} = -k\,(T - S) \text{ where } k > 0 \text{ is a constant.} \qquad (1)$$

We know from the problem that S is constant.

So

$$(1) \quad \Rightarrow \quad \frac{dT}{T-S} = -k\,.\,dt$$

$$\Rightarrow \quad \int\frac{dT}{T-S} = \int(-k)dt \Rightarrow \log\,(T - S) = -kt + c \qquad (2)$$

Now, given that T(0) = 150

So, (2) $\Rightarrow \log(150 - S) = 0 + C = C$ (3)

$\therefore$ (2) and (3) imply $\log(T - S) = -kt + \log(150 - C)$

$\Rightarrow \quad \log(T - S) - \log(150 - C) = -kt$

$$\Rightarrow \quad \log\frac{T-S}{150-S} = -kt$$

$$\Rightarrow \quad \frac{T-S}{150-S} = e^{-kt}$$

$$\Rightarrow \quad T = (150 - S)\,e^{-kt} + S$$

which is the required function.

Example 2 : The population of a country increases at a rate proportional to its population. If the population doubles in 30 years, in how many years will it treble?

Solution : Let x be the population in t years.

Then $\frac{dx}{dt} = kx\,.\,(k > 0) \quad \Rightarrow \quad \frac{dx}{x} = kdt$

$$\Rightarrow \quad \log x = k\,t + \log c \Rightarrow \log x - \log c = kt$$

$$\Rightarrow \quad \log\frac{x}{c} = kt$$

$$\Rightarrow \quad \frac{x}{c} = e^{kt} \quad \Rightarrow \quad x = ce^{kt} \qquad (1)$$

Let $x(0) = x_0$. Then from (1), $x_0 = c$

So $x = x_0e^{kt}$ (2)

Now by hypothesis of the problem, $x(30) = 2x_0$ (3)

(as the population doubles in 30 years).

Now (2) and (3) imply $x_0e^{30k} = 2x_0 \Rightarrow e^{30k} = 2$ (4)

Let after T year x_0 becomes $3x_0$ (i.e. treble)

i.e. $x(T) = 3x_0$

i.e., $x_0e^{kt} = 3x_0$ from (2) $\Rightarrow e^{kt} = 3$ (5)

$$\left.\begin{array}{l}(4) \text{ and } (5) \Rightarrow 30k = \log_e 2 \\ kT = \log_e 3\end{array}\right\} \Rightarrow \frac{T}{30} = \frac{\log_e 3}{\log_e 2} \Rightarrow T = 30 \ (\log_e 3 / \log_e 2) \text{ years.}$$

Example 3: After how many years will a sum of money, invested for interest at 5% continuously compounded double itself?

Solution : Let x be the amount of money after t years.

Then we have $\frac{dx}{dt} = \frac{5}{100}x \Rightarrow \frac{dx}{x} = \frac{dt}{20}$.

$$\Rightarrow \quad \int\frac{dx}{x} = \int\frac{dt}{20} \Rightarrow \quad \log_e x = \frac{t}{20} + \log C$$

$$\Rightarrow \quad \log\frac{x}{C} = \frac{t}{20} \Rightarrow x = Ce^{t/20} \qquad (1)$$

Let $x(0) = x_0 \Rightarrow x_0 = C$ (2)

(1) and (2) $\Rightarrow x = x_0 e^{t/20}$ (3)

Let the amount double itself after T years.

Then $x(T) = 2x_0 \Rightarrow 2x_0 = x_0 e^{T/20}$

$$\Rightarrow \quad 2 = e^{T/20}$$

$$\Rightarrow \quad \frac{T}{20} = \log_e 2 \Rightarrow T = 20 \log_e 2 \text{ years.}$$

Hence the amount doubles itself after 20 $\log_e 2$ years.

Example 4 : A particle with initial velocity u, moves in a straight line with uniform acceleration f. Express the velocity v and the distance s after time t as functions of time variable t.

Solution : From the definitions, velocity $v = \frac{ds}{dt}$ and acceleration $= \frac{dv}{dt} = \frac{d^2s}{dt^2}$.

By hypothesis we have

$$\frac{dv}{dt} = f \Rightarrow dv = kf \cdot dt$$

$$\Rightarrow \quad \int dv = \int f \cdot dt$$

$\Rightarrow$ v(t) = ft + c (1)

By hypothesis v(0) = u

So from (1), u = v(0) = f. 0 + c

$\Rightarrow$ c = u (2)

(1) and (2) $\Rightarrow$ v(t) = ft + u (3)

$$\Rightarrow \frac{ds}{dt} = ft + u \Rightarrow \int ds = \int (ft+u) \cdot dt$$

$\Rightarrow$ $s(t) = \frac{ft^2}{2} + ut + c'$ (4)

But by hypothesis s(0) = 0

So (4) $\Rightarrow$ 0 = s(0) = 0 + 0 = c′ $\Rightarrow$ c′ = 0 (5)

(4) and (5) imply $s(t) = \frac{ft^2}{2} + ut$ (6)

Hence (6) and (3) give the required expression for s and t respectively.

Exercises for Self Evaluation

1. A hot body cools at a rate proportional to the difference in temperature between it and its surroundings. The body is heated to 110° C and placed in air at 10°C. After 1 hour its temperature is 60° C. How much additional time is required for it to cool to 30° C ?
2. A population grows at a rate of 5% per year. How long does it take for the population to double ?
3. A spherical rain drop evaporates at a rate proportional to its surface area. If its radius originally is 4/3 mm and 1 hour later has been reduced to 3mm, find an expression for the radius of the rain drop at any time ?

4. After how many years will Rs.100/- invested at the rate of 5% interest continuously compounded, amount to Rs.1000/- ?

Answers :

1. $\dfrac{\log_e 5}{\log_e 2} - 1$ hours.

2. 20 $\log_e 2$ years

3. r = 4 – t

4. $\dfrac{\log_e 10}{20}$ years.

MATHEMATICAL INDUCTION

Introduction

Principle of Mathematical Induction is a method of proof for propositions which are statements for a general natural number n. Many a theorems which are otherwise complicated to be proved can be easily proved by Principle of Mathematical Induction.

Any proposition involving the positive integer 'n' if true for a large number of positive integers, cannot be assumed to be true for all positive integers.

For eg. "10n + 3 (where n is a positive integer) is a prime number" for positive integers 'n' which are not divisible by three.

This statement is true for many values of n. But it fails, for example, n =14 (not divisible by 3). Then 10n+3 = 143 is not a prime number.

Notation

Any proposition or statement in n is denoted by P(n).

Eg. 1. The statement "n(n+1) is an even integer" is denoted by P(n).

If n = 1. then n(1+1) = 2 an even integer is denoted by P(1).

Similarly, the statement for n = 2 is denoted by P(2) etc.

2. Consider the statement P(n)

i.e. P(n) : " $n^3 + n$ is not divisible by 3".

P(1) : $1^3 + 1 = 2$ not divisible by 3.

$\therefore$ P(1) is true.

P(2) : $2^3 + 2 = 10$ not divisible by 3.

$\therefore$ P(2) is true.

But P(3) : $3^3 + 3 = 30$ divisible by 3.

$\therefore$ P(3) is not true.

$\therefore$ The method of Mathematical Induction does not suit such 'statements' which are not true for all positive integers n.

The Principle of Mathematical Induction

First Step :

In this method, let P(n) be a statement to be proved to be True for all $n \geq a$, a being certain positive integer. Verify that P(a) is true.

II Step :

The proposition is assumed for a positive integer say $n = k$.

i.e. P(k) is assumed.

III Step :

To prove that P(k + 1) is true whenever P(k) is true i.e. for the next integer $n - k + 1$.

IV Step : Conclusion

Hence, by the Principle of M.1, P(n) holds for all $n \geq a$.

Note: While working problems, we have to follow the above four steps.

Examples :

1. P(n) is the statement that $3^n > n$ for all $n \in N$.

   $P(n) : 3^n > n$

I Step :

$n = 1 \quad P(1) : 3 > 1$ true.

II Step :

P(n) is assumed for $n = k$ i.e. P(k) is assumed.

$P(k) \; : 3^k > k \qquad (1)$

III Step :

To prove that P(k + 1) is true

i.e. to prove that $3^{k+1} > k + 1$

Multiplying (1) both sides by 3, we get

$3^{k+1} > 3k \; [= k + k + k)]$

$> k + (k + 1)$

$> k + 1$

$\Rightarrow 3^{k+1} > k + 1$ which is P(k + 1).

IV Step :

By principle of Mathematical Induction P(n) is true for all positive integers n.

2. Prove that 7 divides $2^{3n}-1$ for all positive integers

Let P(n) be the statement 7 divides $2^{3n} - 1$.

I Step :

P(1) : 7 divides $2^3 - 1$

$\Rightarrow$ 7 divides 8 – 1.

$\Rightarrow$ 7 divides 7 which is true.

II Step :

P(k) is assumed i.e. for n = k, i.e. 7 divides $2^{3k} - 1$ is assumed.

$\Rightarrow 2^{3k} - 1 = 7m$ where $m \in N$ (natural numbers).

III Step :

To prove P(k + 1)

i.e. to prove that 7 divides $2^{3(k+1)} - 1$.

i.e. to prove that $2^{3(k+1)} - 1$ is a multiple of 7.

Consider

$$
\begin{aligned}
& 2^{3(k+1)} - 1 \\
= \quad & 2^{3k} . 2^3 - 1 \\
= \quad & 8 . 2^{3k} - 8 + 7 \\
= \quad & 8(2^{3k} - 1) + 7 \\
= \quad & 8 . 7m + 7 \\
= \quad & 7(8m + 1) \text{ which is a multiple of 7.}
\end{aligned}
$$

IV Step :

$\therefore$ By principle of Mathematical Induction, P(n) is true for all $n \in N$.

3. Prove that

$$1 + 4 + 7 + \ldots\ldots + (3n - 2) = \frac{n(3n-1)}{2}.$$

Let P(n) denote the given proposition

$$\therefore P(n) : 1 + 4 + 7 + \ldots. + (3n - 2) = \frac{n(3n-1)}{2}.$$

I Step :

$$P(1) : 1 = \frac{1(3-1)}{2}$$

$$= \frac{1.2}{2}$$

$$= 1$$

P(1) : 1 = 1 which is true i.e. P(1) is true.

II Step :

Let P(k) be true

i.e. P(k) : $1 + 4 + 7 + \ldots + (3k-2) = \frac{k\,(3k-1)}{2}$ is assumed.

III Step :

To prove that P(k + 1) is true i.e. to prove that

$$1 + 4 + 7 + \ldots + (3k-2) + (3k+1) = \frac{(k+1)\,[3(k+1)-1]}{2} = \frac{(k+1)(3k+2)}{2}.$$

To prove this, we have to add (3k + 1) to both sides of P(k) . ∴ We get,

$$1 + 4 + 7 + \ldots. + (3k-2) + (3k+1) = \frac{k(3k-1)}{2} + (3k+1)$$

$$= \frac{3k^2 - k + 6k + 2}{2}$$

$$= \frac{3k^2 + 5k + 2}{2}$$

$$\Rightarrow \quad 1 + 4 + 7 + \ldots. + (3k+1) = \frac{(k+1)\,(3k+2)}{2}$$ which is the required result

i.e. P(k + 1).

IV Step :

By principle of Mathematical Induction, P(n) is true for all positive integers 'n'.

Note: In the given problem, the terms are in A.P. with common difference 3. Therefore, any term is got by adding 3 to its previous.

4. Prove by induction :

$$2 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 2$$

Let P(n) denote $2 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 2$

I Step :

$P(1) : 2 = 2^{1+1} - 2$

$\Rightarrow 2 = 4 - 2$

$\Rightarrow 2 = 2$ which is true i.e. P(1) is true.

II Step : Let P(k) be assumed.

i.e. P(k) : $2 + 2^2 + 2^3 + \ldots + 2^k = 2^{k+1} - 2$ is assumed.

III Step : To prove P(k +1) i.e. for n = k + 1.

i.e. to prove that $2 + 2^2 + 2^3 + \ldots + 2^{k+1} = k + 1 + 1 = 2 - 2 = 2^{k+2} - 2$

Adding 2^{k+1} to both sides of P(k) we get [Note that 2^{k+1} is the next term after 2^k]

$$\begin{aligned} 2 + 2^2 + 2^3 + \ldots + 2^k + 2^{k+1} &= (2^{k+1} - 2) + 2^{k+1} \\ &= 2 \, . \, 2^{k+1} - 2 \\ &= 2^{1+k+1} - 2 \\ &= 2^{k+2} - 2 \end{aligned}$$

$\Rightarrow 2 + 2^2 + 2^3 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 2$

is the required result i.e. P(k +1) is true.

IV Step : Conclusion

∴ By principle of Mathematical Induction, P(n) is true for all positive integral values of n.

5. Prove that 1.2 + 2.3 + 3.4 + ...upto n terms = $\dfrac{n\,(n+1)\,(n+2)}{3}$.

Consider: 1.2 + 2.3 + 3.4 + ... upto n terms.

In this, each term contains 2 factors. Observing the first factor in each term, we see that they are consecutive integers. Therefore in the n^{th} term, the first factor will be n.

Second factor in each term is one greater than the first factor. Therefore, in the n^{th} term, the 2^{nd} factor will be (n + 1). Hence the n^{th} term will be n(n+1).

Therefore, the problem is to prove that

$$1.2 + 2.3 + 3.4 + \ldots + n(n+1) = \frac{n(n+1)(n+2)}{3}.$$

Let this proposition be denoted by P(n).

I Step :

$$P(1) : 1.2 = \frac{1(1+1)(1+2)}{3}$$

$$\Rightarrow 2 = \frac{2 \times 3}{3} \Rightarrow 2 = 2 \text{ which is true i.e. } P(1) \text{ is true.}$$

II Step : P(k) is assumed i.e. n = k.

i.e. $P(k) : 1.2 + 2.3 + 3.4 + \ldots + k.(k+1) = \frac{k(k+1)(k+2)}{3}$ is assumed.

III Step :

To prove that P(k + 1).

i.e. to prove that

$$1.2 + 2.3 + 3.4 + \ldots + (k+1)(k+2) = \frac{(k+1)(k+2)(k+3)}{3}.$$

To prove this result, add (k +1) (k+2) to both sides of P(k).

i.e. $1.2 + 2.3 + 3.4 + \ldots + k(k+1) + (k+1)(k+2) = \frac{k(k+1)(k+2)}{3} + (k+1)(k+2).$

$$= (k+1)(k+2)\frac{[k+3]}{3}$$

i.e. $1.2 + 2.3 + 3.4 + \ldots + (k+1)(k+2) = \frac{(k+1)(k+2)(k+3)}{3}$

i.e. P(k + 1) is true.

IV Step :

By Principle of Mathematical Induction, P(n) is true for all positive integers 'n'.

6. Prove that

$$\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+....+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}.$$

Let the given proposition be denoted by P(n).

I Step :

$$P(1):\frac{1}{1.3}=\frac{1}{(2\times 1)+1}=\frac{1}{3}$$

$\Rightarrow \frac{1}{3}=\frac{1}{3}$ is true, i.e. P(1) is true.

II Step :

Let P(k) be assumed.

i.e. $P(k):\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+....+\frac{1}{(2k-1)(2k+1)}=\frac{k}{2k+1}$ is assumed.

III Step : To prove P(k+1)

i.e. to prove that

$$\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+....+\frac{1}{(2k+1)(2k+3)}=\frac{k+1}{2(k+1)+1}=\frac{k+1}{2k+3}.$$

To prove this, add $\frac{1}{(2k+1)(2k+3)}$ to both sides of P(k) . We get

$$\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+....+\frac{1}{(2k-1)(2k+1)}+\frac{1}{(2k+1)(2k+3)}=\frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)}$$

$$=\frac{k(2k+3)+1}{(2k+1)(2k+3)}$$

$$=\frac{2k^2+3k+1}{(2k+1)(2k+3)}$$

$$=\frac{(2k+1)\ (k+1)}{(2k+1)\ (2k+3)}=\frac{k+1}{2k+3}$$

i.e. $\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+....+\frac{1}{(2k+1)(2k+3)}=\frac{k+1}{2k+3}$

i.e. P(k + 1) is proved.

IV Step :

By Principle of Mathematical Induction, P(n) is true for all natural numbers 'n'.

Exercises :

1. Prove by induction

   i) $10^n + 3.4^{n+2} + 5$ is divisible by 9.

   ii) n(n+1) (n+2) is divisible by 3.

   iii) $3^n > 2^n$.

2. Prove the following by mathematical induction.

   i) $1 + 2 + 3 + \ldots + n = \dfrac{n(n+1)}{2}$.

   ii) $1^2 + 2^2 + 3^2 + \ldots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

   iii) $1^3 + 2^3 + 3^3 + \ldots + n^3 = \dfrac{n^2(n+1)^2}{4}$.

   iv) $1.3 + 2.4 + 3.5 + \ldots + n(n+2) = \dfrac{n(n+1)(2n+7)}{6}$.

   v) $1^2 + 3^2 + 5^2 + \ldots$ upto 'n' terms $= \dfrac{n(4n^2-1)}{3}$.

   vi) $\dfrac{1}{1.2} + \dfrac{1}{2.3} + \dfrac{1}{3.4} + \ldots + \dfrac{1}{n(n+1)} = \dfrac{n}{n+1}$.

3. 1.2.3 + 2.3.6 + 3.4.9 + ... upto n terms

   $$= \frac{n(n+1)(n+2)(3n+1)}{4}$$

   Hint : The n^{th} term is n(n +1) . 3n.

LOGARITHMS

Introduction

Logarithms were very useful for the purposes of calculations when we had not heard of calculator or computer.

We know that addition and subtraction are much easier when compared to multiplication and division.

Logarithms convert multiplication and division to addition and subtraction.

Consider the equation $a^x = b$ where a and b are real numbers.

We know that a^x is defined whenever 'a' is positive, in which case a^x is also positive.

However, when a = 1, the equation $a^x = b$ is meaningless unless b = 1.

When a = 1, b = 1, the equation $a^x = b$ becomes the identity (i.e. 1 = 1).

$\therefore$ The solution of $a^x = b$ exists when $a > 0$ and $b > 0$. Then the unique solution of $a^x = b$ is called the logarithm of 'b' to the base 'a' and is written as $x = \log_a b$.

$\therefore$ $a^x = b \Leftrightarrow x = \log_a b$.

$a^x = b$ is called the Index form.

$x = \log_a b$ is called the logarithmic form.

We must be able to write immediately from Index form to logarithmic form and vice versa.

Examples :

1. $2^4 = 16 \Leftrightarrow \log_2 16 = 4$
2. $3^3 = 27 \Leftrightarrow \log_3 27 = 3$
3. $4^2 = 16 \Leftrightarrow \log_4 16 = 2$
4. $\log_5 25 = 2 \Leftrightarrow 5^2 = 25$
5. $\log_{\sqrt{2}} 32 = 10 \Leftrightarrow (\sqrt{2})^{10} = 32 \Rightarrow 2^5 = 32$
6. $a^0 = 1 \Leftrightarrow \log_a 1 = 0$ (Here $a > 0$ and $a \neq 1$)
7. $a^1 = a \Leftrightarrow \log_a a = 1$

8. $\log_a b = b \Leftrightarrow a^b = b$.
9. $a^{-\infty} = \frac{1}{a^{\infty}} = \frac{1}{\infty} = 0 \Rightarrow a^{-\infty} = 0 \Rightarrow \log_a 0 = -\infty$.

In all the above examples, we observe that the base remains as base both in index and logarithmic forms. For example, in (1), 2 is called the base in both the forms. Similarly in other examples,

We shall observe some more examples.

$10^1 = 10 \Leftrightarrow \log_{10} 10 = 1$

$10^2 = 100 \Leftrightarrow \log_{10} 100 = 2$

$10^3 = 1000 \Leftrightarrow \log_{10} 1000 = 3$, etc.

Similarly,

$$10^{-1} = \frac{1}{10} \Leftrightarrow \log_{10}\left(\frac{1}{10}\right) = -1$$

$$10^{-2} = \frac{1}{10^2} \Leftrightarrow \log_{10}\left(\frac{1}{10^2}\right) = -2 \text{ i.e. } \log_{10}\left(\frac{1}{100}\right) = -2$$

$$10^{-3} = \frac{1}{10^3} \Leftrightarrow \log_{10}\left(\frac{1}{10^3}\right) = -3 \text{ i.e. } \log_{10}\left(\frac{1}{1000}\right) = -3$$

In these examples, we observe logarithm of numbers greater than one is positive and logarithm of numbers lying between 0 and 1 is negative.

By knowing these facts, we will be able to draw the graph of the function $y = \log_a x$.

Laws of Logarithms

Rule 1 : $\log_a mn = \log_a m + \log_a n$

Here the product mn is converted to sum

Eg. $\text{Log}_2 2a = \log_2 2 + \log_2 n$

$\text{Log}_a (5 \times 3) = \log_a 5 + \log_a 3$

Similarly, $\log_e a^2 + \log_e b^3 + \log_e c = \log_e (a^2 b^3 c)$

Note $\log_a (m + n) \neq \log_a m + \log_a n$.

Rule 2 : $\log_a \left(\frac{m}{n}\right) = \log_a m - \log_a n$

Eg. $\log_{10}\left(\frac{5}{3}\right) = \log_{10} 5 - \log_{10} 3$

Similarly, $\log_{10} a - \log_{10} b = \log_{10}\left(\frac{a}{b}\right)$

Combining Rule (1) and (2), we get

$\log_a\left(\frac{mn}{pq}\right) = \log_a m + \log_a n - \log_a p - \log_a q.$

Eg. $\log_e\left(\frac{2ab}{3xy}\right) = \log_e 2 + \log_2 a + \log_e b - \log_e 3 - \log_3 x - \log_e y.$

Note : $\frac{\log m}{\log n} \neq \log\left(\frac{m}{n}\right).$

Rule 3 : $\log_a (m^n) = n \log_a m$

Eg. 1. ~~$\log_e (a^5) = 5 \log_e a.$~~

2. $\log_a (a^5) = 5 \log_a a$

$= 5$

3. $\log_{10} \sqrt{2} = \log_{10} 2^{1/2} = ½ \log_{10} 2$

Note: $(\log_a m)^n \neq n \log_a m.$

Rule 4 : (Change of Base) $\log_a m = \frac{\log_b m}{\log_b a}.$

Eg. $\text{Log}_3 5 = \frac{\log_a 5}{\log_a 3}$

Note : $\log_a m = \frac{\log_m m}{\log_m a} = \frac{1}{\log_m a}.$

$\therefore \log_a m = \frac{1}{\log_m a}.$

Eg. i) $\log_{10} 5 = \frac{1}{\log_5 10}$.

ii) $\log_7 10 = \frac{1}{\log_{10} 7}$

Examples :

1. Find the values of

i) $\log_{\sqrt{2}} 32$

Putting it in the index form

$(\sqrt{2})^x = 32$

$\Rightarrow 2^{x/2} = 2^5$

$\Rightarrow \frac{x}{2} = 5$

$\Rightarrow x = 10$

ii) $\log_{0.1} 10$

Let $x = \log_{0.1} 10$

$\Rightarrow (0.1)^x = 10$

$\Rightarrow \left(\frac{1}{10}\right)^X = 10$

$\Rightarrow (10^{-1})^x = 10$

$\Rightarrow 10^{-x} = 10^1$

$\Rightarrow -x = 1 \Rightarrow x = -1.$

iii) $\frac{\log 625}{\log 5}$

When the base is not mentioned, we may take any common base say 'a'. It may be mentioned or not.

$$\frac{\log 625}{\log 5} = \frac{\log_a (5^4)}{\log_a 5} = \frac{4 \log_a 5}{\log_a 5} = 4$$

Note: $\log_a 5^4$ is of the form $\log_a m^n = n \log_a m$.

iv) $\log_{3\sqrt{2}} 5832$

Let $x = \log_{3\sqrt{2}} 5832$

By definition

$(3\sqrt{2})^x = 5832$ [Index form]

Squaring both sides

$(9 \times 2)^x = (5832)^2$

$18^x = (9^3 . 2^3)^2$

$= (9 \times 2)^6$

$\Rightarrow 18^x = 18^6$

$\Rightarrow x = 6$

2. Find the value of x if

i) $\log_{10}(x - 9) = 2$

Putting it in the index form

$X - 9 = 10^2$

$\Rightarrow \quad x = 9 + 200$

$\Rightarrow \quad x = 109$

ii) $\log_x 16 = \frac{2}{5}$

Writing it in the index form,

$26 = x^{2/5}$

$\Rightarrow 4^2 = (x^{1/5})^2$

$\Rightarrow 4 = x^{1/5}$

$\Rightarrow x^{1/5} = 4$

$\Rightarrow x = 4^5 \Rightarrow x = 1024$

iii) $\log_7 [\log_5 (\sqrt{x+5} + \sqrt{x}) = 0$

Let $y = \log_5 (\sqrt{x+5} + \sqrt{x})$

$\therefore$ Problem becomes

$\log_7 y = 0$

Writing it in the index form,

$y = 7^0$

$\Rightarrow \quad y = 1$

$\Rightarrow \quad \log_5 (\sqrt{x+5} + \sqrt{x}) = 1$

Let $\sqrt{x+5} + \sqrt{x} = z$

$\therefore \log_5 z = 1$

Writing, in the index form,

$z = 5^1$

$\Rightarrow \sqrt{x+5} + \sqrt{x} = 5$

$\Rightarrow \sqrt{x+5} = 5 - \sqrt{x}$

Squaring both sides,

$(\sqrt{x+5})^2 = (5 - \sqrt{x})^2$

$\Rightarrow \quad x + 5 = 25 + x - 10\sqrt{x}.$

$\Rightarrow \quad 10\sqrt{x} = 20$

$\Rightarrow \quad \sqrt{x} = 2$

Squaring both sides

$\Rightarrow x = 4$

iv) $\log_x 9 = 2$

Writing in the index form

$9 = x^2$

$\Rightarrow \quad x = 3,\ x \neq -3$ (cannot be negative).

3. Prove that

i) $\log \frac{x}{y} + \log \frac{y}{z} + \log \frac{z}{x} = 0$

Each term on LHS is of the form $\log \frac{m}{n}$ which is $\log m - \log n$.

$\therefore$ LHS $= \log \frac{x}{y} + \log \frac{y}{z} + \log \frac{z}{x}$

$= (\log x - \log y) + (\log y - \log z) + (\log z - \log x)$

$= 0 =$ RHS

ii) $2\log \frac{16}{15} + \log \frac{25}{24} - \log \frac{32}{27} = 0$

Consider each term separately.

$\log \frac{16}{15} = \log 16 - \log 15$ (using Rule 2).

$= \log(2^4) - [\log(5 \times 3)]$ [Writing each term as powers of lowest integer or product of lowest integers possible).

$\log \frac{16}{15} = 4\log 2 - [\log 5 + \log 3]$ [Using Rule 3 and 1 respectively]

$\therefore 2\log \frac{16}{15} = 8\log 2 - 2\log 5 - 2\log 3$ (1)

Consider the 2nd term namely

$\log \frac{25}{24} = \log 25 - \log 24$

$= \log(5^2) - \log(8 \times 3)$

$= 2 \log 5 - [\log 8 + \log 3]$

$= 2 \log 5 - [\log 2^3 + \log 3]$

$= 2 \log 5 - [3 \log 2 + \log 3]$

$\log \frac{25}{24} = 2 \log 5 - 3 \log 2 - \log 3$. (2)

Consider the 3rd term

i.e. $\log \frac{32}{27} = \log 32 - \log 27$ [using $\log \frac{m}{n} = \log m - \log n$]

$= \log(2^5) - \log_3 3$

$\log \frac{32}{37} = 5 \log 2 - 3 \log 3$ (3)

LHS $= 2 \log \frac{16}{15} + \log \frac{25}{24} - \log \frac{32}{27}$.

Using 1, 2 and 3, we get

LHS $= (8 \log 2 - 2 \log 5 - 2 \log 3) + (2 \log 5 - 3 \log 2 - \log 3) - (5 \log 2 - 3 \log 3)$

$= 0 =$ RHS

iii) $2 \log \frac{6}{7} + \frac{1}{2} \log \frac{81}{16} + \log \frac{196}{27} = \log 12.$

LHS $= 2[\log 6 - \log 7] + \frac{1}{2}[\log 81 - \log 16] + [\log 196 - \log 27]$

$= 2 [\log (2\times3) - \log 7] + \frac{1}{2}[\log_3 4 - \log_2 4] + [\log 14^2 - \log 3^3]$

$= 2[\log 2 + \log 3 - \log 7] + \frac{1}{2}[4 \log 3 - 4 \log 2] + [2 \log 14 - 3 \log 3]$

$= 2 \log 2 + 2 \log 3 - 2 \log 7 + 2 \log 3 - 2 \log 2 + 2 \log 14 - 3 \log 3$

$= \log 3 - 2 \log 7 + 2 \log 14$

$\log 3 - 2 \log 7 + 2 [\log (2 \times 7)]$

$= \log 3 - 2 \log 7 + 2[\log 2 + \log 7]$

$= \log 3 - 2 \log 7 + 2 \log 2 + 2 \log 7$

$= 2 \log 2 + \log 3$

$= \log_2 2 + \log 3 = \log [2^{2'} \times 3] = \log (4 \times 3) = \log 12 = \text{RHS}$

4. i) Show that $\log_7 4, \log_5 7, \log_2 5 = 2$

$\text{LHS} = \log_7 4 \times \log_5 7 \times \log_2 5$

Using the Rule 4 i.e. changing the base, we get

$$\text{LHS} = \frac{\log_a 4}{\log_a 7} \times \frac{\log_a 7}{\log_a 5} \times \frac{\log_a 5}{\log_a 2}$$

$$= \frac{\log_a 4}{\log_a 2} \text{ [After cancellation]}$$

$$= \frac{\log_a 2^2}{\log_a 2}$$

$$= \frac{2 \log_a 2}{\log_a 2} \text{ [Using Rule 3]}$$

$= 2$

ii) If $x = \log_7 27$, $y = \log_5 7$ and $z = \log_3 5$, prove that $xyz = 3$

$y = \log_5 7$

$x = \log_7 27$ $\qquad \Rightarrow y = \dfrac{\log_a 7}{\log_a 5}$

$= \dfrac{\log_a 27}{\log_a 7}$ (rule 4) $\qquad z = \log_3 5$

$= \dfrac{\log_a 3^3}{\log_a 7}$ $\qquad \Rightarrow z = \dfrac{\log_a 5}{\log_a 3}$

$$\Rightarrow \quad x = \frac{3 \log_a 3}{\log_a 7} \quad \text{[Rule 7]}$$

$\therefore$ LHS = xyz

$$\therefore \quad = \frac{3 \log_a 3}{\log_a 7} \times \frac{\log_a 7}{\log_a 5} \times \frac{\log_a 5}{\log_a 3}.$$

On cancelling

$= 3 = \text{RHS}$

iii) Show that $\log_3 4 \times \log_4 5 \times \log_6 7 \times \log_2 8 \times \log_8 9 = 2$

$\text{LHS} = \log_3 4 \times \log_4 5 \times \log_5 6 \times \log_6 7 \times \log_7 8 \times \log_8 9$

Using Rule 4, we get,

$$\text{LHS} = \frac{\log_a 4}{\log_a 3} \times \frac{\log_a 5}{\log_a 4} \times \frac{\log_a 6}{\log_a 5} \times \frac{\log_a 7}{\log_a 6} \times \frac{\log_a 8}{\log_a 7} \times \frac{\log_a 9}{\log_a 8}$$

$$= \frac{\log_a 9}{\log_a 3} \text{ [After canceling]}$$

$$= \frac{\log_a (3^2)}{\log_a 3}$$

$$= \frac{2 \log_a 3}{\log_a 3} \text{ [Using Rule 3]}$$

= 2 = RHS

Note : The common base 'a' need not be mentioned.

iv) Show that $\log_a 2 \times \log_b y = \log_b x \times \log_a y$

LHS = $\log_a x \times \log_b y$

$= \frac{\log_b x}{\log_b a} \times \frac{\log_b y}{\log_b b}$ [using Rule 4 and choosing 'b' as the common base because on RHS, we have $\log_b x$].

$$= \frac{\log_b x}{\log_b a} \times \frac{\log_b y}{1}$$

$$= \log_b x \times \frac{\log_b y}{\log_b a}$$

= $\log_b x \times \log_a y$ [Using Rule 4)

= RHS

5. Prove that

i) $\frac{1}{\log_a n} + \frac{1}{\log_b n} = 1$ if n = ab

$$\text{LHS} = \frac{1}{\log_a n} + \frac{1}{\log_b n}$$

= $\log_n a + \log_n b$ [Note using Under Rule 4]

= $\log_n(ab)$ [Rule 1]

$= \log_{ab} ab$, because n = ab [using $\log_a a = 1$]

$= 1 =$ RHS

ii) $\log_{ab} x = \dfrac{\log_b x}{1 + \log_b a}$

LHS $= \log_{ab} x$

$= \dfrac{\log_b x}{\log_b ab}$

$= \dfrac{\log_b x}{\log_b a + \log_b b}$

$= \dfrac{\log_b x}{\log_b a + 1}$

$= \dfrac{\log_b x}{1 + \log_b a}$ = RHS

iii) $\log_2 [\log_2 \{ \log_3 (\log_3 27^3 \}] = 0$

LHS $= \log_2 [\log_2 \{ \log_3 (\log_3 27^3\}]$

$= \log_2 [\log_2 \{ \log_3 (\log_3 3^9) \}]$

$= \log_2 [\log_2 \{ \log_3 (9 \log_3 3) \}]$ [Rule 3]

$= \log_2 [\log_2 \{ \log_3(9) \}]$ [$\log_a a = 1$]

$= \log_2 [\log_2 \{ \log_3 3^2 \}]$

$= \log_2 [\log_2 \{ 2 \log_3 3\}]$ [Rule 3]

$= \log_2 [\log_2 \{2\}]$ [$\because \log_3 3 = 1$]

$= \log_2 (\log_2 2)$

$= \log_2 1$ [$\because \log_2 2 = 1$]

$= 0$ [$\because \log_a 1 = 0$]

= RHS

6. If $x = \log_c ab$, $y = \log_a bc$, $z = \log_b ca$

Prove that

$$\frac{1}{1+x} + \frac{1}{1+y} + \frac{1}{1+z} = 1$$

We shall calculate each term separately.

Consider

$$\frac{1}{1+x} = \frac{1}{1+\log_c ab} = \frac{1}{\log_c c + \log_c a + \log_c b}$$

[$\because \log_c c = 1$ and using Rule 1]

$$= \frac{1}{\log_c (cab)} = \frac{1}{\log_c (abc)}$$

$= \log_{abc} c$

Similarly, $\frac{1}{1+y} = \log_{abc} a$ and $\frac{1}{1+z} = \log_{abc} b$

$\therefore$ LHS $= \frac{1}{1+x} + \frac{1}{1+y} + \frac{1}{1+z}$

$= \log_{abc} c + \log_{abc} a + \log_{abc} b$

$= \log_{abc} (cab)$

$= \log_{abc} (abc)$

$= 1 =$ RHS

7. If $a^{1/3} + b^{1/3} + c^{1/3} = 0$, show that

$$\log \frac{a+b+c}{3} = \frac{1}{3} (\log a + \log b + \log c)$$

Let $a^{1/3} = A$, $b^{1/3} = B$, $c^{1/3} = C$

$\therefore$ The data becomes $A + B + C = 0$

$\therefore$ $A^3 + B^3 + C^3 = 3ABC$

$\Rightarrow$ $(a^{1/3})^3 + (b^{1/3})^3 + (c^{1/3})^3 = 3a^{1/3} b^{1/3} c^{1/3}$

$\Rightarrow$ $\frac{a+b+c}{3} = (abc)^{1/3}$

Taking logarithm on both sides,

$$\log \left(\frac{a+b+c}{3}\right) = \log (abc)^{1/3}$$

$$= \frac{1}{3} \log (abc) \quad \text{[Rule 3]}$$

$$= \frac{1}{3} [\log a + \log b + \log c]$$ [Corollary of Rule 1]

$$\Rightarrow \log \left(\frac{a+b+c}{3}\right) = \frac{1}{3} (\log a + \log b + \log c)$$

8. If $a^2 + b^2 = 7ab$, show that $\log \frac{a+b}{3} = \frac{1}{2}(\log a + \log b)$

Data is $a^2 + b^2 = 7ab$

Adding 2ab to both sides to make LHS a perfect square

$a^2 + b^2 + 2ab = 7ab + 2ab$

$\Rightarrow \quad (a + b)^2 = 9ab$

$$\Rightarrow \quad \frac{(a+b)^2}{9} = ab$$

$$\Rightarrow \quad \left(\frac{a+b}{3}\right)^2 = ab$$

Taking logarithm on both sides,

$$\log \left(\frac{a+b}{3}\right)^2 = \log ab$$

$$\Rightarrow 2 \log \left(\frac{a+b}{3}\right) = \log a + \log b \quad \text{[Rule 3 and 1]}$$

$$\Rightarrow \log \left(\frac{a+b}{3}\right)^2 = \frac{1}{2}(\log a + \log b)$$ is the required result.

Note: Starting with data, we get the required result. Also note that $\log (a + b) \neq \log a + \log b$

9. If $a^2 + b^2 = c^2$, show that $\frac{1}{\log_{c+b} a} + \frac{1}{\log_{c-b} a} = 2$

$$\text{LHS} = \frac{1}{\log_{c+b} a} + \frac{1}{\log_{c-b} a}$$

$= \log_a (c + b) + \log_a (c - b)$ [Using note under Rule 4]

$= \log_a [(c + b)(c - b)]$ [Rule 1]

$= \log_a (c^2 - b^2)$

$= \log_a a^2$ [Using data $a^2 + b^2 = c^2$]

$= 2 \log_a a$ [Rule 3]

$= 2 = \text{RHS}$

10. If $f(x) = \log \frac{1+x}{1-x}$, prove that $2f(x) = f\left(\frac{2x}{1+x^2}\right)$

By data $f(x) = \log \frac{1+x}{1-x}$

Replacing x by $\frac{2x}{1+x^2}$ on both sides, we get

$$f\left(\frac{2x}{1+x^2}\right) = \log \left[\frac{1+\frac{2x}{1+x^2}}{1-\frac{2x}{1+x^2}}\right]$$

$= \log \left[\frac{1+x^2+2x}{1+x^2-2x}\right]$ [Cancelling $1 + x^2$, after taking LCM to Nr and Dr)

$= \log \frac{(1+x)^2}{(1-x)^2}$

$= \log \left(\frac{1+x}{1-x}\right)^2$

$\Rightarrow \quad f\left(\frac{2x}{1+x^2}\right) = 2 \log \frac{1+x}{1-x}$ [Rule 3]

$= 2 f(x)$ [By data]

$\Rightarrow 2f(x) = f\left(\frac{2x}{1+x^2}\right)$ is the required result.

11. If $a^{3-x} . b^{5x} = a^{5+x} . b^{3x}$, show that $x \log \frac{b}{a} = \log a$.

By data $a^{3-x} . b^{5x} = a^{5+x} . b^{3x}$

Taking logarithm on both sides, we get

$\log [a^{3-x} b^{5x}] = \log [a^{5+x} . b^{3x}]$

$\Rightarrow \quad \log a^{3-x} + \log b^{5x} = \log a^{5+x} + \log b^{3x}$ [Rule 1]

$\Rightarrow \quad (3-x) \log a + 5x \log b = (5+x) \log a + 3x \log b$ [Rule 3]

$\Rightarrow \quad 3 \log a - x \log a + 5x \log b = 5 \log a + x \log a + 3x \log b$

$\Rightarrow \quad 2x \log b = 2x \log a + 2 \log a$

$\Rightarrow \quad x \log b - x \log a = \log a$

$\Rightarrow \quad x [\log b - \log a] = \log a$

$\Rightarrow \quad x\left[\log \frac{b}{a}\right] = \log a$ is the required result.

12. Solve for x :

i) $(\log_{10} x)^2 - \log_{10} x - 6 = 0$

Let $\log_{10} x = a$. $\therefore$ the equation becomes,

$a^2 - a - 6 = 0$

$\Rightarrow \quad (a-3)(a+2) = 0$

$\Rightarrow \quad a = 3$ or $a = -2$

$\Rightarrow \quad \log_{10} x = 3$ or $\log_{10} x = -2$

$\Rightarrow \quad x = 10^3$ or $x = 10^{-2}$

$\Rightarrow \quad x = 1000$ or $x = \frac{1}{100}$

$x = 0.01$

ii) $\log_2(x+5) + \log_2(x-2) = 3$

$\Rightarrow \quad \log_2[(x+5)(x-2)] = 3$ [Rule 1]

$\Rightarrow \quad (x+5)(x-2) = 2^3$ [Writing in the index form]

$\Rightarrow \quad x^2 + 3x - 10 = 8$

$\Rightarrow \quad x^2 + 32 - 18 = 0$

$\Rightarrow \quad (x+6)(x-3) = 0$

$\Rightarrow \quad x = -6$ or 3

$x \neq -6$ because $(x+5)$ and $(x-2)$ can not be negative. $\therefore$ $x = 3$ is the required solution.

Note : Logarithm of a negative number is not defined.

iii) $\log_2 x + \log_4 x^2 + \log_8 x^3 = 6$

$\Rightarrow \quad \log_2 x + 2\log_4 x + 3\log_8 x = 6$ [Rule 3]

$\Rightarrow \quad \frac{\log x}{\log 2} + 2\frac{\log x}{\log 4} + 3\frac{\log x}{\log 8} = 6$ [Rule 4]

$\Rightarrow \quad \frac{\log x}{\log 2} + 2\frac{\log x}{\log 2} + 3\frac{\log x}{\log 2} = 6$

$$\Rightarrow \quad \frac{\log x}{\log 2} + \frac{\log x}{\log 2} + \frac{\log x}{\log 2} = 6$$

$$\Rightarrow \quad 3\,\frac{\log x}{\log 2} = 6$$

$$\Rightarrow \quad \frac{\log x}{\log 2} = 2$$

$\Rightarrow$ $\log x = 2 \log 2$

$\Rightarrow$ $\log x = \log 2^2$ (Rule 3)

$\Rightarrow$ $\log x = \log 4$

$\Rightarrow$ $x = 4$ is the required solution.

iv) $5^{\log x} - 3^{\log x - 1} = 3^{\log x+1} - 5^{\log x - 1}$, the base is 10.

Let $5^{\log x} = a$, $3^{\log x} = b$.

Given problem can be written first as

$5^{\log x} - 3^{\log x} \cdot 3^{-1} = 3^{\log x} \cdot 3 - 5^{\log x} \cdot 5^{-1}$ [Indices]

$$\Rightarrow \quad a - \frac{b}{3} = 3b - \frac{a}{5}$$

$$\Rightarrow \quad a + \frac{a}{5} = 3b + \frac{b}{3}$$

$$\Rightarrow \quad \frac{6a}{5} = \frac{9b + b}{3}$$

$$\Rightarrow \quad \frac{6a}{5} = \frac{10b}{3}$$

$$\frac{3a}{5} = \frac{5b}{3}$$

$$\frac{9a}{25} = b$$

$$\Rightarrow \quad \frac{9}{25} = \frac{b}{a}$$

$$\Rightarrow \quad \frac{3^2}{5^2} = \frac{3^{\log x}}{5^{\log x}}$$

$$\Rightarrow \quad \left(\frac{3}{5}\right)^2 = \left(\frac{3}{5}\right)^{\log x}$$

$\Rightarrow$ $2 = \log x$ [Since base is the same on both sides]

$\Rightarrow \quad \log_{10} x = 2 \quad$ [given base is 10]

$\Rightarrow \quad x = 10^2$ [Index form]

$\Rightarrow \quad x = 100$ is the required solution

Exercises

1. Find the values of

   i) $\log_4 64$ ii) $\log_{0.1} 100$ iii) $\log_{2\sqrt{2}} 512$

2. Find the value of x if

   i) $\log_x 16 = \frac{2}{5}$ ii) $\frac{\log_3 64}{\log_9 8} = x$

3. Prove that

   i) $\frac{2}{3} \log 8 - \frac{1}{2} \log \frac{1}{4} - 3 \log 2 = 0$

   ii) $\log \frac{81}{8} - 2 \log \frac{3}{2} + 3 \log \frac{2}{3} + \log \frac{3}{4} = 0$

4. Prove that

   i) $\log_b x \times \log_a b = 1$

   ii) $\frac{\log_3 64}{\log_9 8} = 4$

   iii) $\log_4 2 + \log_8 2 - \log_{16} 2 = \frac{7}{12}$.

   iv) $\mathrm{Log}_{10} 1600 = 2 + 4 \log_{10} 2$

   v) $\frac{\log_8 2}{\log_{64} 16 \log_8 256} = \frac{3}{16}$

5. Show that $\frac{1}{\log_6 24} + \frac{1}{\log_{12} 24} + \frac{1}{\log_8 24} = 2$

6. If $x = 1 + \log_a bc$, $y = 1 + \log_b ca$, $z = 1 + \log_c ab$

   Show that $xyz = xy + yz + zx$.

7. If $\log_a m = p$ and $\log_b m = q$, prove that

   $\frac{p-q}{p+q} = \frac{\log b - \log a}{\log b + \log a}$ for every permissible base.

8. Solve for x :

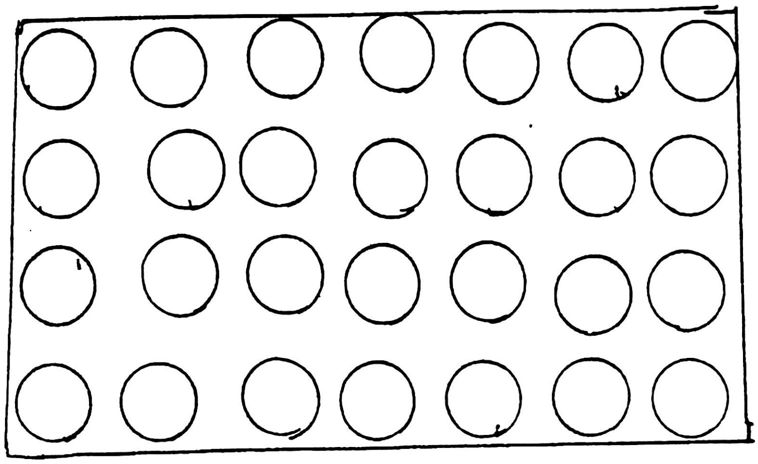
i) $9^x - 5 \times 3^x + 4 = 0$

ii) $\log_x 3 + \log_x 9 + \log_x 729 = 9$

iii) $\log_4 x + \log_2 x = 6$

PERMUTATIONS AND COMBINATIONS

In our daily life, we come across some counting problems. For example, consider the number of circles in the following figure.



By counting one by one, we can say the total number of circles in the above figure is 28. We observe that there are four rows and in each row there are seven circles. So, by multiplying the number of rows by number of circles in each row, we get the total number of circles as 28. Therefore, we can find the total number of objects without actually counting. Here we discuss some of such techniques for determining the total number of objects without direct enumeration.

Fundamental Principle of Counting : If some procedure can be performed in m different ways and another procedure can be performed in n ways, then the number of ways the two procedures can be performed in the order is m n ways.

Example 1 Suppose a car number plate contains two distinct letters. How many different car number plates can be printed with letters ?

Solution : The first letter can be printed in 26 different ways, the second letter in 25 different ways (since the letter printed first cannot be chosen for the second letter). Hence by using above fundamental principle of counting, the different number plates is equal to 26 x 25 i.e. 650.

Example 2 : Suppose a car number plate contains two distinct letters followed by three digits with the first digit not zero. How many different car number plates can be printed ?

Solution : The first letter can be printed in 26 different ways, the second letter in 25 different ways, the first digit in 9 ways and each of the other two digits in 10 ways. Hence,

$$26.25.9.10.10 = 585000$$

different plates can be printed.

From the example 2, we observe that the fundamental principle of counting can be extended to any number of procedures as follows :

If some procedure can be performed in n_1 different ways, and if, following the procedure, a second procedure can be performed in n_2 different ways, and if, following this second procedure, a third procedure can be performed in n_3 different ways, and so forth; then the number of ways the procedures can be performed in order indicated is the product $n_1 . n_2 . n_3 ....$

The product 1.2.3....n of the positive integers from 1 to n is denoted by the symbol $\underline{|n}$ or n! (read "n factorial").

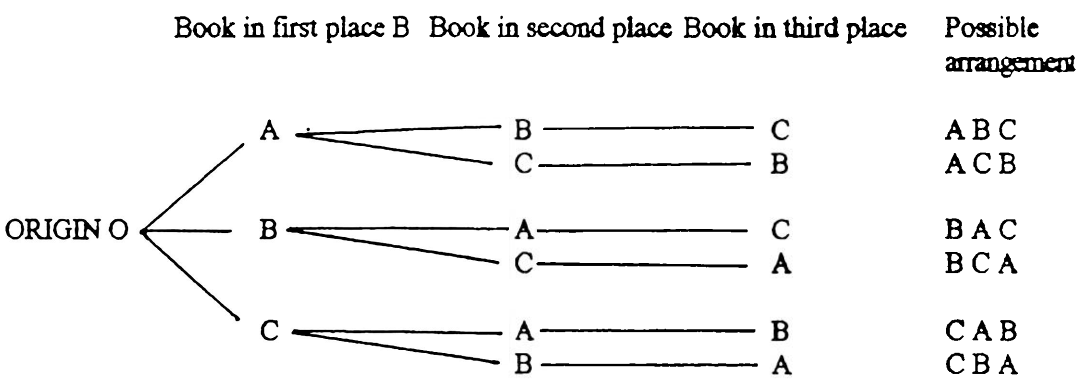
So $\underline{|n}$ = 1.2.3....n

and therefore $\underline{|2}$ = 1.2 = 2, $\underline{|3}$ = 1.2.3 = 6 and so on .

It is also convenient to define $\underline{|0}$ = 1.

Example 3 : In how many ways can 3 books denoted by A, B and C, be arranged in order on a shelf ?

Solution 1 : One way to solve this problem is to list the possible arrangements and count them as shown in the following tree diagram.



The initial point, or origin, is denoted by O. If we follow all possible branches from O to the right hand edge of the tree, we get the six possible arrangements listed in the column on the extreme right. Note that the tree diagram takes order into account. Thus ABC and ACB count as different arrangements of the 3 books because they are in different orders. A change in order yields a different arrangement.

Solution 2 : A more convenient solution to the example 3 is as follows :

The problem requires us to fill 3 places, which can be represented as

In the first space, we can put A or B or C. Hence the first space can be filled in three ways :

3		

For each of the 3 ways of filling the first space, we have 2 ways of filling the second space, because either of the 2 remaining books can be used :

3	2	

Thus we can fill the first 2 spaces in 3x2 or 6 ways. For each of the 6 ways of filling the first 2 spaces, we have one way of filling the third place, because only one book remains. Therefore, we can fill the 3 spaces in 6x1 or 6 ways. We can indicate the number of ways of filling each of the 3 spaces thus

3	2	1

Now we can obtain the total number of arrangements by multiplication.

$$3 \times 2 \times 1 = 6$$

In the above example, each arrangement is different from the other and we call each arrangement as a permutation of these three books.

An arrangement of a set of n objects in a given order is called a permutation of the objects. An arrangement of any $r \leq n$ of these objects in a given order is called an r-permutation or a permutation of the n objects taken r at a time.

The number of permutations of n objects taken r at a time will be denoted by P_r, or P(n,r).

Example 4 : Find the number of 'three letter words' with distinct letters that can be formed from the five letters S, T, U, D, Y.

Solution : Let the general three letter word be represented by three boxes.

$$\boxed{}\quad\boxed{}\quad\boxed{}$$

Now the first letter can be chosen in 5 different ways; following this, the second letter can be chosen in 4 different ways, and following this, the last letter can be chosen in 3 different ways. Write each number in the appropriate box as follows :

$$\boxed{5}\quad\boxed{4}\quad\boxed{3}$$

Thus by fundamental principle of counting, these are 5.4.3 = 60 possible three letter words without repetitions from the five letters or there are 60 permutations of 5 objects taken 3 at a time. That is $^5P_3 = 60$.

Now consider the case of permutations of any r ($\leq$ n) objects of n objects i.e. r-permutation.

The first object in an r-permutation of n objects can be chosen in n different ways ; following this, the second object in the permutation can be chosen in n-1 ways; and following this, the third object in the permutation can be chosen in n-2 ways. Continuing in this manner, we have that the rth (last) object in the r-permutation can be chosen in n- (r-1) = n - r + 1 ways. Thus,

$$^nP_r = n\,(n-1)\,(n-2)\,\ldots\ldots.\,(n-r+1) = \frac{\underline{|n}}{\underline{|n-r}}$$

From the above formula we can see that

$$^nP_0 = \frac{\underline{|n}}{\underline{|n-0}} = 1$$

$$^nP_n = \frac{\underline{|n}}{\underline{|n-n}} = \frac{\underline{|n}}{\underline{|0}} = \underline{|n}$$

That is, there are $\underline{|n}$ permutations of n objects taken all at a time.

Permutations with Repetitions : Frequently we want to know the number of permutations of objects in which some objects are alike as in the following example.

Example 5 : In how many ways can the letters of the word assess be arranged, all at a time ?

Solution : Let the unknown total number of permutations of the letters of the word assess be x. Now consider any one of these permutations; for example,

s s s s a e

In this arrangement, if we replace the four s' s by s_1, s_2, s_3, s_4, the original arrangement gives rise to $\lfloor\underline{4}$ arrangements by permitting the four s's with subscripts (now different) without disturbing the other letters. In the same way, each of the original x permutations gives rise to $\lfloor\underline{4}$ permutations. Thus the total number of permutations is $x\lfloor\underline{4}$. Since the 6 letters

s_1, s_2, s_3, s_4, a, e

are now all different, $x\lfloor\underline{4}$ is the number of permutations of 6 different letters, taken all together. Therefore, $x\lfloor\underline{4} = \lfloor\underline{6}$ or $x = \lfloor\underline{6} / \lfloor\underline{4}$

We can at once generalize this reasoning to show that the number of permutations of a set of n objects, taken all together, where r of the objects are alike and the rest are different, is $\lfloor\underline{n} / \lfloor\underline{r}$.

Repeated applications of this principle yield the following result.

Given a set of n objects having n_1 objects alike of one kind, and n_2 objects alike of another kind, and n_3 objects alike of a third kind, and so on fro k kinds of objects; then the number of permutations of n objects, taken all together, is

$$\frac{\lfloor\underline{n}}{\lfloor\underline{n_1}\,\lfloor\underline{n_2}\cdots\cdots\lfloor\underline{n_k}}$$

where $n = n_1 + n_2 + ... + n_k$

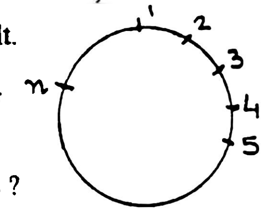
Example 6 : How many distinct permutations can be formed from all the letters of the word mathematics ?

Solution : There are 11 letters of which 2 are m, 2 are a, 2 are t and hence by using the above result, the required number of permutations = $\frac{\lfloor\underline{11}}{\lfloor\underline{2}\,\lfloor\underline{2}\,\lfloor\underline{2}} = 4\,9\,8\,9\,6\,0$.

Circular Permutations : So far we were considering the arrangements of objects in a line. Suppose we consider arrangements of objects in the form of a circle, instead of a line. In such a case, the permutations are called circular permutations whereas in the earlier case we call them as linear permutations.

Suppose n objects are arranged in a circular way as shown in the following figure. Each circular permutation of this kind corresponds to n linear permutations depending on where (out of the n positions) we start. Since there are exactly $\underline{|n}$ linear permutations, there are exactly $\frac{\underline{|n}}{n} = \underline{|n-1}$ circular permutations. Hence we state the following result.

The number of circular permutations of n differnt objects is $\underline{|n-1}$.



Example 7 : In how many ways can 10 students be arranged in a circle ?

Solution : The 10 students can be arranged in a circle in $\underline{|10-1} = \underline{|9}$ ways (by using the above result).

Example 8 : Three boys and three girls are to be seated around a circular table. Among them, the boy X does not want any girl neighbour the girl Y does not want any boy neighbour. How many such arrangements are possible ?

Solution : Since boy X does not want any girl neighbour, all the boys should sit side by side and boy X should be in the middle. Similarly all the girls also should sit side by side and the girl Y should be in the middle. Therefore, this is possible one way. But the remaining two boys can interchange their positions without disturbing the middle (boy X) boy. Similarly the two remaining girls can also interchange their seats. Hence the total number of possible arrangements with the specified restrictions is

$2 \times 2 = 4$ ways.

Combinations

Consider the following example :

Example : In how many ways can a reader select 3 books , without regard to their order, from a set of 4 different books denoted by A, B, C and D ?

We have seen that the number of permutations of 4 different books, taken 3 at a time, is

$^4P_3 = 4 \times 3 \times 2 = 24$

In these permutations, or arrangements, the order of the books counts. An entirely different problem arises if we wish to make a selection of 3 books from A, B, C and D without taking order into account. There are only 4 possible selections.

ABC, ABD, ACD, BCD

For example, we do not list ACB because the selection ACB is the same selection as ABC, since order does not count. Each selection in the above list is called a combination of 4 books taken 3 at a time. The total number of such combinations is denoted by $^4C_3$ or $\binom{4}{3}$ each of which is read "number of combinations of 4 things taken 3 at a time".

A combination of n objects taken r at a time is any subset of r objects.

The number of combinations of n objects taken r at a time will be denoted by ^nC_r or C(n,r) or $\binom{n}{n}$.

Now let us find the number of combinations of a set of n different objects, taken r at a time.

Each combination of r objects can be arranged in $\lfloor r$ ways, and therefore, gives rise to $\lfloor r$ permutations which permutations. Hence, $\lfloor r$ permutation of each of the ^nC_r combinations yield $^nC_r \times \lfloor r$ is the total number of permutations, since each permutation of r objects arises from some combination of r objects. Therefore,

$$^nC_r \times \lfloor r = {}^nP_r = \frac{\lfloor n}{\lfloor n-r}$$

or, $$^nC_r = \frac{\lfloor n}{\lfloor r \lfloor n-r}$$

Thus, the number of combinations of a set of n different objects, taken r at a time is

$$^nC_r = \frac{\lfloor n}{\lfloor r \lfloor n-r}$$

Example 9 : In how many ways can 6 sportsmen be selected from a group of 10 ?

Solution : The required number is $^{10}C_6 = \frac{\lfloor 10}{\lfloor 6 \lfloor 10-6} = \frac{\lfloor 10}{\lfloor 6 \lfloor 4} = 210$.

Identities :

i) $\binom{n}{k} = \binom{n}{n-r}$

ii) $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$

iii) $\sum_{r=0}^{k} \binom{m}{r} \binom{n}{k-r} = \binom{m+n}{k}$

iv) $\binom{n}{r} = \frac{n-r+1}{r} \binom{n}{r-1}$

v) $\binom{n}{r} = \frac{n}{n-r} \binom{n-1}{r}$

vi) $n \binom{n-1}{r} = (r+1) \binom{n}{r+1}$

vii) $\binom{n}{r} = \sum_{i=1}^{r+1} \binom{n-i}{r-i+1}$

viii) $\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$

ix) The total number of combinations of (p + q) things taken any number at a time when p things are alike of one kind and q things are alike of a second kind is (p + 1)) q + 1) - 1.

Exercises

1. How many permutations are there of the letters in the word a) GREAT b) GREET.

2. How many numbers greater than 1000, but not greater than 4000 can be formed with the digits 0,1,2,3,4 repetition of digits being allowed ?

3. Find the total number of 9 digit numbers which have all different digits.

4. Eight chairs are numbered 1 to 8. Two women and three men wish to occupy one chair each. First the women choose the chairs from amongst the chairs marked 1 to 4; and the men select the chairs from amongst the remaining. Find the number of possible arrangements.

5. There are six students A, B, C, D, E, F. In how many ways can a committee of four be formed so as to always include C but exclude D ?

6. Show that the number of diagonals of a polygon of n sides is $\frac{n(n-3)}{2}$.

7. Find the number of ways in which 5 identical balls can be distributed among 10 identical boxes, if not more than one ball can go into a box.

8. Five balls of different colours are to be placed in three boxes of different sizes. Each box can hold all five balls. In how many different ways can we place the balls so that no box remains empty ?

9. Six teachers and 6 students have to sit around a circular table in such a way that there is a teacher in between any two boys. In how many ways this can be done ?

10. If ${}^{n}C_{r} + {}^{n}C_{r+1} = {}^{n+1}C_{x}$ find x.

11. The letters of the word 'TOSS' are permuted in all possible ways and the word thus formed are arranged as in a dictionary. What is the rank of the word 'TOSS' ?

12. A man has four sons and six schools within his reach. Find the number of ways he can send his sons to school if no two of his sons are to read in the same school.

13. The polygon has 35 diagonals. Find the number of its sides.

14. There are 12 points in a plane out of which 7 are in a straight line. Find the number of triangles which can be formed with vertices of these points.

15. If 3n articles can be divided into three equal groups in 280 ways, then what is the value of n ?

16. The number of straight lines that can be formed from 10 points in a plane of which 4 lie on a line is ________.

17. The sides AB, BC and CA of a triangle ABC have 3, 4 and 5 interior points respectively. Find the number of triangles that can be connected using three interior points as vertices.

18. How many numbers are there between 100 and 1000 such that 7 is in the unit place ? At least one of their digits is 7 ? Exactly one of their digits as 7 ?

Answers

1. 120, 60
2. 375
3. 3265920
4. $4\,P_2 . 4\,P_3 = 288$
5. 4
7. $\frac{10!}{5!\,5!}$
8. 150
9. $5!\,6!$
10. r + 1
11. 10

12. $6P_4$

13. 10

14. 185

15. 3

16. 40

17. $12C_3 - 3C_3 - 4C_3 - 5C_3 = 205$

18. 90, 252, 225

References

1. Seymour Lipschutz : Probability, McGraw - Hill Book Company, 1981
2. Frederick Mosteller and Others : Probability and Statistics.

Problems in Permutations and Combinations

Problem 1. Write down all the permutations of xyz.

xyz, xzy, yxz, yzx, zxy, zyx.

Problem 2. How many permutations are there of the letters pqrs?

$4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$

Problem 3. a) How many different arrangements are there of the letters of the word numbers? (Ans. 7! = 5,040)

b) How many of those arrangements have b as the first letter?

Set b as the first letter, and permute the remaining 6. Therefore, there are 6! such arrangements.

c) How many have b as the last letter -- or in any specified position?

The same. 6!.

d) How many will have n, u, and m together?

Begin by permuting the 5 things -- num, b, e, r, s. They will have 5! permutations. But in each one of them, there are 3! rearrangements of num. Consequently, the total number of arrangements in which n, u, and m are together, is $3! \cdot 5! = 6 \cdot 120 = 720$.

Problem 4. a) How many different arrangements (permutations) are there of the digits 01234?

5! = 120

b) How many 5-digit numbers can you make of those digits, in which the first digit is not 0?

Since 0 cannot be first, remove it. Then there will be 4 ways to choose the first digit. Now replace 0. It will now be one of 4 remaining digits. Therefore, there will be 4 ways to fill the second spot, 3 ways to fill the third, and so on. The total number of 5-digit numbers, then, is $4 \cdot 4! = 4 \cdot 24 = 96$.

c) How many 5-digit odd numbers can you make?

Again, 0 cannot be first, so remove it. Since the number must be odd, it must end in either 1 or 3. Place 1, then, in the last position. _ _ _ _ 1. Therefore, for the first position, we may choose either 2, 3, or 4, so that there are 3 ways to choose the first digit. Now replace 0. Hence, there will be 3 ways to choose the second position, 2 ways to choose the third, and 1 way to choose the fourth. Therefore, the total number of odd numbers that end in 1, is $3 \cdot 3 \cdot 2 \cdot 1 = 18$. The same analysis holds if we place 3 in the last position, so that the total number of odd numbers is $2 \cdot 18 = 36$.

Problem 5.

a) If the five letters a, b, c, d, e are put into a hat, in how many different ways could you draw one out? 5

b) When one of them has been drawn, in how many ways could you draw a second? 4

c) Therefore, in how many ways could you draw two letters? $5 \cdot 4 = 20$

This number is denoted by 5P2.

d) What is the meaning of the symbol 5P3?

The number of permutations of 5 different things taken 3 at a time.

e) Evaluate 5P3. $5 \cdot 4 \cdot 3 = 60$

Problem 6. Evaluate

a) 6P3 = 120 b) 10P2 = 90 c) 7P5 = 2520

Problem 6. Express with factorials.

a) nPk n!

(n − k)!

b) 12P7 12! 5!

c) 8P2 8! 6!

d) mP0 m! m!

Exercises 1:

1. 7 people take part in a panel discussion. Each person is to shake hands with all of the other participants at the beginning of the discussion. How many handshakes take place? List them all. (Ans. 28).
2. 8 points are arranged in a circle and each point is joined to each other point by a line. How many lines are needed? (Ans. 28).
3. Linda lives in a neighborhood where the streets either go North and South or East and West, forming rectangular blocks. All the streets go all the way through the neighborhood. Linda lives 3 blocks South and 4 blocks West of her school. She enjoys a little diversity in her life, and so she tries to take a different route to school each day. How many different routes can she take which involve moving only North or East?
4. A basketball team has 11 players on its roster. Only 5 players can be on the court at one time. How many different groups of 5 players can the team put on the floor? (C(11 , ,5))

5. In problem 2, assume that there is no point inside the circle where three of the line meet. How many points of intersection are there inside the circle?

6. The triangular numbers are obtained by adding up consecutive whole numbers starting with one. List the first 10 triangular numbers.

7. A college has 10 basketball players. A 5-member team and a captain will be selected out of these 10 players. How many different selections can be made? (Ans. 6 x C(10 , 6))

8. Badri has 9 pairs of dark Blue socks and 9 pairs of Black socks. He keeps them all in a same bag. If he picks out three socks at random what is the probability he will get a matching pair? (Ans. 1)

9. How many words of 4 consonants and 3 vowels can be made from 12 consonants and 4 vowels, if all the letters are different?

   (Ans. C(12 , 4) x C(4 , 3).)

10. If the letters of the word CHASM are rearranged to form 5 letter words such that none of the word repeat and the results arranged in ascending order as in a dictionary what is the rank of the word CHASM? (Ans. 32)

11. How many four letter distinct initials can be formed using the alphabets of English language such that the last of the four words is always a consonant?

12. When four fair dice are rolled simultaneously, in how many outcomes will at least one of the dice show 3? (Ans. 546)

13. In how many ways can the letters of the word EDUCATION be rearranged so that the relative position of the vowels and consonants remain the same as in the word EDUCATION? (Ans. 2880)

14. How many ways can 10 letters be posted in 5 post boxes, if each of the post boxes can take more than 10 letters? (Ans. 5^{10})

15. How many numbers are there between 100 and 1000 such that at least one of their digits is 6? (Ans. 217)

16. A team of 8 students goes on an excursion, in two cars, of which one can seat 5 and the other only 4. In how many ways can they travel? (Ans, C(12 , 4))

17. There are 12 yes or no questions. How many ways can these be answered? (Ans. 2^{12})

18. How many words can be formed by re-arranging the letters of the word ASCENT such that A and T occupy the first and last position respectively? (Ans. 24)

19. Four dice are rolled simultaneously. What is the number of possible outcomes in which at least one of the die shows 6? (Ans. 546)

20. How many alphabets need to be there in a language if one were to make 1 million distinct 3 digit initials using the alphabets of the language? (Ans. 13)

21. In how many ways can the letters of the word MANAGEMENT be rearranged so that the two As do not appear together? (9! / 2)

22. There are 5 Rock songs, 6 carnatic songs and 3 Hindi pop songs. How many different albums can be formed using the above repertoire if the albums should contain at least 1 Rock song and 1 Carnatic song? (Ans. $\frac{3}{2}(3^{12}-1)$)

23. What is the value of 1x1! + 2x2! + 3x3! + ... +nxn!, where n! means n factorial or n(n-1)(n-2)...1 ?

24. How many number of times will the digit '7' be written when listing the integers from 1 to 1000? (Ans. 251)

25. What do the fractions in the numerator and denominator mean? Where does the formula come from?

26. A bin of computer disks contains a supply of disks from four different manufacturers. In how many ways can you choose 6 disks from the bin? (Ans. 6^4)

27. How many images would be possible on a hypothetical TV screen.

28. Find a formula for the number of combinations of the letters in a name, however many times one letter appears in that name.

29. A club has 8 male and 8 female members and is choosing a committee of 6 members, 3 male and 3 female. How many different committees can be chosen? (Ans. C(8,3) x C(8,3))

30. A multiple-choice test has 30 questions, each with five choices. How many answer keys are possible? (Ans. 5 x (30)!)

31. If a polygon has 42 sides, how many diagonals does it have? (Ans. C(42 , 2) -42)

32. How many possible combinations can a person make with the letters a-z and 1-9 starting from 1 digit and ending up with up to 8 digits? Ans. 1)

33. There are seven dice. Each die has six faces. How many different combinations are there of these seven dice? (Ans. 6^7)

34. How many phone numbers are there given the following restrictions on certain digits of the number? i) the number should contain 7 digits ii) Last digit should be 0. (Ans. 9×10^5)

35. How many squares are there in an 8 x 8 square? How many rectangles are there?

36. What's the probability that a man can draw the numbers 2 and 7 from a hat containing the numbers 1-8?

37. Why are Pascal's triangle and the binomial coefficients the same?

38. There's a connection between Pascal's Triangle and the Tower of Hanoi game but I can't remember what it is!

39. Is there a formula I can use to quickly get the number of possible arrangements of five, six or seven different letters?

40. The letters 'CFOSU' are arranged in dictionary order. What is the rank of the word 'FOCUS' in this order?

41. Is there a systematic way to come up with a schedule for a round robin tournament for up to 32 teams, where each team plays every other team once?

42. What is the equation for the number of squares in a rectangle (like the chessboard puzzle)?

43. How many ways are there to list the numbers one through ten so that no number appears in its own position (i.e. 1 is not first in the list, 2 is not second...)?

44. I have been looking for a recursive algorithm to find all possible derangements of a set n. Find.

45. Is there a number that has only three prime divisors (3, 5, and 7) and that has a total of 18 divisors?

46. How can I reduce the number of arrangements of the word ARRANGEMENT by using the probability of an occurrence?

47. How many arrangements of six 0's, five 1's, and four 2's are there in which i) the first 0 precedes the first 1? ii) the first 0 precedes the first 1, precedes the first 2?

48. In how many ways can the letters in UNUSUAL be arranged? For those arrangements, how many have all 3 U's together?

49. In how many ways can the letters in MISSISSIPPI be arranged? Suppose the 2 P's must be separated?... and other such problems.

50. Six marbles are placed in one of three different boxes. What is the probability that each box contains two marbles?

51. Lucy has four vases--blue, yellow, red, and purple. How many ways can she arrange her vases in a row on a shelf?

52. You have 2t + 1 balls to put into 3 boxes, but the sum of the balls in 2 of the boxes should be more then the balls in the other box...

53. How many possible batting orders are there for a team of nine players?

54. How many different arrangements of 3 red and 3 blue beads on a bracelet are there?

55. I am trying to figure out how many different groups 50 people can be partitioned into.

56. I am looking for the formula for the number of different groups we can split a group of n different items into - order does not matter.

57. In a row of 20 seats, in how many ways can 3 blocks of consecutive seats with 5 seats in each block be selected? .

58. Janny wants to buy three doughnuts, and there are five varieties to choose from. She wants each doughnut to be a different variety. How many combinations are there?

59. a) A door can be opened only with a security code that consists of five buttons: 1, 2, 3, 4, 5. A code consists of pressing any one button, or any two, or any three, or any four, or all five.

How many possible codes are there? (You are to press all the buttons at once, so the order doesn't matter.)

b) If, to open the door you must press three codes, then how many possible ways are there to open the door?

Assume that the same code may be repeated.

60. How many ways can you arrange 7 different books, so that a specific book is on the third place?

61. In how many ways can you take 3 marbles out of a box with 15 different marbles?

62. In a firm are 20 workmen and 10 employees. In how many ways can you have a committee with 3 workmen and 2 employees?

63. In how many ways can you take 5 cards, with at least 2 aces, out of a game of 52 cards?

64. In how many ways can you split a group of 13 persons in 3 persons and 10 persons?

65. How many diagonals are there in a convex n-polygon?

66. How many numbers consisting of 3 figures, can you make with the figures 0,1,2,3,4 ?

67. How many subset are there in a set of 10 elements?

68. Calculate the term with x^2 in the expansion of $(x^3 + 1/2x)^{10}$

69. In how many ways can you arrange m identical stones into k piles so that each pile has at least 1 stone in it.

70. By removing one stone from each pile, this is the number of ways you can arrange m-k identical stones into k (possibly empty) piles. ..

Now, view the k piles as a numbered set .

Write on each stone the number of a chosen pile and order the stones accordingly.

The numbered stones constitute a combination with repetition of k elements (the numbers) choose m-k (the stones). This can be done in

i. $C'(k , m-k) = C(m-1,m-k) = \dfrac{(m-1)!}{(m-k)!\,(k-1)!}$

71. How many strictly positive integer solutions (x, y , z) are there, such that x + y + z = 100

72. This is the same problem as 71.

In how many ways can you arrange 100 identical stones into 3 piles so that each pile has at least 1 stone in it.

From previous problem the answer is C(99,97) = 4851 ways

73. How many terms are contained in $(a + b + c)^{20}$.

All terms can be written as $A.a^p .b^q .c^r$ with p + q + r = 20.

The number of terms is the number of solutions of the equation

b) p + q + r = 20 with p, q, r as positive integer unknowns.

Now regard (p,q,r) as three ordered elements.

Point 20 times one of these elements, and order these elements in the same order as the given elements. This corresponds with one solution of p + q + r = 20 and it is a combination with repetition of 3 elements choose 20. The number of terms is the number of such combinations = C'(3,20) = C(22,20) = C(22,2) = 231

74. The term $A.a^{10} b^3 c^7$ is contained in $(a + b + c)^{20}$. Calculate A.

b) he number of terms a^{10} b^3 c^7 is the number of permutations with repetition of the elements

c) a,a,a,a,a,a,a,a,a,a,b,b,b,c,c,c,c,c,c,c

This number is

$$\frac{(20)!}{(10)!3!\,7!}$$

Exercises 2:

1. Find the number of permutations of length 6 of 9, 5 of which are alike and the rest all different. (Ans. 1044)

2. Show that $\binom{2n}{2} = 2\binom{n}{2} + n^2$ without simplification.

3. Set A has 3 elements and set B has 6 elements. What can be the minimum number of elements in the set $A \cup B$? (Ans. 6)

4. Let A be a set of n distinct elements. Find the total number of distinct functions from A to A. Also find how many of these functions are onto.

   (Ans. $n!$)

5. In how many ways can a collection of 3n distinct objects be divided into n triplets, each having 3 objects? (Ans. $\frac{(3n)!}{6^n n!}$)

6. Find the total number of ways in which six '+' and four '-' signs can be arranged in a line so that no two '-' signs can occur together. (Ans. 35)

7. Find how many palindromes of length n can be formed from an alphabet of k letters. (Ans. $k^m \quad if\ n = 2m$; $k^{m+1} \quad if\ n = 2m+1$)

8. Prove that the number of ways to put r identical objects into n distinct boxes is $\binom{r+n-1}{r}$. What if we further require that no box be empty. (Ans. $\binom{r-1}{r-n}$)

9. A person has 3 sons. He owns 101 shares of a company. He wants to give these to his sons so that no son should have more shares than the combined total of the other two. In how many ways can he do so? (Ans. $\binom{103}{2} - 3\binom{52}{2}$)

10. Find the number of ways to seat m men and n women in a circle so that no two women are seated together. (Ans. $(m-1)!\binom{m}{n}$)

11. Find how many different circular bracelets can be formed using 6n blue and 3 red beads, where n is a positive integer. (Ans. $3n^2+3n+1$)

12. A city has m parallel roads going east –west and n parallel roads going north-south. How many rectangles can be formed with their sides along these roads? If the distance between every consecutive pair of parallel roads is the same, how many shortest possible paths are there to go from one corner of the city to its diagonally opposite corner? (Ans. $\binom{m}{2}\binom{n}{2}$; $\binom{m+n-2}{m-1}$)

13. Suppose there are 20 players of different heights. These are to be divided into two teams, A and B, of 10 players each so that for every i = 1,2, ….., 10. the i-1 th tallest player in A will be taller than the i-th tallest player in B. In how many ways can this be done? (Ans. $\frac{1}{11}\binom{20}{10}$)

14. Prove that for every positive integer n, the number of monotonically increasing functions from the set {1,2, ….., n} to itself with the property that f(x) ≥ x for all x = 1, 2, ….,n equals the Catalan number $\frac{1}{n+1}\binom{2n}{n}$.

15. How many diagonals does a convex n-gon have? If no three of them are concurrent except possibly at a vertex, find the number of segments into which these diagonals are decomposed by one another.
(Ans. $\frac{1}{2}n(n-3)$; $2\binom{n}{4}+\frac{1}{2}n(n-3)$)

16. In a college Of 300 students, every student reads 5 news papers and every newspaper is read by 60 students. Find the number of news papers. (Ans. 25)

17. In how many ways can the four walls of a room be painted with three colours so that no two adjacent walls have the same colour? (Ans. 18)

18. The number of permutations of n different things, taken not more than r at a time, when each thing may occur any number of times

$$= n + n^2 + n^3 + \dots\dots\dots\dots + n^n = \frac{n(n^r - 1)}{n - 1}$$

19. Show that P(n , r) = n P(n-1 , r-1) without using the formula. Hence deduce the value of P(n,r).

20. Show that the number of permutations of n different things taken r at a time in which s particular things always occur is P(n-s , r-s) P(r , s).

21. Show that from first principles P(n , r) = P(n-1, r) + r P(n-1, r-1).

(Hint: i) Consider those that contain a particular things and ii) Those that do not contain the particular things. Sum these cases to get total.)

22. Find n and r if P(n , r) =7920.

23. Show that the number of permutations of n things taken all at a time when p of them are all alike and the rest all different is $\frac{n!}{p!}$.

24. Prove that the number of circular permutations of n different things taken all at a time around a circle is (n-1)!.

25. How many different arrangements can be made out of the letters in the expression $a^2b^2c^4$.

26. Nine articles are to be placed in nine boxes one in each box. Five of them are too big for three of the boxes. Find the number of possible arrangements.

(Ans. 17280).

27. Find the sum of all the numbers formed by taking all the digits from {2,3,4,5}.

(Ans. 93324).

28. Find the number of ways in which 6 boys and 4 girls are to sit for a dinner at a round table so that no two girls are to sit together. (Ans. 5! P(6 , 4).

29. Show that the product of r consecutive positive integers is divisible by r!.

30. Show that C(n , r) = $\frac{n}{r}$ C(n-1 , r-1).

31. Prove that C(n , r) = C(n , n-r).

32. Prove that C(n , r) = C(n-1 , r) + C(n-1 , r-1). This is known as Pascal's formula.

33. If C(n , r) = C(n , s) then prove that either r = s or r + s = n.

34. Show that C(n , r) is greatest if

i) $r = \frac{n}{2}$ when n is even and

ii) $r = \frac{n-1}{2}$ or $\frac{n+1}{2}$ when n is odd.

35. Show that the number of circular permutations of n things taken r at a time is $\frac{P(n,r)}{r}$.

36. Show that the number of ways in which (m + n) things can be divided into two different groups of m and n things respectively is $\frac{(m+n)!}{m!\,n!}$.

37. Show that the total number of combinations of (p + q) things taken any number at a time when p things are alike of one kind and q things are alike of a second kind is (p + 1) (q + 1) – 1.

38. m persons enter a theatre hall and are to sit in n seats placed in a row. In how many ways can they be seated, so that no two persons are seated in adjacent seats?

(Sol. The n seats can be separated as m occupied seats and n-m vacant seats. Between any two of these n-m seats are created n –m + 1 spaces and in these spaces, the m persons can be seated in P(n – m +1 , m) ways. For this to possible it is necessary that m ≤ n – m +1 i.e. $m \le \frac{n+1}{2}$).

39. If the letters of the word MOTHER is permuted among themselves and the words so formed are arranged as in a dictionary. What is the rank of the word MOTHER? (Ans. 309).

40. Five balls of different colours are to be place in three boxes of different sizes. Each box can hold all the five balls. In how many different ways can the balls be placed so that no box is empty? (Ans. 150).

41. 2n persons are to be seated: n on each side of a long table. r (< n) particular persons desire to sit on one side ; and s (< n) other persons desire to sit on the other side. In how many ways can the persons be seated? (Ans. C(2n – r –s , n – r) n! n!).

42. If n distinct objects are arranged in a circle, show that the number of ways of selecting three of these n things so that no two of them is next to each other is $\frac{n(n-4)(n-5)}{6}$.

43. Fifteen persons amongst whom are A, B and C are to speak at a function. Find how many ways can the speech be done if A wants to speak before B and B is to speak before C? (Ans. 12! C(13 , 1) + 2. C(13 , 2) + C(13 , 3)).

44. There are four addressed envelops and 4 letters. In how many ways could all the letters be put into the wrong envelops?

(Ans. $n!\left[1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\ldots\ldots\ldots\ldots+\frac{(-1)^n}{n!}\right]$)

45. Show that the total number of permutations of n different things taken not more than r at a time, each being allowed to repeat any number of times is $\frac{n^r - 1}{n - 1}$.

46. How many different numbers can be formed to satisfy all the following conditions:

    i) The number is less than 2×10^8

    ii) The number is formed by the digits 0 , 1, 2

    iii) The number is divisible by 3 ?

    (Ans. 4373)

47. How many seven digit numbers are there which read the same way from either side? (Ans. 9×10^3).

48. Using permutation or otherwise, prove that $\frac{n^2!}{(n!)^n}$ is an integer, where n is a positive integer. (Hint. Distribute n^2 objects in n groups).

49. Each of 8 chairs numbered as 1,2,3,.....,8. Two women and three men wish to occupy one chair each. First the women choose the chairs from amongst the chairs marked 1 to 4; and the men select the chairs from amongst the remaining. How many ways these persons can sit ? (Ans. 35).

50. An n-digit number is a positive number with exactly n digits. Nine hundred distinct n-digit numbers are to formed using only the three digits 2, 5 and 7. Then find the smallest value of n for which this is possible. (Ans. 7).

51. How many positive integral solutions are there for the equation x+y+z+w=20? (Ans. C(19 , 3).

52. Show that the number of ways in which n different books can be arranged on a shelf so that a particular two books of them are not together is (n – 2) (n – 1)!).

53. A candidate is required to answer 7 out of 15 questions which are divided into three groups A, B, C each containing 4, 5, 6 questions respectively. He is required to select at least 2 questions from each group. In how many ways can he make up his choice? (Ans. 2700).

54. A student is allowed to select at most n books from a collection of (2n + 1) books. If the total number of ways in which he can select a book is 63, find the value of n. (Ans. n=3).

55. If P(n , r) = P(n , r+1) and C(n , r) = C(n , r-1) find n and r. (Ans. n=3, r=2)

56. If C(n,r-1) =36, C(n , r) = 84 and C(n , r+1) = 126, find the values of n and r. (Ans. n = 9 , r = 3)

57. Evaluate $\frac{C(20,r)}{C(25,R)}$ when both numerator and denominator have their greatest values. (Ans. $\frac{143}{4025}$)

58. How many different nine digit numbers can be formed from the number 223355888 by rearranging its digits so that the odd digits occupy even positions? (Ans. 60)

59. m men n women are to be seated in a row so that no two women sit together. If m greater than n, then show that the number of ways in which they can be seated is $\frac{m!\ n!}{(m-n+1)!}$.

60. Every one in a room shakes hands with every one else. The total number of hand shakes is 66. Find the total number of persons in the room. (Ans. 12).

61. Determine the number of rectangles that can be formed on a 8 x 8 chess board. (Ans. C(9 , 2) C(9 , 2) =1296).

62. Show that the number of ways in which m + n things can be divided into two groups containing m and n things respectively = $\frac{(m+n)!}{m!\, n!}$.

63. If m =n in the above problem, show that the number of different ways

$$= \frac{(2m)!}{m!\, m!\, 2!}.$$

64. Prove that the number of divisions of m + n + p things into three groups of m, n and p things respectively = $\frac{(m+n+p)!}{m!\, n!\, p!}$.

65. If 3m things are divided into three equal groups, then show that the number of divisions = $\frac{(3m)!}{m!\, m!\, m!\, 3!}$

66. Show that the number of ways of selecting some or all out of p + q + r things where p are alike of one kind, q alike of second kind, r alike of a third kind

$$= (p + 1)(q + 1)(r + 1) - 1 .$$

67. There are four balls of different colours and four boxes of colours same as those of the balls. Find the number of ways in which the balls, one each in a box, could be placed such that a ball does not go to a box of its own colour. (Ans. 9).

Binomial Theorem

The binomial **theorem** provides a simple method for determining the coefficients of each term in the expansion of a binomial with the general equation $(A + B)^n$. Developed by Isaac Newton, this theorem has been used extensively in the areas of probability and **statistics**. The main argument in this theorem is the use of the combination formula to calculate the desired coefficients.

The question of expanding an equation with two unknown variables called a binomial was posed early in the history of **mathematics**. One solution, known as **Pascal's triangle**, was determined in China as early as the thirteenth century by the mathematician Yang Hui. His solution was independently discovered in **Europe** 300 years later by Blaise Pascal whose name has been permanently associated with it since. The binomial theorem, a simpler and more efficient solution to the problem, was first suggested by Isaac Newton. He developed the theorem as an undergraduate at Cambridge and first published it in a letter written for Gottfried Leibniz, a German mathematician.

Expanding an equation like $(A + B)^n$ just means multiplying it out. By using standard **algebra** the equation $(A + B)^2$ can be expanded into the form $A^2 + 2AB + B^2$. Similarly, $(A + B)^4$ can be written $A^4 + 4A^3B + 6A^2B^2 + 4AB^3 + B^4$. Notice that the terms for A and B follow the general pattern $A^nB^0, A^{n-1}B^1, A^{n-2}B^2, A^{n-3}B^3, \ldots, A^1B^{n-1}$, A^0B^n. Also observe that as the value of n increases, the number of terms increases. This makes finding the coefficients for individual terms in an equation with a large n value tedious. For instance, it would be cumbersome to find the **coefficient** for the term A^4B^3 in the expansion of $(A + B)^7$ if we used this algebraic approach. The inconvenience of this method led to the development of other solutions for the problem of expanding a binomial.

One solution, known as Pascal's triangle, uses an array of numbers (shown below) to determine the coefficients of each term.

Pascal's triangle

						1						
					1		1					
				1		2		1				
			1		3		3		1			
		1		4		6		4		1		
	1		5		10		10		5		1	
1		6		15		20		15		6		1

This triangle of numbers is created by following a simple rule of **addition**. Numbers in one row are equal to the sum of two numbers in the row directly above it. In the fifth row the second term, 4 is equal to the sum of the two numbers above it, namely 3 + 1. Each row represents the terms for the expansion of the binomial on the left. For example, the terms for $(A+B)^3$ are $A^3 + 3A^2B + 3AB^2 + B^3$. Obviously, the coefficient for the terms A^3 and B^3 is 1. Pascal's triangle works more efficiently than the algebraic approach, however, it also becomes tedious to create this triangle for binomials with a large n value.

Binomial theorem

We'll prove that

$(a + b)^n = a^n + C(n,1)a^{n-1} b + C(n,2)a^{n-2} b^2 + C(n,3)a^{n-3} b^3 + ... + C(n,n) b^n$

To prove this theorem we use mathematical induction.

It is easy to verify that the theorem holds for n = 2 .

Now, assume it holds for n = k. We'll show it holds for n= k+1.

$(a + b)^{k+1} = (a + b).(a + b)^k$

$=(a+b).(a^k + C(k,1)a^{k-1} b + C(k,2)a^{k-2} b^2 + C(k,3)a^{k-3} b^3 + ... C(k,k)b^k)$

$= a^{k+1} + C(k,1)a^k b + C(k,2)a^{k-1} b^2 + C(k,3)a^{k-2} b^3 + ... C(k,k)a\, b^k +$

$a^k b + C(k,1)a^{k-1} b^2 + C(k,2)a^{k-2} b^3 + ...+ C(k,k-1)a\, b^k + C(k,k)\, b^{k+1}$

Since C(k,k)= C(k+1,k+1)= 1 and appealing on Pascal's formula

C(n,p) = C(n-1,p) + C(n-1,p-1) , we find

$= a^{k+1} + C(k+1,1)a^k b + C(k+1,2)a^{k-1} b^2 + C(k+1,3)a^{k-2} b^3 + ...$

$+ C(k+1,k+1)\, b^{k+1}$

This proves the theorem.

The Binomial Theorem: Examples

- **Expand $(x^2 + 3)^6$**

Students trying to do this in their heads tend to mess up the powers. This isn't the time to worry about that square on the *x*. we need to just apply the Theorem. The first term in the binomial is "x^2", the second term in "3", and the power *n* is 6, so, counting from 0 to 6, the Binomial Theorem gives :

$$(x^2 + 3)^6 = {}_6C_0\,(x^2)^6(3)^0 + {}_6C_1\,(x^2)^5(3)^1 + {}_6C_2\,(x^2)^4(3)^2 + {}_6C_3\,(x^2)^3(3)^3$$

$$+ {}_6C_4\,(x^2)^2(3)^4 + {}_6C_5\,(x^2)^1(3)^5 + {}_6C_6\,(x^2)^0(3)^6$$

Then simplifying gives

$$(1)(x^{12})(1) + (6)(x^{10})(3) + (15)(x^8)(9) + (20)(x^6)(27)$$

$$+ (15)(x^4)(81) + (6)(x^2)(243) + (1)(1)(729)$$

$$= \mathbf{x^{12} + 18x^{10} + 135x^8 + 540x^6 + 1215x^4 + 1458x^2 + 729}$$

- **Expand $(2x - 5y)^7$**

We will plug "$2x$", "$-5y$", and "7" into the Binomial Theorem, counting up from zero to seven to get each term. (Don't forget the "minus" sign that goes with the second term in the binomial.)

$$(2x - 5y)^7 = {}_7C_0\,(2x)^7(-5y)^0 + {}_7C_1\,(2x)^6(-5y)^1 + {}_7C_2\,(2x)^5(-5y)^2$$

$$+ {}_7C_3\,(2x)^4(-5y)^3 + {}_7C_4\,(2x)^3(-5y)^4 + {}_7C_5\,(2x)^2(-5y)^5$$

$$+ {}_7C_6\,(2x)^1(-5y)^6 + {}_7C_7\,(2x)^0(-5y)^7$$

Then simplifying gives :

$$(1)(128x^7)(1) + (7)(64x^6)(-5y) + (21)(32x^5)(25y^2) + (35)(16x^4)(-125y^3)$$

$$+ (35)(8x^3)(625y^4) + (21)(4x^2)(-3125y^5) + (7)(2x)(15625y^6)$$

$$+ (1)(1)(-78125y^7)$$

$$= \mathbf{128x^7 - 2240x^6y + 16800x^5y^2 - 70000x^4y^3 + 175000x^3y^4 - 262500x^2y^5 + 218750xy^6 - 78125y^7}$$

You may be asked to find a certain term in an expansion, the idea being that the problem will be way easy if you've memorized the Theorem, but will be difficult or impossible if you haven't. So memorize the Theorem and get the easy marks.

- **What is the fourth term in the expansion of $(3x - 2)^{10}$?**

Let's take a look about the expansion :

$$(3x-2)^{10} = {}_{10}C_0\,(3x)^{10-0}(-2)^0 + {}_{10}C_1\,(3x)^{10-1}(-2)^1 + {}_{10}C_2\,(3x)^{10-2}(-2)^2$$

$$+\ \mathbf{{}_{10}C_3\,(3x)^{10-3}(-2)^3} + {}_{10}C_4\,(3x)^{10-4}(-2)^4 + {}_{10}C_5\,(3x)^{10-5}(-2)^5$$

$$+\ {}_{10}C_6\,(3x)^{10-6}(-2)^6 + {}_{10}C_7\,(3x)^{10-7}(-2)^7 + {}_{10}C_8\,(3x)^{10-8}(-2)^8$$

$$+\ {}_{10}C_9\,(3x)^{10-9}(-2)^9 + {}_{10}C_{10}\,(3x)^{10-10}(-2)^{10}$$

So the fourth term is not the one where we counted up to 4, but the one where we have counted up just to 3.

Note that, in any expansion, there is one more term than the number in the power. For instance:

$(x + y)^2 = x^2 + 2xy + y^2$ (second power: three terms)

$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$ (third power: four terms)

$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$ (fourth power: five terms)

The expansion in this problem, $(3x - 2)^{10}$, has power of $n = 10$, so the expansion will have eleven terms, and the terms will count up, not from 1 to 10 or from 1 to 11, but from 0 to 10. This is why the fourth term will not the one where I'm using "4" as my counter, but will be the one where I'm using "3".

$$_{10}C_3\,(3x)^{10-3}(-2)^3 = (120)(2187)(x^7)(-8) = \mathbf{-2099520x^7}$$

- **Find the tenth term in the expansion of $(x + 3)^{12}$.**

To find the tenth term, I plug x, 3, and 12 into the Binomial Theorem, using the number $10 - 1 = 9$ as our counter:

$$_{12}C_9\,(x)^{12-9}(3)^9 = (220)x^3(19683) = \mathbf{4330260x^3}$$

- **Find the middle term in the expansion of $(4x - y)^8$.**

Since this binomial is to the power 8, there will be nine terms in the expansion, which makes the fifth term the middle one. So we will plug $4x$, $-y$, and 8 into the Binomial Theorem, using the number $5 - 1 = 4$ as our counter.

$$_8C_4\,(4x)^{8-4}(-y)^4 = (70)(256x^4)(y^4) = \mathbf{17920x^4y^4}$$

You might be asked to work backwards.

- **Express $1296x^{12} - 4320x^9y^2 + 5400x^6y^4 - 3000x^3y^6 + 625y^8$ in the form $(a + b)^n$.**

We know that the first term is of the form a^n, because, for whatever n is, the first term is $_nC_0$ (which always equals 1) times a^n times b^0 (which also equals 1). So $1296x^{12} = a^n$. By the same reasoning, the last term is b^n, so $625y^8 = b^n$. And since there are alternating "plus" and "minus" signs, we know from experience that the sign in the middle has to be a "minus". (If all the signs had been "plusses", then the middle sign would have been a "plus" also. But in this case, we are really looking for "$(a - b)^n$".)

We know that, for any power n, the expansion has $n + 1$ terms. Since this has 5 terms, this tells me that $n = 4$. So to find a and b, we only have to take the 4th root of the first and last terms of the expanded polynomial:

Example 1.

a) The term a^8b^4 occurs in the expansion of what binomial?

Answer. $(a + b)^{12}$. The sum of 8 + 4 is 12.

b) In that expansion, what number is the coefficient of a^8b^4?

Answer. It is the combinatorial number,

$$_{12}C_4 = \frac{12 \cdot 11 \cdot 10 \cdot 9}{1 \cdot 2 \cdot 3 \cdot 4} = 495$$

Note again that the lower index, 4, is the exponent of b.

This same number is also the coefficient of a^4b^8, since $_{12}C_8 = {}_{12}C_4$.

Example 2. Expand $(a - b)^5$.

Solution. We found the binomial coefficients to be 1 5 10 10 5 1. The difference with $(a - b)$ is that the signs of the terms will alternate:

$(a - b)^5 = a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$.

For, $a - b = a + (-b)$, therefore each term will have the form $a^{5-k}(-b)^k$.

When k is even, $(-b)^k$ will be positive. But when k is odd, $(-b)^k$ will be negative.

Each odd power of b will have a negative sign.

Example 3. In the expansion of $(x - y)^{15}$, calculate the coefficients of x^3y^{12} and x^2y^{13}.

Solution. The coefficient of x^3y^{12} is positive, because the exponent of y is even. That coefficient is ${}^{15}C_{12}$. But ${}^{15}C_{12} = {}^{15}C_3$, and so we have

$$\frac{15\cdot 14\cdot 13}{1\cdot 2\cdot 3} = 455.$$

The coefficient of x^2y^{13}, on the other hand, is negative, because the exponent of y is odd. The coefficient is $-\,{}^{15}C_{13} = -\,{}^{15}C_2$. We have

$$-\frac{15\cdot 14}{1\cdot 2} = -105$$

Example 4. Write the first four terms of $(a + b)^n$. Do not use factorials.

Solution.

$$(a+b)^n = {}^nC_0\, a^n + {}^nC_1\, a^{n-1}b + {}^nC_2\, a^{n-2}b^2 + {}^nC_3\, a^{n-3}b^3$$

$$= a^n + na^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3$$

Notice: Each coefficient contains the previous coefficient. The coefficient

$$\frac{n(n-1)}{1\cdot 2}$$

contains the previous coefficient, n.

The next coefficient, $\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}$ contains the previous one.

Each coefficient is the previous coefficient *multiplied* by the exponent of a, and divided by the *next* exponent of b.

The next coefficient would be this one

$$\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}$$

multiplied by $(n-3)$, and divided by 4:

$$\frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}$$

Example 5. Use the binomial theorem to expand $(a+b)^8$.

Solution. The expansion will begin :

$(a+b)^8 = a^8 + 8a^7b$

The first coefficient is always 1. The second is equal to the exponent of the expansion, which here is 8.

Now, the next coefficient will be 8 times the exponent of a -- 7 -- divided by 2. It will be 56/2 = 28. We have

$(a+b)^8 = a^8 + 8a^7b + 28a^6b^2$

The next coefficient is 28· 6 -- divided by 3:

28· 6 / 3 = 28· 2 = 56.

$(a+b)^8 = a^8 + 8a^7b + 28a^6b^2 + 56a^5b^3$

The next coefficient is 56· 5 -- divided by 4:

56· 5 / 4 = 14· 5 = 70.

$(a + b)^8 = a^8 + 8a^7b + 28a^6b^2 + 56a^5b^3 + 70a^4b^4$

And now we have come to the point of symmetry. For, the coefficient of a^3b^5 is equal to the coefficient of a^5b^3, which is 56. And so on for the remaining coefficients. Here is the complete expansion:

$$(a + b)^8 = a^8 + 8a^7b + 28a^6b^2 + 56a^5b^3 + 70a^4b^4 + 56a^3b^5 + 28a^2b^6 + 8ab^7 + b^8.$$

Example 6. Write the 5th term in the expansion of $(a + b)^{10}$.

Solution. In the 1st term, $k = 0$. In the 2nd term, $k = 1$. And so on.

The index k of each term is one less than the <u>*ordinal number*</u> *of the term.*

Thus in the <u>5th</u> term, $k = 4$. The exponent of b is 4. The 5^{th} term is

$$_{10}C_4 = \frac{10\cdot 9\cdot 8\cdot 7}{1\cdot 2\cdot 3\cdot 4}a^6b^4 = 210\,a^6b^4$$

Example 7. Expand $(x - 1)^6$.

Solution. According to <u>Pascal's triangle</u>, the coefficients are

1 6 15 20 15 6 1.

In the binomial, x is "a", and -1 is "b". The <u>signs</u> will alternate:

$$(x - 1)^6 = x^6 - \underline{6}x^5\cdot 1 + \underline{15}x^4\cdot 1^2 - \underline{20}x^3\cdot 1^3 + \underline{15}x^2\cdot 1^4 - \underline{6}x\cdot 1^5 + 1^6$$
$$= x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$$

Example 8. Expand $(x + 2)^3$.

Solution. The coefficients are <u>1 3 3 1</u>. x is "a", and 2 is "b".

$$(x + 2)^3 = x^3 + 3x^2\cdot 2 + 3x\cdot 2^2 + 2^3$$
$$= x^3 + 6x^2 + 12x + 8$$

Problems:

1. Assume q is a positive integer with q ≤ 50. If the sum $\binom{98}{30}+2\binom{97}{30}+3\binom{96}{30}+\ldots\ldots+68\binom{31}{30}+69\binom{30}{30}$ *equals* $\binom{100}{q}$ then find q.

(Ans. 32)

2. Show that

a) $\binom{n}{0}-\binom{n}{1}+\binom{n}{2}+\ldots\ldots+(-1)^k\binom{n}{k}+\ldots\ldots+(-1)^n\binom{n}{n}=0$

b) $\binom{n}{0}+i\binom{n}{1}-\binom{n}{2}-i\binom{n}{3}\ldots\ldots+\binom{n}{4}+i\binom{n}{5}+\ldots\ldots=(1+i)^n$

c) $({}^{2n}C_0)^2-({}^{2n}C_1)^2+({}^{2n}C_2)^2-\ldots\ldots+({}^{2n}C_{2n})^2=(-1)^n\ {}^{2n}C_n$

d) $\binom{n}{1}+2\binom{n}{2}x+3\binom{n}{3}x^2+\ldots\ldots+n\binom{n}{n}x^{n-1}=n(1+x)^{n-1}$

e) $\binom{n}{0}\frac{1}{1}-\binom{n}{1}\frac{1}{2}+\binom{n}{2}\frac{1}{3}+\ldots\ldots+\binom{n}{n}\frac{1}{n+1}=\frac{1}{n+1}(2^{n+1}-1)$

f) $C_0^2-C_1^2+C_2^2-\ldots\ldots\ldots+(-1)^nC_n^2=0$

g) $C(n,1)+C(n,2)+\ldots\ldots\ldots+C(,n)=2^{n-1}$

h) $\binom{n}{m}+\binom{n-1}{m}+\binom{n-2}{m}+\ldots\ldots\ldots+\binom{m+1}{m}+\binom{m}{m}=\binom{n+1}{m+1}$

i) $C_0+2C_1+3C_2+ \ldots\ldots\ldots\ldots + (n+1)C_n = (n+2)\,2^{n-1}$

j) $C_1+2C_2+3C_3+ \ldots\ldots\ldots\ldots + nC_n = n\,2^{n-1}$

k) $$\frac{C(n,1)}{C(n,0)}+2\frac{C(n,2)}{C(n,1)}+3\frac{C(n,3)}{C(n,2)}+\ldots\ldots\ldots\ldots+n\frac{C(n,n)}{C(n,n-1)}=\frac{n(n+1)}{2}$$

l) $$\frac{1}{1!(n-1)!}+\frac{1}{3!(n-3)!}+\frac{1}{5!(n-5)!}+\ldots\ldots\ldots\ldots=\frac{2^{n-1}}{n!}$$

m) $$\frac{C_0}{2}-\frac{C_1}{3}+\frac{C_2}{4}-\frac{C_3}{5}+\ldots\ldots\ldots\ldots+\ldots=\frac{1}{(n+1)(n+2)}$$

n) $$\sum_{r=0}^{n}\frac{(-1)^r}{C_r}=0 \quad when\ n\ is\ odd\ integer.$$

o) $$(C_0+C_1)(C_1+C_2)(C_2+C_3)\ldots\ldots\ldots\ldots(C_{n-1}+C_n)=\frac{C_0C_1C_2\ldots\ldots\ldots C_{n-1}}{n!}(n+1)^n$$

p) $C_r + 2\,n\,C_{n-1} + n\,C_{r-2} = C(n+2\,,\,r)$

q) $C(n-1\,,\,3) + C(n-4) > C(n,3)$ if $n > 7$.

3. Prove that the coefficient of x^n in $(1 + x)^{2n}$ is twice the coefficient of x^n in $(1 + x)^{2n-1}$.

4. Find the coefficient of x^{50} in

$$(1+x)^{1000}+2x(1+x)^{999}+3x^2(1+x)^{998}\big)+\ldots\ldots\ldots\ldots+1001\,x^{1000}$$

(Ans. C(1002,50))

5. Find the term independent of x in the expansion of $(3x - \frac{2}{x^2})^{15}$.

(Ans. (3003) $3^{10}\, 2^5$)

6. Find the coefficient of x^7 in $(ax^2 + \frac{1}{bx})^{11}$ *and of* x^{-7} *in* $(ax - \frac{1}{bx^2})^{11}$

Also find the relation between a and b so that these coefficients are equal .

(Ans. C(11,6) $\frac{a^5}{b^6}$. When coefficients are equal)

7. Show that the coefficients of x^p and x^q (where p and q are positive integers) in the expansion of $(1+x)^{p+q}$ are equal.

8. Which term in the expansion of the binomial $[\sqrt[3]{(\frac{a}{\sqrt{b}})} + \sqrt{(\frac{b}{\sqrt[3]{a}})}\,]^{21}$ contains a and b to one and the same power?

(Ans. $T_{r=1}$= C(21,r) $a^{7-r/2}$ $b)^{(2r/3)-7/2}$, 10^{th} term.)

9. The sixth term in the expansion of binomial $\{\frac{1}{x^{\frac{8}{3}}} + x^2 \log_{10} x\}^8$ is 5600. Prove that x = 10.

10. The third, fourth and fifth terms in the expansion of $(x+a)^n$ are in AP. Find n.

(Ans. n= 7, 14)

11. Let $(1+x^2)^2(1+x)^n = \sum_{k=0}^{n+4} a_k x^k$. If a_1 , a_2 and a_3 are in A.P. , find n.

(Ans. n=2,3,4)

12. If a_1 , a_2 , a_3 and a_4 are coefficients of any four consecutive terms in the expansion of $(1+x)^n$, prove that $\frac{a_1}{a_1 + a_2} = \frac{a_3}{a_3 + a_4} = \frac{2a_2}{a_2 + a_3}$

13. If $(2+\sqrt{3})^n = I + f$ when I and n are positive integral and 0 < f < 1, show that I is an odd integer and (1-f)(1+f) = 1.

14. If 3^{rd} , 4^{th} , 5^{th} and 6^{th} terms in the expansion of $(x+p)^n$ be respectively a, b, c and d, prove that $\frac{b^2 - ac}{c^2 - bd} = \frac{5a}{3c}$.

15. Find the sum of the rational terms in the expansion of $(\sqrt{2}+\sqrt[5]{3})^{10}$.

(Ans. 41)

16. In the expansion of $(7^{\frac{1}{3}} + 11^{\frac{1}{9}})^{6561}$, prove that there will be only 730 terms.

Sequences and Series

A Lecture – Synopsis

1. A sequence is a function of natural numbers with codomain as the set of real numbers (or complex numbers).

   A sequence $f : N \rightarrow R$ is a real sequence.

   For $n \in N$, $f(n) = t_n$ denoted by

   $\{ t_n \} = \{ f(1), f(2), \ldots f(n) \ldots \} = \{ t_1, t_2, \ldots t_n \ldots\ldots \}$

   The sequence is a finite or infinite according on whether it has finite number of terms or otherwise.

   t_n is called the nth term (or the general term) of the sequence $\{ t_n \}$

   eg.

   1. If $t_n = \dfrac{n}{n+1}$, then

      $\{ t_n \} = \dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \ldots\ldots$

      t_{10} = the tenth term of the sequence = $\dfrac{10}{11}$.

2. Series : Given a sequence $\{ t_n \}$. The expression $t_1 + t_2 + \ldots t_n + \ldots$ is called a series. In other words, the terms of series form a sequence. We denote a series briefly by $\sum_{n=1}^{k} t_n = t_1 + t_2 + \ldots t_k$.

The series is a finite or infinite according as the number of terms of the series is finite or infinite. In the latter case, the infinite series is $\sum_{n=1}^{k} t_n = t_1 + t_2 + \ldots t_n + \ldots t_\infty$.

Examples of Sequences using Recursive Relations

1. If $t_n = t_{n-1} + t_{n-2}$ $n > 2$ with $t_1 = t_2 = 1$.

   Then, $t_3 = t_2 + t_1 = 2$.

   $t_4 = t_3 + t_2 = 2 + 1 = 3$

   $t_5 = t_4 + t_3 = 3 + 2 = 5$ and so on.

   Here the sequence $\{ t_n \}$ is defined by the Recursion formula.

   $t_n = t_{n-1} + t_{n-2}$, $n > 2$.

2. If $t_n = \sqrt{t_{n-1} , t_{n-2}}$, $n > 2$ with $t_1 = t_2 = 1$.

$t_1 = 1, t_2 = 1, t_3 = \sqrt{t_3\ t_1} = \sqrt{1} = 1.$

$t_n = \sqrt{t_3 . t_2} = \sqrt{1.1} = 1,\ t_5 = \sqrt{t_4\ t_3} = 1$ and so on.

3. Special Sequences/ Series

a) Arithmetic Progression/ Arithmetic Series

A sequence { t_n } such that $t_n - t_{n-1}$ = a constant (=d) and (say) s called an Arithmetic Progression (A.P.). d is called the Common Difference (C.D.) of the A.P. A typical Arithmetic Progression looks like a, a+d, a+2d,.... Here a is the first term and d is the common difference (C.D.) of the Arithmetic Progression.

The series whose terms are in A.P. is called an Arithmetic Series.

Then $\sum t_n = t_1 + t_2 + ...$ is an Arithmetic Series if { t_n } is an A.P.

We are familiar with the following :

i) If a and d are known, the A.P. is determined.

ii) For an A.P. $\sum t_n$, $t_n = a\ (n-1)\ d$

iii) For an Arithmetic Series

$$\sum_{n=1}^{n} t_n = \frac{n}{2}(a + t_n) = \frac{n}{2}[\ 2a + (n-1)\ d\ ]$$

iv) $\sum_{n=1}^{n} n = 1 + 2 + ... + n = \frac{n\ (n+1)}{2}$

v) $\sum_{n=0}^{n} 2n + 1 = n^2$

b) Geometric Progression/ Geometric Series

A sequence { t_n } such that $\frac{t_n}{t_{n-1}}$ = constant = r is called a Geometric Progression (G.P.) r is called the common ratio (C.R) of the G.P.

A typical G.P. is a, ar, ar^2, ...

The series $a + ar + ar^2 + ...$ is called a Geometric Series.

We know

i) a and r determine the G.P.

ii) $t_n = ar^{n-1}$

iii) $S_n = \sum_{n=1}^{n} t_n = \frac{a(r^n - 1)}{(r-1)}$

iv) $S_\infty = \sum_{n=1}^{\infty} t_n = \frac{a}{1-r}$, if $|r| < 1$.

The sum $\frac{a}{1-r}$ is called the sum of the Infinite Geometric Series.

c) A sequence $\{t_n\}$ is called a Harmonic Progression (H.P) if the sequence $\left\{\frac{1}{t_n}\right\}$ is an Arithmetic Progression.

A typical Harmonic Progression: $\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \ldots, \frac{1}{a(n-1)d}$.

Means between the two given numbers a and b.

a) Given a,b, n numbers $a_1, a_2, \ldots, a_n$ between a and b such that $a, a_1, a_2, \ldots, a_n, b$ is an A.P. and called the n Arithmetic Means (A.Ms) between a and b.

b) Given a,b > 0, numbers $g_1, g_2, \ldots g_n$ between a and b such that $a, g_1, g_2, \ldots, g_n, b$ is a G.P. are called the n Geometric Means (G.Ms) between a and b.

c) Given a,b n numbers $h_1, h_2, \ldots, h_n$ between a and b such that $a, h_1, h_2 \ldots, h_n$ b is a H.P. are called the n Harmonic Means (H.Ms) between a and b.

d) Single A.M. /G.M. / H.M. between a, b > 0.

A = The A.M.between a and b = $\frac{a+b}{2}$

G = The G.M. between a and b = $\sqrt{ab}$

H = The H.M. between a and b = $\frac{2ab}{a+b}$

Properties of A, G, H

If a > b > 0, then (i) A, G, H are in Geometric Progression so that $G^2 = AH$ and (ii) a > A > G > H > b.

e) Single A.M./ G.M./ H.M. of n given not $a_1, a_2, \ldots, a_n$.

i) $A = \frac{a_1 + a_2 + \ldots + a_n}{n}$ is called the A.M. of $a_1, a_2, \ldots, a_n$.

ii) Let $a_1, a_2, \ldots, a_n$ be n positive numbers ....

$G = (a_1 a_2 \ldots a_n)^{1/n}$ is called the G.M. of $a_1, a_2, \ldots a_n$.

iii) Given n numbers $a_1, a_2, \ldots, a_n$ if H is such that $\frac{1}{H} = \frac{1}{n}\left(\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}\right)$

is called the H.M. of $a_1, a_2, \ldots, a_n$.

Sum to n terms of the Series :

i) $\sum n = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$

ii) $\sum n^2 = 1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$

iii) $\sum n^3 = 1^3 + 2^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}$

Note : $\sum n^3 = (\sum n)^2$

Arithmetico-Geometric (A.G.) Series

A series of the form :

$a + (a+d)r + (a+2d)r^2 + \ldots$ is called an Arithmetico – Geometric Series.

Sum to n terms :

Let $S_n = a(a+d)r + (a+2d)r^2 + \ldots + [a + (n-1)d]r^{n-1}$

$rS_n = ar + (a+d)r^2 + \ldots + [a + (n-2)d]r^{n-1} + [a + (n-1)d]r^n$

$(1-r)S_n = a + dr + dr^2 + \ldots + dr^{n-1} - [a + (n-1)d]r^n$.

$\therefore S_n = \frac{a}{(1-r)} + \frac{dr(1-r^{n-1})}{(1-r)^2} - \frac{[a + (n-1)d]r^n}{(1-r)}$

Sum to Infinity :

If $|r| < 1$, then

$S_\infty = a + (a+d)r + (a+2d)r^2 + \ldots$ to ∞

$= \frac{a}{1-r} + \frac{dr}{(1-r)^2}$

Some Properties of A.P. / G.P.

1. A given A.P. continuous to be an A.P. under each operation given below:
   a) Adding/ Subtracting a constant k to (from) each term.
   b) Multiplying/ dividing by a constant k ($\neq 0$) each term.

c) A given G.P. continues to be a G.P. under the operation multiplying/ dividing by a constant k ($\neq 0$) each term.

Note : In the above, the resulting sequence is not the same on the such sequence.

Eg :

1. Given a,b,c is an A.P, b+c, c+a, a+b is an A.P. a,b,c is an A.P.

   Subtract (a + b + c) frome ach term

   $\therefore$ - (b + c) , - (c + a), - (a + b) is an A.P.

   Multiplying by –1 each term

   (b + c), (c + a), (a + b) is an A.P.

Illustrations and examples

1. A sequence defined by $a_1 = a_2 = 1$, $a_n = a_{n-1} + a_{n-2}$ (n > 2) is called Fibonacci (discovered by Italian Mathematician) Leonardo Fibonacci (1170- 1250 A.D.) sequence. Find the first five terms of the sequence.

Solution :

$a_1 = 1$, $a_2 = 1$, $a_3 = a_2 + a_1 = 2$, $a_4 = a_3 + a_2 = 2 + 1 = 3$ and $a_5 = a_4 + a_3 = 3 + 2 = 5$.

Thus, 1,1,2,3,5 are the first five terms of the sequence.

The 12[th] and 20[th] term of an A.P. are 56 and 96. Find the sum of the Arithmetic Series upto 25[th] term.

Solution :

In the usual notation,

$t_{12} = a + 11d = 56$ (i) $\therefore$ (ii) - (i) = 40 = 8d $\therefore$ d = 5

$t_{20} = a + 19\,d = 96$ (i) $\therefore$ a = 56 – 11 d = 56 – 55 = 1

$$S_{25} = \frac{25}{2}\,[\,2.1 + (25 - 1)\,5]$$

$$= \frac{25}{2}\,[2 + 120\,] = 25 \times 61 = 1525$$

3. In an A.P., $t_m = 1/n$, $t_n = 1/m$. Find t_{mn}.

Solution :

$t_m = a\,(m - 1)\,d = 1/m$

$t_n = a(\,n - 1)\,d = 1/n$

Subtract : $(m - n)\,d = 1/m - 1/n = + \dfrac{m-n}{mn}$

$\Rightarrow$ d = $\frac{+1}{mn}$ $\therefore$ a = $\frac{1}{n} - (m-1)d = \frac{1}{n} + \frac{m-1}{mn} = \frac{1}{mn}$

$\therefore$ a = $\frac{1}{mn}$

$t_{mn} = \frac{1}{mn} + (mn-1)d = \frac{1}{mn} + (mn-1)/mn = \frac{1+mn-1}{mn} = 1$

$\therefore$ $t_{mn} = 1$

4. If a,b,c are such that a + b + c $\neq$ 0.

And $\frac{b+c}{a}, \frac{c+a}{b}, \frac{a+b}{c}$ is an A.P., then S.T. a,b,c are in H.P.

$\frac{b+c}{a}, \frac{c+a}{b}, \frac{a+b}{c}$ is an A.P. [adding throughout]

$\therefore$ $1+\frac{b+c}{a}, 1+\frac{c+a}{b}, 1+\frac{a+b}{c}$ is an A.P.

$\Rightarrow$ $\frac{a+b+c}{a}, \frac{b+c+a}{b}, \frac{c+a+b}{c}$ is an A.P. [$\div$ by a + b + c $\neq$ 0 throughout].

$\Rightarrow$ $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ is an A.P.

$\Rightarrow$ a,b,c is a H.P.

5. In an A.P., t_p= a, t_q = b, t_r = c. such that $\sum a(q-r) = 0$

Solution :

Let α = The 1st term, d = C.D. of the A.P.

$\therefore$ $t_p = \alpha = (p-1)d = a$

$t_q = \alpha + (q-1)d = b$

$t_r = \alpha(r-1)d = c$

$\therefore$ $a(q-r) = [\alpha + (p-1)d](q-r)$

$\therefore$ $\sum a(q-r) = \alpha\sum(q-r) + d\sum(p-1)(q-r)$

$= \alpha \times 0 + d \times 0 = 0$

6. The sums of the first n terms of two Arithmetic Series are as 2n – 3 : 3n – 2. Find the ratio of their tenth terms.

Solution : Let a, d, t_n and S_n be the first term, C.D., nth term of an A.P. while a′, d′, t_n' and S_n' are those of the second A.P.

Then $\frac{S_n}{S_n'} = \frac{2n-3}{3n-2}$

Taking n = 2m – 1, $\frac{S_{2m-1}}{S'_{2m-1}} = \frac{2(2m-1)-3}{3(2m-1)-2} = \frac{4m-5}{6m-5}$ (1)

$S_{2m-1} = \frac{2m-1}{2}\ [\,2a + (2m-1-1)\,d\,] = (2m-1)\,(a + \overline{a + m - 1d}\,) = (2m-1)\,t_m$

$\therefore \quad \frac{S_{2m-1}}{S'_{2m-1}} = \frac{(2m-1)\,t_m}{(2m-1)\,t'_m} = \frac{t_m}{t'_m} = \frac{4m-S}{6m-S}$ (from (i))

$\therefore \quad \frac{t_{10}}{t'_{10}} = \frac{4\times 10-5}{6\times 10-5} = \frac{35}{55} = \frac{7}{11}$

$\therefore \quad \frac{t_{10}}{t'_{10}} = \frac{7}{11}$

7. The natural numbers are grouped as (1) + (2 + 3 + 4) + (5 + 6 + 7 + 8 + 9) +....

Find the sum of the numbers in the nth group.

Solution :

Group Number	No. of terms
1	1
2	3
3	5
∴ n	(2n – 1)

∴ In the nth group, there are (2n – 1) terms.

∴ The number of terms upto (including) the (n-1) th group

$= 1 + 3 + 5 + \ldots + (2n-1) = (n-1)^2 = N$ (say)

∴ The sum of the series upto terms in the nth group

$= \frac{N\,(N+1)}{2} = \frac{n^2\,(n^2+1)}{2}$ (i)

∴ The terms in the nth group are $(n-1)^2 + 1, (n-1)^2 + 2, \ldots (2n-1)$ terms.

∴ The sum of the numbers in the nth group

$= (\,(n-1)^2 + 1\ ((n-1)^2 + 2) + \ldots.(2n-1)\,)$ terms

$= \frac{2n-1}{2}\ [2\,(n-1)^2 + 1) + (\,(2n-1) - 1)\,.\,1\,]$

$$= \frac{2n-1}{2}\ [\,2\,(n-1)^2 + 2 + 2n - 2\,]$$

$$= (2n-1)\ [\,(n-1)^2 + n$$

$$= (2n-1)\ [\,n^2 - n + 1\,]$$

$$= n^3 + (n-1)^3$$

8. The sum of a series upto n terms is $\frac{8}{3}\left(1-\frac{1}{n}\right)$ such that the series is not a geometric series.

Solution :

$S_n = \frac{8}{3}\left(1-\frac{1}{n}\right)$ = sum to n terms of the series.

t_n = The nth term = $S_n - S_{n-1}$

$$= \frac{8}{3}\left[\left(1-\frac{1}{n}\right)-\left(1-\frac{1}{n-1}\right) = \frac{8}{3}\left[\frac{1}{n-1}-\frac{1}{n}\right]\right]$$

$$= \frac{8}{3}\left(\frac{1}{n\,(c-1)}\right)$$

$$\frac{t_n}{t_{n-1}} = \frac{\frac{8}{3}\cdot\frac{1}{n\,(n-1)}}{\frac{8}{3}\ \frac{1}{(n-1)\,(n-2)}} = \frac{n-2}{n} = 1-\frac{2}{n}$$

If the series is a geometric series, $\frac{t_n}{t_m}$ must be a constant (i.e. independent of n).

However, this is not so. Hence the series is not a geometric series.

9. Find four numbers in G.P. if the sum of the 1st and 3rd is 15 while the sum of the other four is 30.

Solution :

Taking the numbers as a, ar, ar^2, ar^3

$$a + ar^2 = 15;\ \ ar + ar^3 = 30$$

$$\Rightarrow a\,(1+r^2) = 15;\ \ ar\,(1+r^2) = 30$$

$$\therefore\ \frac{a\,r\,(1+r^2)}{a\,(1+r^2)} = r = \frac{330}{15} = 2$$

$$\therefore\quad a + a\,r^2 = a + 4a = 15$$

$$\therefore\quad a = 3$$

$\therefore$ The numbers are 3, 6, 12, 24.

10. If $S_1 = 1 + a + a^2 + ...$ to ∞ , $|a| < 1$

$S_2 = 1 + b + b^2 + ...$ to ∞, $|b| < 1$

Find $1 + ab + (ab)^2 + ...$ to ∞.

Solution :

$$S_1 = \frac{1}{1-a} ; \quad S_2 = \frac{1}{1-b}$$

Let $S_3 = 1 + ab + (ab)^2 + ...$t n $= \dfrac{1}{1-ab}$

$$1 - a = \frac{1}{S_1} \Rightarrow a = 1 - \frac{1}{S_1} , \quad b = 1 - \frac{1}{S_2}$$

$$\therefore \quad S_3 = \frac{1}{1-(1-S_1)(1-S_2)} = \frac{1}{1-(1-\frac{1}{S_2})(1-\frac{1}{S_2})}$$

$$= \frac{S_1 S_2}{1-(S_1-1)(S_2-1)} = \frac{S_1 S_2}{S_1+S_2-1}$$

$$\therefore \quad S_3 = \frac{S_1 S_2}{S_1+S_2-1}$$

11. Evaluate : $\sqrt{3\sqrt{3\sqrt{3\sqrt{3\sqrt{.......... to \infty}}}}}$

Solution :

Let $x = \sqrt{3\sqrt{3\sqrt{3\sqrt{3\sqrt{.......... to \infty}}}}}$

$$\therefore \quad x = \sqrt{3} \, . \sqrt[4]{3} \, . \sqrt[8]{3} \, ... = 3^{1/2 + 1/4 + 1/8 + ... \text{to } n}$$

$$\therefore x = 3(\frac{1/2}{1-1/2}) = 3^1 = 3 \qquad \therefore \qquad (x = 3)$$

12. The ratio $\dfrac{S_m}{S_n} = \dfrac{m^2}{n^2}$ in an arithmetic series such that

$$\frac{t_m}{t_n} = \frac{2m-1}{2n-1}$$

$S_m = \dfrac{m}{2}[2a + \overline{n-1}d]$; $\qquad S_n = \dfrac{n}{2}[2a + \overline{m-1}d]$

$$\therefore \quad \frac{S_m}{S_n} = \frac{m^2}{n^2} = \frac{m}{n}\left[\frac{2a + \overline{m-1}d}{2a + \overline{n-1}d}\right]$$

$$\Rightarrow \quad \frac{2a + \overline{m-1}d}{2a + \overline{n-1}d} = \frac{m}{n} \text{ changing } m \to 2m - 1$$

$$n \to 2n - 1$$

$$\frac{2a + \overline{m-1-1}d}{2a + \overline{2n-1-1}d} = \frac{2m-1}{2n-1}$$

$$\Rightarrow \quad \frac{a + \overline{m-1}d}{a + \overline{n-1}d} = \frac{2m-1}{2n-1} \Rightarrow \frac{t_m}{t_n} = \frac{2m-1}{2n-1}$$

13. If $\dfrac{a^n + b^n}{a^{n-1} + b^{n-1}}$ is the Arithmetic Mean between a and b, find n.

Solution : $\dfrac{a^n + b^n}{a^{n-1} + b^{n-1}} = \dfrac{a+b}{2}$

$\Rightarrow 2a^n + 2b^n = (a + b)(a^{n-1} + b^{n-1})$

$\Rightarrow 2a^n + 2b^n = a^n + b^n + ab^{n-1} + a^{n-1} b$

$\Rightarrow a^n + b^n = ab^{n-1} + a^{n-1} b$

$\Rightarrow a^n - ab^{n-1} = a^{n-1} - b^n$

$\Rightarrow a(a^{n-1} - b^{n-1}) = b(a^{n-1} - b^{n-1})$

$\Rightarrow (a - b)(a^{n-1} - b^{n-1}) = 0$

Since $a \neq b$, $a^{n-1} - b^{n-1} = 0 \Rightarrow n = 1$.

14. Four numbers are such that the first and the fourth are equal first three are in G.P., the last 3 are in A.P. with common difference 6. Find the numbers

Let the numbers be a,b, b + 6, b + 12 and a = b + 12

Also $b^2 = a(b + 6)$

$\Rightarrow b^2 (b + 12)(b + 6) = b^2 + 18b + 72$

$\therefore$ b = -4, $\therefore$ a = 8

$\therefore$ The numbers are 8, \-4, 2, 8

15. $S_1, S_2, \ldots S_k$ are the sums of n terms of k A.P's with 1st terms : 1,2,3,..., k; C.Ds : 1,3,5,...(2k – 1) respectively such that

$$S_1 + S_2 + \ldots + S_k = \frac{1}{2} nk\,(nk + 1)$$

$$S_1 = \frac{n}{2}\,[2.1 + \overline{n-1}\,.\,1] = \frac{n}{2}\,[2+(n-1)\,1\,]$$

$$S_2 = \frac{n}{2}\,[2.2 + \overline{n-1}\,.\,3] = \frac{n}{2}\,[4+(n-1)\,3\,]$$

..

..

..

$$S_k = \frac{n}{2}\,[2.k + \overline{n-1}\,\,(2k-1)\,\,]$$

$$\therefore\; S_1 + S_2 \ldots + S_k = \frac{n}{2}\Big[\,2\sum k + (n-1)\sum\,(2k-1)\Big]$$

$$= \frac{n}{2}\left[\,2\left(\frac{k\,(k+1)}{2}\right) + (\,n-1).k^2\right]$$

$$= \frac{n\,k}{2}\,\,[\,k + 1 + nk - k\,] = \frac{n\,k}{2}\,\,[nk\,+1]$$

$$\therefore \qquad S_1 + S_2 + \ldots + S_k = \frac{n\,k}{2}\,\,(nk + 1)$$

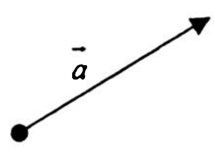
VECTORS

Introduction : The study of vectors is greatly motivated by problems of a physicist and engineer. The notion of vectors and their methods are being used in a variety of branches of knowledge – mechanics, electricity and magnetism, etc.

Basic Terminology and Notations :

1. Representation of Vector:

A vector has both magnitude and direction. It is represented by a directed line segment as shown in the figure.



This vector is denoted by $\overrightarrow{OA}$ or $\vec{a}$. It has both magnitude and direction. The length of OA denotes the magnitude and the arrow indicates the direction of the vector $\overrightarrow{OA}$. 'O' is the initial point and A is the terminal point of vector $\overrightarrow{OA}$. The magnitude of $\overrightarrow{OA}$ is denoted by $|\overrightarrow{OA}|$ or $|\vec{a}|$. By definition, the magnitude of a vector is always a non-negative real number i.e. $|\vec{a}| \geq 0$.

Note (i) A directed line segment being a part of a straight line, the straight line itself is called the *support of the vector*.

(ii) Any two directed segments of the same length drawn in the same direction represent the same vector. Such vectors are called "free vectors".

(iii) In the course of the text, we deal with "free vectors" meaning that a vector is free to move about under parallel displacement.

(iv) In representing a free vector, any point of the plane can be taken as the initial point.

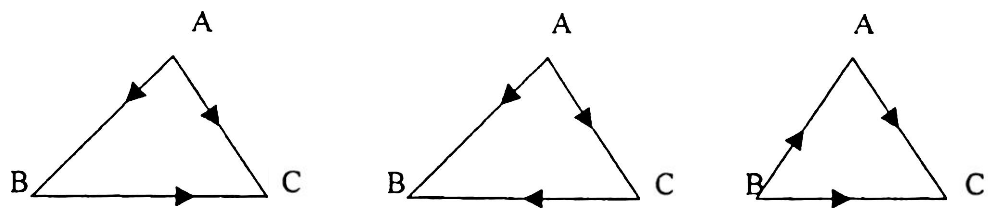
2. Types of Vectors : The concepts of like vectors, unlike vectors, equal vectors, null vector, unit vector, unit vector in the direction of a given vector, coinitial vectors, coterminal vectors, collinear vectors and coplanar vectors are very clear from the different textbooks available.

3. Operations on Vectors :

a) Addition of Vectors : Let $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{AB} = \vec{b}$ be two vectors. Then the vector represented by $\overrightarrow{OB}$ is called the sum of $\vec{a}$ and $\vec{b}$. Then we write

$\overrightarrow{OB} = \vec{a} + \vec{b}$ i.e. $\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$.

This leads to the statement of triangular law of addition of vectors. "If two vectors are represented by the two sides of a triangle taken in order, then their sum is denoted by the third side of the triangle in the reverse order".



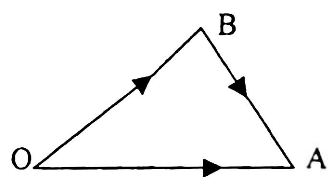
$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ $\overrightarrow{AC} + \overrightarrow{CB} = \overrightarrow{AB}$ $\overrightarrow{BA} + \overrightarrow{AC} = \overrightarrow{BC}$

b) Subtraction of Vectors: If $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$, According to Triangular law,

$\overrightarrow{OB} + \overrightarrow{BA} = \overrightarrow{OA}$

$\therefore \overrightarrow{BA} = \overrightarrow{OA} + (-\overrightarrow{OB})$

$= \vec{a} + (-\vec{b}) = \vec{a} - \vec{b}$



Thus subtraction of a vector is the same as the addition of negative vector to the given vector.

c) Multiplication of a vector by a scalar

Let $\vec{a}$ be a vector and k a scalar (real number). A vector $\vec{b} = k.\vec{a}$ is a vector whose magnitude is | k | times that of $\vec{a}$ and the direction depends upon the sign of k.

If k > 0, $\vec{a}$ and $\vec{b}$ are like vectors as shown in the figure.

If k < 0 $\vec{a}$ and $\vec{b}$ are unlike vectors as shown in the figure.

Note: Two vectors $\vec{a}$ and $\vec{b}$ are collinear (||^{le}) iff for some real number k, $\vec{b}$ =k$\vec{a}$

The important properties of above operations can be understood clearly.

If V is a set of vectors and addition is a Binary operation defined on it, then

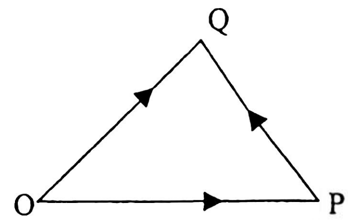
1. for $\vec{a}, \vec{b} \in V$, $\vec{a} + \vec{b} \in V$ (Closure Property)
2. $\vec{a} + \vec{b} = \vec{b} + \vec{a}$ (Commutative) for all $\vec{a}, \vec{b} \in V$
3. $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$ (Associative) for all $\vec{a}, \vec{b}, \vec{c} \in V$
4. $\vec{0} \in \forall$ and $\vec{a} + \vec{0} = \vec{0} + \vec{a} = \vec{a}$ $\forall \vec{a} \in V$ ($\vec{0}$ is called additive identity element.
5. For each $\vec{a} \in V$, $-\vec{a} \in V$, and $\vec{a} + (-\vec{a}) = \vec{0} = (-\vec{a} + \vec{a})$, ($-\vec{a}$ is called the negative of $\vec{a}$).
6. $m(\vec{a} + \vec{b}) = m\vec{a} + m\vec{b}$.
7. $m(n\vec{a}) = mn\vec{a}$.
8. $(m+n)\vec{a} = m\vec{a} + n\vec{a}$

(6–8) for all $\vec{a}, \vec{b} \in V$ and for all Scalars m and n.

4. Position vector of a point w.r.t. origin.

Let P be a point and O be the origin or point of reference, then the vector $\overrightarrow{OP}$ is called the position vector of P w.r.t. O. Given the position vectors of the points P and Q w.r.t. O by $\overrightarrow{OP} = \vec{a}$, $\overrightarrow{OQ} = \vec{b}$ then, we can express $\overrightarrow{PQ}$ in terms of $\vec{a}$ and $\vec{b}$.

By Triangular Law



$\overrightarrow{OP} + \overrightarrow{PQ} = \overrightarrow{OQ}$

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

$= \vec{b} - \vec{a}$

$\overrightarrow{PQ}$ = P.V. of terminal point – P.V. of initial point.

In particular, the P.V. of the origin is always $\vec{0}$.

Note:

1. This concept is to be emphasized to enable the students to express the given vector in terms of P.V. of the points.
2. At this stage some problems can be worked out as application of vector methods.

Problems :

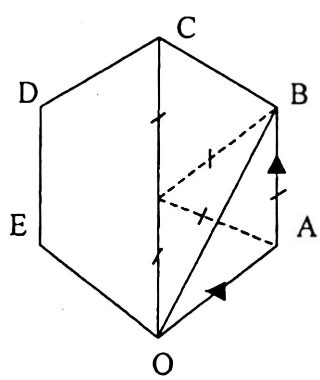
1. $\vec{a}$ and $\vec{b}$ are adjacent sides of a regular hexagon OABCDE. Express the remaining sides in terms of $\vec{a}$ and $\vec{b}$.

Solution :

Let $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{AB} = \vec{b}$.

(Here $|\vec{a}| = |\vec{b}|$ but $\vec{a} \neq \vec{b}$ why?)

hen $\overrightarrow{OB} = \vec{a} + \vec{b}$



$\overrightarrow{OC} = 2\vec{b}$ (clear from the figure)

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

$= 2\vec{b} - \vec{a} - \vec{b}$

$\overrightarrow{BC} = \vec{b} - \vec{a}$

Since CD and OA are equal and parallel.

$\overrightarrow{CD} = -\vec{a}$

Similarly, $\overrightarrow{DE} = -\vec{b}$

and $\overrightarrow{EO} = -\overrightarrow{BC} = -(\vec{b} - \vec{a}) = \vec{a} - \vec{b}$

Hence, $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{AB} = \vec{b}$, $\overrightarrow{BC} = \vec{b} - \vec{a}$, $\overrightarrow{CD} = -\vec{a}$

$\overrightarrow{DE} = -\vec{b}$ and $\overrightarrow{EO} = \vec{a} - \vec{b}$.

2. ABCDE is a pentagon. Prove that the resultant of forces $\overrightarrow{AB}, \overrightarrow{AE}, \overrightarrow{BC}, \overrightarrow{DC}, \overrightarrow{ED}$ and $\overrightarrow{AC}$ is $3\overrightarrow{AC}$.

Solution :

$$\vec{R} = \overrightarrow{AB} + \overrightarrow{AE} + \overrightarrow{BC} + \overrightarrow{DC} + \overrightarrow{ED} + \overrightarrow{AC}$$

$$= (\overrightarrow{AB} + \overrightarrow{BC}) + (\overrightarrow{AE} + \overrightarrow{ED} + \overrightarrow{DC}) + \overrightarrow{AC}$$

$$= \overrightarrow{AC} + \overrightarrow{AC} + \overrightarrow{AC}$$

$$= 3\,\overrightarrow{AC}.$$

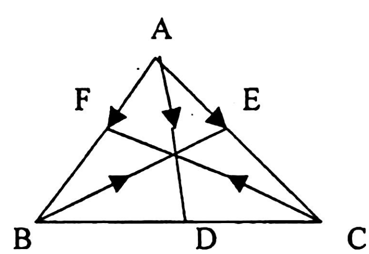
3. Prove that sum of three vectors determined by the medians of a triangle directed from the vertices is zero.

Solution :

AD, BE and CE are the medians of the triangle ABC.

As D is the mid point of BC, w.r.t. A as reference point,

Then DB = DC. $\therefore\ \overrightarrow{DB} = -\overrightarrow{DC}$.



$$\overrightarrow{AD} + \overrightarrow{DB} = \overrightarrow{AB}$$

$$\overrightarrow{AD} + \overrightarrow{DC} = \overrightarrow{AC}$$

$\therefore\quad 2\overrightarrow{AD} + \overrightarrow{DB} + \overrightarrow{DC} = \overrightarrow{AB} + \overrightarrow{AC}$

$\therefore\quad 2\overrightarrow{AB+}\ - \overrightarrow{DC} + \overrightarrow{DC} = \overrightarrow{AB} + \overrightarrow{AC}$

$\therefore\quad 2\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{AC}$

$\therefore\ \overrightarrow{AD} = \dfrac{\overrightarrow{AB} + \overrightarrow{AC}}{2}$ (mid point formula)

Similarly, $\overrightarrow{BE} = \dfrac{\overrightarrow{BC} + \overrightarrow{BA}}{2}$ and $\overrightarrow{CF} = \dfrac{\overrightarrow{CA} + \overrightarrow{CB}}{2}$

$$\therefore \overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF} = \frac{1}{2}[\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{BC} + \overrightarrow{BA} + \overrightarrow{CA} + \overrightarrow{CB}]$$

$$= \frac{1}{2}[\overrightarrow{AB} + \overrightarrow{BA} + \overrightarrow{AC} + \overrightarrow{CA} + \overrightarrow{BC} + \overrightarrow{CB}]$$

$$= \frac{1}{2}[\vec{0} + \vec{0} + \vec{0})]$$

$$\therefore \overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF} = \vec{0}.$$

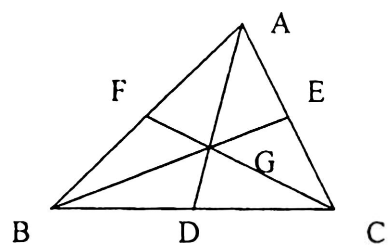
4. If $\vec{a}, \vec{b}$ and $\vec{c}$ are the position vectors of the vertices of a triangle ABC and G is the centroid of the triangle, prove that position vector of the centroid of the triangle is $\dfrac{\vec{a}+\vec{b}+\vec{c}}{3}$.

Solution :

Given $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$ and $\overrightarrow{OC} = \vec{c}$. AD is the median. G is the centroid of the triangle. By geometry, we know that AG : GD = 2 : 1.

$\therefore \dfrac{AG}{GD} = \dfrac{2}{1}$. $\therefore$ AG = 2GD.

$$\overrightarrow{AG} = 2\overrightarrow{GD}$$



$$\therefore \overrightarrow{OG} - \overrightarrow{OA} = 2(\overrightarrow{OD} - \overrightarrow{OG})$$

$$\therefore \overrightarrow{OG} - \overrightarrow{OA} = 2\overrightarrow{OD} - 2\overrightarrow{OG}$$

$$\therefore \overrightarrow{OG} + 2\overrightarrow{OG} = 2\overrightarrow{OD} - \overrightarrow{OA}$$

$3\overrightarrow{OG} = 2 . \dfrac{\overrightarrow{OB} + \overrightarrow{OC}}{2} + \overrightarrow{OA}$ [$\because$ is the mid point of BC]

$$= \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OA}$$

$$= \vec{b} + \vec{c} + \vec{a}$$

$$\therefore \quad \overrightarrow{OG} = \frac{\vec{a} + \vec{b} + \vec{c}}{3}$$

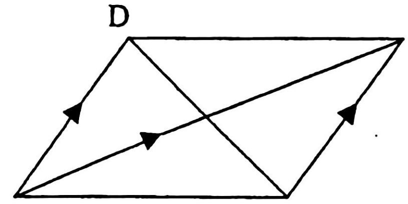
5. Prove that the diagonals of a parallelogram bisect each other and conversely if the diagonals of a quadrilateral bisect each other, it is a parallelogram.

Solution:

Let A be taken as origin and the P.V. of B, C and D be taken as $\vec{b}$, $\vec{c}$ and $\vec{d}$.

Now $\overrightarrow{BC} = \vec{c} - \vec{b}$ and $\overrightarrow{AD} = \vec{d}$.

But BC is parallel and equal to AD.



$\therefore \quad \overrightarrow{BC} = \overrightarrow{AD}$

$\therefore \quad \vec{c} - \vec{b} = \vec{d}$

$\therefore \quad \vec{c} = \vec{b} + \vec{d}$

or $\quad \frac{\vec{c}}{2} = \frac{\vec{b} + \vec{d}}{2}$

i.e. mid point of diagonal AC is the same as the mid point of diagonal BD. Hence the diagonals bisect each other.

Converse : We are given that diagonals bisect each other i.e. $\frac{\vec{c}}{2} = \frac{\vec{b} + \vec{d}}{2}$ or

$\vec{c} = \vec{b} + \vec{d}$ or $\vec{c} - \vec{b} = \vec{d}$. $\quad \therefore \overrightarrow{BC} = \overrightarrow{AD}$

$\therefore$ AD is parallel and equal to BC.

Hence the figure is a parallelogram.

Self Evaluation Tests

1. ABCD is a quadrilateral and E the point of intersection of the lines joining the middle points of opposite sides. Show that the resultant of $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ *and* $\overrightarrow{OD}$ is equal to $4\overrightarrow{OE}$ where O is any point.

2. ABC is a triangle and P any point on BC. If $\overrightarrow{PQ}$ is the resultant of $\overrightarrow{AP}$, $\overrightarrow{PB}$ and $\overrightarrow{PC}$ show that ABQC is a parallelogram.

3. If G is the centroid of the triangle ABC, show that

   $\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} = \bar{0}$ and conversely if

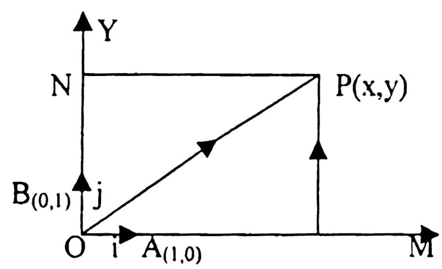
   $\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} = \bar{0}$, then G is the centroid of the triangle ABC.

4. Prove that the straight line joining the mid points of the diagonals of a trapezium is parallel to the parallel sides and half of their sum by vector method.

5. Prove that the figure formed by joining the mid points of the sides of a quadrilateral taken in order is a parallelogram.

5. Vectors in the Cartesian Plane and in Cartesian Space:

a) Position Vector in Cartesian plane



Let P(x,y) be any point in the Cartesian Plane. Let $\hat{i}$ denote the unit vector along x-axis and $\hat{j}$ denote the unit vector along y-axis. Draw PM ⊥ OX and PN ⊥ OY.

Now, OM = x, $\therefore\ \overrightarrow{OM} = x\hat{i}$

MP = y, $\therefore\ \overrightarrow{MP} = y\hat{j} = \overrightarrow{ON}$

Now $\overrightarrow{OP} = \overrightarrow{OM} + \overrightarrow{MP}$

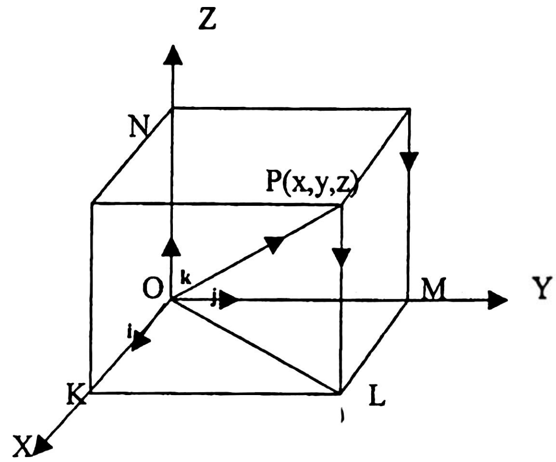
$\overrightarrow{OP} = x\hat{i} + y\hat{j}$

$\therefore$ Position vector of a point P in the Cartesian Plane is $\overrightarrow{OP} = xi + yj = (x, y)$.
Thus an ordered pair of real numbers (x,y) denotes the position vector of a point in a plane.

Magnitude of $\overrightarrow{OP} = |\overrightarrow{AP}| = \sqrt{x^2 + y^2}$

Direction of $\overrightarrow{OP}$ is given by $\tan\theta = \frac{y}{x}$ where θ is the angle of inclination of OP with the positive direction of x-axis.

6. Position Vector in Cartesian Space:



Let P(x,y,z) be any point in the Cartesian space. Let $\hat{i}, \hat{j}$ *and* $\hat{k}$ be the unit vectors along OX, OY and OZ respectively.

Draw PQ ⊥r to XOY plane, QL ⊥r to x-axis and QM ⊥r y-axis. Join OP and OQ.

Now OL = x, $\overrightarrow{OL} = x\hat{i}$, OM = y, $\overrightarrow{OM} = y\hat{j}$, QP = z, $\overrightarrow{QP} = z\hat{k}$. Position vector of the point P(x,y,z) in space is $\overrightarrow{OP}$.

$\overrightarrow{OP} = \overrightarrow{OQ} + \overrightarrow{QP}$

$= \overrightarrow{OL} + \overrightarrow{LQ} + \overrightarrow{QP}$

$= \overrightarrow{OL} + \overrightarrow{OM} + \overrightarrow{QP}$

$\therefore \quad \overrightarrow{OP} = xi + yj + zk$

Thus an ordered triplet of real numbers (x, y, z) denotes the position vector of a point in space.

Magnitude of $\overrightarrow{OP} = |\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$

Direction of $\overrightarrow{OP}$ is given by the angles made by OP with x, y and z axis respectively. Let α, β and γ be the angles made by $\overrightarrow{OP}$ with positive direction of coordinate axes respectively. Then cos α, cos β and cos γ are called **the direction cosines of $\overrightarrow{OP}$.** It can be shown that $\cos\alpha = \frac{x}{|\overrightarrow{OP}|}$, $\cos\beta = \frac{y}{|\overrightarrow{OP}|}$, $\cos\gamma = \frac{z}{|\overrightarrow{OP}|}$.

Note: The approach to the study of vectors can be either on a geometric or an analytic basis. We have considered geometrical approach at the beginning to motivate the students.

Now addition and subtraction can be easily explained with algebraic approach with vectors as an ordered pair of real numbers and ordered triplet of real numbers in a Cartesian plane and space respectively.

Examples :

1. If the P.V. of A is 2i – 3j + 4k and that of B is 3i + 4j – 2k find $\overrightarrow{AB}$ and its length.

Solution : Given $\overrightarrow{OA}$ = 2i – 3j + 4k

$\overrightarrow{OB}$ = 3i + 4j – 2k, O being the origin.

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

= (3i + 4j – 2k) – (2i – 3j + 4k)

$\overrightarrow{AB}$ = i + 7j – 6k

Length of $\overrightarrow{AB} = |\overrightarrow{AB}| = \sqrt{1^2 + 7^2 + 6^2} = \sqrt{1 + 49 + 36} = \sqrt{86}$.

2. Find the magnitude, direction cosines and unit vector in the direction of the vector (2, -1, 2).

Solution : Let $\vec{a}$ = (2, -1, 2) = 2i – j + 2k

$|\vec{a}| = \sqrt{2^2 + 1^2 + 2^2} = 3$

Unit vector in the direction of $\vec{a} = \frac{2i - j + 2k}{3}$.

Direction cosines of $\vec{a}$ are $\cos\alpha = \frac{2}{3}$, $\cos\beta = -\frac{1}{3}$ and $\cos\gamma = \frac{2}{3}$.

3. If $\vec{a} = 2i + j - 3k$ and $\vec{b} = I + j - 2k$

Find I) $|2\vec{a} - \vec{b}|$ ii) a vector of magnitude $\sqrt{38}$ units in the opposite direction of $(\vec{a} + \vec{b})$.

Solution :

(i) $2\vec{a} - \vec{b} = 2(2i + j - 3k) - (I + j - 2k)$

$= 4i + 2j - 6k - I - j + 2k$

$= 3i + j - 4k$ $\therefore |2\vec{a} - \vec{b}| = \sqrt{9 + 1 + 16} = \sqrt{26}$

ii) $\vec{a} + \vec{b} = 2i + j - 3k + I + j - 2k$

$= 3i + 2j - 5k$

$|\vec{a} + \vec{b}| = \sqrt{9 + 4 + 25} = \sqrt{38}$

Unit vector along $(\vec{a} + \vec{b})$ is $\frac{3i + 2j - 5k}{\sqrt{38}}$.

$-\sqrt{38} \cdot \frac{(3i + 2j - 5k)}{\sqrt{38}} = -3i - 2j + 5k$ is the required vector.

Self Evaluation Questions

1. If $\vec{a} = (2,3)$, $\vec{b} = (-1, 5)$ find (i) $2\vec{a} + 3\vec{b}$ ii) $|\vec{a} - 2\vec{b}|$ iii) a unit vector along $2\vec{a} - \vec{b}$.

2. If $\vec{a} = (1, 1, -1)$, $\vec{b} = (4, 1, 2)$ and $\vec{c} = (0, 1, -2)$ show that $(\vec{a} - \vec{b}) = 3(\vec{c} - \vec{a})$.

3. I) Find $\vec{a}$ if $3\vec{a} + (1, -2, 5) = 2\vec{a} - (1, 1, 0)$.

   ii) Find $\vec{a}$ and $\vec{b}$ if $\vec{a} + \vec{b} = (1, 2, 5)$ and $\vec{a} - \vec{b} = (3, 6, -3)$.

4. If the position vectors of P and Q are 2i – 3j + k and 3i – j – 2k find $\overrightarrow{PQ}$ and its direction cosines.

5. If A is (1, -2, 1), B is (2, 1,3) and C is (1, 1, 0) find a unit vector parallel to I) $\overrightarrow{AB} - 2\overrightarrow{BC}$ ii) $\overrightarrow{AC} - \overrightarrow{BC}$.

6. Position vectors of the points A, B, C and D are respectively 2i + 3j + 5k, i + 2j + 3k, -5i + 4j – 2k and i + 10j + 10k. Show that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are parallel.

7. Show that the vectors 2i – j + k, i – 3j – 5k and 3i – 4j – 4k form a right angled triangle.

8. Show that the points (2,3,4) (5,6,8) and (8,9,12) are collinear.

9. Show that the vector (sin α cos β, cos α cos β, sin β) is a unit vector.

10. Find a unit vector along the resultant of $\vec{a}$ = 2i + 4j – 5k, $\vec{b}$ = i + 2j+ 3k, $\vec{c}$ = I - j + k.

7. Multiplication of Vectors

There are two types of Product of two vectors.

i) Scalar Product or dot (•) Product

ii) Vector Product or cross (×) Product

8. Scalar Product (• Dot product)

Let $\vec{a}$ and $\vec{b}$ be two vectors which include an angle θ, ($0 \le \theta \le \pi$). We define the scalar product of $\vec{a}$ and $\vec{b}$ as $\vec{a} . \vec{b} = |\vec{a}|\ |\vec{b}| \cos\theta$. It is also called as Dot Product.

Properties of Dot Product

1. $\vec{a} . \vec{b}$ is a scalar.

2. Since $\vec{a} . \vec{b} = |\vec{a}|\ |\vec{b}| \cos\theta$, we have $\cos\theta = \dfrac{\vec{a} . \vec{b}}{|\vec{a}|\,|\vec{b}|}$.

3. If $\theta = 0$, $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|$ since $\cos 0 = 1$ In particular $\boxed{\vec{a} \cdot \vec{a} = |\vec{a}| . |\vec{a}| = |\vec{a}|^2}$ where $\vec{a} \cdot \vec{a}$ also by $\vec{a}^2$. Then $\vec{a}^2 = \vec{a} \cdot \vec{a} = |\vec{a}|^2$ (Collinear Vectors).

4. Let $\vec{a}$, $\vec{b}$ be non-zero vectors. If $\theta = 90^\circ$, $\vec{a} \cdot \vec{b} = 0$ since $\cos 90 = 0$, $\vec{a}$ and $\vec{b}$ are perpendicular vectors and conversely .

5. Since $\cos(-\theta) = \cos\theta$. We have $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$ (Commutative).

6. Scalar product of unit vectors i, j and k : i, j, k are the unit vectors along x, y and z axis respectively. They are mutually at right angles. The multiplication table gives the scalar product of i, j and k.

.	i	j	k
i	1	0	0
j	0	1	0
k	0	0	1

7. If $\vec{a} = x_1 i + y_1 j + z_1 k$, $\vec{b} = x_2 i + y_2 j + z_2 k$ then

$\vec{a} \cdot \vec{b} = (x_1 i + y_1 j + z_1 k) . (x_2 i + y_2 j + z_2 k)$

$\boxed{\vec{a} \cdot \vec{b} = x_1 x_2 + y_1 y_2 + z_1 z_2}$

8. For any three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$ (Distributive).

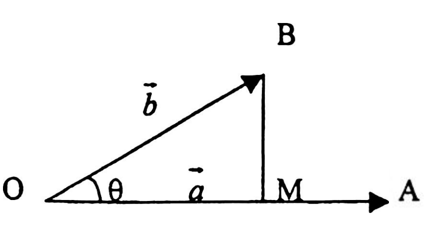
Geometrical Interpretation:

Let $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$ so that $A\hat{O}B = \theta$. Draw BM ⊥r to OA. Then OM is the projection of $\vec{b}$ on $\vec{a}$ and $OM = OB \cos\theta = |\vec{b}| \cos\theta$.

Now $\vec{a} \cdot \vec{b} = |\vec{a}|\ |\vec{b}| \cos\theta$

$= |\vec{a}| . OM$

$\therefore \overrightarrow{OM} = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$



i.e. projection of $\vec{b}$ on $\vec{a}$ = $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$

In particular, if $\vec{a}$ is a unit vector, then $|\vec{a}| = 1$ so that *projection of* $\vec{b}$ on a unit vector $\vec{a} = \vec{a}\cdot\vec{b}$

Projection of $\vec{a}$ on $\vec{b}$ = $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$.

10. Application of Dot Product

Let $\vec{a}$ be a force under which a particle on which it acts, moves through a distance $\vec{b}$ (displaces in particular direction). Then the component of the force $\vec{a}$ in the direction of $\vec{b}$ is $|\vec{a}| \cos\theta$. The work done by the force $\vec{a}$ in moving the particle in the direction of the force is $|\vec{a}| \cos\theta . |\vec{b}|$.

i.e. $|\vec{a}|\,|\vec{b}| \cos\theta = \vec{a}.\vec{b}$.

Hence work done (W by a force $\vec{a}$ displacing a point through $\vec{b} = \vec{a}.\vec{b}$.

i.e. $W = \vec{a}.\vec{b}$

Vector Product of two vectors

Let $\vec{a}$ and $\vec{b}$ be two vectors and θ be the angle between them. We define the vector product of $\vec{a}$ and $\vec{b}$ as

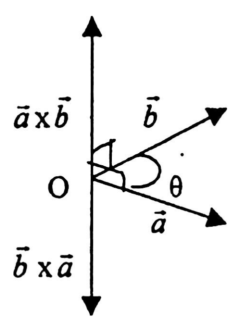
$$\vec{a} \times \vec{b} = (|\vec{a}|\,|\vec{b}| \sin\theta . \hat{n}$$

where i) θ is the angle between $\vec{a}$ and $\vec{b}$

ii) $\hat{n}$ is the unit vector at right angles to $\vec{a}$ and $\vec{b}$.

iii) $\vec{a}, \vec{b}$ and $\vec{a} \times \vec{b}$ form a right handed system

This product is also known as Cross Product.



Note: $\vec{a} \times \vec{b}$ denotes the direction of the movement of the screw when the screw is turned from $\vec{a}$ towards $\vec{b}$. This is taken as positive. On the other hand, when the screw is turned from $\vec{b}$ towards $\vec{a}$ the direction of $\vec{b} . \vec{a}$ is downwards. It is taken as negative.

Properties of Vector Product of $\vec{a}$ and $\vec{b}$.

1. $\vec{a} \times \vec{b}$ is a vector.

2. $|\vec{a} \times \vec{b}| = |\vec{a}|\ |\vec{b}| \sin\theta \quad \therefore \text{Sin } \theta = \dfrac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}$.

3. If $\theta = 0$, $\sin\theta = 0$ $\quad \therefore\ \vec{a} \times \vec{b} = 0$ In particular, $\vec{a} \times \vec{a} = 0$.

   More generally, for any two non zero vectors $\vec{a} \times \vec{b}$, $\vec{a} \parallel \vec{b}$ (collinear) iff $\vec{a} \times \vec{b} = 0$.

4. If $\theta = 90^\circ$, Sin $90^\circ = 1$ $\quad \therefore |\vec{a} \times \vec{b}| = |\vec{a}|\ |\vec{b}|$ (and conversely).

5. $\vec{b} \times \vec{a} = |b|\ |a| \sin(-\theta)\ \hat{n}$.

   $= -|\vec{b}|\ |\vec{a}| \sin\theta\ \hat{n}$

   $= -|\vec{a}|\ |\vec{b}| \sin\theta\ \hat{n}$

   $\vec{b} \times \vec{a} = -\ \vec{a} \times \vec{b}$ ($\because$ Cross Product is not commutative).

6. For any three vectors $\vec{a}, \vec{b}, \vec{c}$, $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$ (distributive)

7. As $\vec{a} \times \vec{b}$ is a vector ⊥r to both $\vec{a}$ and $\vec{b}$ and $\dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$ is the unit vector along $\vec{a} \times \vec{b}$. Hence a unit vector ⊥r to the plane containing $\vec{a}$ and $\vec{b}$ is $\dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$.

8. Vector Product of Unit Vectors $\hat{i}, \hat{j}, \hat{k}$:

   $\hat{i} \times \hat{j} = |\hat{i}|\ |\hat{j}|\ \sin 90 .\ \hat{k}$.

   $= 1 . 1 . 1 . k$

   $\hat{i} \times \hat{j} = \hat{k}$ and $\hat{j} \times \hat{i} = -\hat{k}$.

Product is clear from the following table.

×	$\hat{i}$	$\hat{j}$	$\hat{k}$
$\hat{i}$	0	$\hat{k}$	$-\hat{j}$
$\hat{j}$	-k	0	$\hat{i}$
$\hat{k}$	j	$-\hat{i}$	0

9. Let $\vec{a} = x_1i + y_1j + z_1k$, $\vec{b} = x_2i + y_2j + z_2k$ then

$\vec{a} \times \vec{b} = (x_1i + y_1j + z_1k) \times (x_2i + y_2j + z_2k)$

$= (x_1y_2 - x_2y_1)\ \hat{k} + (x_2z_1 - x_1z_2)\ \hat{j} + (x_1y_1 - x_2y_1)\hat{i}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

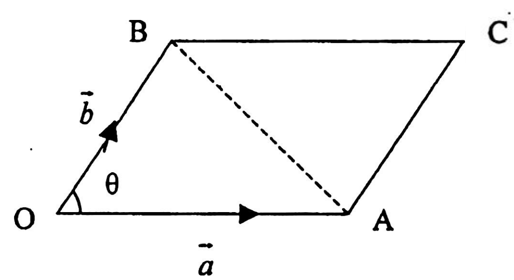
10. Geometrical Interpretation of $|\vec{a} \times \vec{b}|$. Let $\overline{OA} = \vec{a}$ and $\overline{OB} = \vec{b}$.

The area of parallelogram with OA and OB as adjacent sides is

| OA . OB . Sin θ | where θ is the angle between the sides.

= | OA | . | OB | Sin θ |

$= |\vec{a} \times \vec{b}|$



Area of OACB = $|\vec{a} \times \vec{b}|$

Hence area of a parallelogram whose adjacent sides are $\vec{a}$ and $\vec{b}$ is $|\vec{a} \times \vec{b}|$.

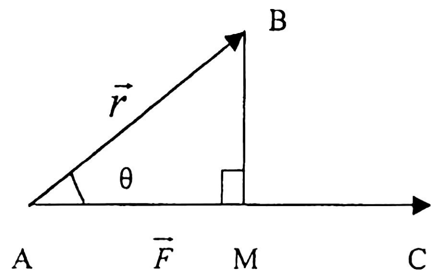
Or $\frac{1}{2}|\vec{a} \times \vec{b}|$ denotes the area of triangle formed by $\vec{a}$ and $\vec{b}$.

Area of ΔOAB=Area of ΔACB = $\frac{1}{2} \mid \vec{a} \times \vec{b} \mid$

11. Physical meaning of Cross Product

The moment of a force $\vec{F}$ acting at a point A about a point B is $\overrightarrow{BA} \times \vec{F}$ or $\vec{F} \times \overrightarrow{AB}$. [Moment or torque of a force about a point is defined in physics as the product of the force and the $\perp$r distance of the point from line of action of force].

Let $\vec{F} = \overrightarrow{AC}$ be the force acting at A. Let B be the given point about which moment is to be determined. Draw BM $\perp$r AC. Then the magnitude of the moment of $\vec{F}$ about B



= AC . BM

= AC .AB sin θ

= | F | . | r | . Sin θ where θ is the angle between $\vec{F}$ and $\vec{r}$.

$= \mid \vec{F} \times \vec{r} \mid$

$= \mid \vec{r} \times \vec{F} \mid$

As the moment of a force is a vector, we consider it as $\vec{r} \times \vec{F}$.

Thus $\overrightarrow{BA} \times \vec{F}$ i.e. $\vec{r} \times \vec{F}$ is the moment of the force $\vec{F}$ acting through a point A about a point B, where $\vec{r}$ denotes the vector $\overrightarrow{AB}$.

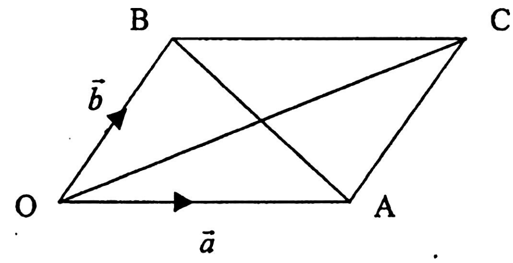
Examples :

1. Prove that in a rectangle, the diagonals are equal.

Solution : Let $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$ be the adjacent sides of rectangle OACB.

Now OA $\perp$r OB $\therefore \vec{a} . \vec{b} = 0$

$\overrightarrow{OC} = \vec{a} + \vec{b}$ and $\overrightarrow{BA} = \vec{a} - \vec{b}$.



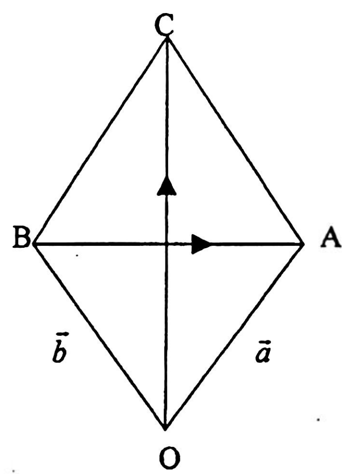
$| \overrightarrow{OC} |^2 = (\vec{a} + \vec{b})^2 = | a^2 | + | b |^2 + 2 \vec{a} . \vec{b} = | a |^2 + | b |^2$

$| \overrightarrow{BA} |^2 = (\vec{a} - \vec{b})^2 = |a|^2 + | b |^2 - 2 \vec{a} . \vec{b} = | a |^2 | | b |^2$

$\therefore \quad | \overrightarrow{OC} |^2 = | BA |^2 \Rightarrow | \overrightarrow{OC} | = | \overrightarrow{BA} |$ diagonals are equal.

2. Prove that diagonals of a Rhombus intersect at right angles.

Solution :



As explained above

$$\overrightarrow{OC} = \vec{a} + \vec{b} \text{ and } \overrightarrow{BA} = \vec{a} - \vec{b}$$

are the diagonals of a rhombus. It is given that $| \vec{a} | = | \vec{b} | \quad \therefore \; | \vec{a} |^2 = | \vec{b} |^2$.

Now $(\vec{a} + \vec{b}) . (\vec{a} - \vec{b}) = | \vec{a} |^2 - | \vec{b} |^2 = 0$

$\therefore$ Diagonals cut at right angles.

3. In any triangle ABC prove that $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$.

Solution : In any triangle ABC

$$\vec{a} + \vec{b} + \vec{c} = 0$$

$\therefore \quad \vec{a} = -(\vec{b} + \vec{c})$

$\therefore \quad \vec{a} \cdot \vec{a} = (\vec{b} + \vec{c})(\vec{b} + \vec{c})$

$\therefore \quad |\vec{a}|^2 = |\vec{b}|^2 + |\vec{c}|^2 + 2\,\vec{b} \cdot \vec{c}$

$\therefore \quad a^2 = b^2 + c^2 + 2 \,.\, |\vec{b}| \,.\, |\vec{c}| \cos(\pi - A)$

$\therefore \quad a^2 = b^2 + c^2 - 2bc \cos A$

$\therefore \quad \cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

4. If $\vec{a}, \vec{b}$ *and* $\vec{c}$ are the P.V. of A, B and C show that the area of triangle is

$$-\frac{1}{2}\,|\vec{b} \times \vec{c} + \vec{c} \times \vec{a} + \vec{a} \times \vec{b}\,|.$$

Solution : In triangle ABC, $\overrightarrow{AB} = \vec{b} - \vec{a}$ and $\overrightarrow{AC} = \vec{c} - \vec{a}$.

Area of Triangle ABC $= \dfrac{1}{2}\,|\,\overrightarrow{AB} \times \overrightarrow{AC}\,|$ (1)

$\overrightarrow{AB} \times \overrightarrow{AC} = (\vec{b} \times \vec{a}) \times (\vec{c} \times \vec{a})$

$= \vec{b} \times \vec{c} - \vec{b} \times \vec{a} - \vec{a} \times \vec{c} + \vec{a} \times \vec{a}$

$= \vec{b} \times \vec{c} + \vec{a} \times \vec{b} + \vec{c} \times \vec{a} + 0$ (2)

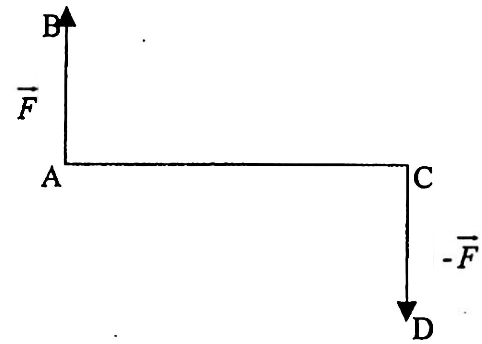
$\therefore$ From (1) and (2)

Area of Triangle $\dfrac{1}{2}\,|\vec{b} \times \vec{c} + \vec{c} \times \vec{a} + \vec{a} \times \vec{b}\,|$.

5. Prove that the moment of a couple about a point is independent of the point.

Solution :

Two equal unlike parallel forces constitute a couple. Let the line of action of forces of the couple be AB and CD.



Then AB || CD. If $\vec{F}$ denotes one force of the couple then - $\vec{F}$ is the other force.

Let O be any reference point in the plane. The sum of the moments of the forces of the couple about any point O is

$= \overrightarrow{OA} \times \vec{F} + \overrightarrow{OD} \times -\vec{F}$

$= (\overrightarrow{OA} - \overrightarrow{OD}) \times \vec{F}.$

$= \overrightarrow{DA} \times \vec{F}.$

So the moment sum is independent of O. This constant moment sum of a couple is called moment of the couple. Moment of a couple is a vector perpendicular to the plane of the couple and the magnitude of the couple is the area of parallelogram ACDB. Hence moment of the couple is equal to the vector product of either force with the ⊥r distance between the lines of action of the forces.

6. Find the moment about the point i + 2j – k of a force represented by i + 2j + 3k acting through the point 2i + 3j + k

Solution : Let $\vec{F}$ = i + 2j + 3k. Let Q be the point i + 2j – k and P be the point 2i + 3j + k.

We are required to find the moment M of $\vec{F}$ about Q

i.e. $\vec{M} = \overrightarrow{QP} \times \vec{F}$

Now $\overrightarrow{QP} = \overrightarrow{OP} - \overrightarrow{OQ}$ = (2i + 3j + k) – (i + 2j – k).

= i + j + 2k

$$\vec{M} = \vec{QP} \times \vec{F} = \begin{vmatrix} i & j & k \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{vmatrix} = -i - j + k.$$

7. A force of 15 units acts in the direction of the vector i - 2j + 2k and passes through a point 2i – 2j + 2k. Find the moment of the force about the point i + j + k.

Solution: The unit vector in the direction of i - 2j + 2k is

$$\frac{i - 2j + 2k}{\sqrt{1+4+4}} = \frac{i - 2j + 2k}{\sqrt{9}} = \frac{1}{3}(i - 2j + 2k)$$

$$\therefore \vec{F} = 15 \times \frac{1}{3}(i - 2j + 2k) = 5i - 10j + 10k.$$

Let Q be the point i + j + k and P be the point 2i – 2j + 2k

Then $\vec{r} = \vec{QP}$ = (2i – 2j + 2k) – (i + j + k)

= i - 3j + k

Required Moment = $\vec{r} \times \vec{F}$

$$= \begin{vmatrix} i & j & k \\ 1 & -3 & 1 \\ 5 & -10 & 10 \end{vmatrix} = -20i - j + 5k.$$

Self Evaluation Tests

1. Find the angle between $\vec{a}$ = 2i + 3j + k and $\vec{b}$ = -i + 2j – 6k.
2. Find the values of p so that (p – 1) i + 2j + k and pi + (p – 1) j + 2k are at right angles.
3. Show that the points in p.v. 3i – 2j + k, i - 3j + 5k and 2i + j – 4k form a right angled triangle.
4. Find the angle which 3i – 6j + 2k makes with the coordinate axes.
5. Find the projection of i + 2j – k on 4i – j + 2k.
6. In any triangle ABC, prove that $c^2 = a^2 + b^2 - 2ab \cos C$.
7. Find the unit vector perpendicular to the vectors 2i – 6j + 3k and 4i + 3j – j.
8. Prove that $\Sigma\, \vec{a} \times (\vec{b} + \vec{c}) = 0$.
9. Show that $|\vec{a} \times \vec{b}|^2 + |\vec{a} \cdot \vec{b}|^2 = |a|^2 \cdot |b|^2$.
10. Find the area of the triangle whose vertices are A(3, -1, 2), B(1, -1, -3), C(4, 3, -1).
11. Find the work done by the force $\vec{F}$ = i - j + 2k in moving a particle through

3i - 2j + k.

12. A particle acted on by constant forces 4i + j + 3k and 3i + j + k is displaced from the point i + 2j + 3k to the point 5i + 4j + k. Find the total work done by the forces.

13. A force $\vec{F}$ = 2i + j – k acts at a point A whose P.V. is 2i – j. Find the moment of $\vec{F}$ about the origin. If the point of application of $\vec{F}$ moves from the point A to the point B with P.V. 2i + j, find the work done by $\vec{F}$.

14. Constant forces $\vec{P}$ = 2i – 5j + 6k and $\vec{Q}$ = -i + 2j – k act on a particle. Find the work done when the particle is displaced from a point A (4, -3, -2) to the point B(6, 1, -3).

15. Two equal forces $\vec{F}$ = i + 2j + 3k are acting at A and B as shown in the figure. If the position vectors of A and B are i + 2j – k and 2i – j + 3k find the magnitude of the moment of the couple.

16. Find the moment about the point i + 2j + 3k of a force represented by i + j+ k acting through the point –2i + 3j + k.

Triple product of Vectors

Scalar Triple Product of Three Vectors :

Given three vectors, a scalar product of one vector with the cross products of two other vectors is called a scalar triple product of three vectors.

If $\vec{a}, \vec{b}, \vec{c}$ are any three vectors, then $\vec{a} \bullet (\vec{b} \times \vec{c})$ is scalar triple product of $\vec{a}, \vec{b}, \vec{c}$. Likewise $(\vec{a} \times \vec{b}).\vec{c}$, is also a scalar triple product. In fast, both these products are equal. So, sometimes we call this product $(\vec{a} \times \vec{b}).\vec{c}$ as box product and denote it by $[\vec{a}, \vec{b}, \vec{c}]$.

Properties of Scalar Triple Products :

1. If $\vec{a}, \vec{b}, \vec{c}$ are any three vectors, then the box product $[\vec{a}, \vec{b}, \vec{c}]$ is a scalar.

   A formula for $[\vec{a}, \vec{b}, \vec{c}]$

2. Let $\vec{a} = x_1i + y_1j + z_1k$, $\vec{b} = x_2i + y_2j + z_2k$ and $\vec{c} = x_3i + y_3j + z_2k$, then

$$\vec{a} \cdot (\overrightarrow{b} \times \vec{c}) = (x_1 i + y_1 j + z_1 k) \cdot \begin{vmatrix} i & j & k \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = (x_1 i + y_1 j + z_1 k) \bullet ((y_2 z_3 -$$

$y_3 z_2)\, i + (x_3 z_2 - x_2 z_3)\, j + (x_2 y_3 - x_3 y_2) \cdot k)$

$= \quad x_1(y_2 z_3 - y_3 z_2) + y_1(x_3 z_2 - x_2 z_3) + z_1(x_2 y_3 - x_3 y_2)$

$$= \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

$$= \quad \vec{a} \cdot (\overrightarrow{b} \times \vec{c}) = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

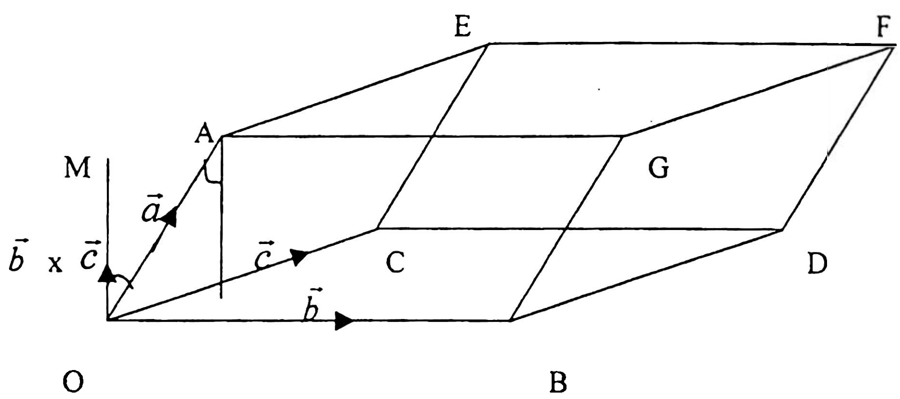
$[\vec{a}\ \vec{b}\ \vec{c}\,] = -[\vec{a}\ \vec{c}\ \vec{b}\,]$

(Interchanging any two vectors in the product, changes the sign of the product).

3. $\vec{a} \cdot (\overrightarrow{b} \times \vec{c}) = (\overrightarrow{a \times b}) \cdot \vec{c}$. Prove.

4. **Cyclic Property :** $[\vec{a}, \vec{b}, \vec{c}\,] = [\vec{b}, \vec{c}, \vec{a}\,] = [\vec{c}, \vec{a}, \vec{b}\,]$ is clear from the property of determinant.

5. Geometrical meaning of a scalar triple product



Let $\overrightarrow{OA}$, $\overrightarrow{OB}$ *and* $\overrightarrow{OC}$ be the three concurrent edges of a parallelopiped OADCEFGAO. $\overrightarrow{OM}$ denotes $\vec{b} \times \vec{c}$. Let θ be the angle between OA and OM. Draw AP $\perp$r to the plane OBDC. Then $O\hat{A}P = A\hat{O}M = \theta$. Join OP.

Now $|\vec{a}\,.(\vec{b}\times\vec{c})| = |\vec{a}|\;|\vec{b}\times\vec{c}|\,.\,|\cos\theta|$

$= |\vec{b}\times\vec{c}|\,.\,OA\,|\cos\theta|$

$= |\vec{b}\times\vec{c}|\,.\,AP$

= Area of OBDC × height

= Area of Base × Height

$|\vec{a}\,.(\vec{b}\times\vec{c})|$ = Volume of the parallelopiped with $\vec{a}, \vec{b}, \vec{c}$ as three concurrent sides.

Hence V = Volume of the ||piped with $\vec{a}, \vec{b}, \vec{c}$ as coinitial edges.

∴ $\boxed{V = |[\vec{a}\;\vec{b}\;\vec{c}]|}$

Note :

1. If $[\vec{a}, \vec{b}, \vec{c}] = 0$ then the three vectors are coplanar.
2. $[\vec{a}, \vec{a}, \vec{b}] = 0$
3. $[\vec{a}, k\,\vec{a}, \vec{b}] = 0$, k being a scalar.

8. **Vector Triple Product** : Let $\vec{a}, \vec{b}, \vec{c}$ be any three non-zero vectors.

Product of the type $\vec{a}\times(\vec{b}\times\vec{c})$ and $(\vec{a}\times\vec{b})\times\vec{c}$ are called vector triple product of $\vec{a}, \vec{b}$ *and* $\vec{c}$.

1. In the product $\vec{a}\times(\vec{b}\times\vec{c})$, $(\vec{b}\times\vec{c})$ is a vector ⊥r to $\vec{b}$ *and* $\vec{c}$. $\vec{a}\times(\vec{b}\times\vec{c})$ is also a vector ⊥r to $\vec{a}$ and $(\vec{b}\times\vec{c})$ [which itself is ⊥r to $\vec{b}$ *and* $\vec{c}$].

   ∴ $\vec{a}\times(\vec{b}\times\vec{c})$ is in the plane of $\vec{a}$ and $\vec{c}$. Hence a unit vector in the direction of $\vec{a}\times(\vec{b}\times\vec{c})$ is $\dfrac{\vec{a}\times(\vec{b}\times\vec{c})}{|\vec{a}\times(\vec{b}\times\vec{c})|}$.

2. The vector triple product of three vectors $\vec{a}, \vec{b}$ *and* $\vec{c}$ can be expressed in terms of $\vec{a}, \vec{b}, \vec{c}$ as

   i) $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\,.\vec{c})\,\vec{b} - (\vec{a}\,.\,\vec{b})\,\vec{c}$

   ii) $(\vec{a}\times\vec{b})\times\vec{c} = (\vec{a}\,.\vec{c})\,\vec{b} - (\vec{b}\,.\,\vec{c})\,.\,\vec{a}$

The results can be verified for any three vectors.

Now, $(\vec{b}\times\vec{c})\times\vec{a} = -[\vec{a}\times(\vec{b}\times\vec{c})]$ (∵ cross product)

$= -\,[(\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}]$

$$= -(\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$$

$$\therefore\ (\vec{b}\times\vec{c})\times\vec{a} = (\vec{a}.\vec{b})\vec{c} - (\vec{a}.\vec{c})\vec{b}$$

Examples :

1. Prove that the four points whose P.V are A(4,5,1), B(0, -1,-1),C(3,9,4) and D(-4, 4, 4) are coplanar.

Solution :

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (0, -1, -1) - (4,5,1) = (-4, -6, -2)$$

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = (3,9,4) - (0, -1, -1) = (3, 10, 5)$$

$$\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC} = (-4, 4, 4) - (3, 9, 4) = (-7, -5, 0)$$

A, B, C and D are coplanar, if $\overrightarrow{AB}$, $\overrightarrow{BC}$,$\overrightarrow{CB}$ are coplanar for which we should have $[\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CD}] = 0$. But $[\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CD}] =$

$$\begin{vmatrix} -4 & -6 & -2 \\ 3 & 10 & 5 \\ -7 & -5 & 0 \end{vmatrix} = -100 + 210 - 110 = 0$$

Hence the result.

2. S.t. $[\vec{a}\times\vec{b}, \vec{b}\times\vec{c}, \vec{c}\times\vec{a}] = [\vec{a}\ \vec{b}\ \vec{c}]^2$.

LHS $= [\vec{a}\times\vec{b}, \vec{b}\times\vec{c}, \vec{c}\times\vec{a}]$

$$= \vec{a}\times\vec{b}\ .\ [(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a})]$$

$$= (\vec{a}\times\vec{b}).[\ \vec{P}\ \times(\vec{c}\times\vec{a})] \text{ where } \vec{b}\times\vec{c} = \vec{P}.$$

$$= (\vec{a}\times\vec{b})\ .\ [(\vec{P}.\vec{a})\ \vec{c} - (\vec{P}.\vec{c})\ .\ \vec{a}]$$

$$= (\vec{a}\times\vec{b}).[(\vec{b}\times\vec{c}.\ \vec{a}) - (\vec{b}\times\vec{c}.\ \vec{c})\ \vec{a}]$$

$$= [(\vec{a}\times\vec{b}).\ \vec{c}][\ \vec{b}\times\vec{c}\ .\vec{a}] \quad \text{(Property)}$$

$$= [\ \vec{a}.\vec{b}\times\vec{c}]\ [\ \vec{b}.\vec{c}\times\vec{a}] \quad \text{(Property)}$$

$$= [\vec{a}\ \ \vec{b}\ \ \vec{c}]\ \ [\vec{b}\ \ \vec{c}\ \ \vec{a}] \quad \text{(Property)}$$

$$= [\vec{a}\ \ \vec{b}\ \ \vec{c}]\ [\vec{a}\ \ \vec{b}\ \ \vec{c}] \quad \text{(Property)}$$

$$= [\vec{a}\ \ \vec{b}\ \ \vec{c}]^2 = \text{RHS.}$$

3. Find a unit vector coplanar with i + j + 2k, i + 2j + k and perpendicular to i +j + k.

Solution : Let $\vec{a}$ = i + j + 2k, $\vec{b}$ = i + 2j + k and $\vec{c}$ = i + j + k.

It is required to find a unit vector in the direction of $\vec{c} \times (\vec{a} \times \vec{b})$ as this vector is coplanar with $\vec{a}, \vec{b}$ and ⊥r to $\vec{c}$.

$$\vec{c} \times (\vec{a} \times \vec{b}) = (\vec{c} \cdot \vec{b})\vec{a} - (\vec{c} \cdot \vec{a})\vec{b}$$

$$= (1 + 2 + 1)\,\vec{a} - (1 + 1 + 2)\,\vec{b}$$

$$= 4\,(i + j + 2k) - 4(i + 2j + k)$$

$$= 4i + 4j + 8k - 4i - 8j + 4k$$

$$= -4j + 4k$$

$$\text{Required unit vector is } = \frac{\vec{c} \times (\vec{a} \times \vec{b})}{|\vec{c} \times (\vec{a} \times \vec{b})|}$$

$$= \frac{-4j + 4k}{\sqrt{16 + 16}} = \frac{-j + k}{\sqrt{2}}.$$

4. If $\vec{a}$ = 2i – j + k, $\vec{b}$ = i + 2j + 3k and $\vec{c}$ = i + 3j – k find a unit vector coplanar with $\vec{b}$ and $\vec{c}$ but perpendicular to $\vec{a}$.

Solution : $\vec{a}$ = 2i – j + k, $\vec{b}$ = i + 2j + 3k, $\vec{c}$ = i + 3j – k

$$\text{Required unit vector} = \frac{\vec{a} \times (\vec{b} \times \vec{c})}{|\vec{a} \times (\vec{b} \times \vec{c})|}$$

$$= \vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\,\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$$

$$= (2 - 3 - 1)\,\vec{b} - (2 - 2 + 3)\,\vec{c}$$

$$= -2\,(i + 2j + 3k) - 3(i + 3j - k)$$

$$= -2i - 4j - 6k - 3i - 3j + 3k$$

$$= -5i - 7j - 3k$$

$$\therefore \quad \text{Required Unit vector} = \frac{-5i - 7j - 3k}{\sqrt{35 + 49 + 9}}$$

$$= \frac{-(5i + 7j + 3k)}{\sqrt{93}}$$

5. Find the volume of the box whose coterminus edges are represented by

$\vec{a}$ = 2i – 3j + k, $\vec{b}$ = i - j + 2k and $\vec{c}$ = 2i + j – k.

The volume is given by

$$V = |\ \vec{a} \cdot (\vec{b} \times \vec{c})|$$

$$= \begin{vmatrix} 2 & -3 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & -1 \end{vmatrix} = -14$$

$\therefore$ V = 14 cubic units.

Self Evaluation Exercise

1. Find m so that $\vec{a} = 2i - j + k$, $\vec{b} = i + 2j - 3k$, $\vec{c} = 3i + mj + 5k$ are coplanar.
2. Find λ so that $2i - j - 2k$, $-i + 3j + k$ and $i + 2j - \lambda k$ are coplanar.
3. Prove that the following four points are coplanar.
   i) (4,5,1), (0, -1, -1), (3,9,4) and (-4, 4, 4)
   ii) (1,2,3), (1, -2,3), (3,4,-2) and (1, -6, 6).
4. Find m if the vectors $i + 2j + 3k$, $3i - j + 2k$, $4i - 4j + 2k$ and $-2i + mj + k$ are coplanar.
5. Find a unit vector coplanar with $2i + j + k$, $i - j + 3k$ and perpendicular to $i + 2j + 3k$..
6. Find a unit vector perpendicular to $\vec{a} = i - j + k$ coplanar with $2i + 3j + k$ and $2i - j + 3k$.
7. Simplify $(\vec{a} + \vec{b}) . (\vec{a} + \vec{c}) \times (\vec{b} + \vec{c})$.
8. Show that $2\vec{a} = i \times (\vec{a} \times i) + j \times (\vec{a} \times j) + k \times (\vec{a} \times k)$
9. Prove that $\vec{r} = (\vec{r} . i)\, i + (\vec{r} . j)\, j + (\vec{r} . k)\, k$ where $\vec{r}$ is any vector.
10. If $\vec{a} = 2i + j + k$, $\vec{b} = i + 2j + k$ and $\vec{c} = i + j + 2k$ find $\vec{a} \times (\vec{b} \times \vec{c})$ and $(\vec{a} \times \vec{b}) \times \vec{c}$.

Trigonometry

Coverage of the Content – Units (T-functions)

Unit 1 : Trigonometric Functions – Identities and Graphs
Angle - Units of measurement, sector of a circle
T – functions
Values of T – functions for some standard angles (0°, 30°, 45°, 60°, 90°)
Variation of T-functions over R.
Graphs of T-functions.

Unit 2 : Trigonometric functions of sum / difference of two angles
Compound Angles Formula
Multiple/Sub-multiple angle formula.
Transformation formulas
a) Sum/difference to product
b) Product to sum/difference
Trigonometric equations

Unit 3 : Inverse Trigonometric Functions
Domain, Range, Principal Value Branches
Graphs of inverse T-functions
Identities involving Inverse T functions

Introduction

Direct measurements in some physical situations will either be impossible or next to it. Finding the width of a river, height of a mountain, or depth of a lake – are such tasks. Knowing some data, with the help of methods and concepts of Trigonometry, these measurements can be calculated. The shadow method of measuring heights indirectly came into use centuries ago. Thales (Greece), it is believed, measured the height of Great Pyramid in 600 B.C. indirectly as about 480 ft, by the Shadow method.

The study of Trigonometry figures in the works of Ancient Indian Mathematicians – Aryabhatta (476 A.D.), Brahma Gupta (598 A.D.), Bhaskara I (600 A.D.) and Bhaskara II (1114 A.D.) (vide Historical note in the text). Trigonometric functions and their properties are studied extensively in advanced centres of Calculus and applied in various branches of applied sciences.

In earlier treatises on trigonometry, the definitions of T-Ratios were confined to acute angles. However, the definitions of T-functions in modern books are for any angle. The graph of these functions and their inverse functions reveal very interesting properties and can be studied both analytically and graphically. A graph of a function tells every aspect of the function. In this development, additional inputs to those areas of trigonometry which are relatively harder or going slightly beyond the scope are given.

The manual being intended to strength en the teachers knowledge beyond the textbook, it is suggested, the teacher uses it along the textbook. To a great extent, repetition of the textual matter is deliberately avoided, at the same time, reference to the text content articlewise and pagewise are given, wherever it is necessary. Tips and exercises / illustrations are given wherever necessary. Important results are either put in a box [ ] or starred, *

Objectives : To enable the teacher to develop the following abilities in his/her pupils –

i) sees the relationship between various measures of a given angle.

ii) Calculate the length of arc/ perimeter/ area of a sector of a circle.

iii) Finds the values of T-functions for given angles.

iv) Studies the features of a graph of a T-function

v) Draws the graph of a T-function in a given domain

vi) Proves T-identities

vii) Given the value of T-function for an angle, calculate the values of the remaining T-functions (or expressions).

viii) Use Trigonometric methods to solve problems in daily life (height and distance).

ix) Using the values of T-functions of angles as 30°, 45°, 60°, etc. evaluates the expressions involving T-functions of these angles.

x) Given the T-functions of two/three angles, finds the values of T-functions of sum/difference of the given angles.

xi) Uses multiple angles, formulas, evaluates the T-functions of multiple angles.

xii) Puts the transformation formulas to use, improving conditional identities.

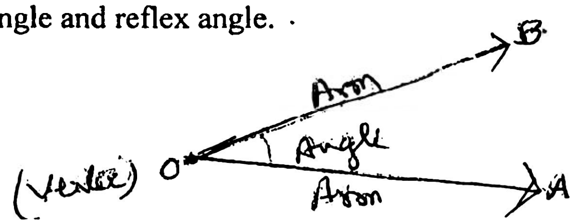
xiii) Solves trigonometric equations.

xiv) Proves identities involving inverse trigonometric functions

xv) Solves equations involving Inverse trigonometric functions.

Unit 1 :

Earlier there were many practices and conventions in measuring angles. Accordingly, angles were classified as – acute angle, obtuse, right angle, straight angle and reflex angle.



The angle formed around a point O was of measures of four right angles. In the textbook an angle can be of any measure. Also according to the conventions – angles measured in the counter clockwise direction are positive angles while those measured in the clockwise direction are negative angles.

In daily life we use this convention. Accordingly, we have screws of both the types. The directions in which taps are turned to open or close follow this convention.

Units of measure of an angle

Notation : Degree → o and Radian measures 30^o = 30 degree, $\frac{\pi}{10}^c = \frac{\pi}{10}$ Radians.

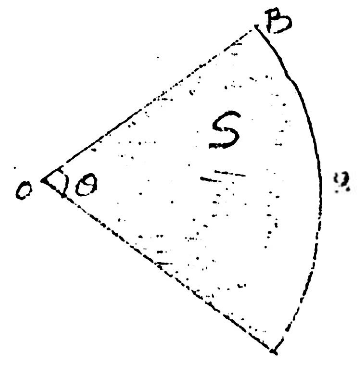
[Read Article 3.2 – 3.2.1, 3.2.2, 3.2.3 and 3.2.4, Pages 49 to 55].

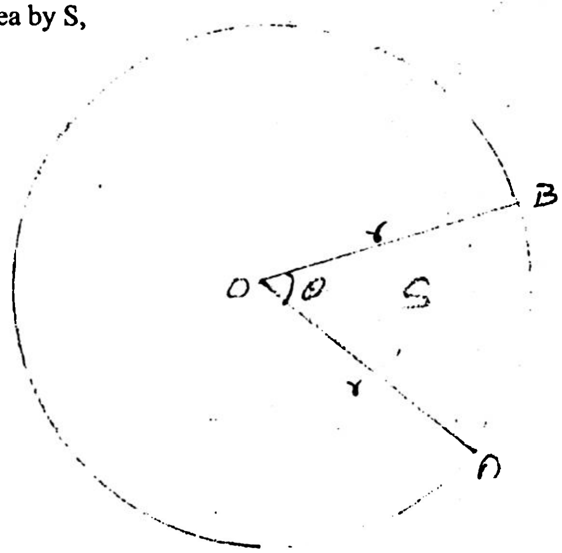
Highlights :

$\theta^o = \left(\frac{\pi}{180}\,0\right)^c$	and	$1^c = 57^o\,16'$ (approx)
$\theta^c = \left(\frac{180}{\pi}\,0\right)^o$	and	$1^o = 0.001746^c$ (approx)

Sector of a Circle

A sector of a circle is a part of the circle bounded by two radii and the arc of the circle connecting the ends of the radii. Denoting the radius by r, the angle subtended by the arc of the sector at the vertex of the sector by θ, in Radians, perimeter by p, arc length by s and the area by S,





$s = r\theta$ — (1)

$p = 2r + s = (2 + \theta)\, r$

and $S = \frac{1}{2} r^2 \theta$ — (2)

The circumference of a circle of radius $r = 2\pi r$.

Caution : In all these formulas, θ is in Radians.

If $\theta = 60^\circ$, say, then $\theta = \frac{\pi}{3}$ in Radians, $\pi = 3.14$.

Conversion formula:

$$\text{Degrees} \xrightarrow{\times \frac{\pi}{180}} \text{Radians}$$

$$\text{Radians} \xrightarrow{\times \frac{180}{\pi}} \text{Degress}$$

Examples :

1. Express the angles of a ΔABC in both degree and Radian measures if $\hat{A} : \hat{B} : \hat{C} = 2 : 3 : 5$.
2. A circular wire of radius 8 cm, is cut and bent into a circular arc of radius 4 cm. What is the angle at the centre in circular measure ?

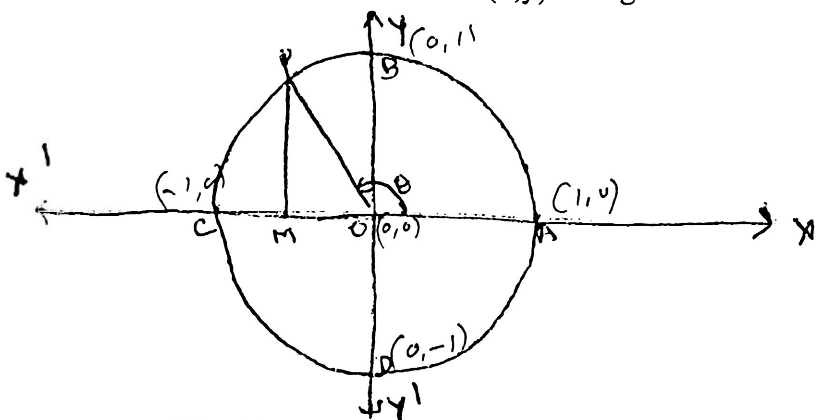
3. Find the angles of a regular polygon of n sides in circular measure.

4. Prove that the diameter of a pulley, which is driven at 360 revolutions per minute by a belt running at 40 ft. per sec. is approximately 2.12 ft.

5. A sector of a circle has its area 24 $(cm)^2$ while its arc length is 6cms, find the radius of the circle.

Trigonometric Functions

A bead (P) moves on a unit circle with centre O, starting from A on the circle. As P moves round the circle the coordinates of P= (x,y) change.



	At A	From A to B	At B	From B to C	At C	From C to D	At D	From D to A
x	1	> 0 and ↓	0	< 0 and ↓	-1	< 0 and ↑	0	> 0 and ↑
Y	0	> 0 and ↑	1	> 0 and ↓	0	< 0 and ↓	-1	< 0 and ↑

↑ = Increasing; ↓ = Decreasing.

After one revolution, these changes repeat in the same order for every revolution. This nature is called Periodic nature.

Also $-1 \le x, y \le 1$ i.e. $-1 \le \cos\theta, \sin\theta \le +1$

Definitions :

$$x = \cos\theta \qquad \ldots \quad I$$
$$y = \sin\theta$$

These are called the cosine and sine functions of θ.

Other T-functions

$$\text{Tan}\ \theta = \frac{\text{Sin}\ \theta}{\cos\theta}$$

$$\text{Sec}\ \theta = \frac{1}{\text{Cos}\ \theta} \qquad \ldots \qquad \text{II}$$

$$\text{Cosec}\ \theta = \frac{1}{\text{Sin}\theta}$$

And $$\text{Cot}\ \theta = \frac{1}{\text{Tan}\ \theta} = \frac{\text{Cos}\ \theta}{\text{Sin}\ \theta}$$

Basic Identities

For all (x,y), $x^2 + y^2 = 1$

$\Rightarrow \cos^2\theta + \sin^2\theta = 1$

Further $1 + \text{Tan}^2\theta = \sec^2\theta$ ... Basic T-identities.

And $1 + \cot^2\theta = \text{cosec}^2\theta$

All the T-functions are periodic functions.

Values of T-functions for $\theta = 0^\circ, 90^\circ, 180^\circ, 270^\circ$

θ →	0°	90°	180°	270°
Sin	0	1	0	-1
Cos	1	0	-1	0
Tan	0	Not defined	0	Not defined
Cosec	Not defined	1	Not defined	-1
Sec	1	Not defined	-1	Not defined
Cot	Not defined	0	Not defined	0

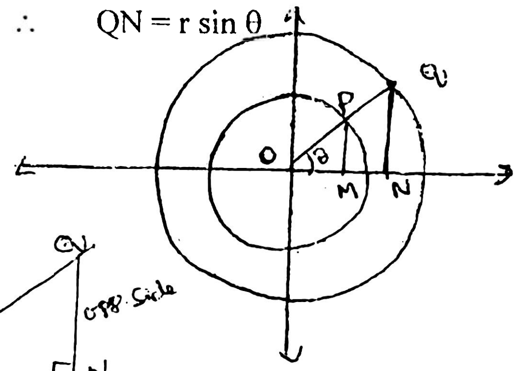
(Read Art 2.3 Pages 55, 56 and 57).

Old and New Definitions

Consider two concentric circles with centre O, taking one to be unit circle and the other of radius r. Suppose OP = 1, OQ = r and $P\hat{O}Q = \theta$, Δ OPM ||| ΔOQN.

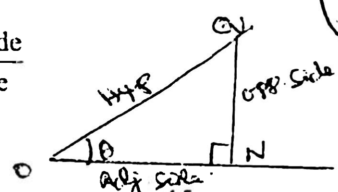
$$\therefore \quad \frac{OM}{OP} = \text{Cos}\ \theta = \frac{ON}{OQ} = \frac{ON}{r} \qquad \therefore \quad ON = r\cos\theta$$

$$\frac{PM}{OP} = \sin\theta = \frac{QN}{OQ} = \frac{QN}{r} \qquad \therefore \quad QN = r\sin\theta$$



$$\text{Sin}\ \theta = \frac{\text{Opposite side}}{\text{hypotenuse}}$$

$$\text{Cos}\ \theta = \frac{\text{Adjacent side}}{\text{hypotenuse}}$$



Examples :

1. θ being acute and $\sin\theta = \frac{15}{17}$, find the values of the remaining T – functions of θ.

2. P.T. $\sqrt{\frac{1+\sin x}{1-\sin x}} = \sec x + \text{Tan}\ x$

3. P.T. $(1 + \text{Tan}\ x - \sec x)(1 + \cot x + \text{cosec}\ x) = 2$

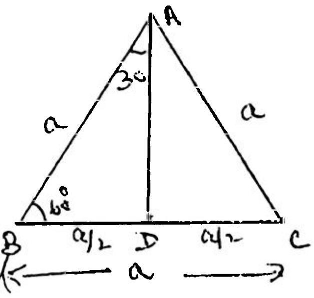
4. If $x = a\cos\theta\cos\phi$, $y = b\cos\theta\sin\phi$, $z = c\sin\theta$

   S.T. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

5. If $2(\text{Tan}\ \theta + \sec\theta) = 3$, P.T. $\sec\theta = \frac{13}{12}$ and $\text{Tan}\ \theta = \frac{5}{12}$.

Values of T-functions for 30°, 60° and 45°.

a) 60° and 30°



In an equivalent triangle ABC, the vertical angle bisector is also the perpendicular bisector of the base.

Therefore, in triangle ABD, if AB = a, then BD = $\frac{a}{2}$, $\hat{B} = 60^\circ$ and $\hat{A} = 30^\circ$ while $\hat{D} = 90^\circ$.

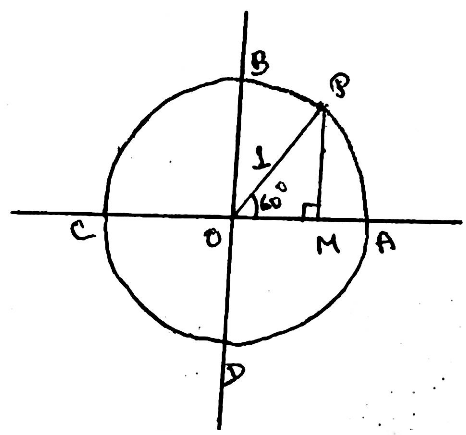
$\therefore \quad AD = \sqrt{a^2 - \frac{a^2}{4}} = \frac{\sqrt{3}}{2}a$

In the unit circle, let P be such that $A\hat{O}P = 60^\circ$. Then $OM = ½$ and $PM = \frac{\sqrt{3}}{2}$.

Also $P = (\cos 60^\circ, \sin 60^\circ) = (OM, PM) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

$\therefore \quad \text{Cos } 60^\circ = \frac{1}{2}; \quad \text{Sin } 60^\circ = \frac{\sqrt{3}}{2}$

For $60^\circ = \left(\frac{\pi}{3}\right)^c$.

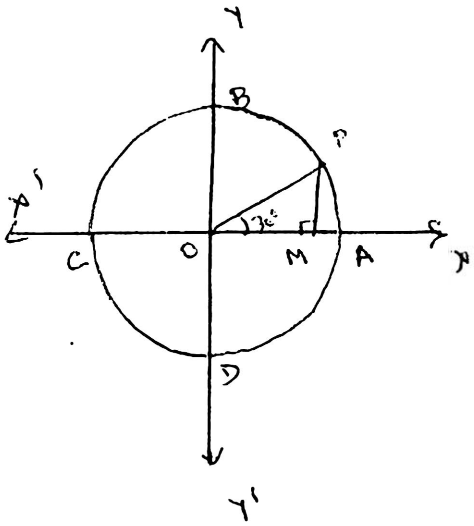


T-functions ↓	$60^\circ = \left(\frac{\pi}{3}\right)^c$
Sin	$\frac{\sqrt{3}}{2}$
Cos	½
Tan	$\sqrt{3}$
Cosec	$\frac{2}{\sqrt{3}}$
Sec	2
Cot	$\frac{1}{\sqrt{3}}$

In the unit circle, OP makes 30° with OA.

$\therefore \quad P = (\text{Cos } 30^o, \ \sin 30^o) = (OM, PM)$

But $PM = ½, \ OM = \frac{\sqrt{3}}{2}$.



$\therefore \quad P = (\text{Cos } 30^o, \ \sin 30^o) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

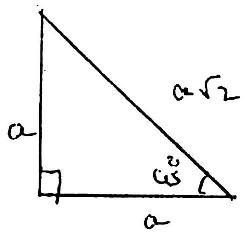
$\therefore \quad \text{Cos } 30^o = \frac{\sqrt{3}}{2}; \ \text{Sin } 30^o = \frac{1}{2}.$

For $30^o = \left(\frac{\pi}{6}\right)^c$

	$30^o = \left(\frac{\pi}{6}\right)^c$
Sin	½
Cos	$\frac{\sqrt{3}}{2}$
Tan	$\frac{1}{\sqrt{3}}$
Cosec	2
Sec	$\frac{2}{\sqrt{3}}$
Cot	$\sqrt{3}$

b) For 45^o

In a right angled isosceles triangle, two smaller angles being 45^o each, the smaller sides are equal.

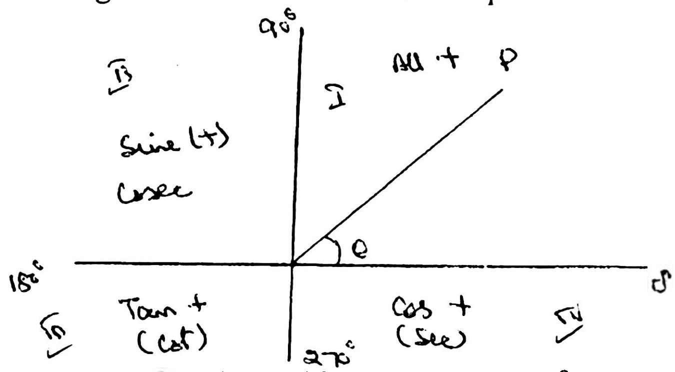


2.	Cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
3.	Tan	O	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Not defined (N.D.)
4.	Cosec	N.D.	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
5.	Sec	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	N.D.
6.	Cot	N.D	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	O

Variation of T-functions of θ as it changes from 0° to 360°.

(Ref Art.3.3. Pages 57, 58 and 59).

1. Signs of T-functions in various quadrants.

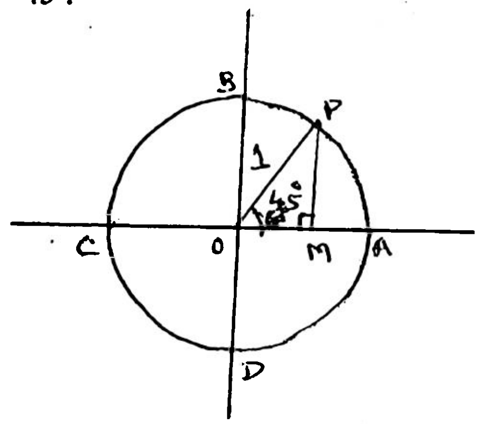


a) All T-functions of θ are positive for $0° < \theta < 90°$ (i.e. I Quadrant).

b) The table below gives the signs of the T-functions of θ in the other quadrant.

Quadrant	II ($90° < \theta < 180°$)	III ($180° < \theta < 270°$)	IV ($270° < \theta < 360°$)
Sin θ & cosec θ	+	-	-
Tan θ & cot θ	-	+	-
Cos θ & sec θ	-	-	+

Let P be a point on the unit circle when $P\hat{O}X = 45^\circ$.



Then $OM = PM = \dfrac{1}{\sqrt{2}}$

$\therefore \quad P = (\text{Cos } 45^\circ, \text{ Sin } 45^\circ) = \left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$

$\therefore \quad \text{Cos } 45^\circ = \text{Sin } 45^\circ = \dfrac{1}{\sqrt{2}}$

For $45^\circ = \left(\dfrac{\pi}{4}\right)^c$

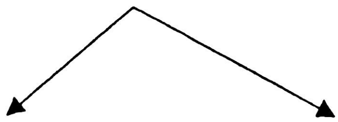
	$45^\circ = \left(\dfrac{\pi}{4}\right)^c$
Sin	$\dfrac{1}{\sqrt{2}}$
Cos	$\dfrac{1}{\sqrt{2}}$
Tan	1
Cosec	$\sqrt{2}$
Sec	$\sqrt{2}$
Cot	1

A way to remember the values of T-functions for 0°, 30°, 45°, 60° and 90°.

Write →	0	1	2	3	4
÷ by 4 →	$\dfrac{0}{4} = 0$	$\dfrac{1}{4}$	$\dfrac{2}{4} = \dfrac{1}{2}$	$\dfrac{3}{4}$	$\dfrac{4}{4} = 1$
Take $\sqrt{\ }$ →	0	$\dfrac{1}{2}$	$\dfrac{1}{\sqrt{2}}$	$\dfrac{\sqrt{3}}{2}$	1

		$0^\circ = 0^c$	$30^\circ = \left(\dfrac{\pi}{6}\right)^c$	$45^\circ = \left(\dfrac{\pi}{4}\right)^c$	$60^\circ = \left(\dfrac{\pi}{3}\right)^c$	$90^\circ = \left(\dfrac{\pi}{2}\right)^c$
1.	Sin	0	½	$\dfrac{1}{\sqrt{2}}$	$\dfrac{\sqrt{3}}{2}$	1

Among the T-functions



Sine	cosine
Sec	cosecant
Tan	cotangent
Called Ratios	Called Co-Ratios

a) Any T-ratio of $90^o \pm \theta$, $270^o \pm \theta$ becomes the corresponding co-ratio and vice-versa.

Eg.

i) $\sec(90^o + \theta)$ becomes $\operatorname{cosec}\theta$, but for the sign

ii) $\cot(270^o - \theta)$ becomes $\tan\theta$, but for the sign

b) Any T-Ratio of $180^o \pm \theta$, $360^o \pm \theta$ do not change.

e.g. i) $\operatorname{cosec}(180^o + \theta)$ remains as $\operatorname{cosec}\theta$, but for the sign.

ii) $\tan(360^o - \theta)$ remains as $\tan\theta$, but for the sign.

c) For any T function $T(\theta)$, $T(360^o + \theta) = T(\theta)$

Caution : While expressing the given T-function in terms of those of θ, fix the sign and write the T-function of θ.

Illustrations

1. Simplify (express as a T-function of θ) or find the value of

a) $\sin(270^o + \theta)$

$\sin(270^o + \theta) \rightarrow$ sign - because $270^o + \theta$ lies in IV quadrant.

$\rightarrow \cos\theta$ because by the tip * (Ratio becomes co-Ratio)

$\therefore \quad \sin(270^o + \theta) = -\cos\theta$

Remember A ll S tate T ransport C orporation

All +ve sin +ve Tan +ve Cosine +ve

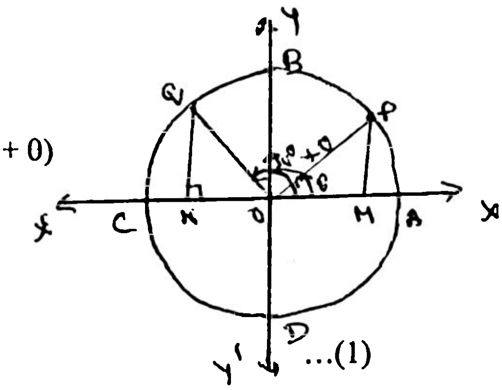
2. Expressing T-Ratios of $90^0 \pm \theta$; $180^o \pm \theta$, $270^o \pm \theta$ and $360^o \pm \theta$ in terms of the T-ratios of θ.

An illustration to express the T-functions of $(90^o + \theta)$ in terms of those of θ.

P is on the unit circle such that OP makes θ with OX and Q is on the unit circle when OQ makes $(90^o + \theta)$ with OX.

Therefore, P = (OM, PM) = $(\cos\theta, \sin\theta)$

And Q = (-ON, QM) = $(\cos(90^o+\theta), \sin(90^o+0)$



Hence $\cos(90^o+\theta) = -$ ON

And $\sin(90^o+\theta) =$ QN ...(1)

$\Delta OPM \equiv \Delta OQN$ with ON = PM = Sin θ and QN = OM = Cos θ ...(2)

$\therefore$ $\text{Cos}(90^o+\theta) = -\text{Sin}\,\theta$ and $\text{Sin}(90^o+\theta) = \text{Cos}\,\theta$

Other T-functions of $(90^o + \theta)$

$\text{Tan}(90^o+\theta) = -\cot\theta$

$\text{Cosec}(90^o+\theta) = \sec\theta$

$\text{Sec}(90^o+\theta) = -\text{cosec}\,\theta$

And $\text{Cot}(90^o+\theta) = -\tan\theta$

Ex : Do similar constructions and get the remaining T-functions, in terms of those of θ.

A tip to remember these results.

In expressing the T-ratio, in terms of those of θ, firstly fix the proper sign. Next write the proper T-functions of θ.

b) cosec (180° + θ) → sign - because (180° + θ) lies in III quadrant.

→ cosec θ because by the tip * (No change in the Ratio).

∴ cosec (180° + θ) = - cosec θ

c) $\text{Cos } 150° = \text{Cos}(180° – 30°)$ [because 150° lies in II quadrant]

$= - \cos 30°$ [cos 30° because, no change in the ratio]

$= - \dfrac{\sqrt{3}}{2}$

Aliter $\cos 150° = \cos (90° + 60°)$

$= \sin 60°$

$= - \dfrac{\sqrt{3}}{2}$

d) $\sec (240°) = \sec (180° + 60°)$ [because 240° lies in III quadrant and no change in the ratio.]

$= - \sec 60°$

$= -2$

or $\sec (240°) = \sec (270° – 30°) = - \text{cosec } 30° = -2$

(e) $\cot 330° = \cot (360° – 30°) = - \cot 30° = - \sqrt{3}$

or $\cot 330° = \cot (270° + 60°) = - \text{Tan } 60° = - \sqrt{3}$

Note :

1. 360° - θ is taken as -θ and
2. sin (-θ) = sin (360° - θ) = - sin θ.

Thus, a) T (360° - θ) = T(-θ) for any T-functions.

Sin 330° = sin (360° – 30°) = sin (-30°) = - ½

Hence

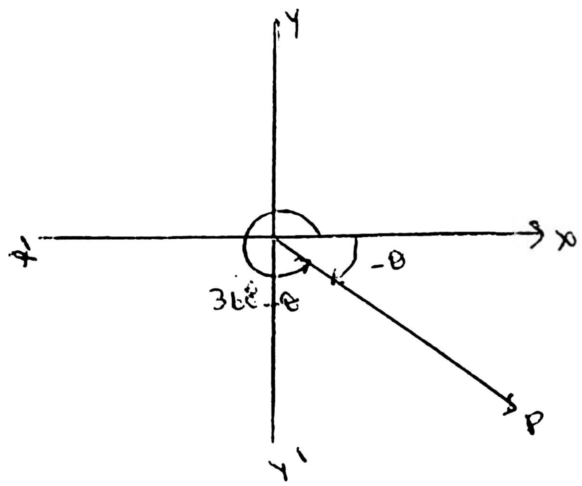
Sin (-θ) = - sin θ
Cos (-θ) = cos θ
Tan (-θ) = - Tan θ
Cosec (-θ) = - cosec θ
Sec (-θ) = sec θ
Cot (-θ) = - cot θ

If you recall the definition of odd and even functions, you will observe

Note: of T functions : Sin θ, Tan θ, Cosec θ, Sec θ are odd functions

Cos θ, Sec θ are even functions

T-functions	**(360° - θ) or -θ**
Sin	- sin θ
Cos	cos θ
Tan	- tan θ
Cosec	- Cosec θ
Sec	Sec θ
Cot	- cot θ



Examples :

1. Express a) Tan (270° + θ) b) sec (360° - θ) c) sin (180° - θ) d) cosec (180° + θ) and e) cot (270° - θ) in terms of the T-functions of θ.

2. Find the values of a) sin 120°, b) cos 150°, c) sec 210°, d) cot 240° e) cosec (330°) f) Tan (870°)

3. S.T. $\dfrac{\text{Tan}^2\frac{2\pi}{3}\ \text{Cot}^2\frac{4\pi}{3} - \text{Cot}^2\frac{7\pi}{6}\ \text{Tan}^2\frac{4\pi}{3}}{\text{Tan}\frac{\pi}{4}\ \text{Cos}\frac{3\pi}{2} - \sec\frac{8\pi}{3}\ \text{Cosec}\frac{3\pi}{4}} = -2\sqrt{2}$

4. Prove : Tan 315° cot (-405°) + cot (495°) Tan (-585°) = 2

5. S.T. Tan 1° Tan 2° ......Tan 89° = 1

Graphs of Standard Trigonometric functions.

The chart below gives the variation of various T-functions

(Notation : ↑ = increases; ↓ = decreases)

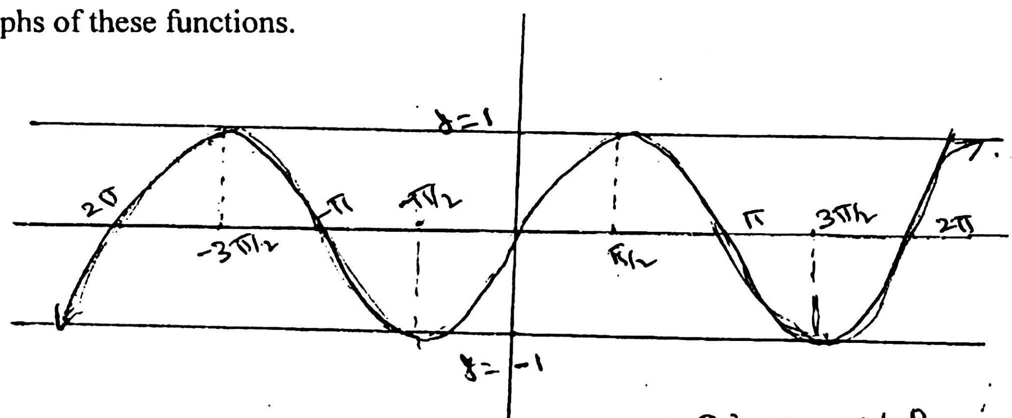
	Range of θ → T-functions of θ ↓	**0° – 90°**	**90° – 180°**	**180° – 270°**	**270° – 360°**
1.	Sin θ	↑ from 0 to 1 (+)	↓ from 1 to 0 (+)	↓ from 0 to –1 (-)	↑ from –1 to 0 (-)
2.	Cos θ	↓ from 1 to 0 (+)	↓ from 0 to –1 (-)	↑ from –1 to 0 (-)	↑ from o to 1 (+)
3.	Tan θ	↑ from 0 to ∞ (+)	↑ from -∞ to 0 (-)	↑ from 0 to +∞ (+)	↑ from -∞ to 0 (-)

4.	Cosec θ	↓ from +∞ to 1 (+)	↑ from 1 to + ∞ (+)	↑ from -∞ to –1 (-)	↓ from –1 to -∞ (-)
5.	Sec θ	↑ from 1 to -∞ (+)	↑ from -∞ to –1 (-)	↓ from –1 to -∞ (-)	↓ from +∞ to 1 (+)
6.	Cot θ	↓ from ∞ to 0 (+)	↓ from 0 to -∞ (-)	↓ from +∞ to 0 (-)	↓ from 0 to +∞ (-)

Special Features

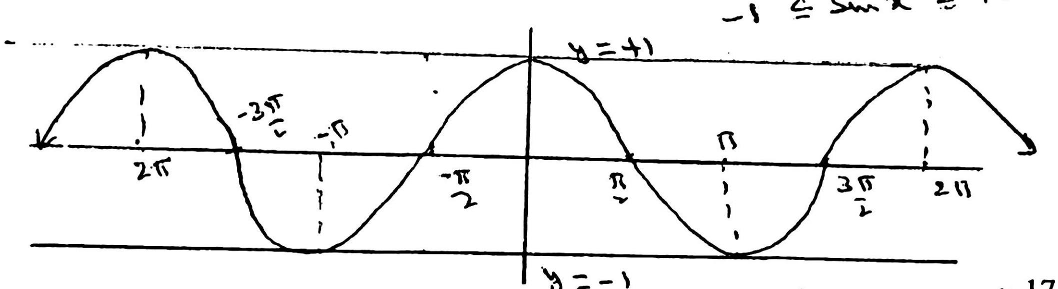
		Domain	Range	Discontinuities	Max/ Min Values
1.	Sin θ	R	[-1, +1]	Nil	Max = 1
2.	Cos θ	R	[-1, +1]	Nil	Min = -1
3.	Tan θ	$R - \left\{ (2n+1)\frac{\pi}{2} \right\}$ n ∈ Z	R	$\left\{ (2n+1)\frac{\pi}{2} \right\}$ n ∈ Z	--
4.	Cosec θ	R – { n π, n ∈ Z }	R – (-1, + 1)	n π , n ∈ R	Local min =1
5.	Sec θ	$R - \{ (2n+1)\frac{\pi}{2}$, n ∈ Z }	R – (-1, +1)	$(2n+1)\frac{\pi}{2}$, n ∈ Z	Local max = -1
6.	Cot θ	R – {n π, n∈Z }	R	nπ, n ∈ Z	.

All the facts seen earlier (like how the various T-functions of θ changes as x changes from 0° to 360° and other features) can be inferred by looking at the graphs of these functions.



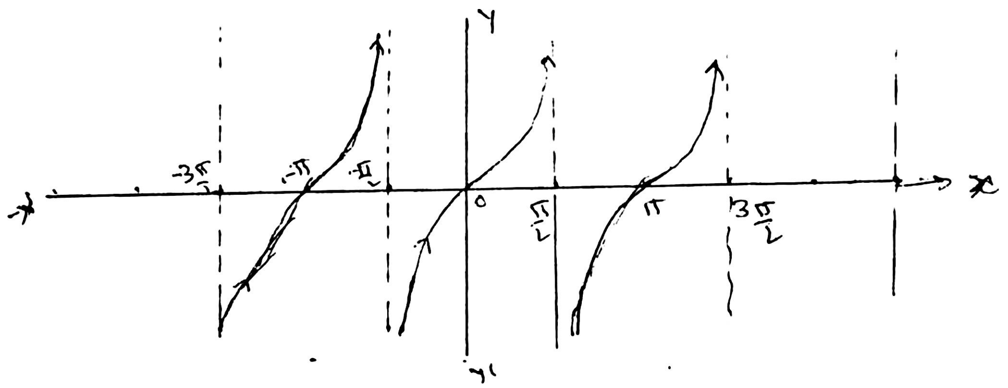
y = Sin x , x ∈ R

-1 ≤ Sin x ≤ +1

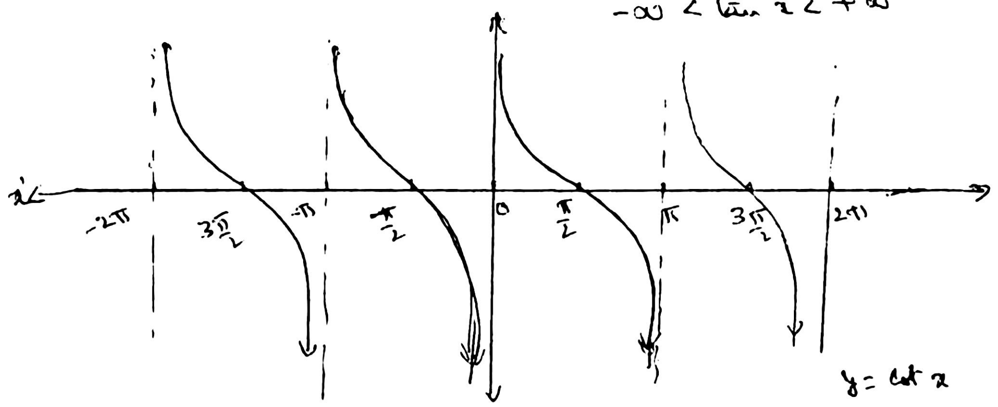


y = cos x , x ∈ R17

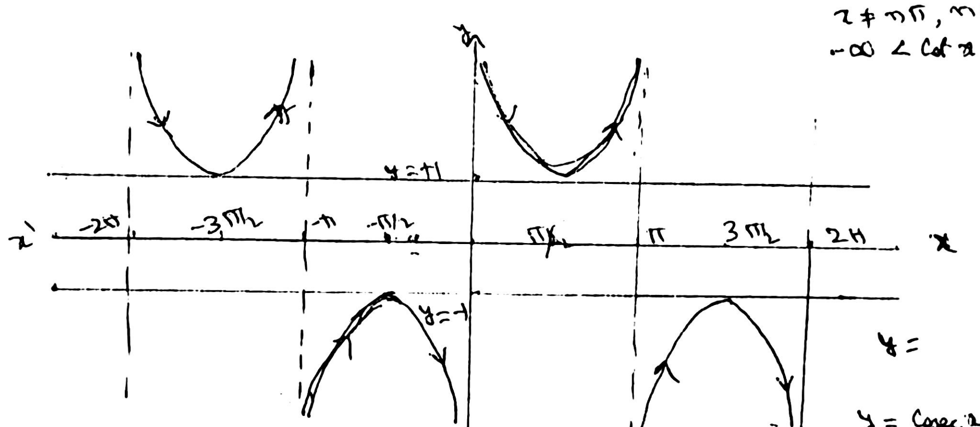
-1 ≤ cos x ≤ +1



y = tan x
x ≠ (2n+1) π/2 , n ∈ Z
−∞ < tan x < +∞

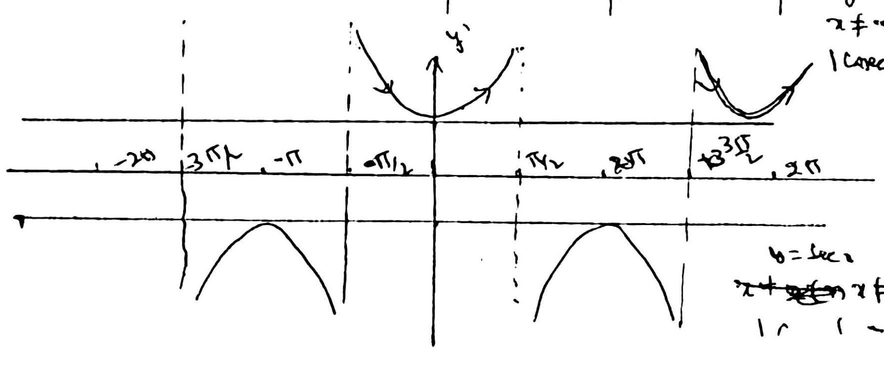


y = cot x
x ≠ nπ , n ∈ Z
−∞ < cot x < +∞



y =

y = Cosec x
x ≠ nπ , n ∈ Z
|Cosec x| ≥ 1



y = sec x 18
x ≠ (2n+1) π/2 , n ∈ Z

Exercises :

Looking at the graphs of T-functions carefully, answer

1. Find the values of x for which T functions of x vanish.
2. Which graphs lie in the strip formed by Y = -1 and Y = +1 ?
3. Which graphs do not lie in the strip formed by Y = -1 and Y = +1 ?
4. Which functions take all real values ?
5. At what points (values of x), the graphs of T-functions have cuts ?
6. Choose the domains of T-functions, in each case, so that they become a one-one function.
7. How do you get the graph of each of the following, from that of y = sin x?

   a) $y = -\sin x$ b) $y = 1 + \sin x$

8. What are the periods of each of the following functions ?

   a) $y = \sin 2x$ b) $y = \cos (x/2)$

   c) $y = \text{Tan } 3x$ d) $y = \sec (x/3)$

9. Find the values of a in the following cases.

   a) $y = \sin ax$, with distance between any two consecutive crossings of the graph with the x-axis

      i) $< \pi$ ii) $> \pi$

   b) $y = \text{Tan } ax$, with the distance between any two consecutive crossings of the graph i) $< \pi/2$ ii) $> \pi/2$

10. Draw the graph of each function.

    a) $y = -\text{Tan } x$ b) $y = \text{Tan } 2x$

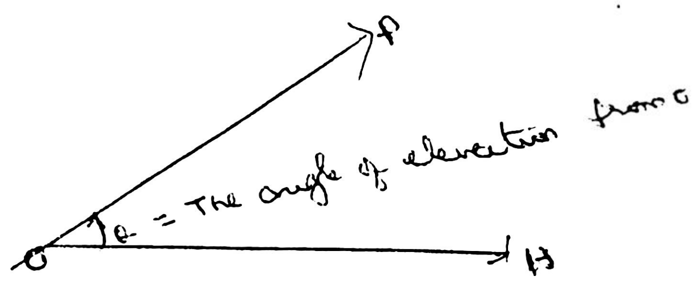
    c) $Y = 2 + \cos x$ d) $y = \sin (x/2) - 1$

    e) $y = 1 + 2 \sin x$

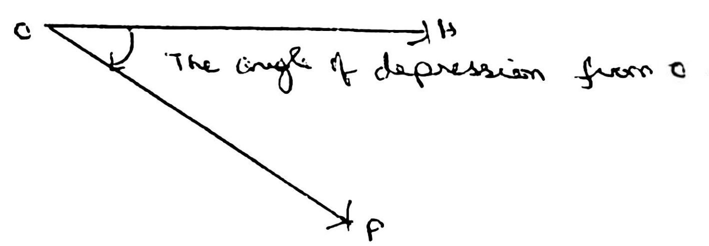
Some Applications of Trigonometry

Terminology

1. If O is the position of an observer and P is an object at a higher elevation than O, then the angle which $\overrightarrow{OP}$ makes with the horizontal line is called the **angles of elevation** of P as seen from O (w.r.t. O).



2. If O is the position of the observer which is at a higher elevation than the object P, then the angle which OP makes with the horizontal line is called the **angle of depression** of P as seen from O (or w.r.t. O).



Examples :

1. A tower and a building stand at the same level. The height of the building being 30m, the angle of elevation of the top of the tower from the bottom and top of the building are 60° and 30° respectively. Find i) the distance between the building and the tower and ii) the height of the tower.

2. A person walks along a straight road to the base of a hill. The angle of elevation of the top of the hill from two points on the road, separated by 1 km are 30° and 45° respectively. Find the height of the hill.

3. A man observes two ships from a lighthouse which is 150 m above the sea level. The ships being on the same side of the lighthouse and the line joining them passing through the bottom of the light house, if the angles of depression of the ships from the observer are 30° and 60°, find the distance between the ships.

4. A person standing on a vertical column observes that the top of a tree on the bank of a lake and its reflection in the lake as seen by him have angles of elevation and angle of depression as 30° and 60° respectively. Show that the height of the tree is double the height of the column.

5. A helicopter travelling along a horizontal path which passes right above an observer on the ground is such that as it approaches the observer, the angles of elevation of the helicopter at two points separated by 800 m are 30° and 45°. Find the altitude of the helicopter above the observer.

Unit 2 : **(Read articles 3.4, pages 64,65, 66 and 67)**

Compound angle Formulas

If sin $(\theta + \phi)$ is to be found, we have the intuition that sin $(\theta + \phi)$ = sin θ + sin ϕ.

Likewise for other T-functions of sum/difference.

Let us put $\theta = 45^\circ$ and $\phi = 30^\circ$.

Then sin $(\theta + \phi)$ = sin 75°

And sin θ + sin ϕ = sin 45° + sin 30°

$$= \frac{1}{\sqrt{2}} + \frac{1}{2} = \frac{\sqrt{2}+1}{2} > 1$$

Therefore, if sin $(\theta + \phi)$ = sin θ + sin ϕ, then sin $75^\circ > 1$ which is absurd

because $0 \le \sin\theta \le 1$, for all θ.

Hence sin $(\theta + \phi) \neq$ sin θ + sin ϕ

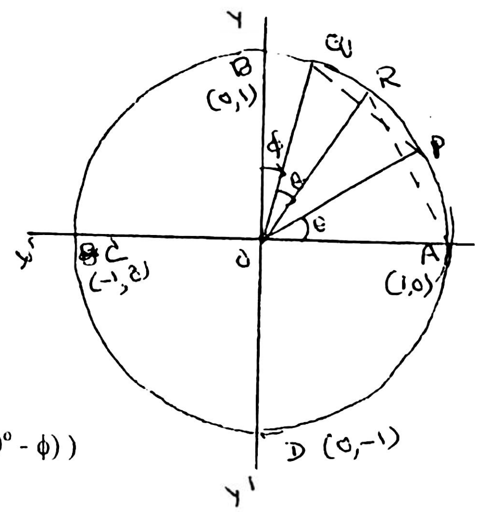
Similarly cos $(\theta + \phi) \neq$ cos θ + cos ϕ and so on.

Can you give a counter example to this ?

To find a formula for sin $(\theta + \phi)$

Taking P, Q, R on the unit circle with centre at O, such that $A\hat{O}P = \theta$, $B\hat{O}Q = \phi$

and $Q\hat{O}R = \theta$,



$A\hat{O}R = 90^\circ - B\hat{O}R$

$= 90^\circ - (\theta + \phi)$ ... (i)

$Q\hat{O}P = 90^\circ - (A\hat{O}P + B\hat{O}Q)$

$= 90^\circ - (\theta + \phi)$ ... (ii)

$\therefore \quad \frac{A\hat{O}R}{AR} = \frac{P\hat{O}Q}{PQ}$ ... (iii)

Since $A\hat{O}P = \theta$, P = (cos θ, sin θ) ... (iv)

Since $A\hat{O}Q = 90^\circ - \phi$, Q = (Cos $(90^\circ - \phi)$, Sin $(90^\circ - \phi)$)

$\therefore$ Q = (sin ϕ, Cos ϕ) ... (v)

$A\hat{O}R = 90^\circ - (\theta - \phi)$ $\therefore R = (\cos(90^\circ - (\theta + \phi)), \sin(90^\circ - (\theta + \phi)))$

$\therefore$ $R = (\sin(\theta + \phi), \text{Cos}(\theta + \phi))$ .... (vi)

$\therefore$ By distance formula,

$\overline{AR} = \sqrt{(\sin(\theta + \phi) - 1)^2 + \cos^2(\theta + \phi)}$;

$\overline{PQ} = \sqrt{(\text{Cos}\,\theta - \sin\phi)^2 + (\sin\theta - \cos\phi)^2}$

$\therefore$ From (iii)

$(\sin(\theta + \phi) - 1)^2 + \cos^2(\theta + \phi) = (\cos\theta - \sin\phi)^2 + (\sin\theta - \cos\phi)^2$

➔ $1 + 1 - 2\sin(\theta + \phi) = 1 + 1 - 2(\cos\theta\sin\phi + \sin\theta\cos\phi)$

➔ $\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$ ... (1)

Other formulas :

In (1), replace ϕ by $-\phi$.

1. ➔ $\text{Sin}(\theta - \phi) = \sin\theta\cos(-\phi) + \text{Cos}\,\theta\sin(-\phi)$

$\therefore$ $\sin(\theta - \phi) = \sin\theta\cos\phi - \cos\theta\sin\phi$ (2)

In (1), replace θ by $(90^\circ - \theta)$, ϕ by $-\phi$.

Then 1 ➔ $\sin(90^\circ - \theta - \phi) = \sin(90^\circ - (\theta + \phi) = \text{Cos}(\theta + \phi)$

$$RHS = \sin(90^\circ - \theta)\cos(-\phi) + \cos(90^\circ - \theta)\sin(-\phi)$$

$$= \cos\theta\cos\phi - \sin\theta\sin\phi$$

Hence $\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$ .... (3)

In (3) replace ϕ bt $-\phi$

3 ➔ $\cos(\theta - \phi) = \cos\theta\cos(-\phi) - \sin\theta\sin(-\phi)$

$\therefore \cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$ ..... (4)

From (1) and (3)

$\text{Tan}(\theta + \phi) = \dfrac{\sin(\theta + \phi)}{\cos(\theta + \phi)} = \dfrac{\sin\theta\cos\phi + \cos\theta\sin\phi}{\cos\theta\cos\phi - \sin\theta\sin\phi}$ (divide both numerator and denominator by $\cos\theta\cos\phi$).

$\therefore \quad \text{Tan}(\theta + \phi) = \dfrac{\text{Tan}\,\theta + \text{Tan}\,\phi}{1 - \text{Tan}\,\theta\,\text{Tan}\,\phi}$ ..... (5)

In (5), replace ϕ by $-\phi$.

(5) ➔ $\text{Tan}(\theta - \phi) = \dfrac{\text{Tan}\,\theta + \text{Tan}(-\phi)}{1 - \text{Tan}\,\theta\,\text{Tan}(-\phi)}$

$\therefore \quad \text{Tan}(\theta - \phi) = \dfrac{\text{Tan}\,\theta - \text{Tan}\,\phi}{1 + \text{Tan}\,\theta\,\text{Tan}\,\phi}$ ..... (6)

exercises
~~Examples~~ :

1. Construct the geometrical proofs for the formula (2), (3) and (4).
2. Find the values of all the T-functions of 75° and 15°.
3. If $\sin A = \frac{3}{5}$ and $\cos B = \frac{8}{17}$, find the values of T-functions (A+B) and (A-B), if $0^\circ < A, B < 90^\circ$.
4. If $\text{Tan}\,A = \frac{3}{4}$, $180^\circ < A < 270^\circ$ and $\cos B = -\frac{15}{17}$, $90^\circ < B < 180^\circ$, find the T-functions of (A+B) and (A-B).
5. S.T. $\text{Tan}\,70^\circ - \text{Tan}\,20^\circ = 2\,\text{Tan}\,50^\circ$
6. S.T. $\text{Tan}(A + B + C) = \dfrac{\text{Tan}\,A + \text{Tan}\,B + \text{Tan}\,C - \text{Tan}\,A\,\text{Tan}\,B\,\text{Tan}\,C}{1 - \text{Tan}\,A\,\text{Tan}\,B - \text{Tan}\,B\,\text{Tan}\,C - \text{Tan}\,C\,\text{Tan}\,A}$

Hence deduce that in a triangle ABC,

i) $\text{Tan } A + \text{Tan } B + \text{Tan } C = \text{Tan } A \text{ Tan } B \text{ Tan } C$ and

ii) $\Sigma \cot A \cot B = 1$.

II. Multiple Angle Formulas

In (1), (3) and (5), putting $\phi = \theta$, we get the T-functions of 2θ in terms of those of θ.

Accordingly

$$\left.\begin{aligned} \text{Sin}(2\theta) &= 2 \sin\theta \cos\theta \\ \text{Cos}(2\theta) &= \cos^2\theta - \sin^2\theta \\ &= 2\cos^2\theta - 1 \\ &= 1 - 2\sin^2\theta \\ \text{and} \quad \text{Tan}(2\theta) &= \frac{2\,\text{Tan}\,\theta}{1 - \text{Tan}^2\theta}. \end{aligned}\right\} \quad \ldots \quad (7)$$

Further, from the above we get

$$\left.\begin{aligned} 2\cos^2\theta &= 1 + \cos(2\theta) \\ \text{And} \quad 2\sin^2\theta &= 1 - \cos(2\theta) \end{aligned}\right\} \quad \ldots. \quad (8)$$

(Ref articles 15 to 19, pages 67 and 68).

Example :

1. Express the T-functions of θ in terms of $\left(\frac{\theta}{2}\right)$.

2. Find the values of T-functions of 22 ½°.

3. Find sin 2A, cos 2A and Tan 2A, if

   i) $\text{Sin } A = \frac{4}{5}, 90^\circ < A < 180^\circ$

   ii) $\text{Cos } A = -\frac{8}{17}, \ 180^\circ < A < 270^\circ$

Again in (1), (3) and (5), put $\phi = 2\theta$ and use the formula (7) appropriately. (Ref Articles 3.4 – 17, 18, 19; pages 68 and 69).

We get the T-functions of 3θ in terms of those of θ. Accordingly,

$$\left.\begin{aligned} \text{Sin}(3\theta) &= 3\sin\theta - 4\sin^3\theta \\ \text{Cos}(3\theta) &= 4\cos^3\theta - 3\cos\theta \\ \text{And}\quad \text{Tan}(3\theta) &= \frac{3\,\text{Tan}\,\theta - \text{Tan}^3\theta}{1 - 3\,\text{Tan}^2\theta} \end{aligned}\right\} \quad \ldots \quad (9)$$

Values of Tan 22 ½°, sin 18° and cos 36°

Tan 22 ½°

We know that $\text{Tan}(2\theta) = \dfrac{2\,\text{Tan}\,\theta}{1 - \text{Tan}^2\theta}$ .... (i)

Taking $2\theta = 45^\circ$, $\theta = 22\frac{1}{2}^\circ$

∴ Let $x = \text{Tan}\,\theta = \text{Tan}\,22\frac{1}{2}^\circ$

Then (i) ➔ $1 = \dfrac{2x}{1 - x^2}$ or $1 - x^2 = 2x$

➔ $x^2 + 2x + 1 = 0$

or $x = \dfrac{-2 \pm \sqrt{4+4}}{2} = \dfrac{-2 \pm \sqrt{2}}{2}$

∴ $x = -1 + \sqrt{2}$ or $-1 - \sqrt{2}$.

But $x > 0$, ∴ $x = \sqrt{2} - 1$

∴ $\text{Tan}\,22\frac{1}{2}^\circ = (\sqrt{2} - 1)$

Sin 18°

Taking $\theta = 18^\circ$

$5\theta = 90^\circ$

➔ $2\theta \pm 3\theta = 90^\circ$

∴ $2\theta = 90^\circ - 3\theta$

∴ $\sin(2\theta) = \sin(90^\circ - 3\theta) = \cos(3\theta)$

∴ $2\sin\theta\cos\theta = 4\cos^3\theta - 3\cos\theta \quad [\div \cos\theta]$

$\Rightarrow$ $2 \sin \theta = 4 \cos^2 \theta - 3$

$= 4 (1 - \sin^2 \theta) - 3$

$= 1 - 4 \sin^2 \theta$

$\Rightarrow$ $4 \sin^2\theta + 2 \sin \theta - 1 = 0$

$$\therefore \quad \sin \theta = \frac{-2 \pm \sqrt{4+16}}{8} = \frac{-2 \pm 2\sqrt{5}}{8}$$

$$\therefore \quad \sin \theta = \frac{\sqrt{5} - 1}{4} \text{ or } \frac{-\sqrt{5} - 1}{4}$$

Since $\sin \theta = \sin 18^\circ > 0$.

$$\text{Sin } 18^\circ = \frac{\sqrt{5} - 1}{4}$$

Cos 36°

$\text{Cos } 2\theta = 1 - 2 \sin^2 \theta$

$$\therefore \quad \cos 36^\circ = 1 - 2 \sin^2 18^\circ$$

$$= \quad 1 - 2 \left(\frac{\sqrt{5} - 1}{4} \right)^2$$

$$= \quad 1 - \frac{6 - 2\sqrt{5}}{8}$$

$$\therefore \quad \cos 36^\circ = \frac{\sqrt{5} + 1}{8}$$

Example : Get the values of other T-functions of 18° and 36°.

Note : $\sin 54^\circ = \cos 36^\circ = \dfrac{\sqrt{5} + 1}{4}$

And $\cos 72^\circ = \sin 18^\circ = \dfrac{\sqrt{5} - 1}{4}$

Illustrations – Solved Problems

1. S.T. $\cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ = \frac{1}{16}$.

$$LHS = \frac{1}{\sin 20^\circ} (\sin 20 . \cos 20^\circ . \cos 40 . \cos 60^\circ . \cos 80^\circ)$$

$$= \frac{1}{\sin 20^\circ} (½ \sin 40 . \cos 40 . \cos 60 . \cos 80^\circ)$$

$$= \frac{1}{2 \sin 20^\circ} (½ \sin 80^\circ . \cos 60^\circ . \cos 80^\circ)$$

$$= \frac{1}{4 \sin 20^\circ} (\cos 60 . ½ \sin 160^\circ) = \frac{1}{8}\left(\frac{1}{2}\right) . \frac{\sin 160^\circ}{\sin 20^\circ}$$

$$= \frac{1}{16} \frac{\sin (180^\circ - 20^\circ)}{\sin 20^\circ} = \frac{1}{16} \frac{\sin 20^\circ}{\sin 20^\circ} = \frac{1}{16} = RHS$$

2. S.T. $\frac{\sin^3 x + \sin 3x}{\sin x} + \frac{\cos^3 x - \cos 3x}{\cos x} = 3$

$$LHS = \frac{\sin^3 x + 3 \sin x - 4 \sin^3 x}{\sin x} + \frac{\cos^3 x - (4\cos^3 x - 3\cos x)}{\cos x}$$

$$= \frac{3 \sin x - 3 \sin^3 x}{\sin x} + \frac{3 \cos x - 3 \cos^3 x}{\cos x}$$

$$= 3 - 3 \sin^2 x + 3 - 3 \cos^2 x$$

$$= 6 - 3 (\sin^2 x + \cos^2 x) = 6 - 3 = 3 = RHS$$

3. S.T. $\frac{1 + \cos A + \sin A}{1 - \cos A + \sin A} = \cot\left(\frac{A}{2}\right)$

$$LHS = \frac{(1 + \cos A) + \sin A}{(1 - \cos A) + \sin A} = \frac{2 \cos^2 A/2 + 2 \sin A/2 \cos A/2}{2\sin^2 A/2 + 2 \sin A/2 \cos A/2}$$

$$= \frac{2 \cos A/2 (\text{Cos } A/2 + \sin A/2)}{2 \text{ Sin } A/2 (\sin A/2 + \cos A/2)} = \cot A/2 = RHS$$

4. Given $2 \sin A = x + 1/x$, find $x^2 + 1/x^2$ and $x^3 + 1/x^3$.

$$x + \frac{1}{x} = 2 \sin A.$$

$\therefore \quad \left(x+\frac{1}{x}\right)^2 = 4\sin^2 A \rightarrow x^2+\frac{1}{x^2}+2 = 4\sin^2 A$

$\therefore x^2 + \frac{1}{x^2} = (4\sin^2 A - 2)$ ...(i)

Again, $x+\frac{1}{x} = 2\sin A.$

$\therefore \quad \left(x+\frac{1}{x}\right)^3 = 8\sin^3 A$

$\rightarrow x^3+\frac{1}{x^3}+3x.\frac{1}{x}\left(x+\frac{1}{x}\right) = 8\sin^3 A$

$\therefore \quad x^3+\frac{1}{x^3} = 8\sin^3 A - 3.2\sin A$

$= -2(3\sin A - 4\sin^3 A) = -2\sin 3A$

$\therefore \quad x^3+\frac{1}{x^3} = -2\sin 3A.$ (ii)

III. Transformation Formulae

Converting sum/difference to product and product to sum/difference

We know

Sin (A + B) = Sin A Cos B + Cos A Sin B (i)

Sin (A – B) = Sin A Cos B – Cos A Sin B (ii)

Cos (A + B) = Cos A Cos B – Sin A Sin B (iii)

And Cos (A – B) = Cos A Cos B + Sin A Sin B (iv)

Putting A + B = x

A – B = y

$A = \frac{x+y}{2}$ and $B = \frac{x-y}{2}.$

$$\left.\begin{aligned}
&(i)+(ii) \rightarrow \sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)\\
&(i)-(ii) \rightarrow \sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)\\
&(iii)+(iv) \rightarrow \cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)
\end{aligned}\right\} \quad ...(I)$$

(iv) – (iii) → $\quad \text{Cos } y - \text{Cos } x = 2 \text{ Sin}\left(\frac{x+y}{2}\right) \text{Sin}\left(\frac{x-y}{2}\right)$

or $\quad \text{Cos } x - \text{Cos } y = -2 \sin\left(\frac{x+y}{2}\right) \text{Sin}\left(\frac{x-y}{2}\right)$

These formulas enable a sum/difference expressed as a product.

Again,

(i) – (ii) → 2 Sin A Cos B = Sin (A +B) + Sin (A – B)

(i) – (ii) → 2 Cos A sin B = Sin (A + B) – Sin (A – B) ...(II)

(iii) + (iv) → 2 Cos A Cos B = Cos (A + B) + Cos (A – B)

and (iii) – (iv) → 2 Sin A Sin B = Cos (A – B) – Cos (A + B)

These formulas enable products expressed as a sum/difference.

Thus I are of the form :

$\text{Sin } x + \text{Sin } y = 2 \text{ Sin}\left(\frac{\text{Sum}}{2}\right) \text{ Cos}\left(\frac{\text{Difference}}{2}\right)$, etc.

And II are of the form

2 Sin A Cos B = Sin (Sum) + Sin (Difference), etc.

Illustrations : Solved Problems

1. S.T. $\frac{\text{Sin} 3x + \text{Sin} x}{\text{Cos } 3x + \text{Cos } x} = \text{Tan } 2x$

$$\text{L.H.S.} = \frac{2 \text{ Sin}\left(\frac{3x+x}{2}\right) \text{Cos}\left(\frac{3x-x}{2}\right)}{2 \text{ Cos}\left(\frac{3x+x}{2}\right) \text{Cos}\left(\frac{3x-x}{2}\right)} = \frac{2\text{Sin } 2x \text{ Cos } x}{2 \text{ Cos } 2x \text{ Cos } x}$$

$= \text{Tan } 2x = \text{RHS}$

2. Simplify : $\frac{\text{Cos } 2x + \text{Cos } 3x + \text{Cos } 4x}{\text{Sin } 3x + \text{Sin } 3x + \text{Sin } 4x}$

$$\frac{\text{Cos} 2x + \text{Cos } 3x + \text{Cos} 4x}{\text{Sin} 3x + \text{Sin} 3x + \text{Sin} 4x} = \frac{(\text{Cos } 4x + \text{Cos } 2x) + \text{Cos } 3x}{(\text{Sin } 4x + \text{Sin } 2x) + \text{Sin } 3x} = \frac{2 \text{ Cos } 3x \,.\, \text{Cos } x + \text{Cos } 3x}{2 \text{ Sin } 3x \text{ Cos } x + \text{Sin } 3x}$$

$$= \frac{\text{Cos } 3x \ (2 \text{ Cos } x + \text{Sin } x)}{\text{Sin } 3x \ (2 \text{ Cos } x + \text{Sin } x)} = \text{Cot } 3x = \text{RHS}$$

3. Prove: Cos 35° + Cos 85° + Cos 155° = 0

LHS = Cos 155° + Cos 35° + Cos 85°

$$= 2 \text{ Cos} \left(\frac{190^\circ}{2}\right) \text{Cos} \left(\frac{120^\circ}{2}\right) + \text{Cos } 85^\circ$$

= 2 Cos 95° Cos 60° + Cos 85°

$$= 2 \text{ Cos } 95 \left(\frac{1}{2}\right) + \text{Cos } 85^\circ$$

= 2 Cos 90° Cos 5° = 0 (Since Cos 90° = 0)

= RHS

4. P.T. Cos 5A Cos 2A + Sin 2A . Sin A = Cos 4A . Cos 3A.

LHS = ½ (Cos 7A + Cos 3A) + ½ (Cos A – Cos 3A)

= ½ (Cos 7A + Cos A) = ½ . 2 Cos 4 A . Cos 3A

= Cos 4A . Cos 3A = RHS

5. In a triangle ABC, S.T. Σ Sin 2A = 4 Sin A Sin B Sin C

LHS = Sin 2A + Sin 2B + Sin 3C

= 2 Sin (A + B) Cos (A – B) + 2 Sin C Cos C

= 2 Sin (180° – C) Cos (A – B) + 2 Sin C . Cos C

= 2 Sin C [Cos (A – B) + Cos C]

Cos C = Cos (180° – (A + R)) = - Cos (A+B)

= 2 Sin C [Cos (A – B) – Cos (A + B)

= 2 Sin C . 2 Sin A Sin B = 4 Sin A Sin B Sin C = RHS.

6. In a triangle ABC, P.T.

$\Sigma \text{ Cos}^2 (A/2) = 2 + 2 \text{ Sin } (A/2) \ \text{Sin } (B/2) \text{ Sin } (C/2)$

$$\text{LHS} = \frac{1 + \text{Cos A}}{2} + \frac{1 + \text{Cos B}}{2} + \text{Cos}^2 \left(\frac{C}{2}\right) \quad \text{Cos}^2 \frac{\theta}{2} = \frac{1 + \text{Cos}^2 \theta}{2}$$

$$= 1 + \frac{1}{2} \ (\text{Cos A} + \text{Cos B}) + \text{Cos}^2 (C/2)$$

$$= 1 + \frac{1}{2} \ . \text{x Cos}\left(\frac{A+B}{2}\right) \text{Cos}\left(\frac{A-B}{2}\right) + \text{Cos}^2 (C/2)$$

$$= 1 + \text{Cos}\left(\frac{180^\circ - C}{2}\right) \cos\left(\frac{A-B}{2}\right) + \text{Cos}^2 (A/2)$$

$$= 1 + \text{Sin}(C/2) \ . \text{Cos}\left(\frac{A-B}{2}\right) + \frac{1 + \text{Cos C}}{2}$$

$$= 1 + \text{Sin}(C/2) \ . \text{Cos}\left(\frac{A-B}{2}\right) + \frac{2 + 1 + \text{Cos C}}{2}$$

$$= 1 + \text{Sin}(C/2) \ \text{Cos}\left(\frac{A-B}{2}\right) + 1 - \frac{1 - \text{Cos C}}{2}$$

$$= 2 + \text{Sin}(C/2) \ \text{Cos}\left(\frac{A-B}{2}\right) - \frac{2 \text{Sin}^2 C/2}{2}$$

$$= 2 + \text{Sin}(C/2)\left[\text{Cos}\left(\frac{A-B}{2}\right) - \text{Cos}\left(\frac{A+B}{2}\right)\right]$$

$$= 2 + \text{Sin C/2} \ . 2 \text{Sin A/2} \ . \text{Sin B/2}$$

$$= 2 + 2 \text{Sin A/2} \ \text{Sin B/2} \ \text{Sin C/3} = \text{RHS}$$

Exercises

1. P.T. $\text{Sin } 20^\circ \text{ Sin } 40^\circ \text{ Sin } 80^\circ = \frac{3}{16}$.

2. If $\text{Cos A} + \text{Cos B} = \frac{1}{3}$; $\text{Sin A} + \text{Sin B} = \frac{1}{4}$.

   P.T. $\text{Tan}\left(\frac{A+B}{2}\right) = \frac{3}{4}$

3. If $\text{Tan}\left(\frac{\theta}{2}\right) = t$, P.T.

   $\text{Sin } \theta = \frac{2t}{1+t^2}$; $\text{Cos } \theta = \frac{1-t^2}{1+t^2}$ and $\text{Tan } \theta = \frac{2t}{1-t^2}$

4. S.T. $\text{Tan}\left(\frac{\pi}{4} + \frac{\theta}{2}\right) + \text{Tan}\left(\frac{\pi}{4} - \frac{\theta}{2}\right) = 2 \sec \theta$

5. If $\cos\theta = \frac{3}{4}$, find the value of $\sqrt{2+\sqrt{2+2\cos 4\theta}}$.

6. In triangle ABC, S.T.

$\cos 2A + \cos 2B + \cos 2C + 1 + 4\cos A \ \cos B + \cos C = 0.$

7. In Triangle ABC, P.T. $\sum \cos\frac{A}{2} = 4\cos\left(\frac{\pi - A}{4}\right)\cos\left(\frac{\pi - B}{4}\right)\cos\left(\frac{\pi - C}{4}\right)$.

8. In triangle ABC, prove that $\sum\left(\frac{\cot A + \cot B}{\tan A + \tan B}\right) = 1$.

9. If $\cos x = -\frac{1}{3}$, $180^\circ < x < 270^\circ$, $\tan y = -\frac{4}{3}$, $90^\circ < y < 180^\circ$

Find $\sin(x + y)$ and $\cos(x + y)$. In which quadrant $(x + y)$ lies.

10. Find the sum of (i) $\sum_{r=1}^{2n-1} \cos(r\theta)$ ii) $\sum_{r=1}^{2n-1} \sin(r\theta)$

Trigonometric Equations and Their Solutions

An Example :

Consider the equation $\tan x = \frac{1}{\sqrt{3}}$.

An obvious solution is $x = \frac{\pi}{6}$.

$x = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$ is also a solution .

Are there other solutions ? If so, how many ?

These are discussed in what follows :

An equation involving Trigonometric functions of a variable x (say) is called a Trigonometric equation in x. A value of x satisfying the given T-equation in x is called a solution.

Principal Solutions and General Solution of a T-equation

Given a T-equation in x, the solutions $x : 0 \le x \le 2\pi$ are called *principal solutions* of the equation. All solutions of a given equation can be put in a formula called the *general solution* of the equation.

We recall that

$\sin (n\pi) = 0 \ \forall \ n \in Z$

$\tan (n\pi) = 0 \ \forall \ n \in Z$

And $\cos (2n + 1) \pi/2 = 0 \ \forall \ n \in Z$

To find the general solution of a given T-equation, given a solution of the equation, we use the known formulas for general solutions of the three basic T-functions [Ref : Art 3.5 Pages 74 and 75).

Summary

	The T-equation	General Solution
1.	$\sin x = K, \mid K + \le 1$	$x = n\pi + (-1)^n y, n \in Z$ y being a given solution.
2.	$\cos x = K, \mid K \mid \le 1$	$x = 2n\pi \pm y$, $n \in Z$, y being a given solution.
3.	$\tan x = K, K \in R$	$x = n\pi + y$, $n \in Z$, y being a given solution.

In numerical problems,

a) the given T-equation, breaks up into equations of the types given above. Each such equation given a set of solutions. The set of all sets of solutions so got will be the general solution of the given T-equation.

b) It is necessary to identify a solution y, in order to write the general solution (G.S.). It is convenient to take the particular solution as a principal solution. The formula for general solution given in the tabular column above are used to introduce the general solution of the equation.

Illustrations

1. Find the General Solution of the equation, Cos 4x = Cos 2x

Cos 4x = cos 2x

$\Rightarrow$ Cos 4x – Cos 2x = 0 $\Rightarrow$ -2 Sin 3x . Sin x = 0

$\Rightarrow$ Sin 3x = 0, Sin x = 0.

Sin x = 0 $\Rightarrow$ $x = n\pi + (-1)^n 0 = n\pi$, $n \in Z$

$\therefore$ $x = n\pi$ (1), $n \in Z$

Sin 3x = 0 $\Rightarrow$ $3x = n\pi$ $\therefore$ $(x = \frac{n\pi}{3})$ (2), $n \in Z$

(1) and (2) together form the general solution of the equation.

2. Solve : Sin x + Sin 3x + Sin 5x = 0

$\Rightarrow$ (Sin 5x + Sin x) + Sin 3x = 0

$\Rightarrow$ 2 Sin 3x . Cos 2x + Sin 3x = 0

$\Rightarrow$ Sin 3x (2 Cos 2x + 1) = 0

$\Rightarrow$ Sin 3x = 0; 2 Cos 2x + 1 = 0

$3x = n\pi$	Cos 2x = - ½
$\therefore$ n - $\frac{n\pi}{3}$, $n \in Z$	A solution of the equation can be found in the second quadrant.
$\therefore$ cos 2x = - ½	If Cos θ = - ½
$\Rightarrow 2x = 2n\pi \pm 2\frac{\pi}{3}$	Cos (π - θ) = - Cos θ = ½
or $x = n\pi \pm \frac{\pi}{3}$, $n \in Z$	$\therefore$ $\pi - \theta = \frac{\pi}{3}$
	$\therefore$ $\theta = \frac{2\pi}{3}$

(1) and (2) form the general solution of the given equation.

Exercises :

Solve :

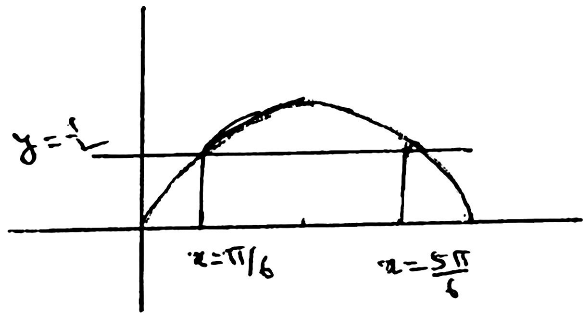
1. $2 \cos^2 \theta + 3 \sin \theta = 0$
2. $\text{Sin } 2x + \text{Sin } 4x + \text{Sin } 6x = 0$
3. $\text{Cos } 3x = \text{Sin } 2x$
4. $\text{Tan}^2 \theta + \text{Cot}^2 \theta = 2$
5. $\text{Tan}^2 \theta + \text{Tan } \theta - \sqrt{3} (\text{Tan } \theta + 1) = 0$
6. $\text{Tan } \theta + \text{Sec } \theta = \sqrt{3}$

Unit 3 : Inverse T-functions and their graphs

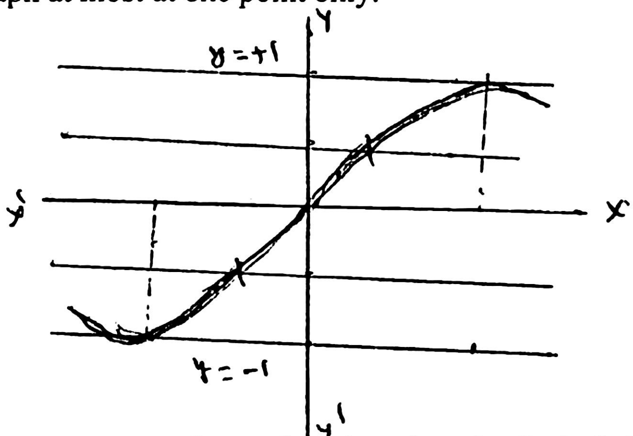
Preliminaries

1. The graph of a one-one function will be cut by a line parallel to the x-axis, at most at one point only. Otherwise, the graph is cut by a parallel to the x-axis in more than one point. At all such points, the values of the function are equal.

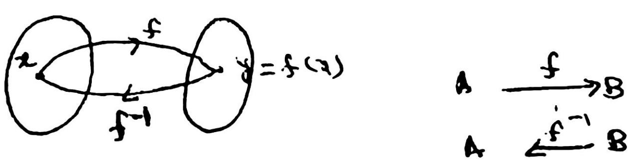
For example, the graph of y = sin x in [0, π] by the line y = ½ is cut at $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$. However, y = 2 does not meet the graph, while y = 1 touches the graph at $x = \frac{\pi}{2}$.



In the case of y = sin x [- π/2, π/2] any parallel line to the x-axis in the strip $-1 \leq y \leq +1$ cuts the graph at most at one point only.



2. Let y = f(x) be a one-one and onto function given by f : A → B.
Then f^1 : B → A such that $(f^1 \circ f)(x) = f^1(f(x)) = x$ and
$(f \circ f^1)(y) = f(f^1(y)) = f(x) = y$ is called the Inverse function of f(x). It is defined by f^1 : B → A when $f^1 = \{ y : y = f(x), x \in A \}$.



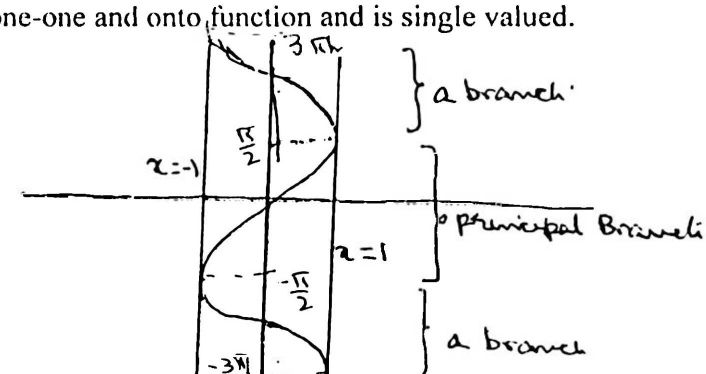
An Example

The function y = sin x is one to one and onto in any of the following domains.

$[-\frac{\pi}{2}, \frac{\pi}{2}]$; $[\frac{\pi}{2}, \frac{3\pi}{2}]$, $[\frac{3\pi}{2}, \frac{5\pi}{2}]$, or $[-\frac{3\pi}{2}, -\frac{\pi}{2}]$; $[-\frac{5\pi}{2}, -\frac{3\pi}{2}]$ and so on.

The range of this function is [-1, +1].

The inverse function y = sin x defined on [-1, +1]. The inverse function of y = Sin x defined on [-1, +1] and range on any one of the above mentioned domains of y = Sin x is a one-one and onto function (so that it is acceptable as a function). Corresponding to each of these intervals we get a branch of the inverse function y = Sin^{-1} x. The branch on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ is called the principal branch of Sin^{-1} x, while others are merely different branches of Sin^{-1} x. Note : In each branch y = Sin^{-1} x is a one-one and onto function and is single valued.



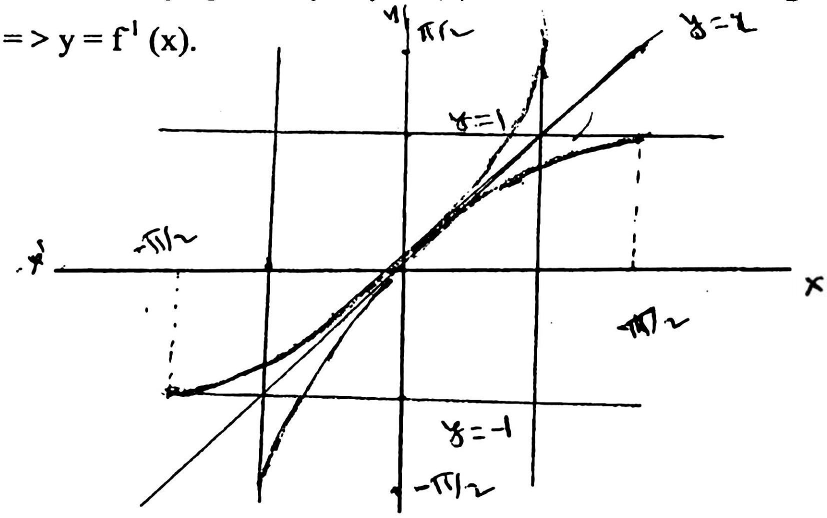
Example : Draw the graphs of Y = Cos x in [0, π]. Is it one-one ? If so, find the graph of y = Cos^{-1} x (the inverse of y = Cos x). Identify the principal branch of y = Cos^{-1} x.

A method to draw the graph of the inverse function of the given function whose graph is known.

Draw the graph of y = f(x) and the line y = x (bisectors of first and third quadrants) flip the graph sheet about the line y = x. the graph of y = $f^{1}(X)$ (inverse of f(x)) is got.

Rationale : Flipping about y = x means taking the reflection of y = f(x) on y = x, i.e. interchanging x and y in y = f(x). When this is done we get

$x = f(y) \Leftrightarrow y = f^{-1}(x)$.



Caution :

1. In $\text{Sin}^{-1} x$, -1 is not to be misunderstood as an exponent i.e.

   $\text{Sin}^{-1} x \neq \frac{1}{\text{Sin } x}$

   On the other hand, $(\text{Sin } x)^{-1} = \frac{1}{\text{Sin } x}$

2. $\text{Sin}^{-1} x$ is also read as *arc sine x*. Similarly, for the other Inverse T-functions.

Domain, Range and Graph of Inverse T-functions

	Inverse T-function	**Domain**	**Range (Principal Value Branch)**
1.	$y = \text{Sin}^{-1} x$	[-1, +1]	$[-\frac{\pi}{2}, +\frac{\pi}{2}]$
2.	$y = \cos^{-1} x$	[-1, +1]	$[0, \pi]$
3.	$y = \text{Tan}^{-1} x$	R	$(-\frac{\pi}{2}, +\frac{\pi}{2})$
4.	$y = \text{cosec}^{-1} x$	R – (-1 + 1)	$[-\frac{\pi}{2}, +\frac{\pi}{2}]$
5.	$y = \sec^{-1} x$	R – (-1 +1)	$[0, \pi] - \{\frac{\pi}{2}\}$

6.	$y = \cot^{-1} x$	R	$(0, \pi)$

Principal Value of an Inverse T-function

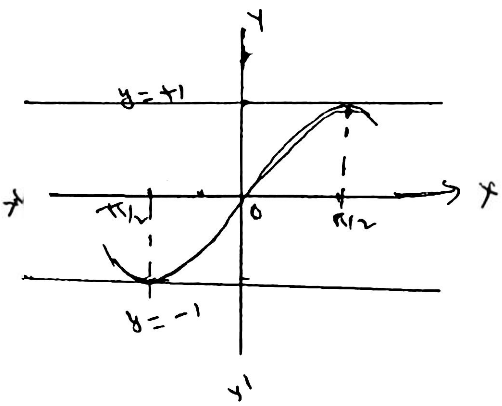
The value of an Inverse T-function which lies in the Principal branches of the function is called the Principal value of that inverse Trigonometric function.

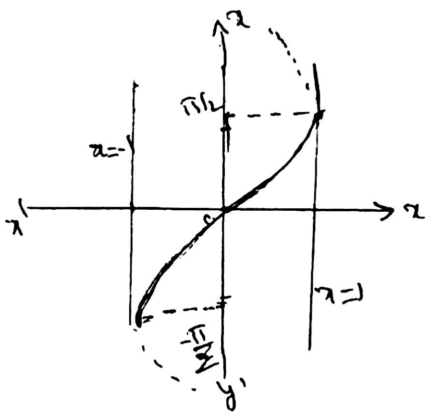
e.g. Consider $\text{Sin}^{-1}\left(\frac{1}{\sqrt{2}}\right) = y. \Rightarrow \text{Sin}\, y = \frac{1}{\sqrt{2}},\ y \in [-\frac{\pi}{2}, +\frac{\pi}{2}]$

$\therefore\ y = \frac{\pi}{4}$

$\therefore\ y = \frac{\pi}{4}$ is the principal value of $\text{Sin}^{-1}\left(\frac{1}{\sqrt{2}}\right)$.

Graphs of T-function and Inverse T-functions (Principal Branches)



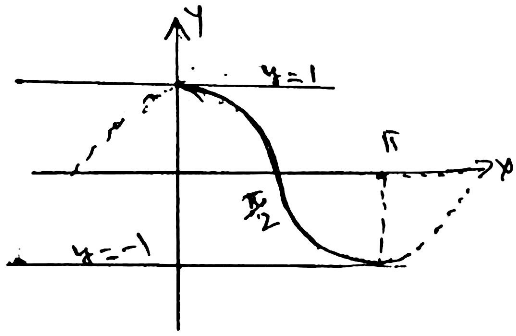


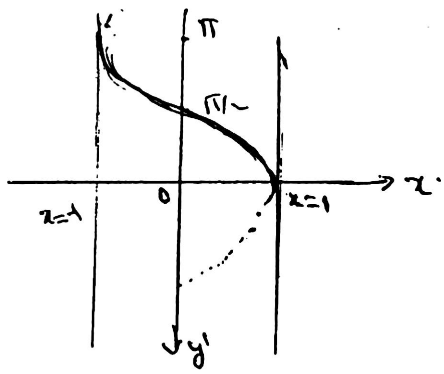
$y = \text{Sin}\, x\ (-\frac{\pi}{2} \le x < \frac{\pi}{2})$

Range : $-1 \le y \le +1$

$y = \text{Sin}^{-1} x\ (-1 \le x \le +1)$

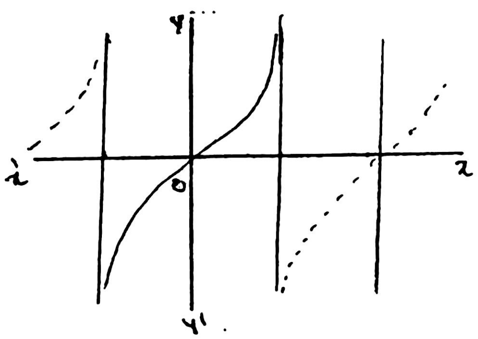
Range : $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

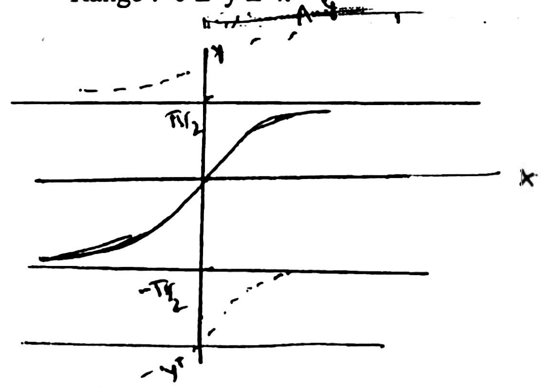




$y = \text{Cos } x \ (0 \le x \le \pi)$ $y = \text{Cos}^{-1} x \ (-1 \le x \le +1)$

Range : $-1 \le y \le +1$ Range : $0 \le y \le \pi$

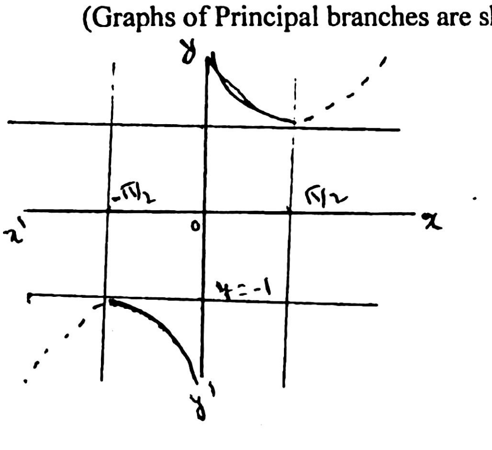


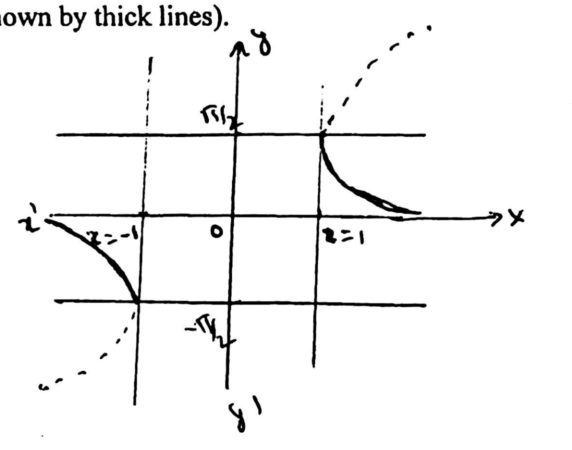


$y = \text{Tan } x \ (-\frac{\pi}{2} \le x \le +\frac{\pi}{2})$ $y = \text{Tan}^{-1} x \ (-\infty < x < +\infty)$

Range : $-\frac{\pi}{2} < y < +\frac{\pi}{2}$)

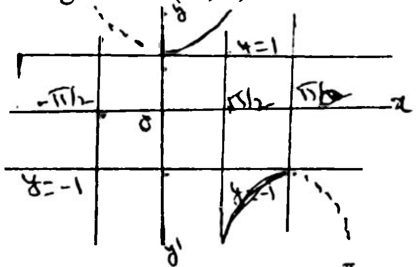
(Graphs of Principal branches are shown by thick lines).





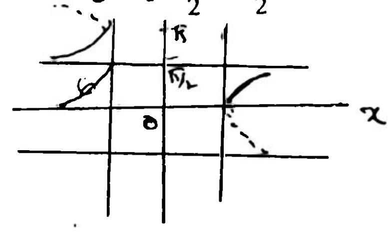
$y = \text{cosec } x \ (-\frac{\pi}{2} \le x \le \frac{\pi}{2}, x \ne 0)$ $y = \text{cosec}^{-1} x, \ |x| \ge 1$

Range : R – (-1, 1)



y = sec x, (0 ≤ x ≤ π, n ≠ $\frac{\pi}{2}$)

Range: [- $\frac{\pi}{2}$, + $\frac{\pi}{2}$]

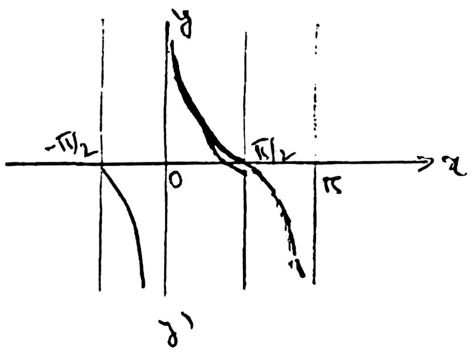


y = $\sec^{-1}$ x

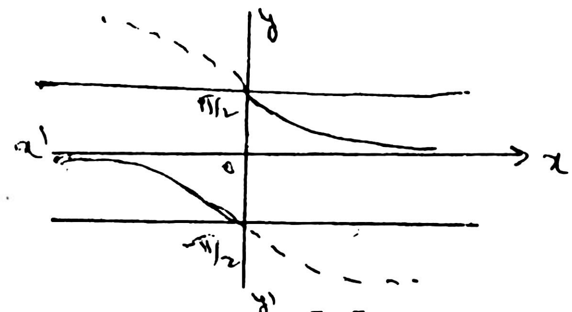
Range : R – (-1, 1)

Range [0, π] – { $\frac{\pi}{2}$}

y = cot x, - $\frac{\pi}{2}$ < x < + $\frac{\pi}{2}$



y = $\cot^{-1}$ x



Range: [- $\frac{\pi}{2}$, $\frac{\pi}{2}$] – {0}

4. Solve for x.

a) $\tan^{-1}(2x) + \tan^{-1}(3x) = \frac{\pi}{4}$.

$$\Rightarrow \quad \tan^{-1}\left(\frac{2x+3x}{1-6x^2}\right) = \frac{\pi}{4}$$

$$\Rightarrow \quad \frac{5x}{1-6x^2} = 1 \Rightarrow 6x^2 + 5x - 1 = 0$$

$$\therefore \quad x = \frac{1}{6} \text{ or } -1$$

However, x = -1 does not satisfy the equation (Check)

$\therefore \quad x = \frac{1}{6}$ is the only solution.

b) Solve for x $\text{Tan}^{-1}\left(\frac{x-1}{x-2}\right) + \text{Tan}^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}$.

$$\Rightarrow \quad \text{Tan}^{-1}\left(\frac{\frac{x-1}{x-2} + \frac{x+1}{x+2}}{1-\left(\frac{x-1}{x-2}\right)\left(\frac{x+1}{x+2}\right)}\right) = \frac{\pi}{4}$$

$$\Rightarrow \quad \text{Tan}^{-1}\left(\frac{(x-1)\ (x+2)\ + (x+1)\ (x-2)}{(x^2-4) - (x^2-1)}\right) = \frac{\pi}{4}$$

$$\Rightarrow \quad \text{Tan}^{-1}\left(\frac{2x^2-4}{-3}\right) = \frac{\pi}{4}$$

$$\Rightarrow \quad \frac{2x^2-4}{-3} = \text{Tan}\,\frac{\pi}{4} = 1$$

$$\Rightarrow \quad 2x^2 = 1 \qquad \therefore \quad x = \pm\,\frac{1}{\sqrt{2}}$$

5. Simplify : $\text{Tan}^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)$

Putting $x^2 = \cos\theta$

$$\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right) = \frac{\sqrt{1+\text{Cos}\,\theta}+\sqrt{1-\text{Cos}\,\theta}}{\sqrt{1+\text{Cos}\,\theta}-\sqrt{1-\text{Cos}\,\theta}}$$

$$= \quad \frac{\sqrt{2}\ (\cos\frac{\theta}{2} + \text{Sin}\,\frac{\theta}{2})}{\sqrt{2}\ (\text{Cos}\,\frac{\theta}{2} - \text{Sin}\,\frac{\theta}{2})} = \frac{1+\text{Tan}\,\frac{\theta}{2}}{1-\text{Tan}\,\frac{\theta}{2}}$$

$$= \quad \text{Tan}^{-1}\ (\frac{\pi}{4}+\frac{\theta}{2})$$

$$\therefore \quad \text{Tan}^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right) = \frac{\pi}{4}+\frac{\theta}{2}$$

But $x^2 = \cos\theta$ $\quad\therefore\ \theta = \text{Cos}^{-1}\,x^2$

$\therefore \quad$ G.E. $= \frac{\pi}{4} + \frac{1}{2} \cos^{-1}(x^2)$

exercises
~~Examples~~ :

1. Evaluate :

   a) $\text{Sin}\left(\text{Cos}^{-1}\left(\frac{3}{5}\right)\right)$

   b) $\text{Cos}(\text{Tan}^{-1}(3/4))$

   c) $\text{Sin}(\pi/3 - \sin^{-1}(-½))$

   d) $\text{Cos}(\text{Sin}^{-1}(3/5) + \text{Sin}^{-1}(5/13))$

2. P.T.

a) $\text{Tan}^{-1}(1/4) + \text{Tan}^{-1}(2/9) = ½\ \text{Cos}^{-1}(3/5)$

b) $\text{Cos}^{-1}(4/5) + \text{Cos}^{-1}(12/13) = \text{Cos}^{-1}(33/65)$

c) $\text{Tan}^{-1}(1/3) + \text{Tan}^{-1}(1/5) + \text{Tan}^{-1}(1/7) + \text{Tan}^{-1}(1/8) = \pi/4$

d) $2\ \text{Tan}^{-1}(1/2) + \text{Tan}^{-1}(1/7) = \text{Tan}^{-1}(31/17)$

3. Solve for x.

a) $\text{Sin}^{-1} x + \text{Sin}^{-1}(2x) = \pi/3$

b) $\text{Tan}^{-1}\left(\frac{1-x}{1+x}\right) - \frac{1}{2}\text{Tan}^{-1} x = 0, \ x > 0.$

4.

a) If $\text{Tan}^{-1} x + \text{Tan}^{-1} y + \text{Tan}^{-1} z = \pi$, then

b) If $\text{Cos}^{-1} x + \text{Cos}^{-1} y + \text{Cos}^{-1} z = \pi$, then

5. Draw the graph of

a) $y = \text{Tan}^{-1}(-x), x \in R$

b) $y = \text{Sin}^{-1}(2x), \ |x| \le ½$

Properties of Inverse T-functions

1. a) $\text{Sin}^{-1}(1/x) = \text{Cosec}^{-1} x$

   b) $\text{Cos}^{-1}(1/x) = \text{Sec}^{-1} x$

   c) $\text{Tan}^{-1}(1/x) = \text{Cot}^{-1} x$

2. a) $\text{Cos}^{-1} x + \sin^{-1} x = \pi/2$

   b) $\text{Sec}^{-1} x + \text{Cosec}^{-1} x = \pi/2$

   c) $\text{Tan}^{-1} x + \text{Cot}^{-1} x = \pi/2$

3. a) $\text{Sin}^{-1}(-x) = -\text{Sin}^{-1} x$

   b) $\text{Cos}^{-1}(-x) = \pi - \text{Cos}^{-1} x$

   c) $\text{Tan}^{-1}(-x) = -\text{Tan}^{-1} x$

4. a) $\text{Tan}^{-1} x + \text{Tan}^{-1} y = \text{Tan}^{-1}\left(\frac{x+y}{1-xy}\right)$ if $xy < 1$.

   b) $\text{Tan}^{-1} x + \text{Tan}^{-1} y = \pi + \text{Tan}^{-1}\left(\frac{x+y}{1-xy}\right)$ if $xy > 1$.

   c) $\text{Tan}^{-1} x - \text{Tan}^{-1} y = \text{Tan}^{-1}\left(\frac{x-y}{1+xy}\right)$

   d) $2\,\text{Tan}^{-1} x = \text{Tan}\left(\frac{2x}{1-x^2}\right)$; if $|x| < 1$.

   Also $2\,\text{Tan}^{-1} x = \text{Sin}^{-1}\left(\frac{2x}{1+x^2}\right)$ and

   $\text{Tan}^{-1} x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

5. a) $\text{Sin}^{-1}(\text{Sin}\, x) = x$

   b) $\text{Cos}^{-1}(\text{Cos}\, x) = x$

   c) $\text{Tan}^{-1}(\text{Tan}\, x) = x$

   d) $\text{Cosec}^{-1}(\text{Cos}\, x) = x$

   e) $\text{Sec}^{-1}(\text{Sec}\, x) = x$

   f) $\text{Cot}^{-1}(\text{Cot}\, x) = x$

References

1. Trigonometry, Hall and Knight.

2. Trigonometry, S.L.Loney

3. Senior Secondary School Mathematics – Agarwal

4 XI St. Text NCERT

II. Complex Numbers

A major inadequacy of the set $\Re$ of real numbers is that with $\Re$ we cannot solve several algebraic equations, the simplest of them being $x^2 + 1 = 0$. There is no $x \in \Re$ which is a solution of this equation, because $x^2 = -1$ is an impossible relation for any $x \in \Re$. So, a new number called *i* is defined such that $i^2 = -1$. It is called 'imaginary' because it is not in the real number system and so it is not real. At the same time, there is nothing 'imaginary' about *i* either. Having defined *i* in this way we arrive at the general complex number $z = x + iy$ where x and y are real numbers.

Thus a'$z = x + iy$' is a complex number where x and y are reals. x is called the *real part* and y is called the *imaginary part* of the complex number z. Some examples of complex numbers are :

$$\sqrt{3} - i\sqrt{2}, \quad -\frac{1}{5} + i; \quad \cos\theta + i\sin\theta.$$

Real part of the complex number is denoted by Re(z and imaginary part is denoted by $I_m(z)$. Thus, $x = Re(z)$, $y = I_m(z)$. If $y = 0$, then the number $a + i.0 = a$ is called a purely real number and if $x = 0$, then the number $0 + i\,b = i\,.\,b$ is called a purely imaginary number.

1. **Equality of Complex Numbers** :Two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are said to be equal if their real and imaginary parts are identical, i.e. $x_1 = x_2$ and $y_1 = y_2$. Thus
   $z = x+iy = 0$ if $x = 0$ and $y = 0$.
2. **Addition and Subtraction of Complex Numbers** : Given $z_1 = x_1 + iy_1$ and $z_2 = x_2+iy_2$, addition of z_1 and z_2 is denoted by $z_1 + z_2$ is defined as
   $z_1 + z_2 = (x_1 + x_2) + i\ (y_1 + y_2)$.
   $z_1 - z_2$ is defined by
   $z_1 - z_2 = (x_1 - x_2) + i\ (y_1 - y_2)$.
   It can be shown that the set of all complex numbers **C** is an abelian group with respect to 'addition'. The additive identity being the

complex numbers z = 0, and the additive inverse of z = x + iy is –z = -x + i(-y).

3. **Multiplication in C** : For z_1 $(x_1 + iy_1)$ and $z_2 = (x_2 + iy_2)$ define

$z_1z_2 = (x_1x_2 - y_1y_2) + i\,(x_1y_2 + x_2y_1)$.

This operation is a binary operation in . It is easy to verify that

$z_1z_2 = z_2z_1$

$z_1(z_2 + z_3) = z_1z_2 + z_1z_3$

$z \cdot 1 = z$

For complex numbers z, z_1, z_2, and z_3.

4. **Division in C** : If z = x + iy is a non-zero complex number then either x ≠ 0 ox y ≠ 0. Then

$$\frac{1}{z} = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2} = \frac{x - iy}{x^2 + y^2}$$

is the multiplicative inverse of Z. For,

$$z \times \frac{1}{z} = (x + iy)\left(\frac{x - iy}{x^2 + y^2}\right) = \left(x \times \frac{x}{x^2 + y^2}\right) + i\left(\frac{-xy + xy}{x^2 + y^2}\right)$$

$$= \frac{x^2 + y^2}{x^2 + y^2} + i\,.\,0 = 1.$$

For $z_1, z_2 \in$ .... with $z_2 \neq 0$, $\frac{z_1}{z_2}$ is defined as :

$\frac{z_1}{z_2} = z_1 \times \frac{1}{z_2}$. Hence division by a non-zero complex number is allowed. It follows that the set of all non-zero complex numbers forms a commutative under multiplication, the identity element being 1 = 1 + i .0. Thus in ...... we can perform all the algebraic operations that we do in $\Re$.

5. **Conjugate of a complex number** z = x + iy is the complex number x – iy denoted by $\bar{z}$. Thus $\bar{z} = x - iy$.

i) $\bar{\bar{z}} = z$, for $\bar{z}$ = 1x – iy and $\bar{\bar{z}} = \overline{x - iy}$ = x + iy = z.

ii) z + $\bar{z}$ = x + iy + x – iy and 2x = 2Re(z).

iii) z - $\bar{z}$ = x + iy – (x – iy) = 2iy = 2I $I_m(z)$

iv) $z\bar{z} = (x + iy)(x - iy) = x^2 + y^2$ and so $z\bar{z}$ is a non-negative number such that $z\bar{z} = 0$ if and only if $x^2 + y^2 = 0$ since $x^2 + y^2 = 0$ if and only if $x = 0 = y$.

v) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ *and* $\overline{z_1\, z_2} = \overline{z_1} \,.\, \overline{z_2}$

6. **Modulus of a complex number** $z = x + iy$

$z = x + iy$ is defined to be $\sqrt{x^2 + y^2}$ and it is denoted by $| z |$.

Thus, $| z | = | x + iy | = \sqrt{x^2 + y^2}$

Note that $| z |^2 = x^2 + y^2 = Z\overline{Z}$.

Also, $| \overline{Z} | = | x - iy | = \sqrt{x^2 + y^2} = | z |$

$-\sqrt{x^2 + y^2} \leq x \leq \sqrt{x^2 + y^2}$ and so $-| z | \leq \text{Re } z \leq | z |$

Similarly, $-| z | \leq I_m\, z \leq | z |$.

7. **The triangle inequality**

For complex numbers z_1 and z_2

i) $| z_1 + z_2 | \leq | z_1 | + | z_2 |$ (called the triangle inequality)

ii) $| z_1 - z_2 | \geq | |z_1| - |z_2| |$

We have,

$| z_1 + z_2 |^2 = (z_1 + z_2)(\overline{z_1 + z_2})$

$= (z_1 + z_2)(\bar{z}_1 + \bar{z}_2)$

$= z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_1\bar{z}_2$

$= | z_1 |^2 + |z_2|^2 + 2 \text{ Re } z_1\bar{z}_2$

$\leq |z_1|^2 + |z_2|^2 + 2 | z_1 \bar{z}_2 |$

$\leq | z_1|^2 + |z_2|^2 + 2 |z_1| |z_2|$

$\leq (| z_1 | + | z_2 |)^2$

Thus $| z_1 + z_2 | + \leq | z_1 | + | z_2 |$. This proves (i).

Now using the triangle inequality, we get,

$| z_1 | = | z_1 - z_2 + z_2 | \leq | z_1 - z_2 | + | z_2 |$

$| z_1 | - | z_2 | \leq | z_1 - z_2 |$

Similarly, $| z_2 | - | z_1 | \leq | z_2 - z_1 | = | z_1 - z_2 |$

Thus, $| |z_1| - |z_2| | \leq | z_1 - z_2 |$ which proves (ii).

Example :

1. Write the complex number $z = \dfrac{2+i}{(1+i)(1-2i)}$ in the form x + iy and

hence find | z |.

$$z = \frac{2+i}{(1+i)(1-2i)} = \frac{2+i}{3-i} = \frac{(2+i)(3+i)}{(3-i)(3+i)} = \frac{5+5i}{10} = \frac{1}{2} + \frac{i}{2}.$$

$$\therefore \ |z| = \sqrt{\frac{1}{4} + \frac{1}{4}} = \frac{1}{\sqrt{2}}.$$

Geometrical Representation of Complex Numbers :

We represent z = x +iy as a point P with coordinates (x,y) on the coordinate plane. This representation is called the *Argand Diagram Representation*. A point on the x-axis (now called the *Real axis)* is (x,0) and so represents the purely real number x. A point on the y-axis (now called the *imaginary axis)* is (0,y) and so represents the purely imaginary number iy.

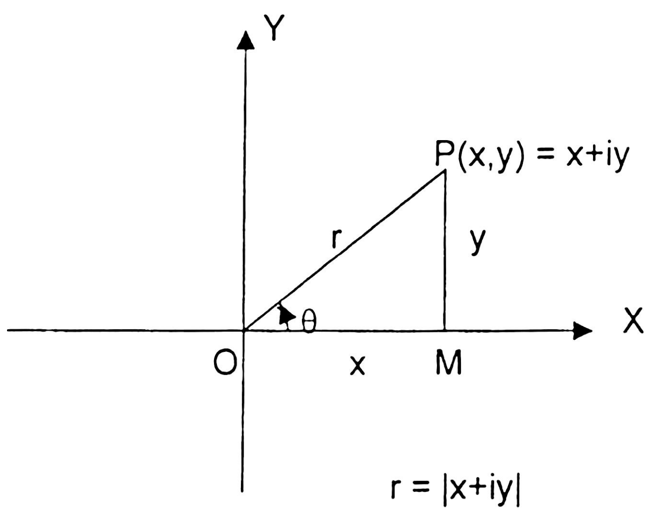


fig.1

If P is the point (x,y), it represents the complex number z = x +iy. We have the following geometrical interpretation of | z |. In Fig. 1, the distance OP is equal to $x^2 + y^2$ and thus, | z | is equal to the square root of the distance between the points O and P where P represents the complex number z in the coordinate plane, now called the complex plane.

Now if PM is perpendicular to the real axis as in Fig. 1 then x = OP cos θ and y = OP sin θ. If we put OP = r, then we have,

$x = r\cos\theta$ and $y = r\sin\theta$. Thus,

$Z = x + iy = r\cos\theta + i\ r\sin\theta$

$= r(\cos\theta + i\ \sin\theta)$.

This representation of the complex number z is called the *polar form of z.*
$z = x + iy$ is called the *Cartesian* or *rectangular* form of the complex number z.

If $z = r(\cos\theta + i\ \sin\theta)$, then $r = OP = |z|$ is the modulus of the complex number z. The angle θ which OP makes with the positive direction of the real axis is called the *argument* of z and is written *arg z.* Since $\sin(2n\pi + \theta) = \sin\theta$, and $\cos(2n\pi + \theta) = \cos\theta$, where n is any integer, θ is not unique. The value of θ satisfying $-\pi < \theta \leq \pi$ is called the *principal value* of the argument.

If $z = r(\cos\theta + i\ \sin\theta) = x + iy$.

Then $|z| = x^2 + y^2 = r$ and $\tan\theta = \dfrac{r\sin\theta}{r\cos\theta} = \dfrac{y}{x}$.

Thus, $\theta = \arg z = \tan^{-1}\dfrac{y}{x}$.

Example 1 : Represent the following complex numbers in the polar form.

i) $1 + i\sqrt{3}$ ii) $\dfrac{1}{\sin\theta + i\cos\theta}$

(i) $1 + i\sqrt{3} = x + iy \Rightarrow x = 1, y = \sqrt{3}$

$\Rightarrow x^2 + y^2 = 4$

$\Rightarrow r = \sqrt{x^2 + y^2} = 2$

$\tan\theta = \dfrac{y}{x} = \sqrt{3} \quad \therefore \theta = \dfrac{\pi}{3}$

$\therefore 1+i\sqrt{3} = 2\left(\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}\right)$

(ii) $z = \dfrac{1}{\sin\theta + i\cos\theta} = \dfrac{\sin\theta - i\cos\theta}{(\sin\theta + i\cos\theta)} \times \dfrac{1}{(\sin\theta - i\cos\theta)}$

$= \dfrac{\sin\theta - i\cos\theta}{\sin^2\theta + \cos^2\theta}$

$= \sin\theta - i\cos\theta$

$= \cos\left(\frac{\pi}{2} - \theta\right) - i\sin\left(\frac{\pi}{2} - \theta\right)$

$= \cos\left(\theta - \frac{\pi}{2}\right) + i\sin\left(\theta - \frac{\pi}{2}\right)$

Thus $r = 1$ and $\arg z = \theta - \frac{\pi}{2}$.

Example 2 : If z_1 and z_2 are non-zero complex numbers, then

i) $\arg(z_1 z_2) = \arg z_1 + \arg z_2$

ii) $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$.

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

Then, $z_1 z_2 = r_1 r_2 \{\cos\theta_1\cos\theta_2 - \sin\theta_1 . \sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \sin\theta_2 \cos\theta_1)$

$= r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$

which implies $|z_1 z_2| = |z_1||z_2|$

and $\arg(z_1 z_2) = \theta_1 + \theta_2$

$= \arg z_1 + \arg z_2$

This proves (i).

To prove (ii), we write z_1 as $z_1 = \frac{z_1}{z_2} . z_2$.

Hence by (i), $\arg z_1 = \arg\left(\frac{z_1}{z_2}\right) + \arg z_2$

and thus $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$

This proves (ii).

We remark that (i) and (ii) may not be valid if we take the principal argument.

De Moivre's theorem and its applications : De Moivre's theorem is an important theorem in complex analysis due to which calculation of large powers of complex numbers becomes quite easy.

De Moivre's theorem : If n is an integer, then

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

And if n is a fraction, then one of the values of $(\cos\theta + i\sin\theta)^n$ is $\cos n\theta + i\sin n\theta$.

Proof : We have seen above that

$(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) = \cos(\alpha+\beta) + i\sin(\alpha+\beta)$.

This can be extended to any number of quantities by induction.

i.e. For $\alpha_1, \alpha_2, \ldots, \alpha_n$.

$(\cos\alpha_1 + i\sin\alpha_1)\ldots(\cos\alpha_n + i\sin\alpha_n) = \cos(\alpha_1 + \ldots + \alpha_n) + i\sin(\alpha_1 + \ldots + \alpha_n)$

Case 1 : Let n be a positive integer. Putting $\alpha_1 = \alpha_2 = \ldots = \alpha_n = \theta$, in the above, we get,

$(\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)$ ..... n times

$= \cos(\theta + \theta + \ldots$ n times$) + i\sin(\theta + \theta + \ldots$ n times$)$.

Hence,

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta).$$

Case 2: Let n be a negative integer, say, $n = -m$, where m is a positive integer. Then,

$$(\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{-m}$$

$$= \frac{1}{(\cos\theta + i\sin\theta)^m}$$

$$= \frac{1}{(\cos m\theta + i\sin m\theta)}, \text{ by case I}$$

$$= \frac{\cos m\theta - i\sin m\theta}{(\cos m\theta + i\sin m\theta)(\cos m\theta - i\sin m\theta)} = \frac{\cos m\theta - i\sin m\theta}{\cos^2 m\theta + \sin^2 m\theta}$$

$$= \cos(-m\theta) + i\sin(-m\theta)$$

$$= \cos n\theta + i\sin n\theta.$$

Case 3 : Let n be a rational fraction, say, $n = \frac{p}{q}$ where p and q are integers.

By case 1 and case 2, we have

$$\left(\cos\left(\frac{p}{q}\theta\right) + i\sin\left(\frac{p}{q}\theta\right)\right)^q = \cos\left(q\cdot\frac{p}{q}\cdot\theta\right) + i\sin\left(q\cdot\frac{p}{q}\cdot\theta\right)$$

$= \cos p\,\theta + i\, \sin p\,\theta$

$= (\cos\theta + i\, \sin\theta)^p$, by case 1.

Thus, $\cos \frac{p}{q}\theta + i\, \sin \frac{p}{q}\theta$ is one of the values of $(\cos\theta + i \sin\theta)^{p/q}$. This completes the proof of the theorem.

Remark: Using De Moivre's theorem, we have

1. If $z = r(\cos\theta + i\, \sin\theta)$ then $z^n = r^n(\cos n\,\theta + i\, \sin n\,\theta)$
2. $(\cos\theta + i \sin\theta)^{-1} = \cos(-\theta) + i\, \sin(-\theta) = \cos\theta - i\, \sin\theta$.
3. If $z = \cos\theta + i\, \sin\theta$, then

$$z^n + \frac{1}{z^n} = 2\cos n\theta; \quad z^n - \frac{1}{z^n} = 2i\, \sin n\,\theta.$$

Application of De Moivre's Theorem to determine roots of complex numbers :

Let $A = \rho(\cos\phi + i \sin\phi)$. We find the nth roots of A where n is any positive integer. The complex number $z = r(\cos\theta + i\, \sin\theta)$ is called an nth root of the number A, if $z^n = A$ i.e.

$$r^n(\cos\theta + i\, \sin\theta)^n = \rho(\cos\phi + i\, \sin\phi)$$

i.e. $\quad r^n(\cos n\,\theta + i\, \sin n\,\theta) = \rho(\cos\phi + i\, \sin\phi)$

Since moduli of equal complex numbers must be equal while their arguments differ by $2\pi k$, k being an arbitrary integer, we have

$r^n = \rho$ and $n\theta = \phi + 2\pi k$, where k is an integer.

Thus, $z = \sqrt[n]{\rho}\left(\cos\frac{\phi + 2\pi k}{n} + i \sin\frac{\phi + 2\pi\, k}{n}\right)$

Giving k the values 0,1,2,..., n-1, we get n different values of the root. Thus all the solutions of the equation $z^n = A$ can be written as follows :

$$z_k = \sqrt[n]{\rho}\left(\cos\left(\frac{\phi}{n} + \frac{2\pi k}{n}\right) + i \sin\left(\frac{\phi}{n} + \frac{2\,\pi\, k}{n}\right)\right)$$

k = 0, 1, 2,...., n-1.

We remark that the modulus of each of the roots of degree n of the non-zero complex number is the same non-negative real number, but arguments differ from each other by $\frac{2\pi}{k}$, where k is some integer. The number of nth roots of a non-zero complex number is n. Hence it follows that the roots $z_0, z_1, \ldots, z_{n-1}$ correspond to the points of the complex plane as vertices of a regular polygon of n sides inscribed in a circle of radius $\sqrt[n]{\rho}$ with centre at the point z = 0.

1. **Square roots of Unity** : Let A = cos 0 + i sin 0, and n = 2 then the square roots of A are :

$$w_k = \cos\left(\frac{0 + 2\pi k}{n}\right) + i\left(\sin\ \frac{0 + 2\pi k}{n}\right)$$

$$= \cos\ \frac{2\pi k}{2} + i\ \sin\ \frac{2\pi k}{2}, \text{ where } k = 0, 1.$$

Then $w_1 = \cos 0 + i\ \sin 0$; $w_2 = \cos \pi + i\ \sin \pi$.

i.e. $w_1 = 1, -1$ are the square roots of unity.

2. **Cube roots of unity** : We have

$$\sqrt[3]{1} = \sqrt[3]{\cos 0^\circ + i \sin 0^\circ} = \sqrt[3]{1}\ .\left(\cos \frac{0^\circ + 2\pi k}{3} + i \sin \frac{0^\circ + 2\pi k}{3}\right)$$

$$= \cos \frac{2\pi k}{3} + i \sin \frac{2\pi k}{3},\ k = 0, 1,2,\ldots$$

Consequently, three cube roots of unity are given by

$$w_1 = \cos\ \frac{0}{3} + i\ \sin\ \frac{0}{3} = \cos 0 + I \sin 0 = 1.$$

$$w_2 = \cos\ \frac{2\pi}{3} + i\ \sin\ \frac{2\pi}{3} = -\ \frac{1}{2} + i\ \frac{\sqrt{3}}{2}$$

$$w_3 = \cos\ \frac{4\pi}{3} + i \sin\ \frac{4\pi}{3} = -\ \frac{1}{2} -\ i\ \frac{\sqrt{3}}{2}$$

Putting w_2 = w, we get

$$w^2 = \cos 2\ .\ \frac{2\pi}{3} + i\ .\ \sin 2\ .\ \frac{2\pi}{3} = w_3$$

Thus, 1, w, w^2 are the cube roots of unity as shown in Fig.2 .

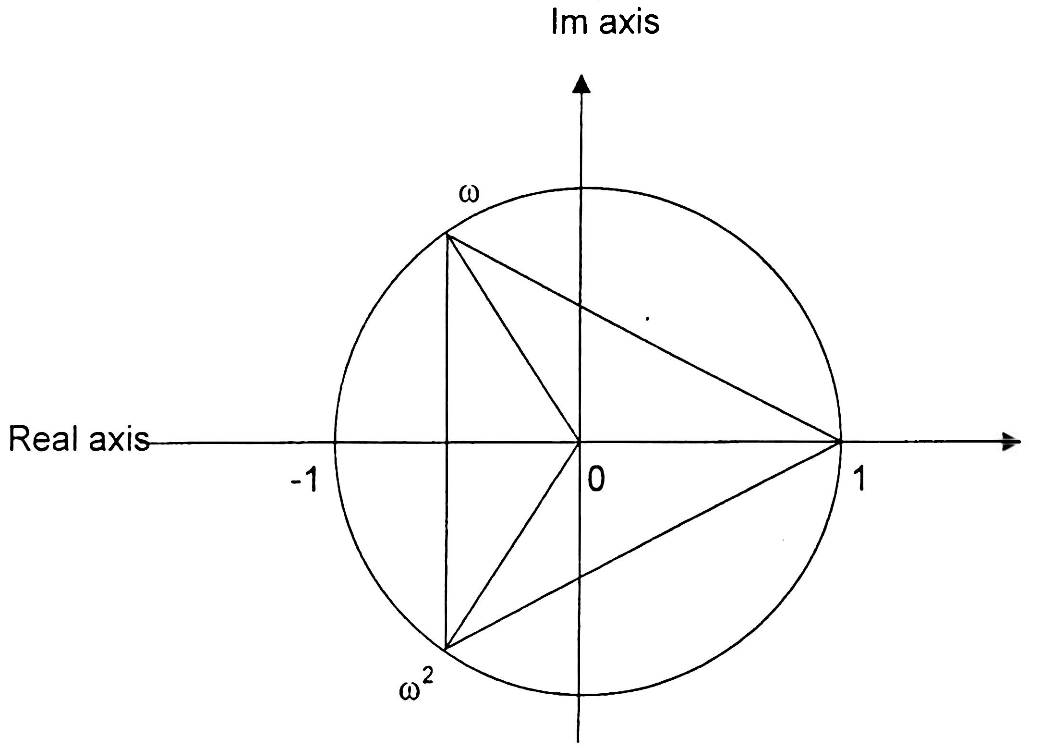


Fig.2

Note that : $1 + w + w^2 = 1 + \left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right) + \left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)$

$= 0$

Example 1 : Find the cube roots of 8

$8 = 8 (\cos 0 + i \sin 0)$

$\therefore w_k = \sqrt[3]{8}\left(\cos\frac{0+2\pi k}{3} + i\sin\frac{0+2\pi k}{3}\right) \qquad k = 0, 1, 2,$

$\therefore$ The roots are

$2 (\cos 0 + i \sin 0 = 2;\ 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = (-1 + i\sqrt{3})$

and $2\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right) = -1 - i\sqrt{3}$.

2. If 1, w, w^2 are the three cube roots of unity, show that

$(1 + w)^3 - (1 + w^2)^3 = 0$

We use the relation $1 + w + w^2 = 0$ and $w^3 = 1$, then,

$(1 + w)^3 - (1 + w^2)^3 = (-w^2)^3 - (-w)^3$

$= -w^6 + w^3$

$= -(w^3)^2 + w^3 = -1 + 1 = 0$

3. Simplify : $\left(\dfrac{(\cos 3\theta + i\sin 3\theta)^5 \ (\cos 2\theta - i\sin 2\theta)^3}{(\cos 4\theta + i\sin 4\theta)^2 \ (\cos 5\theta - i\sin 5\theta)^4}\right)$

We denote cos n θ + i sin n θ by cis n θ for short. By De Moivre' theorem,

$(\cos 3\theta + i \sin 3\theta)^5 = \text{cis } 15\theta$; $(\cos 4\theta + i \sin 4\theta)^2 = \text{cis } 8\theta$.

$(\cos 2\theta - i \sin 2\theta)^3 = (\cos 2\theta + i \sin 2\theta)^{-3} = \text{cis } (-6\theta)$

$(\cos 5\theta - i \sin 5\theta)^4 = \text{cis } (-20\theta)$.

∴ The given expression is

$$\frac{cis\,(15\,\theta)\ \ cis\,(-6\theta)}{cis(8\theta)\ \ cis(-20\theta)}$$

$= \text{cis } (15 - 6 - 8 + 20)\,\theta$

$= \text{cis } 21\,\theta$

$= \cos 21\,\theta + i \sin 21\theta$

4. Find the cube roots of $\sqrt{3} + i$ and represent them on the complex plane.

Let $\sqrt{3} + i = r(\cos\theta + i\sin\theta)$

Then $r = \sqrt{3+1} = 2$, $\theta = \tan^{-1}\dfrac{1}{\sqrt{3}}$. $\quad\therefore\ \theta = \dfrac{\pi}{6}$

Then $\sqrt{3} + i = 2(\cos \pi/6 + i \sin \pi/6)$

$$\therefore\ \sqrt[3]{\sqrt{3}+i} = 2^{1/3}\left(\cos\frac{\frac{\pi}{6}+2k\pi}{3} + i\sin\frac{\frac{\pi}{6}+2k\pi}{3}\right),\ k = 0,1,2.$$

Thus the roots are :

$$2^{1/3}\left(\cos\frac{\pi}{18} + i\sin\frac{\pi}{18}\right),\ 2^{1/3}\left(\cos\frac{13\pi}{18} + i\sin\frac{13\pi}{18}\right)$$

$$2^{1/3}\left(\cos\frac{25}{18}\pi + i\sin\frac{25}{18}\pi\right)$$

The roots divide the circle whose centre at z = 0 of radius $2^{1/3}$ into three equal parts as in Fig. 3.

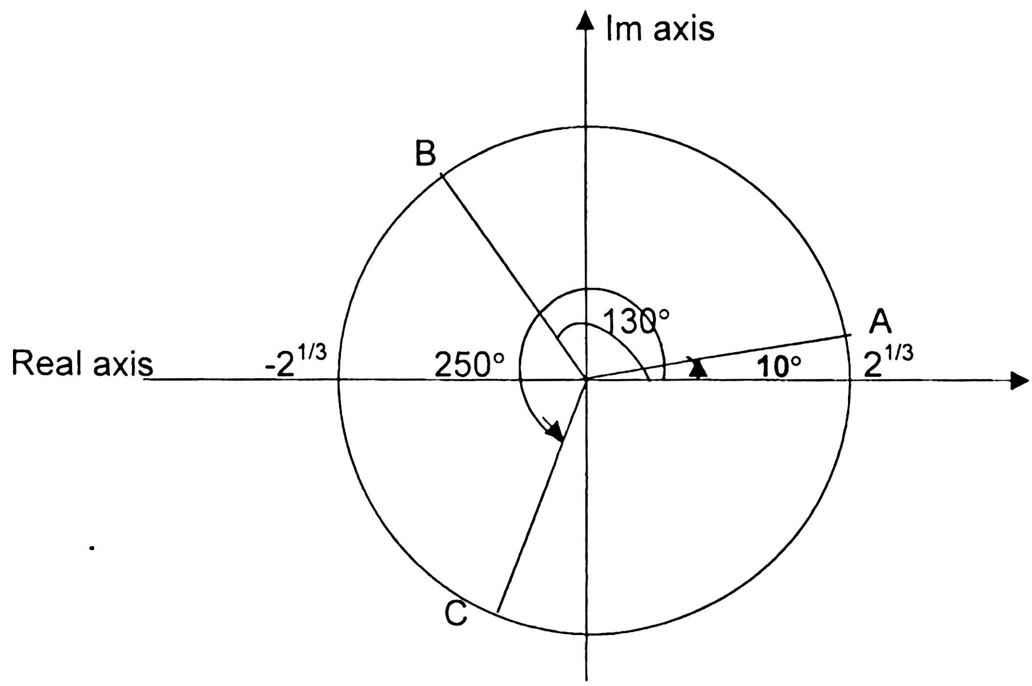


Fig.3

Note : As in $\Re$ we can perform the algebraic operations in **C** such as addition, subtraction, multiplication and division. Thus **C** is a field, called the field of complex numbers, similar to the field of real numbers $\Re$. Both in Algebra and analysis,**C** plays a central role. Unlike in $\Re$, no ordering is possible in **C** which respects addition and multiplication.

You have seen that the roots of the real quadratic equation $ax^2 + bx + c$ are

$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. The roots are real if and only if $b^2 - 4ac > 0$. Since $\Re \subseteq$ **C**, in any case, we can say that the roots are in **C**. Thus roots of any quadratic equation with coefficients in $\Re$ has roots in **C**. In fact, this is true for any algebraic equation, i.e. any polynomial equation $f(x) = 0$ with real coefficients (result is true even when the coefficients are complex numbers) has a root in **C** This result is one of the most important theorems in mathematics, called the Fundamental Theorem of Algebra.

Exercises :

1. Express the following complex numbers in the Cartesian form.

$$\frac{1+2i}{1-2i}+\frac{1-2i}{1+2i},\quad \frac{10}{(1-i)\,(2-i)\,(3-i)},\quad (1+i)^4$$

2. Express the following complex numbers in the polar form.

$$\frac{1-i}{1+i},\quad \left(\frac{2+3i}{2-3i}\right)^2,\quad 5+12\,i,\quad \frac{-2}{1+i\sqrt{3}},\quad (1+i)^3$$

3. If $x + iy = \dfrac{1}{1+\cos\theta+i\sin\theta}$, prove that $4x^2 = 1$.

4. Show that the points 5 + 4I, 3 + 2/i, -4 + 3/i and -2 + 3i from the vertices of a parallelogram.

5. If $x + iy = \sqrt{\dfrac{1+2i}{3+4i}}$, prove that $(x^2 + y^2) = \dfrac{1}{5}$.

6. Simplify : $\left[\dfrac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}\right]^n$.

7. Express $(1 - i)^9$ in the polar and cartesian form.

8. Prove that $(1 + i)^n + (1 - i)^n = 2^{n/2+1}\,.\cos\dfrac{n\pi}{4}$.

9. If 1, w, w^2 are cube roots of unity, show that
$(1 + w)(1 + w^2)(1 + w^4)(1 + w^8) = 1$.

10. i) Find the cube roots of i, -1, 1 – i.

 ii) Find the fourth roots of 1 – i, $\sqrt{3}$ - i .

 iii) Find the fifth roots of 2 + 2i.

DETERMINANTS

Introduction

Every square matrix A = [a_{ij}] is associated with a determinant (a number) of A denoted by | A |. The non-square matrices do not have determinants. A determinant can be evaluated to get a real number whereas a matrix is only an arrangement of numbers within a square bracket.

The concept of determinant is very useful in solving a system of linear equations. The determinants have also been used as a convenient way of expressing certain formulas.

i) The determinant of a 1 by 1 matrix [a] is defined to be | a | = a.

ii) We define the determinant of a 2 × 2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ by } |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc. \ |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

iii) The determinant of a 3 × 3 matrix

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \text{ is denoted by } \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

and is defined to be

$$a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

which can be evaluated further.

Note: Some of the following points are very clear from the textbook of Mathematics for Second PUC.

a) Evaluating the determinant by direct expansion.

b) The properties of determinants which are true for any order.

c) To evaluate a determinant, the method of expanding the determinant is laborious. It is better to use some of the properties to simplify the determinant first and then evaluate.

Examples : Show that

$$\Delta = \begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & ba+3b & 10a+6b+3x \end{vmatrix} = a^3$$

Solution :

$$\Delta = \begin{vmatrix} a & a & a+b+c \\ 2a & 3a & 4a+3b+2c \\ 3a & 6a & 10a+6b+3c \end{vmatrix} + \begin{vmatrix} a & b & a+b+c \\ 2a & 2b & 4a+3b+2c \\ 3a & 3b & 10a+6b+3c \end{vmatrix}$$

[This step is got by splitting the 2nd column. The second determinant becomes zero when 'a' and 'b' are taken as common factor from c_1 and c_2].

$$\therefore \quad \Delta = \begin{vmatrix} a & a & a \\ 2a & 3a & 4a \\ 3a & 6a & 10a \end{vmatrix} + \begin{vmatrix} a & a & b \\ 2a & 3a & 3b \\ 3a & 6a & 6b \end{vmatrix} + \begin{vmatrix} a & a & c \\ 2a & 3a & 2c \\ 3a & 6a & 3c \end{vmatrix}$$

[The first determinant is expressed as sum of 3 determinants using the elements of c_3. Now the second and third determinants are zero].

$$\Delta = a^3 \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 6 & 10 \end{vmatrix} \text{ applying } c_2 - c_1 \text{ and } c_3 - c_1$$

$$= a^3 \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 2 \\ 3 & 3 & 7 \end{vmatrix} = a^3 . 1\,(7-6) = a^3$$

Note: $c_2 - c_1$ means subtracting first column from second column. Thus only second column will change and not the first column. Similar statements regarding rows and columns have their obvious natural meaning. If more than one operation is done in one step, care should be taken to see that a row that is affected in one operation should not be used in another operation. Similarly for columns.

2. Prove that $\begin{vmatrix} 1 & x & x^2-yz \\ 1 & y & y^2-zx \\ 1 & z & z^2-xy \end{vmatrix} = 0$ without expanding the determinant.

Solution :

$$\text{LHS} = \begin{vmatrix} 1 & x & x^2-yz \\ 1 & y & y^2-zx \\ 1 & z & z^2-xy \end{vmatrix}$$

$$= \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} - \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$$

$$= \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} - \frac{1}{xyz}\begin{vmatrix} x & x^2 & xyz \\ y & y^2 & xyz \\ z & z^2 & xyz \end{vmatrix} \begin{matrix} R_1x \\ R_1y \\ R_1z \end{matrix}$$

$$= \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} - \frac{xyz}{xyz}\begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} - \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \begin{matrix} c_2 \leftrightarrow c_3 \\ c_2 \leftrightarrow c_1 \end{matrix} \text{ then}$$

$= 0 = \text{RHS}.$

3. Prove that $\begin{vmatrix} a-x & a+x & a+x \\ a+x & a-x & a+x \\ a+x & a+x & a-x \end{vmatrix} = 4x^2 (x + 3a)$

Note: here RHS is the clue to start the problem. By using properties of determinants, it may be possible to have factors of the RHS as common for a row or column. When it is removed outside we get all entries 1 in a row or column. Then it is easy to operate and expand the get the required answer.

Solution :

$$\text{LHS} = \begin{vmatrix} a-x & a+x & a+x \\ a+x & a-x & a+x \\ a+x & a+x & a-x \end{vmatrix}$$

$$= \begin{vmatrix} x+3a & 0 & a+x \\ x+3a & -2x & a+x \\ x+3a & 2x & a-x \end{vmatrix} \quad \begin{matrix} C_1^1 = C_1 + C_2 + C_3 \\ C_2^1 = C_2 - C_3 \end{matrix}$$

Take out (x + 3a) and 2x from C_1 and C_2

$$= 2x\,(x+3a) \begin{vmatrix} 1 & 0 & a+x \\ 1 & -1 & a+x \\ 1 & 1 & a-x \end{vmatrix}$$

$$= 2x\,(x+3a) \begin{vmatrix} 0 & 1 & 0 \\ 0 & -2 & 2x \\ 1 & 1 & a-x \end{vmatrix} \quad \begin{matrix} R_1^1 = R_1 - R_2 \\ R_2^1 = R_2 - R_3 \end{matrix}$$

Now expand columnwise from 1st column.

$= 2x\,(x + 3a)\ 1 \,.\, 2x$

$= 4x^2\ (x + 3a)$

= RHS

4. Find the value of the determinant without expansion.

$$\begin{vmatrix} b^2-ab & b-c & bc-ac \\ ab-a^2 & a-b & b^2-ab \\ bc-ac & c-a & ab-a^2 \end{vmatrix}$$

$$\begin{vmatrix} b^2-ab & b-c & bc-ac \\ ab-a^2 & a-b & b^2-ab \\ bc-ac & c-a & ab-a^2 \end{vmatrix}$$

$$= \begin{vmatrix} b(b-a) & (b-c) & c(b-a) \\ a(b-a) & (a-b) & b(b-a) \\ c(b-a) & (c-a) & a(b-a) \end{vmatrix}$$

Taking out (b-a) from c_1 and c_3, we get

$$= (b-a)^2 \begin{vmatrix} b & b-c & c \\ a & a-b & b \\ c & c-a & a \end{vmatrix}$$

6. Prove that $\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix}$ = (a-b) (b-c) (c-a))(a+b+c)

Solution :

$$\Delta = \begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix}$$

As C_1 contains all entries as 1, it is easy to get zeros in 1^{st} column by $R_1^1 = R_1 - R_2$, $R_2^1 = R_2 - R_3$.

$$\Delta = \begin{vmatrix} 0 & a-b & a^3-b^3 \\ 0 & b-c & b^3-c^3 \\ 1 & c & c^3 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & (a-b) & (a-b)(a^2+ab+b^2) \\ 0 & (b-c) & (b-c)(b^2+bc+c^2) \\ 1 & c & c^3 \end{vmatrix}$$

Taking out (a-b) and (b-c) as common factors from R_1 and R_2, we get

$$= \quad (a-b)(b-c) \begin{vmatrix} 0 & 1 & a^2+ab+b^2 \\ 0 & 1 & b^2+bc+c^2 \\ 1 & c & c^3 \end{vmatrix}$$

Expanding the determinant from C_1 we get

Δ = (a-b) (b-c) [$(b^2 + bc + c^2) - (a^2 + ab + b^2)$]

= (a-b) (b-c) [$b^2 + bc + c^2 - a^2 - ab - b^2$]

= (a-b) (b-c) [$(c^2 - a^2)$ + b(c-a)]

= (a-b) (b-c) [(c-a) (c+a) + b(c-a)]

= (a-b) (b-c) (c-a) (c+a+b) = RHS

Alternatively,

The given determinant is a homogeneous symmetric expression in a,b,c. In such cases, the result given on RHS is the clue to start the problem. The following technique can be applied. Put a = b in Δ then

$$\Delta = \begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & b & b^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = 0 \quad \text{since } R_1 = R_2$$

∴ (a-b) is a factor of Δ.

By symmetry (b-c) and (c-a) are also factors. Consider the product of principle diagonal elements $1.b.c^{3+} = bc^3$. It is of fourth degree. Hence there exists another factor of first degree in a,b and c i.e. (a+b+c). There may be a scale factor k.

$$\therefore \quad \begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = k\,(a-b)\,(b-c)\,(c-a)\,(a+b+c)$$

To find k, put a = 0, b = 1, c = 2.

$$\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 8 \end{vmatrix} = k\,(0-1)\,(1-2)\,(2-0)\,(0+1+2)$$

$8 - 2 = k\,(-1)\,(-1)\,(2)\,(3)$

$6 = 6k$

∴ k = 1 ∴ Δ = (a-b) (b-c) (c-a) (a+b+c)

After solving some problems on determinants using the properties, we can have a guideline or general rule to evaluate the determinant.

1. Direct expansion of the determinant should be avoided as far as possible.
2. It is always desired that we should try to bring in as many zeros as possible in any row or column before expanding the determinant.
3. In order to bring zeros in any row or column we have to use the properties.

4. If we can get three zeros to the right or left of the principal diagonal elements, the value of the determinant becomes the product of the diagonal elements.
5. At the beginning or after one or two operations, if we find some common factors in any row or column it is better to take out the common factor and proceed further.
6. If a determinant is a homogeneous symmetric expression, it can be solved in general as in example 6.
7. Try to bring unity (1) in a row (or col) by suitable Row / column operations. If this is done, it is easy to reduce the order of the determinant or evaluate the determinant.

Remembering these points one can evaluate the determinant having an eye on the result which is given on RHS in most of the problems.

Self Evaluation Test

Prove the following statements.

1. $$\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2$$

2. $$\begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ca & cb & a^2+b^2 \end{vmatrix} = 4a^2b^2c^2$$

3. $$\begin{vmatrix} b^2+c^2 & a^2 & a^2 \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix} = 4a^2b^2c^2$$

4. $$\begin{vmatrix} a & b-c & c+b \\ a+c & b & c-a \\ a-b & b+a & c \end{vmatrix} = (a+b+c)(a^2+b^2+c^2)$$

5. $$\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc\,(a+b+c)^3$$

6. $$\begin{vmatrix} (a+b)^2 & ca & cb \\ ca & (b+c)^2 & ab \\ bc & ab & (c+a)^2 \end{vmatrix} = 2abc\,(a+b+c)^3$$

7. $$\begin{vmatrix} 1+a^2 & ab & ac \\ ab & 1+b^2 & bc \\ ac & bc & 1+c^2 \end{vmatrix} = 1 + a^2 + b^2 + c^2$$

8. $$\begin{vmatrix} -bc & b^2+bc & c^2+bc \\ a^2+ac & -ac & c^2+ac \\ a^2+ab & b^2+ab & -ab \end{vmatrix} = (bc + ca + ab)^3$$

9. $$\begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ac-c^2 \end{vmatrix} = (a^3 + b^3 + c^3 - 3abc)^2$$

10. $$\begin{vmatrix} -2a & a+b & c+a \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix} = 4(b+c)\,(c+a)\,(a+b)$$

11. Without expanding the determinant, prove that

$$\begin{vmatrix} a^2 & b^2 & c^2 \\ (a+1)^2 & (b+1)^2 & (c+1)^2 \\ (a-1)^2 & (b-1)^2 & (c-1)^2 \end{vmatrix} = 4\begin{vmatrix} a^2 & b^2 & c^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$$

12. If $2S = a + b + c$, prove that

$$\begin{vmatrix} a^2 & (s-a)^2 & (s-a)^2 \\ (s-b)^2 & b^2 & (s-b)^2 \\ (s-c)^2 & (s-c)^2 & s^2 \end{vmatrix} = 2s^3 (s-a)(s-b)(s-c)$$

13. If a + b + c = 0 show that the value of 'x' from the equation

$$\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & (c-x) \end{vmatrix} = 0$$

are x = 0 and $x = \pm \sqrt{\frac{3}{2}(a^2 + b^2 + c^2}$

2. Straight lines and family of concurrent straight lines.

2.1 Slope of a straight line – Parallel and Perpendicular Straight lines

The locus of a point with a fixed direction is a straight line.

Definition : If a straight line *l* makes an angle θ with the positive x-axis (i.e. with ox), $(0 \leq \theta \leq 180^{o})$ then m = Tan θ is called the slope of the straight line *l*.

Special Cases :

1. For straight lines making acute angles
   $(0^{o} < \theta < 90^{o})$ *m is positive.*
2. For straight lights making obtuse angles
   $(\theta > 90^{o})$ *m is negative.*
3. For straight lines ⊥r to the x-axis
   (or parallel to the y-axis) *m is not defined.*

4. For straight lines ||[le] to the x-axis
   (or ⊥r to the y-axis) $(\theta = 0 \text{ or } 180^{o})$
   m = 0.
5. The slope of the x-axis is zero and the slope of the y-axis is not defined.

Note (for the teacher)

In some books the condition for a line parallel to the y-axis is given as m is not defined and $m = +\infty$. This presents learning difficulty to the students because (a) the idea of $+\infty$ (or $-\infty$) being abstract is not easily understood. ($+\infty$ indicates the tendency of a variable and not a value).

(b) Saying m is undefined and at the same time, $m = +\infty$, is not easy to reconcile. Therefore, the condition for a line parallel to the y-axis may be taken as $\frac{1}{m} = 0$

(since $\frac{1}{m} = \cot\theta$ and $\cot 90^{o} = 0$)

5. Case of Parallel lines

If two straight lines l and l' are parallel, then they make the same angle θ with ox. Hence the slopes of the parallel lines are equal. In other words, if m and m′ are the slopes of two lines l and l', then $l \parallel l'$ iff m = m′.

Further, *all parallel straight lines have the same slope.* This condition is called the *parallelism condition.*

6. **Case of perpendicular lines** : Let l and l' be two straight lines none of which is parallel to the y-axis (vertical).

Let their slopes be m and m′. If l makes θ with ox,

Then l' makes $(90° + \theta)$ with ox.

$\therefore$ m = Tan θ

and m′ = Tan $(90° + \theta)$ = -Cot θ

$$\Rightarrow m' = -\frac{1}{Tan\ \theta} = -\frac{1}{m}.$$

Hence, mm′ = -1.

Hence $l \perp r\ l'$ iff mm′ = -1.

This condition is called the perpendicularity condition (or othogonality condition) for two straight lines.

Further, if the slope of a given line is m, then the slope of any line perpendicular to the given line is $-\frac{1}{m}$. (i.e. $m' = -\frac{1}{m}$).

Worked Examples

1. A straight line makes an angle θ with ox. In which case the slope (m) of the line is (a) +ve, (b)-ve, (c) zero, (d) undefined ?

   (i) θ = 135° (ii) θ = 22.5 (iii) θ = 90° (iv) θ = 150°

   (v) θ = 0.

Solution : (i) m < 0 (ii) m > 0 (iii) m undefined (iv) m < 0 (v) m = 0.

2. Find the slope of the line making θ with ox in each of the cases below.

(i) $\sin\theta = \frac{3}{5}$ $(0^\circ < \theta < 90^\circ)$

(ii) $\sec\theta = -\frac{13}{12}$

(iii) $\cot\theta = -1$

(iv) $\text{Cosec}\,\theta = \sqrt{2}$ $(90^\circ < \theta < 180^\circ)$

Solution : (i) m = 3/4; (ii) $m = -\frac{5}{12}$ (iii) m = -1 (iv) m = -1.

3. The slope of a line is a root of the equation $x^2 = \frac{2}{\sqrt{3}}x - 1 = 0$. Find the angle(s) made by the line with ox.

Solution : If m is the slope of the line, then $m^2 - \frac{2}{\sqrt{3}}m - 1 = 0$.

$$\Rightarrow m^2 - \frac{2}{\sqrt{3}}m + \frac{1}{3} - 1 - \frac{1}{3} = 0$$

$$\Rightarrow \left(m - \frac{1}{\sqrt{3}}\right)^2 = \frac{4}{3} \qquad \therefore\ m - \frac{1}{\sqrt{3}} = \pm\frac{2}{\sqrt{3}}$$

$$\therefore\ m = \frac{1 \pm 2}{\sqrt{3}} \text{ or } m = \sqrt{3} \text{ or } -\frac{1}{\sqrt{3}}.$$

If θ is the angle made by the line with ox, then

$\therefore$ Tan θ = $\sqrt{3}$ or Tan θ = $-\frac{1}{\sqrt{3}}$

$\therefore$ θ = 60° or θ = 150°.

4. m and m′ are the slopes of two lines. Find m′ in each case below.

(i) The lines are parallel and $m = \frac{1}{2}$.

(ii) The lines are ⊥r and $m = \frac{1}{2}$.

(iii) The line with slope m is parallel to the y-axis.

(iv) The lines are ⊥r and m = -1.

(v) $m = \text{Tan}\ \theta$ where $\text{Cos}\ \theta = -\dfrac{1}{2}$ and the lines are parallel.

(vi) $m = \text{Tan}\ \theta$ where $\cos\theta = -\dfrac{1}{2}$ and the lines are $\perp$r.

Solution : (i) $m' = ½$ (ii) $m' = -2$ (iii) $m' = 0$

(iv) $m' = 1$ (v) $m' = -\sqrt{3}$ (vi) $m' = \dfrac{1}{\sqrt{3}}$.

5. Find k so that (2k + 1) and (k – 1) are the slopes of two (a) parallel lines, (b) perpendicular lines.

Solution :

(a) For Parallel lines : $2k + 1 = k - 1 \quad \therefore\ k = -2$.

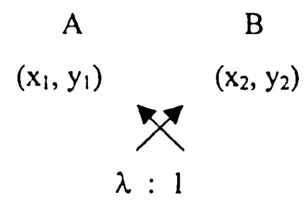
(b) For Perpendicular lines: $(2k + 1)(k - 1) = -1 \Rightarrow 2k^2 - k = 0$ or $k = 0, ½$

2.2 Straight lines – Various forms of equations of a straight line

The equation to a straight line in the xy-plane is a first degree equation in x and y. It is of the form : $ax + by + c = 0$, where a, b, c are constants (not both a, b zero). The equation is called *the general equation* of a straight line and hence called a *linear equation in x and y.*

Every form of the equation of a straight line can be brought to this form. *We first prove that $ax + by + c = 0$* represents a straight line always.

In order to show that $ax + by\ c = 0$ (1) represents a straight line always, we assume that two points $A(x_1, y_1)$ and $B(x_2, y_2)$ lie on (1) and prove that every point on the line AB also lies on (1). If $C(x_3, y_3)$ is any point on the line AB, when AC : CB = $\lambda : 1$ then $C(x_3, y_3) = \left[\dfrac{x_1 + \lambda x_2}{1 + \lambda}, \dfrac{y_1 + \lambda\, y_2}{1 = \lambda}\right]$ (2)



Since A and B lie on (1)

$ax_1 + by_1 + c = 0$ and

$ax_2 + by_2 + c = 0 \quad \times \lambda$

Add : $a(x_1 + \lambda x_2) + b(y_1 + \lambda y_2) + (1 +) = 0$

$$\therefore \quad a\left(\frac{x_1 + \lambda\, x_2}{1 + \lambda}\right) + b\left(\frac{y_1 + \lambda\, y_2}{1 + \lambda}\right) + c = 0$$

$\Rightarrow ax_3 + by_3 + c = 0$ (From (2))

$\therefore$ $C(x_3, y_3)$ also lies on $ax + by + c = 0$.

Thus the collinear points A, B, C lie on (1).

Hence (1) represents a straight line always.

Slope of the line ax + by + c = 0

Let m = Tan θ be the slope of the line $ax + by + c = 0$

Then $\text{Tan}\,(180° - \theta) = \dfrac{OB}{OA}$. (i)

$\Rightarrow -\text{Tan}\,\theta = \dfrac{OB}{OA}$.

$\Rightarrow -m = \dfrac{OB}{OA}$ (ii)

Putting y = 0, in (i) x = -c/a = OA
Putting x = 0, in (i) y = - c/b = OB
} (iii)

$\therefore$ From (ii) and (iii) $-m = \dfrac{-c/b}{-c/a}$

$$\Rightarrow m = -a/b = -\left(\frac{\textit{The coefficient of } x}{\textit{The coefficient of } y}\right)$$

Other forms of the equation to a straight line

a) **The slope intercept form** : $y = mx + c$

where m = the slope of the straight line and c = the intercept of the line.

b) **The intercept form** : $\frac{x}{a} + \frac{y}{b} = 1$ (1)

where a = the x-intercept and

b = the y-intercept of the line.

c) **The normal form** : $x \cos \alpha + y \sin \alpha = p$ (2)

where p = the perpendicular distance of the line from the origin and α is the angle made by the perpendicular to the line with ox.

d) **The point-Slope form** : $(y - y_1) = m(x - x_1)$ (3)

m being the slope of the line and (x_1, y_1) is a point on the line.

e) **The two-point form** : $\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$ (4)

where (x_1, y_1) and (x_2, y_2) are any two points on the straight line.

f) **the Parametric – form (or symmetric form)**

$$\frac{x - x_1}{Cos\ \theta} = \frac{y - y_1}{Sin\ \theta} \quad (5)$$

where m = the slope = Tan θ and (x_1, y_1) is a point on the straight line.

Equivalently, $x = x_1 + r \cos \theta,\ y = y_1 + r \sin \theta \quad (-\infty < r < +\infty)$. (5′)

Other Results

Given the straight line as $ax + by + c = 0$.

a) m = The slope of the line = $-\frac{a}{b}$.

b) The x and the y-intercepts : $-\frac{c}{a}, -\frac{c}{b}$ respectively (got by putting y = 0 and x = 0 in the equation.

c) p = The (⊥r) distance of the line from O = $\frac{|c|}{\sqrt{a^2 + b^2}}$.

d) If (x_1, y_1), (x_2, y_2) are any two points on a straight line, then the slope of the line = m = $\frac{y_2 - y_1}{x_2 - x_1}$.

e) Perpendicular distance from (x_1, y_1) to the line ax + by + c = 0.

Shifting the origin to (x_1, y_1), x = X + x, y = Y + y, and the equation becomes
$a(X + n_1) + b(Y + y_1) + c = 0$ or $aX + bY + (aX_1 + bY_1 + c) = 0$

or aX + bY + k = 0.

∴ The ⊥r distance from (x_1, y_1) (new origin) to the line

$$p = \frac{|k|}{\sqrt{a^2 + b^2}} = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}.$$

$$\text{Or } p = \frac{|ax_1 + by_1 + c)}{\sqrt{a^2 + b^2}}.$$

f) Equations of straight lines parallel to the coordinate axes.

In ax + by + c = 0,

i) If a = 0, the equation reduces to by + c = 0 or y = $-\frac{c}{b}$ = k (say) and m = $-\frac{a}{b} = 0$ which is a line parallel to the x-axis.

∴ Any line parallel to the x-axis (horizontal line) has its equation in the form y = k.

ii) If b = 0, the equation reduces to ax + c = 0 or x = $-\frac{c}{a} = K'$ (say) which is a line parallel to the y-axis.

∴ Any line parallel to the y-axis (vertical line) has its equation in the form x=K.

g) The angle of intersection of two straight lines

Let $m_1 = \text{Tan } \theta_1$, $m_2 = \text{Tan } \theta_2$ be the slopes of two straight lines intersecting at an angle θ at P (See the figure). Then $\theta_1 = \theta + \theta_2$ or $\theta = \theta_1 - \theta_2$.

$\therefore \quad \text{Tan } \theta = \text{Tan } (\theta_1 - \theta_2)$

$$= \frac{Tan\ \theta_1 - Tan\ \theta_2}{1 + Tan\ \theta_1 \theta_2}$$

$$\therefore \quad \text{Tan}\theta = \frac{m_1 - m_2}{1 + m_1 m_2}.$$

For acute angle θ, $\text{Tan } \theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$.

Note: If the equations of the lines are $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, then

$m_1 = -\frac{a_1}{b_1}$, $m_2 = -\frac{a_2}{b_2}$ and using the above formula, θ is got.

h) Equations of $||^{le}$ and $\perp r$ lines

Given a straight line $ax + by + c = 0$

(i) Any $||^{le}$ line to it may be taken by the equation $ax + by + k = 0$.

(ii) Any $\perp r$ line to it may be taken by the equation $bx - ay + k' = 0$.

(i) **Distance between two parallel straight lines**

Let the $||^{le}$ straight lines be

$ax + by + c_1 = 0$ (1)

and $ax + by + c_2 = 0$ (2)

If d = the distance between the lines, then taking a point $P(x_1, y_1)$ on one of the lines, say (1) and drawing PQ $\perp r$ to the lines then PQ = d.

$\therefore d = PQ = \frac{| ax_1 + by_1 + c_2 |}{\sqrt{a^2 + b^2}}$. But $ax_1 + by_1 + c_1 = 0$.

$$\therefore \quad d = \frac{| -c_1 + c_2 |}{\sqrt{a^2 + b^2}} = \frac{| c_1 - c_2 |}{\sqrt{a^2 + b^2}}$$

$$\therefore \; d = \frac{|c_1 - c_2|}{\sqrt{a^2 + b^2}}$$

j) Point of intersection of two straight lines

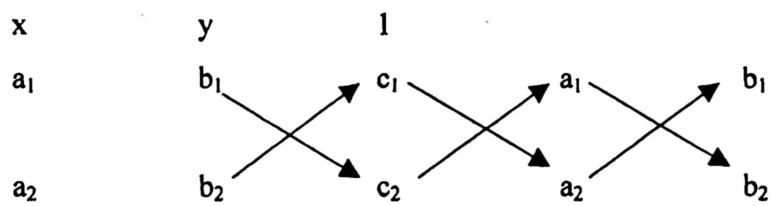
To get the point of intersection of two intersecting straight lines :

$a_1x + b_1y + c_1 = 0$ and (1)

$a_2x + b_2y + c_2 = 0$ (2)

the equations are solved for x and y. The values of x and y so got are the coordinates of the point of intersection.

Cross Multiplication Rule for solving (1) and (2)



$$\therefore \; \frac{x}{b_1 c_2 - b_2 c_1} = \frac{y}{c_1 a_2 - c_2 a_1} = \frac{1}{a_1 b_2 - a_2 b_1}.$$

$$\therefore \; P(x,y) = \left[\frac{b_1 c_2 - b_2 c_1}{a_1 b_2 - a_2 b_1}, \frac{c_1 a_2 - c_2 a_1}{a_1 b_2 - a_2 b_1}\right].$$

k) Concurrency condition for three straight lines: $a_i x + b_i j + c_i = 0$, $i = 1,2,3$

When the given lines are concurrent, they pass through the same point.

$a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$

intersect at $P\left[\frac{b_1 c_2 - b_2 c_1}{a_1 b_2 - a_2 b_1}, \frac{c_1 a_2 - c_2 a_1}{a_1 b_2 - a_2 b_1}\right]$

P lies on $a_3 x + b_3 y + c_3 = 0$

$$\therefore \; a_3 \left(\frac{b_1 c_2 - b_2 c_1}{a_1 b_2 - a_2 b_1}\right) + b_3 \left(\frac{c_1 a_2 - c_2 a_1}{a_1 b_2 - a_2 b_1}\right) + c_3 = 0$$

$\Rightarrow a_3 (b_1c_2 - b_2 c_1) + b_3 (c_1a_2 - c_2 a_1) + c_3 (a_1b_2 - a_2b_1) = 0.$

Hence the concurrency condition for the three lines is

$$\sum(a_1\, b_2 - a_2\, b_1)\; c_3 = 0$$

l) The sides of a straight line in the Cartesian Plan :

A straight line *l* in the Cartesian plane divides the plane into two regions (non overlapping) called the half-planes with the line as common boundary.

The origin side and the non-origin side of a straight line in the plane

If the line *l* does not pass through the origin, then one-half plane contains the origin. That half-plane is called *the origin side* of the line *l*. The other half plane is called the *non-origin side* of the line *l*.

Given two points $A(x_1,y_1)$ *and* $B(x_2, y_2)$ *and a line* $l : ax + by + c = 0$ *not passing through either of the points to find whether A and B belong to the same side (i.e. same half plane) of l or to the opposite sides of l.*

Let AB intersect *l* at C(x,y).

Taking AC : CB = λ : 1

$$C = (x,y) = \left(\frac{x_1 + \lambda\, x_2}{1+\lambda}\, ,\; \frac{y_1 + \lambda\, y_2}{1+\lambda}\right).$$

But C lies on $l : ax + by + c = 0$.

$$\therefore\; a\left(\frac{x_1 + \lambda\, x_2}{1+\lambda}\right) + b\left(\frac{y_1 + \lambda\, y_2}{1+\lambda}\right) + c = 0$$

$$\Rightarrow \lambda = -\left(\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c}\right) \qquad \text{(i)}$$

Case 1 : Where A and B are an opposite sides of *l*, C divides $\overline{AB}$ *internally* in the ratio λ : 1 . Hence λ > 0. (Fig. 1)

∴ From (i) $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ must have *opposite signs.*

Case 2 : When A and B are *on the same side of l*, C divides $\overline{AB}$ *externally* in the ratio λ : 1. Hence λ < 0. (Fig. 2)

∴ From (i) $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ must have the *same sign.*

Summary : Given a line $l : ax + by + c = 0$ and points $A(x_1, y_1)$, $B(x_2, y_2)$

Sign of $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$	**Conclusion**
1. Same sign.	A and B on the *same side of l.*
2. Opposite sign.	A and B on the *opposite side of l.*
3. $ax_1 + by_1 + c = 0$ $ax_2 + by_2 + c = 0$	A lies on l. B lies on l.

A Special Case : Condition that a point $A(x_1, y_1)$ lies on the origin side of the line $l : ax + by + c = 0$, not passing through the origin ($c \neq 0$).

Taking $B = (0,0)$ (i.e. $x_2 = 0, y_2 = 0$)

$A(x_1, y_1)$ belongs to the origin side or does not belong to the origin side according as $ax_1 + by_1 + c$ has *the same sign as C or opposite sign of C.*

Summary :

Sign of $ax_1 + by_1 + c$	**Conclusion**
1. Same as c.	A on the origin side of the line.
2. Opposite to c.	A on the non-origin side of the line.

(m) Angle bisectors of two intersecting lines $l_1 : a_1x + b_1y + c_1 = 0$ and

$l_2 = a_2x + b_2y + c_2 = 0$.

An angle bisector being the locus of a point $P(x_1, y_1)$ equidistant from l_1 and l_2

PM = PN

$$\Rightarrow \frac{|a_1 x_1 + b_1 y_1 + c_1|}{\sqrt{a_1^2 + b_1^2}} = \frac{|a_{21} x_1 + b_2 y + c_2|}{\sqrt{a_2^2 + b_2^2}}.$$

$$\Rightarrow \left(\frac{a_1 x_1 + b_1 y + c_1}{\sqrt{a_1^2 + b_1^2}}\right) = \left(\frac{a_2 x_1 + b_2 y_1 + c_2}{\sqrt{a_2^2 + b_2^2}}\right)$$

$\therefore$ The locus of $P(x_1, y_1)$ is

$$\left(\frac{a_1 x + b_1 y + c_1 |}{\sqrt{a_1^2 + b_1^2}}\right) = \pm \left(\frac{a_2 x + b_2 y + c_1 |}{\sqrt{a_2^2 + b_2^2}}\right)$$

These are the equations of the angle bisectors of the given lines.

(n) Family of Concurrent Lines

Given two intersecting lines $L_1 : a_1 x + b_1 y + c_1 = 0$ and

$L_2 : a_2 x + b_2 y + c_2 = 0$

To find a line through the point of intersection of L_1 and L_2.

Consider,

$L_1 + kL_2 = 0$ .... (1)

$\Rightarrow (a_1x + b_1y + c_1) + k(a_2x + b_2y + c_2) = 0$

$\Rightarrow (a_1 + ka_2) x + (b_1 + kb_2) y + (c_1 + kc_2) = 0$

is a straight line (because it is a linear equation).

If $P(x_1, y_1)$ is the point of intersection of L_1 and L_2, then

$a_1x_1 + b_1y_1 + c_1 = 0 = a_2x_1 + b_2y_1 + c_2$

$\therefore$ (1) passes through P.

Therefore, (1) represents a line through the

Intersection of $L_1 = 0$ and $L_2 = 0$.

Tip : This idea ($L_1 + k L_2 = 0$ represents a line through the intersection of $L_1=0$ and $L_2 = 0$ is very useful in problems when a line through the equation of two given lines L_1 and L_2 is to be found out. To find such a line, it is not necessary to solve $L_1 = 0$ and $L_2 = 0$ for the point of intersection of the lines.

Family of lines through the point of intersection P of $L_1 = 0$ and $L_2 = 0$.

Given $L_1 = 0$, $L_2 = 0$, the equation to every line through P is of the form $L_1 + kL_2 = 0$. As k takes all real values, all the lines through P are given by the equation. Hence given $L_1 = 0$, $L_2 = 0$, $L_1 + k L_2 = 0$ represents the set of all straight lines through intersection point of $L_1 = 0$ and $L_2 = 0$. The set of these lines is called the family of concurrent lines through the intersection of $L_1 = 0$ and $L_2 = 0$. The family of lines is given by $L_1 + kL_2 = 0$ and k is called the parameter of the family of concurrent lines given by $L_1 + kL_2 = 0$.

Worked Examples :

1. Find the equation to the straight line at a distance 5 from the origin and having the slope ¾.

Solution : Let the equation of the line be $ax + by + c = 0$. (1)

The slope is $-\frac{a}{b} = \frac{3}{4}$ $\therefore -4a = 3b$

Or $\frac{a}{-3} = \frac{b}{4} = k$ (say)

$\therefore$ $a = -3k$, $b = 4k$

$\therefore$ $(1) \Rightarrow -3kx + 4ky + c = 0$.

Or $3x - 4y - \frac{c}{k} = 0$ or $3x - 4y + d = 0$, where $d = -\frac{c}{k}$. (2)

The distance of the line from the origin = 5

$\therefore \frac{|d|}{\sqrt{3^2 + 4^2}} = 5$ $\therefore |d| = 25$

$\therefore$ $d = \pm 25$.

Hence the equations of the lines are $3x - 4y \pm 25 = 0$.

2. Find the equation of the line whose intercepts on the axes are positive and are 2:3 and passing through (2,3).

Solution: Take the equation of the line as $\frac{x}{a} + \frac{y}{b} = 1$.

$a : b = 2 : 3$ $\therefore$ Taking $a = 2k$, $b = 3k$

the equation of the line is $\frac{x}{2k} + \frac{y}{3k} = 1$

or 3x + 2y = 6k

The line passes through (2,3). $\therefore$ 6 + 6 = 6k or k = 2.

Hence the equation of the line is 3x + 2y = 12.

3. Find the equation of the line through A(1,5) and B(2,4). In What ratio is the segment $\overline{AB}$ divided by 4x + 3y = 5 ?

Solution : The equation to AB : $\frac{y-5}{x-1} = \frac{4-5}{2-1}$

$\Rightarrow \frac{y-5}{x-1} = -1 \Rightarrow y-5 = 1-x$

or x + y = 6

If c is the point of intersection of AB and the given line

Solving 4x + 3y = 5 and

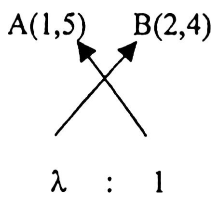
x + y = 6 ,

we get, x = -13

$\therefore$ y = 6 – x = 6 + 13 = 19

$\therefore$ $\therefore$ C = (-13, 19)

Let AC : CB = λ : 1



Then c = $\left[\frac{1+2\lambda}{1+\lambda}, \frac{5+4\lambda}{1+\lambda}\right]$ = (-13, 19)

$\therefore \frac{1+2\lambda}{1+\lambda} = -13; \quad \frac{5+4\lambda}{1+\lambda} = 19$

$\therefore$ 1 + 2λ = -13 - 13λ

Check: $\frac{5 - 4\left(\frac{14}{15}\right)}{1 - \frac{14}{15}} = \frac{75-56}{1} = 19$

$\therefore$ 15λ = -14

$\therefore$ $\overline{AB}$ is divided by the given line in

$\therefore \lambda = -\frac{14}{15}$ the ratio 14 : 15 *externally.*

4. The distance of P(a,b) from the line $\frac{x}{a} + \frac{y}{b} = 1$ is p. Show that $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{p^2}$. Also write down the equation of the ⊥r to the given line through P(a,b).

Solution : $p = \frac{\left|\frac{a}{b} + \frac{b}{a} - 1\right|}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}}$

$\therefore p = \frac{|1|}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}}$ $\therefore \frac{|1|}{p} = \sqrt{\frac{1}{a^2} + \frac{1}{b^2}}$.

Squaring $\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$.

The equation to any ⊥r line to $\frac{x}{a} + \frac{y}{b} = 1$ is of the form $\frac{x}{b} - \frac{y}{a} = k$.

The line passes through (a,b)

$\therefore \frac{a}{b} - \frac{b}{a} = k$.

$\therefore$ The equation of the perpendicular is $\frac{x}{b} - \frac{y}{a} = \frac{a}{b} - \frac{b}{a}$

or $\left(\frac{x-a}{b}\right) - \left(\frac{y-b}{a}\right) = 0$ is the required line.

5. Given the lines 6x + 3y – 2 = 0 and (1 + k) x + ky = 0 find k if (a) the lines are parallel, (b) the lines are perpendicular. Also find the distance between the lines, if they are parallel.

(a) Consider $L_1 : 6x + 3y - 2 = 0$

$L_2 : (1 + k) x + ky = 0$

If $L_1 \parallel L_2$, then $\frac{6}{1+k} = \frac{3}{k}$

$\Rightarrow 2k = 1 + k$

$\therefore \quad k = 1$

(b) If $L_1 \perp L_2$, then $-\frac{6}{3} \times -\frac{1+k}{k} = -1$

$$\therefore \quad \frac{2(1+k)}{k} = -1$$

$$\therefore \quad 2 + 2k = -k \qquad \therefore \; -3k = 2$$

$$\therefore \quad k = -\frac{2}{3}.$$

In (a) if the lines are parallel, then their equations are

$6x + 3y - 2 = 0$ and

$2x + y = 0$

Rewriting the equations $6x + 3y - 2 = 0$ and

$$6x + 3y = 0$$

$$\therefore d = \frac{|-2-0|}{\sqrt{6^2 + 3^2}}.$$

$$= \frac{2}{3\sqrt{10}} = \frac{1}{3}\sqrt{\frac{2}{5}}$$

$\therefore$ Distance between the $||^{le}$ lines $= \frac{1}{3}\sqrt{\frac{2}{5}}$.

6. Find the equations of the altitudes of the triangle whose vertices are A(1,1), B(2, -2) and C(-1,0). Write down the coordinates of the orthocentre of the triangle.

Solution :

Equation of AD : Slope of BC $= \frac{0-2}{-1-2} = -\frac{2}{3}$

$\therefore$ Slope of AD $= \frac{3}{2}$.

$\therefore$ AD : $(y-1) = \frac{3}{2}(x-1)$

$\Rightarrow 2y - 2 = 3x - 3$ or AD : $3x - 2y - 1 = 0$ (1)

BE : Slope of AC $= \frac{1-0}{1+1} = \frac{1}{2}$

$\therefore$ Slope of BE = -2

Equation of BE : $\frac{y+2}{x-2} = -2$ or y + 2 = -2x + 4

Equation of BE : 2x + y – 2 = 0 (2)

OF : Slope of AB = $\frac{-2-1}{2-1} = -3$

$\therefore$ Slope of CF = $\frac{1}{3}$

Equation of CF : $(y - 0) = \frac{1}{3}(x+1)$

$\Rightarrow$ 3y = x + 1 or CF : x - 3y + 1 = 0 (3)

For othocentre of the triangle, solve any two of the equations (1), (2) and (3).

Consider : 3x – 2y – 1 = 0

and 2x + y – 2 = 0

Solving, $x = \frac{5}{7}$.

: $\frac{10}{7} + y - 2 = 0$ $\therefore y = 2 - \frac{10}{7} = \frac{4}{7}$.

$\therefore y = \frac{4}{7}$. $\therefore$ The orthocentre of the triangle = $\left(\frac{5}{6}, \frac{4}{7}\right)$.

7. Find the equation to the line through the intersection of 2x + 3y – 5 = 0 and 3x + 2y + 4 = 0 and (a) $||^{le}$ to x + y + 1 = 0, (b) $\perp$r to x – 2y + 5 = 0.

Solution : L_1 = 2x + 3y – 5 = 0

L_2 : 3x + 2y + 4 = 0

Any line through the intersection of L_1 and L_2 may be taken as $L_1 + kL_2 = 0$.

i.e. (2x + 3y – 5) + k(3x + 2y + 4) = 0

$\Rightarrow$ (2 + 3k) x + (3 + 2k) y + (4k – 5) = 0 (1)

The slope of (1) = $-\left(\dfrac{2+3k}{3+2k}\right)$ (2)

(a) If (1) is $||^{le}$ to x + y + 1 = 0

then, $-\dfrac{2+3k}{3+2k} = -\dfrac{1}{1}$

$\Rightarrow$ 2 + 3k = 3 + 2k

$\therefore$ k = 1

$\therefore$ 5x + 5y – 1 = 0 is the required $||^{le}$ line to x+y+1 = 0.

(b) If (1) is ⊥r to x-2y+5 = 0

then $-\left(\dfrac{2+3k}{3+2k}\right) \times \dfrac{1}{-2} = -1$

$\Rightarrow \dfrac{2+3k}{2(3+2k)} = +1 \Rightarrow 2+3k = 6+4k$

$\therefore$ k = -4

$\therefore$ (1) $\Rightarrow$ (2 + 3 × -4) x + (3 + 2 × -4) y + 4(-4) – 5 = 0

$\Rightarrow$ =10x – 5y – 21 = 0

or 10x + 5y + 21 = 0 is the required line ⊥r to x – 2y + 5 = 0.

8. Find the condition that the lines

ax + by + c = 0, bx + cy + a = 0 and cx+ay +b = 0 are concurrent.

Solution : Solving ax + by + c = 0

bx + cy + a = 0

Using Cross Multiplication Rule

x	y	1		
a	b	c	a	b
b	c	a	b	c

$$\frac{x}{ab-c^2} = \frac{y}{bc-a^2} = \frac{1}{ac-b^2}.$$

$\therefore \quad x = \dfrac{ab - c^2}{ac - b^2}$; $y = \dfrac{bc - a^2}{ac - b^2}$. For concurrency of the given lines

$\left(\dfrac{ab - c^2}{ac - b^2}, \dfrac{bc - a^2}{ac - b^2}\right)$ lies on cx + ay + b = 0.

$$\therefore \quad c\left(\frac{ab - c^2}{ac - b^2}\right) + a\left(\frac{bc - a^2}{ac - b^2}\right) + b = 0$$

$\Rightarrow abc - c^3 + abc - a^3 + bac - b^3 = 0$

$\Rightarrow a^3 + b^3 + c^3 = 3abc$ is the required condition.

9. Find the equations of the lines through (1,1) making 45° with 2x+y+3=0.
Hence find the areas of the Δ^{le} formed by the three lines.

Solution : Let m be the slope of one of the lines making 45° with the given line.
Slope of the given line = -2.

$$\text{Tan } 45° = \left|\frac{m + 2}{1 - 2m}\right|$$

$$\Rightarrow \left|\frac{m + 2}{1 - 2m}\right| = 1$$

$\Rightarrow m + 2 = \pm (1 - 2m)$

$\Rightarrow m + 2 = 1 - 2m$ or $m + 2 = 2m - 1$

$\Rightarrow 3m = -1$ or $m = 3$.

$\Rightarrow m = -\dfrac{1}{3}$, $m = 3$.

$\therefore$ The lines through (1,1) making 45° with the given line are $(y - 1) = -\dfrac{1}{3}(x - 1)$

and $(y - 1) = 3(x - 1)$.

$\Rightarrow \quad 3y - 3 = -x + 1$ and $3x - y - 2 = 0$.

$\Rightarrow \quad x + 3y - 4 = 0$ and $3x - y - 2 = 0$.

Consider $x + 3y - 4 = 0$ ...(1)

$3x - y - 2 = 0$ ...(2)

and $2x + y + 3 = 0$ ...(3)

Solving (1) and (3),

$$x + 3y - 4 = 0$$

$$2x + y + 3 = 0 \quad \times 3$$

Sub: $-5x - 13 = 0 \quad \therefore \; x = -\frac{13}{5}.$

$\therefore \; y = -3 - 2x$

$\therefore \; = -3 + \frac{26}{5} = \frac{11}{5}.$

$\therefore \; B = \left(\frac{-13}{5}, \frac{11}{5}\right)$

Solving (2) and (3)

$$3x - y - 2 = 0$$

$$2x + y + 3 = 0$$

$$5x + 1 = 0$$

$$-\frac{2}{5} + y \; 3 = 0$$

$\therefore \; y = \frac{2}{5} - 3 = -\frac{13}{5}$

$\therefore \; y = -\frac{13}{5}$

$\therefore \; C = \left(-\frac{1}{5}, -\frac{13}{5}\right)$

$\therefore \; BC = \sqrt{\left(-\frac{1}{5} + \frac{13}{5}\right)^2 + \left(-\frac{13}{5} - \frac{11}{5}\right)^2} = \frac{1}{5}\sqrt{144 + 576} = \frac{12}{5}\sqrt{5}.$

$\therefore \; BC = \frac{12}{5}\sqrt{5}.$

$\perp$r distance of BC from A = AD = $\dfrac{2.1+1+3}{\sqrt{4+1}} = \dfrac{6}{\sqrt{5}}$.

$\therefore$ Area Δ ABC = $\dfrac{1}{2}$ BC $\times$ AD = $\dfrac{1}{2}\,\dfrac{12}{5}\sqrt{5} \times \dfrac{6}{\sqrt{5}} = \dfrac{36}{5}$ (unit)^2.

10. Find the equations of the angle bisectors of the lines 2x + 3y − 5 = 0 and 3x+2y+15 = 0. Show that the angle bisectors are perpendicular to each other.

Solution :

The angle bisectors are

$$\frac{2x+3y-5}{\sqrt{2^2+3^2}} = \pm\frac{3x+2y+15}{\sqrt{3^2+2^2}}$$

$\Rightarrow$ (2x + 3y − 5) = ± (3x + 2y + 15)

$\Rightarrow$ 2x + 3y − 5 = 3x + 2y + 15 and 2x + 3y − 5 = -3x − 2y − 5.

$\Rightarrow$ x − y + 20 = 0 and 5x + 5y = 0

Or the angle bisectors are x − y + 20 = 0 and x + y = 0.

The slope of these angle bisectors are $-\dfrac{1}{-1} = 1$ and $\dfrac{-1}{1} = -1$ and $1 \times -1 = -1$

(Product of the slopes).

Hence the angle bisectors are $\perp$r to each other.

11. Show that the point A(4,3) and B(1,1) are on opposite sides of the line 4x − 3y − 6 = 0. Which point is on the origin side of the line ?

Denoting by f(x,y) = 4x − 3y − 6

f(4,3) = 4 × 4 − 3 × 3 − 6 = 16 − 15 = +1 (+ve)

f(1,1) = 4 × 1 − 3 × 1 − 6 = -5 = (-ve)

$\therefore$ A and B lie on opposite sides. Since f(1,1) and the constant term -6 both have the same sign (-) B lies on the origin side of the line.

Exercises (for Self valuation)

1. Find the lines through (-1, -5) (a) ||[le] to $2x + 3y - 5 = 0$,

   (b) ⊥r to $2x + 3y - 2 = 0$

   [(a) $2x+3y + 17 = 0$, (b) $3x - 2y - 7 = 0$)

2. Find k so that the line $3x + 4y + k = 0$ and the coordinate axes form a Δ[le] of area 24(unit)^2. [$k = \pm 24$]

3. P is the distance of $\frac{x}{a} + \frac{y}{b} = 1$ show that $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{p^2}$.

4. Find the ratio in which the line $7x - 6y + 1$ divides the segment joining (1,4) and (-2, -5). [16 : 17, internally]

5. Show that all the lines represented by $(2 + 6\lambda) x - (3 - 5\lambda) y + (2 - 8\lambda) = 0$ are concurrent. Find the point at which the lines are concurrent. [$\left(\frac{1}{2}, 1\right)$].

6. Show that the area of the Δ[le] formed by $y = m_1, x$, $y = m_2 x$ and $y = c$ is

   $$\frac{1}{2} c^2 \left(\frac{1}{m_1} - \frac{1}{m_2}\right).$$

7. Find the lines through (1,2) making 60° with $\sqrt{3}x + y - 2 = 0$.

   [$y = 2$ and $y = x\sqrt{3} + 2 - \sqrt{3}$] $y = \sqrt{3}x + 2 - \sqrt{3}$

8. Find the value of k so that the lines $3x + 4y - 5 = 0$, $2x + 3y - 4 = 0$ and $px + 4y - 6 = 0$ are concurrent. Find the point of concurrency also.

   [p = 2, (-1, 2)]

9. Find the values of a so that (2,3) and (-4, a) are equidistant from the line $3x + 4y = 8$. [$a = \frac{5}{2}, \frac{15}{2}$]

10. Find the angle bisector of the angle between the lines $4x + y - 7 = 0$ and $x - 4y + 3=0$, containing the origin. [$5x - 3y - 4=0$]

3. Pair of Straight Lines

The product of two linear equations (each representing a line) $a_1x + b_1y\ \ c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ is a second degree equation in x and y of the form

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Therefore, under certain conditions, the equation (1) represents a pair (two) of straight lines.

In what follows, we recall all the important results concerning a pair of straight lines.

Let $F(x,y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ (1)

This equation is called general equation of second degree in x and y (a,h,b,g,f,c are real constants). The equation $ax^2 + 2hxy + by^2 = 0$ (2) is called the corresponding homogenous equation of (1) in x and y.

Conditions and Results

1. The general equation (1) represents a pair of straight lines if

   $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

2. The homogeneous equation (2) represents a pair of straight lines through the origin always.

3. If m_1 and m_2 are the slopes of the lines given by (1) or (2) then

   $$m_1 + m_2 = -\frac{2h}{b}$$

   $$m_1\, m_2 = \frac{a}{b}.$$

4. If (1) represents a pair of straight lines, then these lines are parallel to the lines given by (2),.

5. The angle between the lines (1) and (2) is given by $\tan\theta = \dfrac{2\sqrt{h^2 - ab}}{a+b}$.

6. If (1) represents a pair of lines, then the lines are $||^{le}$ if $h^2 = ab$.

Then for the condition for (1) to be a pair of ||le line is $\Delta = 0$ and $h^2 = ab$.

7. The lines (1) and (2) are perpendicular, if $a + b = 0$.

8. If the equation (1) represents intersecting lines, intersecting at P then

$$P = \left[\frac{hf - bg}{ab - h^2}, \frac{hg - af}{ab - h^2}\right].$$

9. If the lines are $||^{le}$, then the distance between the lines is

$$d = 2\sqrt{\frac{g^2 - ac}{a(a+b)}} \ .$$

Tips to Teachers :

1. When the general equation (1) represents a pair of straight lines, to find the separate equations of the lines, factorise the corresponding homogeneous equations into two linear equations : $y = m_1x$, $y = m_2x$ (say). Then the separate equations of the lines (1) may be taken as $y= m_1x + c_1$ and $y = m_2x + c_2$ and c_1, c_2 found.
2. Emphasis that the general equation does not represent a pair of lines always but only when the condition $\Delta = 0$ holds. In general, it represents a conic.
3. The proofs of the results recalled are proved in any PUC Maths textbook and hence can be studied.

Worked Examples :

1. Find k so that $x^2 - 5xy + 4y^2 + x + 2y + k = 0$ represents a pair of straight lines. Find the separate equations of the straight lines and their point of intersection.

Solution : Comparing the given equation with the standard general equation :

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

$a = 1, h = - 5/2, b = 4, g = ½, \ f= 1, c = k.$

The condition for pair of lines is $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

$$\Rightarrow \quad 1 . 4 . k + 2 . 1 . ½ . (-\frac{5}{2}) - 1 . 1^2 - 4 . ¼ - k \frac{25}{4} = 0.$$

$$\Rightarrow \quad 4k - \frac{5}{2} - 1 - 1 - \frac{25}{4}k = 0. \qquad [\ \times 4$$

$$\Rightarrow \quad 16k - 10 - 8 - 25k = 0 \qquad \therefore 9k = -18$$

$\therefore \ k = -2$

The equation becomes $x^2 - 5xy + 4y^2 + x + 2y - 2 = 0 \qquad \ldots \qquad (1)$

The lines are parallel to the lines given by the corresponding homogeneous equations

$x^2 - 5xy + 4y^2 = 0$

$\Rightarrow (x - y)(x - 4y) = 0$ or $x - y = 0$ and $x - 4y = 0$.

$\therefore$ The given lines are

$$x - y + c_1 = 0 \text{ and} \qquad (2)$$
$$x - 4y + c_2 = 0$$

$\therefore$ The combined equation of the pair of lines is

$(x - y + c_1)(x - 4y + c_2) = 0$

$\Rightarrow x^2 - 5xy + 4y^2 + (c_1 + c_2)x - (4c_1 + c_2)y + c_1c_2 = 0 \qquad (3)$

Comparing (1) and (3)

$c_1 + c_2 = 1$ and $4c_1 + c_2 = -2$, $c_1c_2 = -2$

Solving :

$c_1 + c_2 = 1$ and $4c_1 + c_2 = -2$, $c_1c_2 = -2$

Solving : $c_1 + c_1 = 1$

$4c_1 + c_2 = -2$

Sub: $-3c_1 = 3 \quad \therefore c_1 = -1$

$\therefore c_2 = 2$ Check : $c_1c_2 = -1 \times 2 = -2$

$\therefore$ From (2), the separate equations of the lines are

$x - y - 1 = 0$

and $x - 4y + 2 = 0 \qquad (4)$

From the point of intersection of the lines, equation (4) are solved.

From (4) $x = y + 1$

$\therefore x - 4y + 2 = 0 \Rightarrow y + 1 - 4y + 2 = 0$

$\Rightarrow -3y + 3 = 0 \qquad \therefore y = 1,\ x = 1 + 1 = 2$

$\therefore$ The point of intersection of the lines is (2,1).

2. Show that $x^2 + 6xy + 9y^2 + 4x + 12y - 5 = 0$ represents a pair of parallel lines. Find the distance between these parallel lines.

Solution : Here $a = 1$, $2h = 6$, $b = 9$, $2g = 4$, $2f = 12$, and $c = -5$.

$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0.$

$\Rightarrow \quad 1 . 9 . (-5) + 2 . 6 . 2 . 3 - 1.36 - 9.4 + 5.9 = 0.$

$\Rightarrow \quad -45 + 72 - 72 + 45 = 0$

Also, $h^2 = ab$

$\Rightarrow 3^2 = 1.9 \Rightarrow 9 = 9.$

$\therefore$ The equation represents a pair of parallel lines.

I Method :

Each of these lines is parallel to the line (coincident) given by

$x^2 + 6xy + 9y^2 = 0$

$\Rightarrow \quad (x + 3y) = 0$ or $x + 3y = 0.$

$\therefore$ Let the parallel lines be $x + 3y + c_1 = 0$ and $x + 3y + c_2 = 0$

$\therefore$ The combined equation is $(x + 3y + c_1)(x + 3y + c_2) = 0.$

$\Rightarrow x^2 + 6xy + 9y^2 + (c_1 + c_2)x + 3(c_1 + c_2)y + c_1c_2 = 0.$

Comparing this equation with the given equation, we get,

$c_1 + c_2 = 4; \quad 3(c_1 + c_2) = 12, \quad c_1c_2 = -5$

$\therefore \quad c_1 = 5, c_2 = -1.$

$\therefore$ the lines are $x + 3y + 5 = 0$

and $x + 3y - 1 = 0$

$\therefore$ the distance between the $||^{le}$ lines $= \dfrac{|c_1 - c_2|}{\sqrt{9^2 + 3^2}} = \dfrac{|5 + 1|}{\sqrt{10}} = \dfrac{6}{\sqrt{10}}.$

$\therefore$ The distance between the lines $= \dfrac{6}{\sqrt{10}}$ units.

II Method :

Distance between the two parallel lines

$$d = 2\sqrt{\frac{g^2 - ac}{a(a+b)}}$$

$$= 2\sqrt{\frac{(2)^2 + 5}{1(1+9)}}$$

$$= 2\sqrt{\frac{9}{10}}$$

$$= \frac{2 \times 3}{\sqrt{10}}$$

$$= \frac{6}{\sqrt{10}} \text{ units.}$$

3. Find the equation of the angle bisectors of the lines $ax^2 + 2bxy + by^2 = 0$.

Solution : Let the separate lines make angles α and β with ox and an angle bisector make θ with ox. Let P(x,y) be a point on the angle bisector.

Then $A\hat{O}P = P\hat{O}B$.

$\Rightarrow \alpha - \theta = \theta - \beta \Rightarrow 2\theta = \alpha + \beta$.

$\therefore \ \text{Tan}(2\theta) = \text{Tan}(\alpha + \beta)$

$$\Rightarrow \frac{2\, Tan\, \theta}{1 - Tan^2\, \theta} = \frac{Tan\, \alpha + Tan\, \beta}{1 - Tan\, \alpha\;\; Tan\, \beta}$$

$$\Rightarrow \frac{2\,(y/x)}{1-(y/x)^2} = \frac{m_1 + m_2}{1 - m_1\, m_2} \quad \text{Since } m_1 = \text{Tan}\,\alpha, \;\; m_2 = \text{Tan}\,\beta$$

$$\Rightarrow \frac{2\,xy}{x^2 - y^2} = \frac{m_1 + m_2}{1 - m_1\, m_2}. \text{ But } m_1 + m_2 = -\frac{2h}{b} \text{ and } m_1 m_2 = \frac{a}{b}.$$

$$\therefore \frac{2xy}{x^2 - y^2} = \frac{-2h/b}{1 - a^2/b^2}$$

$$\Rightarrow \frac{xy}{x^2 - y^2} = \frac{ab}{a^2 - b^2}$$

$$\Rightarrow \frac{x^2 - y^2}{a - b} = \frac{xy}{h}$$ is the equation of the angle of bisectors of the given lines.

4. Find the equations of the lines through the origin, perpendicular to the lines $ax^2 + 2hxy + by^2$.

Solution: Let the slope of the lines given by $ax^2 + 2hxy + by^2 = 0$ (1)

be m_1 and m_2. Then $m_1 + m_2 = -\frac{2h}{b}$ and $m_1 m_2 = \frac{a}{b}$ (2)

The slopes of the lines ⊥r to the given lines are

$-\dfrac{1}{m_1}$ and $-\dfrac{1}{m_2}$ $\left[\because\ m_1 \times -\dfrac{1}{m_1} = -1 = m_2 \times -\dfrac{1}{m_2}\right]$

$\therefore$ The separate equations of these lines are

$$y = -\frac{1}{m_1}x \quad \text{and } y = -\frac{1}{m_2}x$$

$\Rightarrow$ $x + m_1y = 0$ and $x + m_2y = 0$

$\therefore$ The combined equation of these lines is

$(x + m_1y)(x + m_2y) = 0$

or $x^2 + (m_1 + m_2)xy + (m_1 m_2)y^2 = 0$

$\Rightarrow$ $x^2 + \left(-\dfrac{2h}{b}\right)xy + \left(\dfrac{a}{b}\right)y^2 = 0$ from (2)

$\Rightarrow$ $bx^2 - 2hxy + ay^2 = 0$ is the equation of the lines through (0,0) $\perp$r to the lines given by (1).

5. Find the acute angle between the lines $2x^2 + 5xy + 3y^2 + 6x + 7y + 4 = 0$ and the equations of the lines through (-1, 1) parallel to the given lines.

Solution: The angle 'θ' between the given lines is given by

$$\text{Tan } \theta = \left| 2\frac{\sqrt{h^2 - ab}}{a+b} \right|$$

Here $a = 2$, $h = \dfrac{5}{2}$, $b = 3$

$$\therefore \text{ Tan } \theta = \left| 2\frac{\sqrt{\frac{25}{4} - 2 \times 3}}{2+3} \right| = \frac{1}{5}$$

$$\therefore \theta = \text{Tan}^{-1}\left(\frac{1}{5}\right).$$

The lines are parallel to the lines $2x^2 + 5xy + 3y^2 = 0 \Rightarrow (2x + 3y)(x + y) = 0$.

Hence the separate equations of the required lines are

$2x + 3y + c_1 = 0$ and

$x + y + c_2 = 0$

These pass through (-1, 1)

$\therefore\ 2(-1) + 3(1) + c_1 = 0$

$\therefore\ c_1 = -1$

and $-1 + 1 + c_2 = 0$ $\therefore c_2 = 0$

$\therefore$ the equation of the lines is

$(2x + 3y - 1)(x + y) = 0$

$\Rightarrow\ 2x^2 + 5xy + 3y^2 - x - y = 0$

Exercises (Self Evaluation)

1. Find k, if the slope of the lines $9x^2 + 4xy + ky^2 = 0$ are as 3 : 1. Find the $\therefore$

The lines through (1,1) making 45° with the given line are $(y - 1) = -\frac{1}{3}(x - 1)$ and $(y - 1) = 3(x - 1)$.

$\Rightarrow\ 3y - 3 = -x + 1$ and $3x - y - 2 = 0$.

$\Rightarrow\ x + 3y - 4 = 0$ and $3x - y - 2 = 0$.

separate equations of the lines. K = 1/3, x+y = 0, x+3y = 0

2. Find the separate equations of the lines $2x^2 + 5xy + 2y^2 - x - 5y + k = 0$, (Hint: find k). Also find the angle between the lines and the points of intersection of the lines. [k = -3, 2x +y – 3 = 0 and x + 2y +1 = 0, Tan^{-1} (3/4_ and (7/3, -5/3)

3. Find the conditions that $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of parallel lines . S.T. the distance between them is $2\sqrt{\frac{g^2 - ac}{a(a+b)}}$.

   [$\Delta = 0, h^2 = ab$]

4. Derive the equation of the lines through (0,0), ⊥r to the lines $ax^2+2hxy+by^2=0$.

5. Find the equation of the lines parallel to the lines $2x^2 - 5xy + 3y^2 + 5x - 7y + 2 = 0$ through (1,2).

   [$2x^2 - 5xy + 3y^2 + 6x - 7y + 4 = 0$]

Conics:

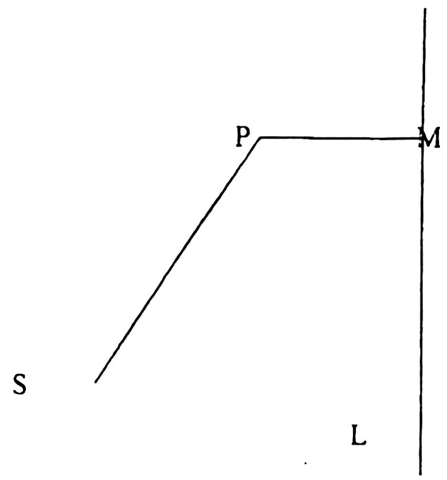
Various Cross Sections of a double-right circular cone give different plane curve. These curves of section are called conics.

No.	Position of the Cross Section	Name of the Conic	Nature of the Conic
1.	Parallel to a generator of the cone.	Parabola (Proper Conics)	Open ended
2.	Parallel to the axis of the cone	Hyperbola (Proper Conics)	Open ended-with Two disconnected branches.
3.	Making an acute (obtuse) angle with the axis.	Ellipse (Proper Conics)	Closed
4.	At right angles to the axis	Circle	Closed
5.	Through the axis	Pair of straight lines	-

Focus-Directrix Property of a Conic.

Let l be a fixed line and S, a fixed point Then the locus of a point P such that the ratio of Distance of P from S to the distance of P from L Is a constant (denoted by e) is a conic.

Ie. $\dfrac{SP}{PM} = e$



S is called a *focus* and L is called a *directrix* of the conic.

e is called the *eccentricity* of the conic.

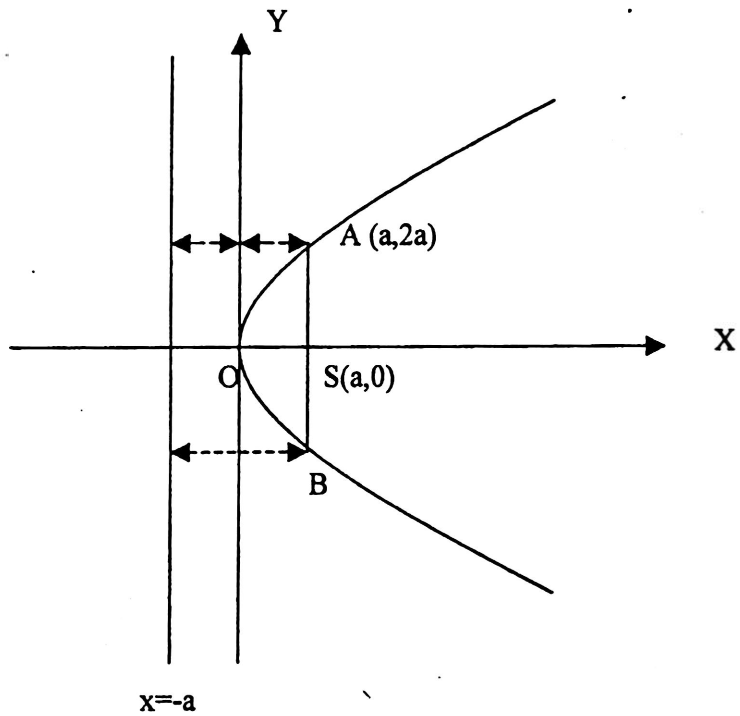
Classification of Conics on the value of e

	Value of e	Conic
1)	e=1	Parabola
2)	O<e<1	Ellipse
3)	e=0	Circle
4)	e>1	hyperbola
5)	e= $\sqrt{2}$	Rectangular hyperbola

5.1 Parabola $(e = 1)$

We recall some important equations and results related to a Parabola. The proofs of these are available in all PUC Text books.

1) Standard Equation of a Parabola: $y^2=4ax$



Properties

1) $y^2=4ax$ $(a>0)$
   exists for $x>0$ only.

2) Y-axis $(x=0)$
   Is the tangent at $(0,0)$

3) $(0,0)$ is the vertex

4) X-axis is the $(y=0)$
   axis is of the parabola

5) AB is the Chord thro'
   The focus $S(a,0)$ $\perp$r to the
   axis. It is called the Latus rectum (L.R)

Length of AB=4a.

6) The concavity (Cup like portion) of the Parabola faces the +ve x-axis.

7) Distance between the directrix and L.R= Semi L.R.=2a

8) The parabola is symmetrical about the x-axis (axis).

Observations

1) The directrix, the tangent at the vertex and the L.R. of a Parabola are ||le and each is ⊥r to the axis of the parabola.

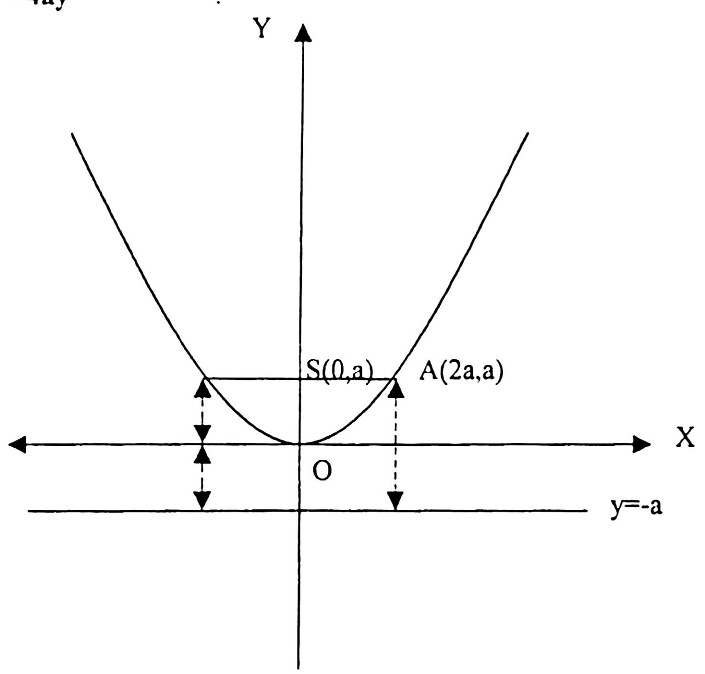
2) The directix and L.R. are on opposite sides of the vertex.

3) The directrix does not meet (or x if) the parabola. While the L.R. intersect it at two points.

Other Important Points Connected with a Parabola:

Other Standard equations of a Parabola

1) $x^2=4ay$

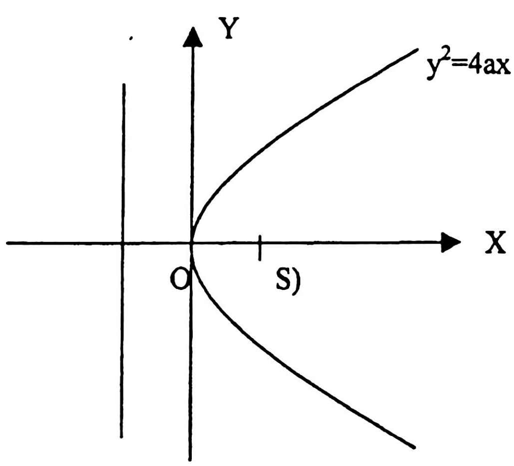


2) **$y=ax^2+bx+c$ ($a\neq0$).**
can be reduced to the form
$(x-h)^2 = 4a(y-k)$
with vertex $v=(h,k)$
and axis $x=k$ (||le to the y-axis)

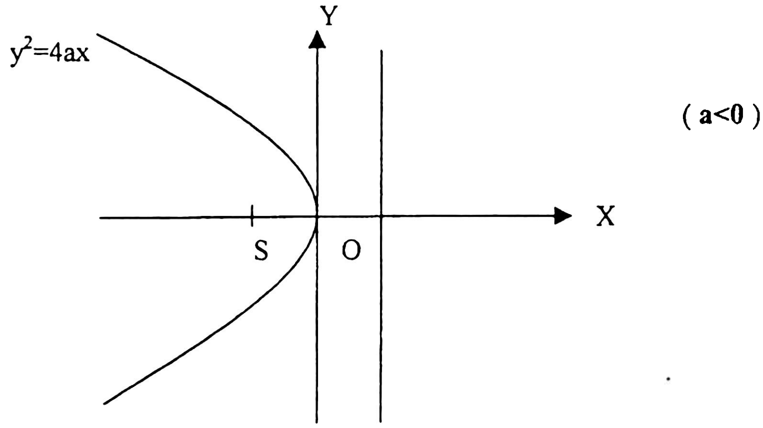
3) **$x=ay^2+by+c$ ($a\neq0$)**
can be reduced to the form
$(y-k)^2=4a(x-h)$
with vertex $v=(h,k)$
and axis is ||le to the x-axis ($y=k$)

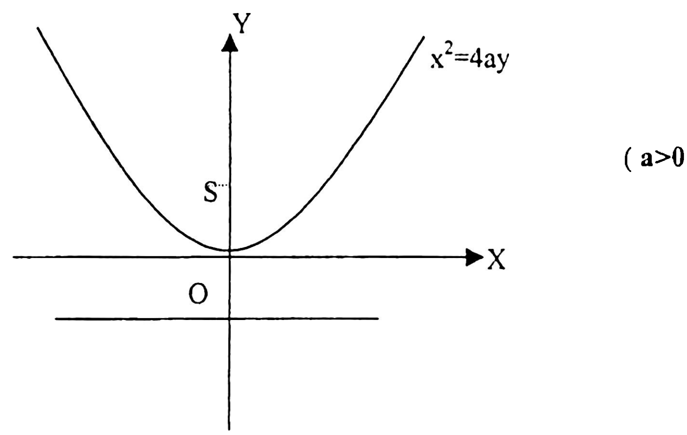
Useful Points to be remembered

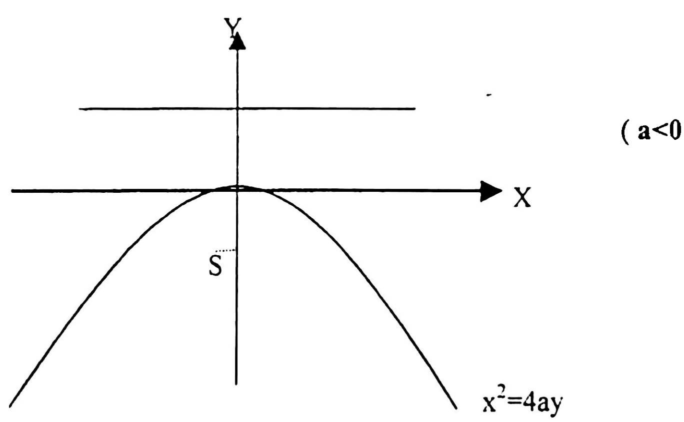
	Equation Aspects	**$y^2=4ax$**	**$x^2=4ay$**	**$(y-k)^2=4a(x-h)$**	**$(x-h)^2=4a(y-k)$**
1	Axis	Y=0(x-axis)	X=0(y-axis)	Y=k	X=h
2	Tangent at the vertex	X=0(y-acis)	Y=0(x-axis)	X=h	Y=k
3	Vertex	V=(0,0)	V=(0,0)	V=(h,k)	V=(h,k)
4	Focus	S=(a,0)	S=(0,a)	S=(a+h,k)	S=(h,a+k)
5	L.R. a) Length b) Equation	 4|a| x=a	 4|a| y=a	 4|a| x=a+h	 4|a| y=a+k
6	Directrix	X= -a	Y= -a	X= -a+h	Y= -a+k

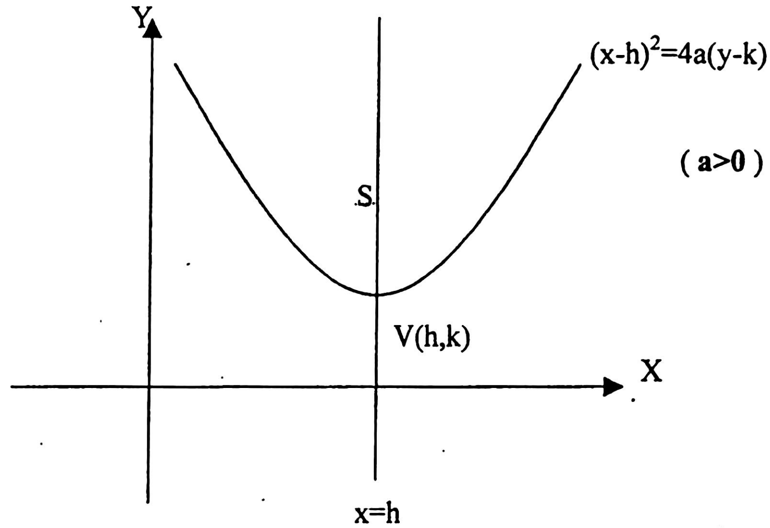


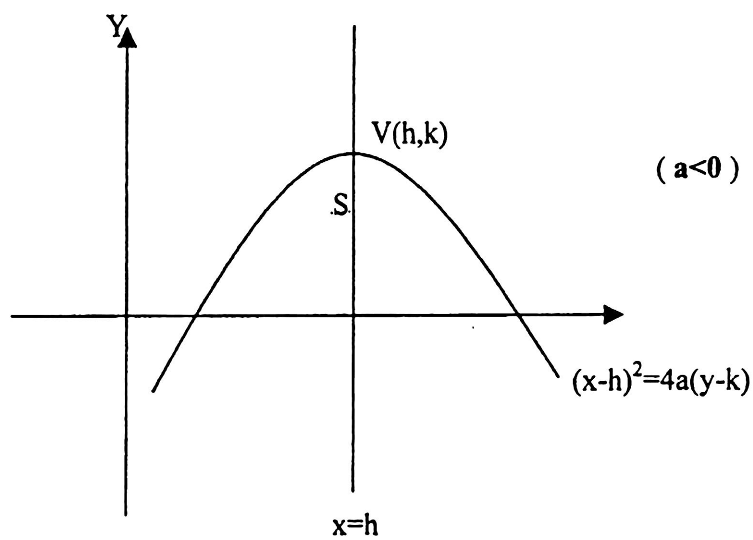
(a>0)

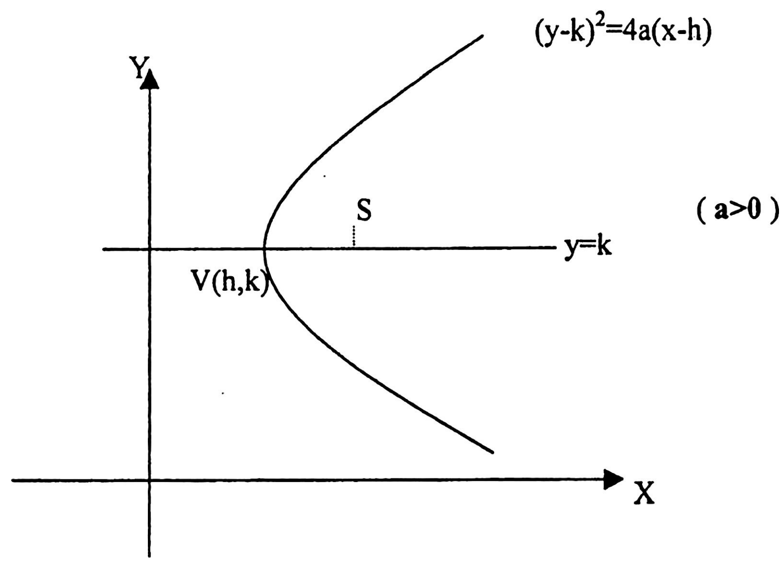

y^2=4ax
Y
X
S
O
(a<0)

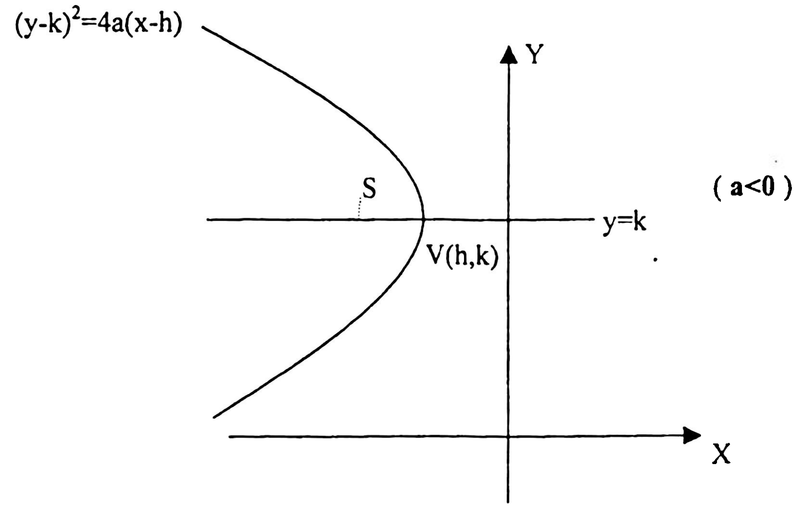

Y
x^2=4ay
(a>0
S
X
O


Y
(a<0
X
S
x^2=4ay


Y
(x-h)²=4a(y-k)
(a>0)
S
V(h,k)
X
x=h


Y
V(h,k)
(a<0)
S
(x-h)²=4a(y-k)
x=h


(y-k)²=4a(x-h)
Y
S
(a>0)
y=k
V(h,k)
X



Concerning a Parabola: $y^2=4ax$

i) Tangency condition for $y=mx+c$ $c = \frac{a}{m}$

a) Tangent with slope m: $y=mx+{}^{a}\!/_{m}$

b) Point of contact: $p = \left(\frac{a}{m^2}, \frac{2a}{m}\right)$

ii) Tangent at $p(x_1,y_1)$: $yy_1 = 2a(x+x_1)$ $yy_1 = 2a(x+x_1)$
Tangent at $p(at^2, 2at)$: $yt = x+at^2$

iii) Normal at $p(x_1y_1)$: $y_1x+2ay=(x_1+2a)y$,
Normal at $p(at^2, 2at)$: $4+xt=(at^2+2a)t$

iv) Parametric equations of the Parabola: $y^2=4ax$

$x=at^2, y=2at$

v) Focal Chord: If $p=(at_1^2, 2at_1)$ & $Q=(at_2^2, 2at_2)$ are the ends of a focal chord (ie the chord passing thro' the focus of the parabola) of $y^2=4ax$, then $t_1 t_2= -1$

vi) A property of a focal chord: The tangent at the end of a focal chord intersect on the directrix at right angle. Coversely, the chord of contact of tangent from a point on the directrix of a parabola is a focal chord and the tangent so drawn are perpendicular.

Worked Examples

1) Find the vertex, axis, focus, directrix, Latus rectum and in ends for the parabolas-

a) $x^2-2x+4y+9=0$ b) $y^2-4y+8x-12=0$

Solution: $x^2-2x+4y+9=0$

$\Rightarrow x^2-2x+1+4y+9-1=0$

$\Rightarrow (x-1)^2+4y+8=0$

$\Rightarrow (x-1)^2=-4(y+2)$

Comparing with $(x-h)^2=4a(y-k)$,

h=1, 4a= -4 or a= -1 k= -2

Vertex: V=(h,k) = **(1, -2)**

Axis: x-1=0 or x=1

Focus: S=(h, k+a)

= (1, -2-1)

S=(1, -3)

Directrix: y=k-a

= -2+1

y = -1

L.R: i) Length = 4|a|

= **4 mts**

ii) Equation: y=k+a

y = -3

iii) Ends of L.R : $(h\pm 2a, k+a)$=

(-1, -3) and (3, -3)

b) $y^2-4y+8x-12=0$

$\Rightarrow y^2-4y+4+8x-12-4=0$

$\Rightarrow (y-2)^2=-8(x-2)$

comparing with : $(y-k)^2=4a(x-h)$

h=2. k=2. 4a= -8 or a= -2

Vertex: v=(h, k) = **h,k= (2,2)**

Focus: S=(h+a, k)=(2-2, 2)= (0,2)

Directrix: x=h-a=2+2=4

or x=4

L.R: (i) Length= 4|a| = 8 mts

(ii) Equation: x=h+a=0

or x=0

(iii) Ends of L.R: (h+a, k ± 2a)

= (2+2, 2 ± 4)

= (0,6), (0, -2)

2) Find the equations of the parabola, given-

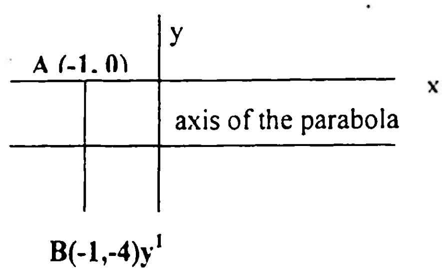
a) The ends of L.R: A(-1,0), B (-1,-4)

b) The ends of L.R: A(-2,3), B (4,3)

a) Focus S=Mid point of AB

$=\left(\frac{-1-1}{2},\frac{0-4}{2}\right)$

$=(-1,-1)$



Let the vertex be V=(h, k), Since the axis is ⊥r to AB (which is vertical) the axis of the parabola is ||le to the x-axis.

∴ Its equation is $(y-k)^2= 4a(x-h)$ (2)
& 4|a| = Length of L.R.=AB=4. ∴ |a|=1or $a=\pm 1$ (3)

and Focus S=(h+a,k)= (-1,-1)

h+a=-1, k=-1

When a=1, h+1=-1 ∴ h = -2, θ = -1

∴ (2) ⇒ $(y+1)^2=4.1\,(x+2)$ or $(y+1)^2=4\,(x+2)$ (5)

when a=-1, h-1=-1 ∴ h=0, k=-1

∴ (2) ⇒ $(y+1)^2 = 4(-1)\,(x-0)$ ⇒ $(y+1)^2 = -4x$ (6)

(5) & (6) are the required parabolas.

(b) A=(-2,3), B=(4,3)

Focus =S=$\left(\frac{-2+4}{2},\frac{3+3}{2}\right)$

$S=(1,3)$ (1)

Let V=(h,k) be the vertex

The axis being $\perp$r to AB the axis of the parabola is $||^{le}$ to y-axis.

$\therefore$ The equation of a parabola is : $(x-h)^2=4a(y-k)$ (2)

$4|a|=AB=6 \quad \therefore |a|=3/2 \quad \therefore$ $a=\pm 3/2$ (3)

$S=(h+a+k)=(1,3) \Rightarrow h=1, a+k=3$

$\therefore$ **h=1, k=3-a** (4)

when $a=3/2, h=1, k=3-3/2=3/2$

$\because (2) \Rightarrow (x-1)^2=4 \times 3/2 \, (y-3/2)$

$\Rightarrow$ $(x-1)^2=3(2y-3)$ (5)

when $a=-3/2, h=1, k=3+3/2=9/2$

$\therefore (2) \Rightarrow (x-1)^2=4\left(\frac{-3}{2}\right)\left(y-\frac{9}{2}\right)$

$=-3(2y-9)$

or $(x-1)^2=-3(2y-9)$ (6)

(5) & (6) are the required parabolas.

(3) Find the equation and the ends of the focal chord of $y^2=8x$ making 45^0 with ox. In what ratio is the chord divided at the focus

Solution: comparing $y^2=8x$ with $y^2=4ax$

$a=2$

$\therefore$ The focus $=S=(a,o)=(2,0)]$

If AB is the focal chord making 45^0

Then the equation of AB is $(y-0)=\text{Tan } 45^0(x-2)$

$\Rightarrow$ $y=x-2$ (1) is the equation of the Focal Chord.

Putting y=x-2 in y^2=8x
We get $(x-2)^2$=8x

$\Rightarrow x^2-4x+4-8x=0$

$\Rightarrow x^2-12x+4=0$

$\Rightarrow x=\frac{12\pm\sqrt{144-16}}{2}=\frac{12\pm\sqrt{128}}{2}$

$\therefore x=6\pm4\sqrt{2}$ $\quad\therefore y=x-2$

$=6\pm4\sqrt{2}-2=4\pm4\sqrt{2}$

$\therefore$ The ends of AB are $(6\pm4\sqrt{2},4\pm4\sqrt{2})$

Let A=$(6+4\sqrt{2})\lambda+(6+4\sqrt{2})$; $B=(6-4\sqrt{2},4-4\sqrt{2})$
If AS:SB=$\lambda:1$

$$S=\left[\frac{(6-4\sqrt{2})\lambda+(6+4\sqrt{2})}{1+\lambda},\frac{(4-4\sqrt{2})\lambda+(4+4\sqrt{2})}{1+\lambda}\right]=(2,0)$$

$\Rightarrow (4-4\sqrt{2})\lambda+(4+4\sqrt{2})=0 \quad \Rightarrow \lambda:1=\frac{\sqrt{2}+1}{\sqrt{2}-1}$

$\therefore$ S divides AB in the ratio: $\boxed{(\sqrt{2}+1):(\sqrt{2}-1)}$

(4) Find the equations of the tangent ||le to 3x-2y+5, to the parabola y^2=6x and the coordinates of the point of contact. Write down the normal at that point.

Solution: Taking the tangent as y=mx+$\frac{a}{m}$. Hence 6=4a

$\therefore$ $\boxed{a=\frac{3}{2}}$

m=slope of the tangent = slope of the given line

$\therefore$ m=$\frac{3}{2}$

$\therefore$ The tangent is y=$\frac{3}{2}x+\frac{\frac{3}{2}}{\frac{3}{2}}$

or y=$\frac{3}{2}x+1$

or **3x-2y+2=0** (1) is the required Tangent.

The point of contact $=\left(\frac{a}{m^2},\frac{2a}{m}\right)=\left(\frac{\frac{3}{2}}{\left(\frac{3}{2}\right)^2},\frac{2.\frac{3}{2}}{\frac{3}{2}}\right)$

$=$ **$P=\left(\frac{2}{3},2\right)$** (2)

The Normal at p is 2x+3y=k

$\Rightarrow 2\left(\frac{2}{3}\right)+3.2=k\Rightarrow k=\frac{22}{3}$

or **6x+9y=22** (3)

(5) Show that perpendicular tangents to a parabola intersect on the directrix and the points of contact of the tangents are the ends of a focal chord of the parabola.

Consider the parabola $y^2=4ax$

(i) The perpendicular tangent may be taken as

$Y=mx+\frac{a}{m}$ (1)

& $y=m^1x+\frac{a}{m^1}$ (2)

When $mm^1=-1$

(Since the tangents are $\perp$r)

$\therefore$ The tangents are

(1) $\Rightarrow y=mx+\frac{a}{m}$

(2) $\Rightarrow y=\frac{-1}{mx}-am$

Sub: $0=x\left(m+\frac{1}{m}\right)+a\left(m+\frac{1}{m}\right)$

$$\Rightarrow (x+a)\left(m+\frac{1}{m}\right)=0$$

$\Rightarrow$ x+a=0

which is the equation of the directrix of the parabola.

Hence $\perp r$ tangent intersect on the directrix.

(ii) Let P, Q be the point of contact.

Then $P=\left(\frac{a}{m^2},\frac{2a}{m}\right)$

& $Q=\left(\frac{a}{m^{1^2}},\frac{2a}{m^1}\right)$

or $P=\left(\frac{a}{m^2},\frac{2a}{m}\right)$

& $Q=\left(am^2,-2am\right)$

$\therefore$ Equation to PQ is

$$\frac{y-\frac{2a}{m}}{x-\frac{a}{m^2}}=\frac{2\frac{a}{m}+2am}{\frac{a}{m^2}-am^2}$$

$$\therefore Eqn\ of\ PQ\ is\ \frac{y-2\frac{a}{m}}{x-\frac{a}{m^2}}=\frac{2a\left(m+\frac{1}{m}\right)}{a\left(\frac{1}{m^2-m^2}\right)}$$

$$\Rightarrow \frac{y-2\frac{a}{m}}{x-\frac{a}{m^2}}=\frac{2}{\frac{1}{m}-m}$$

$$\Rightarrow \frac{m(my-2a)}{m^2x-a}=\frac{2m}{1-m^2}\Rightarrow\frac{my-2a}{m^2n-a}=\frac{2}{1-m^2}$$

Putting x=a, y=0

$$\frac{-2a}{m^2a-a}=\frac{2}{1-m^2}$$

$$\Rightarrow \frac{2}{1-m^2}=\frac{2}{1-m^2}$$

$\therefore$ PQ passes thro' the focus S=(a,0)

In other words PQ is a focal chord.

Exercise for self Evaluation

1) Find the vertex, focus, ends of Latus Rectum and the directrix of the parabola: $y^2-6y-8x-7=0$
[(-2,3), (0,3), (0,7) and (0,1), x+4=0)

2) Find the parabola with Common L.R. whose ends are (-3,1) and (1,1) $[(x+1)^2=4y$ & $(x+1)^2+4(y-2)=0]$

3) If 5x+4y=5 touches the parabola $y^2=4ax$, find a and the point of contact

$$\left[a=-\frac{25}{16},\left(-1,\frac{5}{2}\right)\right]$$

4) Find the tangent and normal at one end of the L.R. of $y^2=8x$
[At the end (2,4), x-y+2=0 & x+y-6=0]

5) Show that $(at^2, 2at)$ and $\left(\frac{a}{t^2},-\frac{2a}{t}\right)$ are the ends of a focal chord of $y^2=4ax$.

5.2 ELLIPSE (0<e<1)

1) Standard Equation:

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 (a>b) (1)$

when $\boxed{b^2=a^2(1-e^2)}$ (2)

e being the eccentricity (0<e<1) of the ellipse.

From (2) , we have

$$\boxed{e = \frac{\sqrt{a^2-b^2}}{a}} \quad (3)$$

Properties (of an ellipse) [For proofs, ref. to any PUC Text books]

(1) Symmetry about the Coord. Axes: The ellipse given by (1) is symmetrical about the Coord. Axes. It crosses the x-axis at A(a,o), A^1(-a,o) and the y-axis at B (o,b), B^1(o,b) These four points are called the *vertices* of the ellipse. The origin is the geometric *Centre* of the ellipse.

AA^1=2a is the longest diameter called the *Major diameter* or *Major axis* while BB^1=2b is the shortest diameter called the *Minor diameter* or *Minor axis* of the ellipse.

The half-lengths are respectively the *Major radius* (=a) & the Minor radius (=b) of the ellipse. For any point P(x,y) on the ellipse $\boxed{b \le OP \le a}$. The major diameter is greater than the Minor diameter always. Due to symmetry, the ellipse has two foci S(ae,0) & S^1(-ae,0), equidistant from the center of the ellipse and two directrices $x=\pm\frac{a}{e}$, equidistant from the center.

(2) Latus rectum: The chord thro' either focus $\perp r$ to the major axis is called a latus rectum. There are two latera rects (L_1L_2 & $L_1^1L_2^1$). Equidistant from the center. Each is of length $\frac{2b^2}{a}$ and the ends of L.R. are $\left(\pm ae, \pm\frac{b^2}{a}\right)$. $\frac{b^2}{a}$ is the length of Semi-L.R.

(3) Focal Distance Property: The sum of the focal distances (ie distances from the foci) of any point on the ellipse is equal to the length of the Major diameter.

i.e. $SP+S^1P=2a$ (constant)

This gives an easy method for free hand drawing of an ellipse.

Observations:

(1) The ellipse is a closed curve, symmetrical about the major and minor axis.

(2) The distance between the directrices is greater than the distance between the foci.

(3) L.R. and directrices are ||le to the minor axis each being to the major axis.

(4) For any point P(x,y) inside the ellipse $\frac{x}{a^2}+\frac{y^2}{b^2}<1$, $\frac{x^2}{a^2}+\frac{y^2}{b^2}<1$

(5) Directrices never intersect the ellipse.

2) Other forms of the equations to an ellipse.

In $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, (a>b) , the major axis is a part of the x-axis, and minor axis a part of y-axis. On the other hand if a<b, then the major axis is a part of the y-axis (as shown in fig.1) and minor axis , a part of the x-axis.

This makes a lot of difference

(see the fig 2)

The results connected with the two cases are tabulated.

When the center of the ellipse is same

Point C(h,k), with the major axis parallel

to a coordinate axis (ie the x-axis or

y-axis), the results are altered. All these are listed below.

Equation →	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$		$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$	
Aspects ↓	a>b	a<b	a>b	a<b
1) Major axis a) Length b) Equation	2a y=0(x-axis)	2b x=0(y-axis)	2a y=k (‖le to the x-axis)	2b x=h (‖le to the y-axis)
2) Minor axis a) Length b) Equation	2b x=0(y-axis)	2a y=0(x-axis)	2b x=h (‖le to the y-axis)	2a y=k (‖le to the x-axis)
3) Centre C	(0,0)	(0,0)	C(h,k)	C(h,k)
4) Vertices	$(\pm a,0),(0,\pm b)$	$(0,\pm b),(\pm a,0)$	$(h\pm a,k),(h,k\pm b)$	$(h,k\pm b),(h\pm a,k)$
5) Eccentricity	$b^2=a^2(1-e^2)$	$a^2=b^2(1-e^2)$	$b^2=a^2(1-e^2)$	$a^2=b^2(1-e^2)$
6) Foci	$(\pm ae,0)$	$(0,\pm be)$	$(h\pm ae,k)$	$(h,k\pm be)$
7) L.R. a) Length b) Equations	$\frac{2b^2}{a}$ $x=\pm ae$	$\frac{2a^2}{b}$ $y=\pm be$	$\frac{2b^2}{a}$ $x=h\pm ae$	$\frac{2a^2}{b}$ $y=k\pm be$
8) Directrices	$x=\pm\frac{a}{e}$	$y=\pm\frac{b}{e}$	$e=h\pm\frac{a}{e}$	$y=k\pm\frac{b}{e}$

Tips:

(a) Remember over the 1st column result

(b) To get the 2nd column results – interchange (i) x and y
(ii) a and b
(iii) x & y coordinates of points

(c) To get the 3rd/4th column results
Add h to x-value & k to y-value in 1st column/2nd column results.

3) Tangent /Normal to an ellipse: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

(a) Tangency condition for y=mx+c: **$c^2=a^2m^2+b^2$**

(i) Tangent with slope m: $y=mx\pm\sqrt{a^2m^2+b^2}$

(ii) Point of contact $=\left(\frac{-ma^2}{c},\frac{b^2}{c}\right)$ when $c=\sqrt{a^2m^2+b}$

(b) Tangent at a point P

(i) At $p(x_1,y_1)$: $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$

(ii) At p(a conθ, b sinθ): $\frac{x\,con\theta}{a}+\frac{y\sin\theta}{b}=1$

(c) Normal at a point p

(i) At $p(x_1,y_1)$: $\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=(a^2-b^2)$

(ii) At p (a con $\theta, b\sin\theta$): ax sec $\theta - by\,\mathrm{cosec}\,\theta=(a^2-b^2)$

4) Other results:

(a) Parametric Equations of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

is **x=a cos θ; y= b sin θ**

θ is called the eccentric angle of p when any point p on the ellipse is taken as p=(a cos θ, b sin θ)

(b) Director Circle of the ellipse: Any two perpendicular tangents intersect on the circle $x^2+y^2=a^2+b^2$, called the Director Circle of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Worked Examples:

(1) Find the axis, center, vertices, eccentricity, foci, L.R. and directrices of the ellipse: $16x^2+25y^2+32x-100y-284=0$

Solution: $16x^2+25y^2+32x-100y-284=0$

$\Rightarrow 16(x^2+2x)+25(y^2-4y)=284$

$\Rightarrow 16(x+1)^2-16+25(y-2)^2-100=284$

$\Rightarrow 16(x+1)^2+25(y-2)^2=400$

$\Rightarrow \frac{(x+1)^2}{25}+\frac{(y-2)^2}{16}=1$

Comparing with $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$

$h=-1$, $k=2$, $a^2=25$, $b^2=16$ or $a=5$, $b=4$ and $a>b$.

Hence

(i) Axis (a) Major axis: Length $=2a$ **=10** , equation; $y=k$ or **y=2**

(b) Minor axis: Length:$2b$ **=8** , equation $x=h$ or **x=-1**

(ii) Centre: C = **(h, θ) = (-1,2)**

(iii) Vertices : $A=(h-a,k)=(-1-5,2)=(-6,2)$

$A^1=(h+a,k)=(-1+5,2)=(4,2)$

$B=(h,k+b)=(-1,2+2)=(-1,6)$

$B^1=(h,k-b)=(-1,2-4)=(-1,-2)$

(iv) Eccentricity (e): $b^2=a^2(1-e^2)$

$$\Rightarrow 16=25(1-e^2)\Rightarrow e^2=\frac{25-16}{25}=\frac{9}{25}$$

$$\therefore \boxed{e=\frac{3}{5}}$$

(v) Foci: $S^1=(h-ae,k)=(-1-3,2)$

or $S^1=(-4,2)$ $\qquad ae=5.\frac{3}{5}=3$

& $S=(h+ae,k)=(-1+3,2)$
or $S=(2,2)$

$$\therefore \boxed{S^1=(-4,2),\ S=(2,2)}$$

(vi) L.R.: Length$=\frac{2b^2}{a}=2\times\frac{16}{5}$ $\boxed{=\frac{32}{5}}$

Equations: $x=h\pm ae$

$$\Rightarrow \boxed{x=2,\ x=-4}$$

(vii) Directrices: $x=\left(h\pm\frac{a}{e}\right)$

$$\Rightarrow x=\left(-1\pm\frac{25}{3}\right) \qquad \frac{a}{e}=\frac{5}{\frac{3}{5}}=\frac{25}{3}$$

$$\Rightarrow \boxed{x=\frac{-28}{3},\ x=\frac{22}{3}}$$

(2) Write down the equation of the ellipse in each case below

(a) in the form: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, given $2ae=8$, $\frac{2a}{e}=12$

(b) Centre $=(2,1)$, Major axis ||le to the x-axis, $e=\frac{1}{2}$, L.R.=4

Solution: (a) $2ae=8$, $\frac{2a}{e}=12\Rightarrow 2ae\times\frac{2a}{e}=8\times12$

$\Rightarrow a^2 = 24,$ $\boxed{a=2\sqrt{6}}$ $\therefore e = \frac{8}{2a} = \frac{4}{a} = \frac{4}{2\sqrt{6}}$

$\therefore e = \frac{2}{\sqrt{6}}$

$\therefore\ b^2=a^2(1-e^2)=24\left(1-\frac{4}{6}\right) = 24\times\frac{1}{3} = 8$

$\therefore$ $\boxed{b^2=8}$ $\therefore$ The ellipse $\boxed{\frac{x^2}{24}+\frac{y^2}{8}=1}$

(b) The equation of the ellipse is of the form $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$

Hence =(h,k)=(2,-1) (a>b)

and L.R.= $\frac{2b^2}{a} = 4$ $\Rightarrow b^2 = 2a$

$\Rightarrow a^2(1-e^2) = 2a$

$\Rightarrow a\left(1-\frac{1}{4}\right) = 2$

$\therefore$ $\boxed{a=\frac{8}{3}}$ $\therefore b^2 = 2\times\frac{8}{3} = \frac{16}{3}$

$\therefore a^2 = \frac{64}{9}, b^2 = \frac{16}{3}$

Hence the ellipse is $\frac{(x-2)^2}{\frac{64}{9}}+\frac{(y+1)^2}{\frac{16}{3}}=1$

$\Rightarrow 9(x-2)^2+12(y+1)^2=64$

(3) Find the equation of the tangent ||le to x+y=0 to the ellipse $9x^2+16y^2=144$. Also find the point of contact and the equation and the normal at that point to the ellipse.

Solution: Rewriting the equation of the ellipse : $\frac{x}{16}+\frac{y^2}{9}=1$

$a^2=16$, $b^2=9$. The tangent being ||le to x+y=1, in slope m=-1 $\therefore$ the equation is $y=mx+\sqrt{a^2m^2+b^2}$

$$\Rightarrow y=-x+\sqrt{16(-1)^2+9}$$

$$\Rightarrow y=-x+5 \text{ or } \boxed{x+y=5}$$

$\therefore$ The equation of the tangent ||le to the given line is $\boxed{x+y=5}$

The point of contact P= $\left(\frac{-a^2m}{c},\frac{b^2}{c}\right)=\left(\frac{-16\times-1}{5},\frac{9}{5}\right)=\boxed{=\left(\frac{16}{5},\frac{9}{5}\right)}$

The Normal can be taken as x-y=k

Since the normal is at p $\left(\frac{16}{5},\frac{9}{5}\right)$, $\frac{16}{5}-\frac{9}{5}=k$

or $k=\frac{7}{5}$ $\therefore$ The Normal at p is $\boxed{x-y=\frac{7}{5}}$

4) p and q are the distances of the foci of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ from a tangent to the ellipse so that $pq=b^2$.

Solution:

The foci are S=(ae,o)
S^1=(-ae,o)

Consider the tangent with slope m.

Its equation is y=mx+c

or mx-y+c=0

when $e^2=a^2m^2+b^2$

$$p=SA=\frac{mae-o+c}{\sqrt{m^2+1}}$$

$$\& \ q=S^1B=\frac{-mae-o+c}{\sqrt{m^2+1}}$$

$$\therefore\ pq = \frac{(c+mae)}{\sqrt{m^2+1}} \cdot \frac{(c-mae1}{\sqrt{m^2+1}}$$

$$= \frac{c^2 - m^2a^2e^1}{m^2+1} = \frac{a^2m^2 + b^2 - m^2a^2e^2}{m^2+1}$$

$$= \frac{m^2a^2(1-e^2)+b^2}{m^2+1} = \frac{m^2b^2+b^2}{m^2+1}$$

$$= \frac{(m^2+1)b^2}{(m^2+1)} = b^2 \quad \therefore \quad \boxed{pq=b^2}$$

5) The Normal at one end of latus rectum of an ellipse of eccentricity e, passes through one end of the minor axis. Prove that $e^4+e^2=1$

Consider the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(a>b)

one end of L.R. is P $\left(ae, \frac{b^2}{a}\right)$

The Normal at $p(x_1,y_1)$ is $\frac{a^2(x-x_1)}{x_1} - \frac{b^2(y-y_1)}{y_1} = 0$

or $\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = (a^2-b^2)$

$\therefore$ The Normal at p$\left(ae, \frac{b^2}{a}\right)$ is

$$\frac{a^2x}{ae} - \frac{b^2y}{\frac{b^2}{a}} = a^2 - b^2$$

$$\Rightarrow \frac{ax}{e} - ay = (a^2-b^2)$$

This passes thro' B (0,-b)

$\therefore$ $0-a(-b)=a^2-b^2$

$\Rightarrow ab=a^2-b^2=a^2e^2$

$\Rightarrow$ $b=ae^2$

Sequencing $b^2=a^2e^4$

$\Rightarrow$ $a^2(1-e^2)=a^2e^4$

$\Rightarrow$ $\boxed{e^4+e^2=1}$

Exercise for self-evaluation

1. The ellipse equation is of the form $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ (a>b)

has in center at (3,1), passes thro' (2,0) and it eccentricity is $\frac{1}{\sqrt{2}}$. Find its equation.

$$\left[\frac{(x-3)^2}{3}+\frac{2(y-1)^2}{3}=1\right]$$

2. Find the center, foci & L.R. of $4x^2+9y^2+16x-18y-11=0$

$\left[C(-2,1), \text{ foci: } (\pm\sqrt{5}-2,1), \text{ L.R.}=\frac{8}{3}\right]$

3. Find the tangent and normal at $\left(-1,\frac{4}{3}\right)$ to $4x^2+9y^2=20$

[Tangent: x-3y+5=0; Normal: 9x+3y+5=0]

4. Derive the equation of the Director circle of the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$

5. S.T for the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$, the area of the Δle formed by the foci and one end of the minor axis is $12(mt)^2$

5.3 HYPERBOLA

I Standard Equation of a Hyperbola and its properties

1) A Hyperbola with equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ has the following characteristics.

(a) It is symmetrical about both the coordinate axis. The origin is the center of the hyperbola. It intersect the x-axis at A (a,0), A^1(-a,0) and these points are called the vertices of the hyperbolas.
AA^1=2a and the sigment $AA^1$is called the Transverse axis. Marking B (0,b), B^1=(0,b) on the y-axis. BB^1=2b, and BB^1 is called the Conjugate axis.

b) Hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ has

(i) two foci S (ae,0) & S^1(-ae,0) on the T-axis and

(ii) two directrices: x=+$\frac{a}{e}$, x=$\frac{-a}{e}$ both $\perp$r to the T-axis.

c) Latus Recta(L.R.). The chords thro' the foci $\perp$r to the T-axis are called the latus recta of the hyperbola. The equations are x=± ae, and length $=\frac{2b^2}{a}$.

d) Hyperbola is open ended (unlike a circle or an ellipse) It has two similar looking disconnected branches on either side of the center of the hyperbola (namely the origin). In other words, the hyperbola exists for |x|>,a (ie x>a>0 & x<-a) and does not exist for in | x | <a (ie-a<x<a). The directrices are nearer to the center than the foci, (unlike in the case of an ellipse) For the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, the cases a>b, a=b, a<b are not distinguished (unlike in the case of an ellipse)

e) Asymptotes of the hyperbola: Tangent to the hyperbola thro' the center are called the asymptotes of the hyperbola. Their combined equation is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$ (differing from the equation of the hyperbola: $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ by a constant (ie 0 in place of 1 on the R.H.S) The point of contact of each asymptote with the hyperbola is at infinity'. The a symptoptes act as boundaries of the region in which the hyperbola lies. They are equally inclined to the T-axis. ($\tan\theta=\frac{b}{a}$)

f) Tangent and normal: Tangency condition for the line y=mx+e, Tangent at a point on the hyperbola: $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and normal at a point on the hyperbola.

(i) The line y=mx+c touches the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ if $c^2=a^2m^2-b^2$. Then the tangent's equation is $y=mx\pm\sqrt{a^2m^2-b^2}$. The point of contact of the tangent is $\left(\frac{-ma}{\sqrt{a^2m^2-b^2}},\frac{-b^2}{\sqrt{a^2m^2-b^2}}\right)$

(ii) The tangent at p(x_1,y_1): $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$

The tangent at p(a secθ , b tan θ): $\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1$

(iii) The Normal at p(x_1,y_1): $\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=(a^2+b^2)$

The Normal at p (a secθ, b tan θ):

(g) The parametric equations of the hyperbola: x=a sec θ, y = b tan θ.

(h) Director circle: Perpendicular Tangents intersect on a circle $x^2+y^2=a^2-b^2$.

(i) The conjugate hyperbola to the hyperbola is $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$

The two hyperbolas $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ & $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ are mutually conjugate. The

Y- axis of one is the e-axis of the other and vice versa. Both hyperbolas have the same asymptotes:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$$

(j) A special hyperbola- Rectangular hyperbola

In $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, if b=a, the hyperbola equation becomes $x^2-y^2=a^2$. This hyperbola is called a rectangular hyperbola.

For a rectangular hyperbola

(i) The length of T-axis = The length of C-axis

(ii) The eccentricity $= \sqrt{2}$

(iii) The asymptotes of rectangular hyperbola are at right angles

(iv) The conjugates of a rectangular hyperbola is also a rectangular hyperbola.

Two forms of equation to a hyperbola

	The equation $\rightarrow \frac{x^2}{a^2}-\frac{y^2}{b^2}=1$		$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$
	Aspects ↓		
1)	T-axis: length Equation	2a y=0	2a y=k
2)	C-axis: Length Equation	2b x=0	2b x=h
3)	Centre	(0,0)	(h,k)
4)	Vertices	$(\pm a,0)$	$(h\pm a,k)$
5)	Eccentricity(e)	$b^2=a^2(e^2-1)$	$b^2=a^2(e^2-1)$
6)	Foci	$(\pm ae,0)$	$(h\pm ae,k)$
7)	L.R. : Length Equation	$\frac{2b^2}{a}$ $x=\pm ae$	$\frac{2b^2}{a}$ $x=h\pm ae$
8)	Directrices	$x=\pm\frac{a}{e}$	$X=h\pm\frac{a}{e}$

Worked Examples:

1. Find the axis, center, vertices, eccentricity, foci, L.R. and directrices of the hyperbola $9x^2-4y^2+18x-8y-31=0$

Solution: $9x^2-4y^2+18x-8y-31=0$

$\Rightarrow \quad 9(x^2+2x)-4(y^2+2y)-31=0$

$\Rightarrow \quad 9(x^2+2x+1-1)-4(y^2+2y+1-1)-31=0$

$\Rightarrow \quad 9(x+1)^2-4(y+1)^2=36$

$$\Rightarrow \quad \frac{(x+1)^2}{4}-\frac{(y+1)^2}{9}=1$$

$\therefore$ $\boxed{(h,k)=(-1,-1), a^2=4\ ,\ b^2=9}$

(i) T-axis : Equation $\boxed{y+1=0}$ length =2a=4

(ii) C-axis: Equation $\boxed{x+1=0}$ length = 2b=6

(iii) Centre: C + (h,k)=(-1,-1)

(iv) Vertices: $(h\pm a,k)=(-1\pm 2,-1)$

= $\boxed{(-3,-1)\ (1,-1)}$

(v) Eccentricity (e): $b^2=a^2(e^2-1)$

$\Rightarrow \quad 9=4(e^2-1)\Rightarrow 4e^2=13$

$\therefore$ $\boxed{e=\frac{\sqrt{13}}{2}}$

(vi) Foci: $(h\pm ae,k)$ $\boxed{=(-1\pm\sqrt{13},1)}$

$ae=2.\frac{\sqrt{13}}{2}=\sqrt{13}$

(vii) L.R. : Length = $\frac{2b^2}{a}=2\times\frac{9}{2}$ $\boxed{=9}$

Equations: $x=h\pm\sqrt{13}$

$\Rightarrow$ $\boxed{x=-1\pm\sqrt{13}}$

(viii) Directrices: $x = h \pm \frac{a}{e}$ $\qquad \frac{a}{e} = \frac{2}{\sqrt{\frac{13}{2}}} = \frac{4}{\sqrt{13}}$

or $\boxed{x = -1 \pm \frac{4}{\sqrt{13}}}$

2) Find the equation of the hyperbola in each case-

(a) In the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, given length of T-axis =16 and e=2

(b) In the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, the hyperbola passes through (2,1) and (4,3)

Solution:

(a) $b^2 = a^2(e^2-1) \therefore b^2 = 8^2 = 64$ $\quad (\therefore 2a = 16)$

$= a^2(4-1) = 3a^2$

$\therefore a^2 = \frac{64}{3}$. $\therefore$ The equation is $\frac{3x^2}{64} - \frac{y^2}{64} = 1$

or $\boxed{3x^2 - y^2 = 64}$

(b) The hyperbola passes thro'

A(2,1) $\therefore \frac{4}{a^2} - \frac{1}{b^2} = 1$ $\qquad$ (i) x 4

& B(4,3) $\therefore \frac{16}{a^2} - \frac{9}{b^2} = 1$ $\qquad$ (ii)

Sub: $\frac{9}{b^2} - \frac{4}{b^2} = 3 \Rightarrow \frac{5}{b^2} = 3$ $\qquad \therefore \boxed{b^2 = \frac{5}{3}}$

$\therefore$ From (i) $\frac{4}{a^2} - \frac{3}{5} = 1$ $\qquad \therefore \frac{4}{a^2} = \frac{8}{5}$

$\therefore \boxed{a^2 = \frac{5}{2}}$

Hence the equation of the hyperbola: $\frac{2x^2}{5} - \frac{3y^2}{5} = 1$

or $\boxed{2x^2-3y^2=5}$

3) Find the tangents parallel to 5x-4y=0 to $\frac{x^2}{16} - \frac{y^2}{9} = 1$ and the distance between the tangents.

Solution: The slope (m) of the tangent=slope of the given line= $\frac{5}{4}$

$\therefore$ The tangent is y=mx+c or $y=\frac{5}{4}x + c$

when $c^2=a^2m^2-b^2$

$=16 \times \frac{25}{16} - 9 = 16 \quad \therefore \quad \boxed{c=\pm 4}$

$\therefore$ The tangents are $y=\frac{5}{4}x \pm 4$

$\Rightarrow 4y = 5x \pm 16$

or $\boxed{5x-4y \pm 16 = 0}$

The distance between the tangents $= \frac{c_1 - c_2}{\sqrt{a^2 + b^2}} = \frac{16+16}{\sqrt{25+16}}$

$\boxed{= \frac{32}{\sqrt{41}}}$

(4) S.T. the difference of focal lengths of a point on a hyperbola is equal to the length of the T-axis of the hyperbola.

Solution: L & L^1 being the directrices & P

any point on the hyperbola while

S, S^1 are the foci,

SP=e.px

$S^1P = e.px'$

$\therefore S^1P - SP = e(px - px^1) = e.xx'$

$\therefore S^1P - SP = e \times \left(\frac{2a}{e}\right)$ $\quad \therefore xx^1 =$ The Distance between the directrices$= \frac{2a}{e}$

$\therefore$ $\boxed{|S^1P - SP| = 2a}$ = Length of T-axis

(5) if e, e^1 are the eccentricities of a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ & its conjugate hyperbola, then S.T. $\frac{1}{e^2} + \frac{1}{e^{1^2}} = 1$

For $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, $b^2 = a^2(e^2 - 1)$

For its conjugate hyperbola

$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$, $a^2 = b^2(e^{1^2} - 1)$

$\therefore a^2 = b^2(e^{1^2} - 1)$

$\therefore a^2 = a^2(e^2 - 1)(e^{1^2} - 1)$

$\Rightarrow 1 = e^2 e^{1^2} - e^2 - e^{1^2} + 1$

$\Rightarrow e^{1^2} + e^2 = e^2 e^{1^2} + 1$

$\Rightarrow e^{1^2} + e^2 = e^2 e^{1^2} \quad \div e^2 e^{1^2}$

$\therefore$ $\boxed{\frac{1}{e^2} + \frac{1}{e^{1^2}} = 1}$

Exercise for self-evaluation

1) Find the center, eccentricity, foci and directrices of the hyperbola: $\frac{(x-2)^2}{16} - \frac{(y-1)^2}{9} = 1$

$$\left[C(2,1), e = \frac{5}{9}; foci = (7,1) \& (-3,1), x = \frac{26}{5}, x = \frac{-6}{5}\right]$$

(2) Find the hyperbola when T-axis is of length 10, eccentricity 2 in the form

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ [$3x^2-y^2=75$]

(3) Find the tangent ||le to y=2x to $\frac{x^2}{36} - \frac{y^2}{9} = 1$

$$\left[y = 2x \pm \sqrt{135}\right]$$

(4) If e, e^1 are the eccentricities of a hyperbola and its conjugate, then prove that

$$\frac{1}{e^2} + \frac{1}{e^{1^2}} = 1$$

(5) Find the asymptotes of the hyperbola $x^2-y^2-6x-2y=0$. What is the name of the hyperbola [x-y-4=0; x+y-2=0].

PROBABILITY

In our day-to-day life we perform certain activities to verify certain known facts or to observe certain phenomena. Such activities usually we call as experiments. In certain experiments, we can predict results exactly before conducting the experiment and in other it will not be possible. The experiments where the results can be predicted exactly are known as deterministic experiments and the experiments where the prediction is not exact are known as non-deterministic or random or probabilistic experiments. For example, . . a train is running at a uniform speed of sixty K.M. per hour, then we can predict with hundred percent surety that it will cover one hundred twenty kilometers after two hours, assuming that it never stopped during these hours. Similarly for a perfect gas, PV = constant (P is pressure, V is volume).

In case of non-deterministic experiments we cannot make predictions with complete reliability. The results are based on some 'chance element'. For example, if we toss a coin, will it show 'head up' or 'tail up' ? Although we cannot predict anything with complete surety, yet if we throw the coin a large number of times, it is very likely that the head will turn up fifty percent of the times and also it is very unlikely that the head turns up in every case.

Consider another example of a trained parachuter who is ready to jump. When he jumps then either his parachute will open or it will not. But experience says that most of the time it opens, though there are occasions on which it does not i.e., the uncertainty associated with opening or not opening of a parachute when a parachuter jumps is lesser as compared with the uncertainty associated with the head or tail coming up when we toss a coin.

How will you proceed in answering the following questions ?

1. How should a businessman order for replenishment (filling once again) of his stocks (inventory) so that he has not carried very large stocks, yet the risk of refusing customers is minimized ? (Inventory problem).

2. At what intervals should a car owner replace the car so that the total maintenance expenses are minimized ? (Replacement problem).

3. How many trainees should a large business organisation recruit and train them in certain intervals so that at any time it does not have a large number of trained persons whom it cannot employ and yet the risk of its being without sufficient persons when needed is minimized ?

4. How should the bus service in a city be scheduled so that the queues do not become too long and yet the gains by the bus company are maximized ? (Queuing problem)

5. How many booking counters should be provided at a station to serve in the best way the interests of both the railways and the travelling public ?. (Queuing problem)

6. What should be the strength of a dam (or a bridge) so that its cost is reasonable and yet the risk of its being swept away by the floods is minimized ?

7. How many telephone exchanges should be established in a given city so as to give the best service at a given cost ?

8. Which variety is the best out of given varieties of wheat, on the basis of yields from experimental farms ?

9. What should be the minimum premia charged by an insurance company so that the chance of its running into loss is minimized ?

10. How to decide whether a given batch of items is defective when only a sample of the batch can be examined ?

Answers for all such questions are based upon certain facts and they try to measure the uncertainty associated with some events which may or may not materialize. The theory of probability deals with the problem of measuring the uncertainty associated with various events rather precisely, making it these by possible today, to a certain extent of course, to control phenomena depending upon chance.

The 'measure of uncertainty' is known as probability.

In our day-to-day vocabulary we use words such as 'probably', 'likely' 'fairly good chances' etc. to express the uncertainty as indicated in the following example. Suppose a father of a X class student wants to know his son's progress in the studies and asks the concerned teacher about his son. Teacher may express to the father about the students' progress in any one of the following sentences.

1. It is certain that he will get a first class.
2. He is sure to get a first class.
3. I believe he will get a first class.
4. It is quite likely that he will get a first class.
5. Perhaps he may get a first class.
6. He may or he may not get a first class.
7. I believe he will not get a first class.
8. I am sure he will not get a first class.
9. I am certain he will not get a first class.

Instead of expressing uncertainty associated with any event with such phrases, it is better and exact if we express uncertainty mathematically. The measure of uncertainty or probability can be measured in three ways and these are known as the three definitions of probability. These methods are

i) Mathematical or Classical or Priori Probability
ii) Statistical or Empirical probability and
iii) Axiomatic probability.

Before discussing those methods, we define some of the terms which are useful in the definition of probability.

i) Experiment : An act of doing something to verify some fact or to obtain some result.

Ex: Throwing a die to observe which number will come up (Die is a six-faced cube).

ii) Trial : Conducting experiment once is known as the trial of that experiment.

Ex: Throwing a die once.

iii) Outcomes : The results of an experiment are known as outcomes.

Ex: In throwing a die, getting '1' or '2' or '6' are the outcomes.

iv) Events : Any single outcome or set of outcomes in an experiment is known as an event.

Ex: 1. Getting '1' in throwing of a die is an event.
Also getting an even number in throwing a die is also an event.
Ex: 2. Drawing two cards from a well shuffled pack of cards is a trial and getting of a king and a queen is an event.

v) Exhaustive Events : The total number of possible outcomes in any trial are known as exhaustive events.

Ex: 1. In tossing a coin there are two exhaustive events.
2. In throwing a die, there are six exhaustive

cases viz. (1,2,3,4,5,6).

vi) <u>Favourable Events (cases)</u> : The number of outcomes which entail the happening of an event are known as the favourable cases (events) of that event.

Ex: In throwing two dice, the number of cases favourable for getting a sum 5 are (1,4), (2,3), (3,2) and (4,1).

vii) <u>Mutually Exclusive Events</u> : Events are said to be mutually exclusive or incompatible if the happening of any one of them precludes or excludes the happening of all others.

Ex: In tossing a coin, the events head and tail are mutually exclusive (because both cannot occur simultaneously).

<u>Mathematical or Classical or 'a priori' probability</u> :

If a trial results in 'n' exhaustive, mutually exclusive and equally likely cases and 'm' of them are favourable to the happening of an event E, then the probability 'p' of happening of E is given by

$$p = \frac{\text{Favourable number of outcomes}}{\text{Total number of outcomes}} = \frac{m}{n}$$

We write p = P(E).

Ex: 1. Probability of getting head in tossing of a coin once is 1/2 because the number of exhaustive cases are 2 and these are mutually exclusive and equally likely (assuming the coin is made evenly) and of these only 1 case is favourable to our event of getting head.

Ex: 2. The probability of getting a number divisible by 3 in throwing of a fair (evenly made) die is 2/6 because the favourable cases are (viz. 3 and 6) and exhaustive cases are 6.

The probability 'q' that E will not happen is given by

$$q = \frac{n-m}{n} = 1 - \frac{m}{n} = 1-p$$

Always $0 \leq p \leq 1$

If P = P(E) = 1, E is called a certain event and if P(E) = 0, E is called an impossible event.

In this method, the mathematical ratio of two integers is giving the probability and therefore, this definition is known as mathematical definition. Here we are using the concept of probability in the form of 'equally likely cases' and therefore, this definition is a classical definition. Before using this definition, we should know about the nature of outcomes (viz. mutually exclusive, exhaustive and equally likely) and therefore, it is also known as 'a priori' probability definition.

The definition of mathematical or classical probability definition breaks down in the following cases : 1. If the various outcomes of the trial are not equally likely. 2. If the exhaustive number of cases in a trial is infinite.

Ex: 1. When we talk about the probability of a pass of a candidate, is not 1/2 as the two outcomes 'pass' and 'fail' are not equally likely.

Ex: 2. When we talk about the probability of a selected real number is to be divided by 10, the number of exhaustive cases are infinite.

In such above mentioned circumstances it is not possible to use mathematical probability definition. Therefore, probability is defined in the other way as below :

Statistical or Empirical Probability

If a trial is repeated a number of times under essentially homogeneous and identical conditions, then the limiting value of the ratio of the number of times an event happens to the number of trials, as the number of trials becomes indefinitely large, is called the probability of happening that event.

Mathematically, we write

$$P = P(E) = \lim_{n \to \infty} (m/n)$$

Here n is the number of trials and m is the number of times of the occurrence of event E. The above limit should be finite.

Ex: When you throw a die 10000 times and if you get 1600 times the number '1', then the probability of getting '1' is 1600/10000. This ratio is nothing but the relative frequency of '1'.

But this definition is also not applicable always because it is very difficult to maintain the identical conditions throughout the experiment. Therefore, the probability is defined in another way by using certain axioms. This definition is known as 'Axiomatic Probability' definition.

Here we define some of the terms which are useful in the 'Axiomatic Probability' definition.

Sample Space : The set of all possible outcomes of an experiment is known as the sample space of that experiment. Usually we denote it by S. Ex: In tossing a coin, S = {H,T}.

Sample Point: Any element of a sample space is known as a sample point.

Ex: In tossing of a coin experiment, H or T is a sample point.

Event : Any subset of a sample space is an event.

Ex: In throwing a die, (1,3,5) or (2,4,6) or (5,6) are the events where S = {1,2,3,4,5,6}.

If A and B are any two events then A, B, A∪B, A ∩ B are also events because they are also subsets of S.

The event S (entire sample space) is known as certain event and the event $\emptyset$ (empty set) is known as impossible event.

Mutually Exclusive Events : Events are said to be mutually exclusive if the corresponding sets are disjoint.

Ex: In throwing of a die experiment, if A = (1,3,5) and B = (2,4,6) then A and B are mutually exclusive because we cannot get both odd number and even number simultaneously. That is, if $A \cap B = \emptyset$, then A and B are mutually exclusive events.

Axiomatic Probability :

Let S be a sample space and ξ be the class of events. Also let P be a real valued function defined on ξ. Then P is called a probability function, and P(A) is called the probability of the event A if the following axioms hold :

i) For every event A, $0 \leq P(A) \leq 1$.

ii) $P(S) = 1$

iii) If A and B are mutually exclusive events, then $P(A \cup B) = P(A) + P(B)$.

iv) If $A_1, A_2, \ldots$ is a sequence of mutually exclusive events, then $P(A_1 \cup A_2 \ldots) = P(A_1) + P(A_2) + \ldots\ldots$

In the above definition axiom (iv) may seem to be not necessary. But it is necessary to stress that axiom (iii) should be extended to more than two events.

Theorem 1 : If $\emptyset$ is the empty set, then $P(\emptyset) = 0$.

Proof: We know that $S = S \cup \emptyset$ and $P(S) = P(S \cup \emptyset) = P(S) + P(\emptyset)$ (because S and $\emptyset$ are disjoint and according to axiom (iii)).

But $P(S) = 1$ and therefore, $1 = 1 + P(\emptyset)$.

$\therefore P(\emptyset) = 0$

Theorem 2 : If $\bar{A}$ is the complement of an event A, then $P(\bar{A}) = 1 - P(A)$.

Proof : $\bar{A} \cup A = S$

$P(\bar{A} \cup A) = P(\bar{A}) + P(A) = P(S)$ ($\bar{A}$ and A are disjoint).

But $P(S) = 1$, therefore,

$P(\bar{A}) + P(A) = 1$

or $P(\bar{A}) = 1 - p(A)$

Theorem 3 : If $A \subseteq B$, then $P(A) \leqslant P(B)$.

Proof: We know that if $A \subseteq B$, then

$B = A \cup (B - A)$ (here we may use the notation B/A)

So, $P(B) = P(A) + P(B-A)$

But from axiom i, $P(B-A) \geqslant 0$

$\therefore P(B) \geqslant P(A)$

Theorem 4 : If A and B are any two events, then

$P(A-B) = P(A) - P(A \cap B)$

Proof: We can write, $A = (A \cap B) \cup (A-B)$

But $(A \cap B)$ and $(A-B)$ are disjoint and according to axiom (iii),

$P(A) = P(A \cap B) + P(A-B)$

or $P(A-B) = P(A) - P(A \cap B)$

Theorem 5 : (Addition Theorem)

If A and B are any two events, then,

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Proof : We can write, $A \cup B = B \cup (A-B)$. But B and (A-B) are disjoint and therefore, by axiom (iii),

$P(A \cup B) = P(B) + P(A-B)$.

Also from theorem 4, $P(A-B) = P(A) - P(A \cap B)$

Hence, $P(A \cup B) = P(B) + P(A-B)$

$= P(B) + P(A) - P(A \cap B)$

This theorem is known as addition theorem and it can be extended to any number of events as follows.

<u>Theorem 6</u> (Addition Theorem in case of n events)

If $A_1, A_2, \ldots, A_n$ are any n events, then

$$P(A_1 \cup A_2 \cup \ldots A_n) = \sum_{\substack{i,j=1\\ i \quad j}}^{n} P(A_i) - \sum_{\substack{i,j=1\\ i<j}}^{n} P(A_i \cap A_j) + \ldots +$$

$$+ \ldots + (-1)^{n-1} P(A_1 \cap A_2 \cap \ldots \cap A_n)$$

<u>Proof</u>: This theorem can be proved by the method of induction.

For the events A_1 and A_2 we have from theorem 5,

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$$

$$= \sum_{i=1}^{2} P(A_i) + (-1)^{2-1} P(A_1 \cap A_2)$$

Hence the theorem is true for n=2.

Now, suppose the theorem is true for n = r (say).

Then,

$$P(A_1 \cup A_2 \cup \cdots \cup A_r) = \sum_{i=1}^{n} P(A_i) - \sum_{\substack{i,j=1\\ i<j}}^{n} P(A_i \cap A_j) + \cdots$$

$$+ (-1)^{r-1} P(A_1 \cap A_2 \cap \cdots \cap A_r)$$

Now,

$$P(A_1 \cup A_2 \cdots \cup A_r \cup A_{r+1}) = P((A_1 \cup A_2 \cup \cdots \cup A_r) \cup A_{r+1})$$

$$= P(A_1 \cup A_2 \cup \cdots \cup A_r) + P(A_{r+1}) - P((A_1 \cup A_2 \cup \cdots \cup A_r) \cap A_{r+1})$$

$$= P(A_1 \cup A_2 \cup \cdots \cup A_r) + P(A_{r+1}) - P((A_1 \cap A_{r+1}) \cup (A_2 \cap A_{r+1}) \cdots \cup (A_r \cap A_{r+1}))$$

$$= \sum_{i=1}^{r} P(A_i) - \sum_{\substack{i,j=1\\ i<j}}^{n} P(A_i \cap A_j) + \cdots + (-1)^{r-1} P(A_1 \cap A_2 \cap \cdots A_{r+1})$$

$$+ P(A_{r+1}) - \left\{ \sum_{i=1}^{r+1} P(A_i \cap A_{r+1}) + \sum_{\substack{i,j=1\\ i<j}}^{r+1} P(A_i \cap A_j \cap A_{r+1}) \right.$$

$$\left. + \cdots + (-1)^{r-1} P(A_1 \cap A_2 \cap \cdots \cap A_{r+1}) \right\}$$

$$= \sum_{i=1}^{r+1} P(A_i) - \sum_{\substack{i,j=1 \\ i>j}}^{r+1} P(A_i \cap A_j) + \ldots\ldots$$

$$+ (-1)^{r} P(A_1 \cap A_2 \cap \ldots\ldots A_{r+1})$$

Hence, if the theorem is true for n=r, it is also true for n=r+1. But we have proved that the theorem is true for n=2. Hence by the method of induction, the theorem is true for all positive integer values of n.

Corollary 1 : If A and B are two mutually exclusive events, then, $P(A \cup B) = P(A) + P(B)$

Corollary 2 : If $A_1, A_2 \ldots A_n$ are n mutually exclusive events, then $P(A_1 \cup A_2 \ldots \cup A_n) = P(A_1) + P(A_2) + \ldots + P(A_n)$.

Conditional Probability :

So far, we have assumed that no information was available about the experiment other than the sample space while calculating the probabilities of events. Sometimes, however, it is known that an event A has happened. How do we use this information in making a statement concerning the outcome of another event B ?

Consider the following examples :

Ex.1: Draw a card from a well-shuffled pack of cards. Define the event A as getting a black card and the event B as getting a spade card. Here $P(A) = 1/2$ and $P(B) = 1/4$. Suppose the drawn card is a black card then what is the probability that card is a spade card ? That is, if the event A has happened then what is the probability of B given that A has already happened. This probability symbolically we write as $P(B/A)$. In the given example,

$$P(B/A) = 1/2 = \frac{P(A \cap B)}{P(A)} = \frac{(1/4)}{(1/2)}$$

because probability of simultaneous occurence of A and B is 1/4 and probability of A is 1/2.

Ex.2: Let us toss two fair coins. Then the sample space of the experiment is $S = \{HH, HT, TH, TT\}$. Let event A = { both coins show same face } and B = { at least one coin shows H }. Then P(A) = 2/4. If B is known to have happened, this information assures that TT cannot happen, and P { A, Conditional on the information that B has happened}

$$= P(A/B) = 1/3 = \frac{1/4}{3/4}$$

$$= \frac{P(A \cap B)}{P(B)}$$

In the above two examples, we were interested to find the probability of one event given the condition that the other event has already happened. Such events based on some conditions are known as conditional events. In the above examples B/A and A/B are the conditional events. The probability of a conditional event is known as conditional probability of that event. We write the conditional probabilities as P(A/B), P(E/F) etc.

<u>Definition of conditional probability</u> : The conditional probability of an event A, given B, is denoted by P(A/B) and is defined by

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

where A, B and $A \cap B$ are events in a sample space S, and $P(B) \neq 0$.

From the definition of conditional probability we know that

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

Therefore, we can write from the above

$$P(A \cap B) = P(B) \cdot P(A/B)$$

Also, we know that $P(A \cap B) = P(B \cap A)$

$$P(B \cap A) = P(A) \cdot P(B/A)$$

Hence we can write

$P(A \cap B) = P(A) \cdot P(B/A)$ or $P(B) \cdot P(A/B)$.

The above result is known as multiplication law of probabilities in case of two events.

<u>Multiplication Theorem of Probabilities:</u> If A and B are any events of a sample space S, then

$P(A \cap B) = P(A) . P(B/A)$ or $P(B) . P(A/B)$.

The above theorem can be extended to any n events as follows: If $A_1, A_2, \ldots, A_n$ are any n events, then

$$P(A_1 \cap A_2 \ldots \cap A_n) = P(A_1)\, P(A_2/A_1) \,.\, P(A_3/A_1 \cap A_2) \ldots \ldots P(A_n/A_1 \cap A_2 \ldots \cap A_{n-1}).$$

This theorem can be proved by method of induction or generalization.

<u>Baye's Theorem</u> : If $E_1, E_2, \ldots, E_n$ are mutually exclusive events with $P(E_i) \neq 0$, $(i = 1,2,\ldots,n)$ then for any arbitrary event A which is a subset of $\bigcup_{i=1}^{n} E_i$ such that $P(A) \geqslant 0$, we have

$$P(E_i/A) = \frac{P(E_i)\, P(A/E_i)}{\sum_{i=1}^{} P(E_i)\, P(A/E_i)} \text{ for all i.}$$

Proof: Since $A \subset \bigcup_{i=1}^{n} E_i$ we have

$$A = A \cap (\bigcup_{i=1}^{n} E_i)$$

$$= \bigcup_{i=1}^{n} (A \cap E_i) \text{ by distributive law).}$$

Since $(A \cap E_i) \subset E_i$, (for $i = 1,2,\ldots n$) are mutually exclusive events, we have by additional theorem of probability

$$P(A) = P\left[\bigcup_{i=1}^{n} (A \cap E_i)\right] = \sum_{i=1}^{n} P(A \cap E_i)$$

$$= \sum_{i=1}^{n} P(E_i)\, P(A/E_i) \text{ by multiplication theorem in case of two events.}$$

Also, we have

$$P(A \cap E_i) = P(A) \quad P(E_i/A)$$

and $$P(E_i/A) = \frac{P(A \cap E_i)}{P(A)} = \frac{P(E_i)\, P(A/E_i)}{P(A)}$$

Hence, $$P(E_i/A) = \frac{P(E_i)\, P(A/E_i)}{\sum_{i=1}^{n} P(E_i)\, P(A/E_i)}.$$

This theorem is very useful in calculating the conditional probabilities in certain situations.

If $P(A \cap B) = P(A).P(B)$, then we see that $P(B/A) = P(B)$ and hence we say that the probability of B is not depending upon the happening of A. That is the conditional probability of B is same as the unconditional probability of B. Such events are called independent events.

Two events A and B are independent if and only if $P(A \cap B) = P(A) \cdot P(B)$.

Ex: Let two fair coins be tossed and let

$A = \{\text{head on first coin}\}$, $B = \{\text{head on the second coin}\}$

Then $P(A), = P\{HH, HT\} = 1/2$

$P(B) = P\{HH, TH\} = 1/2$ and

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{1/4}{1/2} = 1/2 = P(A)$$

Thus,

$$P(A \cap B) = P(A) \cdot P(B)$$

and we know that the probability of getting head on the first coin does not depend upon the probability of getting head on the second coin. Hence A and B are independent. Also we see that the condition $P(A \cap B) = P(A) \cdot P(B)$ is both necessary and sufficient for those events A and B to be independent.

If there are three or more than three events, we will have the situation where every pair of these events are independent or the situation where the events in every set of events are independent. In the first case, we call the events as pairwise independent and in the second case we call as complete or mutual independent events.

PART - II

.2

NATIONAL CURRICULUM FRAMEWORK 2005

POSITION PAPER

NATIONAL FOCUS GROUP ON

TEACHING OF MATHEMATICS

1. Goals of Mathematics Education

What are the main goals of mathematics education in schools? Simply stated, there is one main goal—the **mathematisation** of the child's thought processes. In the words of David Wheeler, it is "more useful to know how to mathematise than to know a lot of mathematics" [1].

According to George Polya, we can think of two kinds of aims for school education: a good and **narrow aim**, that of turning out employable adults who (eventually) contribute to social and economic development; and a **higher aim**, that of developing the inner resources of the growing child[2]. With regard to school mathematics, the former aim specifically relates to numeracy. Primary schools teach numbers and operations on them, measurement of quantities, fractions, percentages and ratios: all these are important for numeracy.

What about the higher aim? In developing a child's inner resources, the role that mathematics plays is mostly about thinking. Clarity of thought and pursuing assumptions to logical conclusions is central to the mathematical enterprise. There are many ways of thinking, and the kind of thinking one learns in mathematics is an ability to handle abstractions.

Even more importantly, what mathematics offers is a way of doing things: to be able to solve mathematical problems, and more generally, to have the right attitude for problem solving and to be able to attack all kinds of problems in a systematic manner.

This calls for a curriculum that is ambitious, coherent and teaches important mathematics. It should be **ambitious** in the sense that it seeks to achieve the higher aim mentioned above rather than (only) the narrower aim. It should be **coherent** in the sense that the variety of methods and skills available piecemeal (in arithmetic, algebra, geometry) cohere into an ability to address problems that come from science and social studies in high school. It should be **important** in the sense that students feel the need to solve such problems, that teachers and students find it worth their time and energy addressing these problems, and that mathematicians consider it an activity that is mathematically worthwhile. Note that such importance is not a given thing, and curriculum can help shape it. An important consequence of such requirements is that school mathematics must be **activity-oriented**.

In the Indian context, there is a centrality of concern which has an impact on all areas of school education, namely that of **universalisation of schooling**. This has two important implications for the discussion on curriculum, especially mathematics. Firstly, schooling is a legal right, and mathematics being a compulsory subject of study, access to quality mathematics education is every child's right. Keeping in mind the Indian reality, where few children have access to expensive material, we want mathematics education that is affordable to every child, and at the same time, enjoyable. This implies that the mathematics taught is situated in the child's lived reality, and that for the system, it is not the subject that matters more than the child, but the other way about.

Secondly, in a country where nearly half the children drop out of school during the elementary stage, mathematics curricula cannot be grounded only on preparation for higher secondary and university education. Even if we achieve our targeted universalisation goals, during the next decade, we will still have a substantial proportion of children exiting the system after Class VIII. It is then fair to ask what eight years of school mathematics offers for such children in terms of the challenges they will face afterwards.

Much has been written about *life skills* and linkage of school education to livelihood. It is certainly true that most of the skills taught at the primary stage are useful in everyday life. However, a reorientation of the curriculum towards addressing the 'higher aims' mentioned above will make better use of the time children spend in schools in terms of the problem solving and analytical skills it builds in children, and prepare them better to encounter a wide variety of problems in life.

Our reflections on the place of mathematics teaching in the curricular framework are positioned on these twin concerns: what mathematics education can do to engage the mind of every student, and how it can strengthen the student's resources. We describe our vision of mathematics in school, attempt to delineate the core areas of concern and offer recommendations that address the concerns, based on these twin perspectives.

Many of our considerations in what follows have been shaped by discussions of Mathematics Curriculum in NCTM, USA[3], the New Jersey Mathematics Coalition[4], the Mathematics academic content standards of the California State Board of Education[5], the Singapore Mathematics Curriculum[6], the Mathematics Learning Area statements of Australia and New Zealand[7], and the national curricula of France, Hungary[8] and the United Kingdom[9]. Ferrini-Mundi et al (eds.) offer an interesting discussion comparing national curriculum and teaching practice in mathematics in France with that of Brazil, Egypt, Japan, Kenya, Sweden and the USA[10].

2. A Vision Statement

In our vision, school mathematics takes place in a situation where:

- Children learn to *enjoy mathematics*: this is an important goal, based on the premise that mathematics can be both used and enjoyed life-long, and hence that school is best placed to create such a taste for mathematics. On the other hand, creating (or not removing) a fear of mathematics can deprive children of an important faculty for life.
- Children learn *important mathematics*: Equating mathematics with formulas and mechanical procedures does great harm. Understanding when and how a mathematical technique is to be used is always more important than recalling the technique from memory (which may easily be done using a book), and the school needs to create such understanding.
- Children see mathematics as something to talk about, to communicate, to discuss among themselves, to work together on. Making mathematics *a part of children's life experience* is the best mathematics education possible.
- Children pose and solve *meaningful problems*: In school, mathematics is the domain which formally addresses problem solving as a skill. Considering that this is an ability of use in all of one's life, techniques and approaches learnt in school have great value. Mathematics also provides an opportunity to make up interesting problems, and create new dialogues thereby.
- Children use abstractions to perceive relationships, to see structure, to reason about things, to argue the truth or falsity of statements. *Logical thinking* is a great gift mathematics can offer us, and inculcating such habits of thought and communication in

children is a principal goal of teaching mathematics.

- Children understand the *basic structure of mathematics*: Arithmetic, algebra, geometry and trigonometry, the basic content areas of school mathematics, all offer a methodology for abstraction, structuration and generalization. Appreciating the scope and power of mathematics refines our instincts in a unique manner.
- **Teachers** expect to *engage every child* in class: Settling for anything less can only act towards systematic exclusion, in the long run. Adequately challenging the talented even while ensuring the participation of all children is a challenge, and offering teachers means and resources to do this is essential for the health of the system.

Such a vision is based on a diagnosis of what we consider to be the central problems afflicting school mathematics education in the country today, as also on what we perceive can be done, and ought to be done.

Before we present the vision, a quick look at the history of mathematics curricular framework is in order.

3. A BRIEF HISTORY

Etymologically, the term 'curriculum' which has been derived from the Latin root means 'race course'. The word race is suggestive of time and course - the path. Obviously, curriculum was seen as the prescribed course of study to be covered in a prescribed time frame. But, evolution of curriculum as a field of study began in 1890's only, albeit of the fact that thinkers of education were interested in exploring the field for centuries. Johann Friedrich Herbart (1776-1841), a German thinker, is generally associated with the evolution of curriculum- field. Herbart had emphasized the importance of 'selection' and 'organization' of content in his theories of teaching/learning. The first book devoted to the theme of curriculum entitled, *The Curriculum* was published in 1918 by Franklin Bobbitt followed by another book *How to make Curriculum* in 1924. In 1926, the *National society for the study of education* in America published the year book devoted to the theme of curriculum-*The Foundation and Technique of Curriculum Construction*. This way the curriculum development movement, from its beginning in 1890s, started becoming a vigorous educational movement across the world.

School systems are a relatively new phenomenon in historical terms, having developed only during the past two hundred years or so. Before then, there existed schools in parts of the West, as an appendage to religious organisations. The purpose of these schools was to produce an educated cleric. Interest in mathematics was rudimentary-'the different kinds of numbers and the various shapes and sufficient astronomy to help to determine the dates of religious rituals'. **However, in India the practice of education was a well established phenomenon.** Arithmetic and astronomy were core components of the course of study. Astronomy was considered essential for determining auspicious times for performing religious rituals and sacrifices. Geometry was taught because it was required for the construction of sacrificial altars and 'havan kunds' of various shapes and sizes. With the arrival of the British, the system of education underwent a major change. Western system of education was introduced to educate Indians on western lines for the smooth functioning of the Empire.

However, much of the curriculum development in mathematics has taken place during the past thirty/forty years. This is because of the new technological revolution which has an impact on society as great as the industrial revolution. Modern technology is therefore causing, and will increasingly cause educational aims to be rethought, making curriculum development a dynamic process. To a scanning eye, mathematics itself is being directly affected by the modern technology as new branches are developed in response to new technological needs, leaving some 'time-hallowed' techniques redundant. In addition, teaching of mathematics also gets affected in order to keep pace with new developments in technology. Moreover, there exists a strong similarity of mathematics syllabi all over the world, with the result that any change which comes from the curriculum developers elsewhere is often copied or tried by others. India, for example, got swayed with the wave of new mathematics. Later, following the trends in other countries, new mathematics also receded here. To conclude, the various trends in curriculum development we observe no longer remain a static process, but a dynamic one. Its focus from 'selection' and 'organisation' of the informational material shifts to the development of a curriculum that 'manifests life in its reality'.

In 1937, when Gandhiji propounded the idea of basic education, the Zakir Husain committee was appointed to elaborate on this idea. It recommended: 'Knowledge of mathematics is an essential part of any curriculum. Every child is expected to work out the ordinary calculations required in the course of his craft work or his personal and community concerns and activities.' The Secondary Education Commission appointed in 1952 also emphasised the need for mathematics as a compulsory subject in the schools.

In line with the recommendations of the National Policy on Education, 1968, when the NCERT published its "Curriculum for the Ten Year School", it remarked that the 'advent of automation and cybernatics in this century marks the beginning of the new scientific industrial revolution and makes it all the more imperative to devote special attention to the study of mathematics'. It stressed on an 'investigatory approach' in the teaching of mathematics.

The National Policy on Education 1986 went further:

> Mathematics should be visualized as the vehicle to train a child to think, reason, analyze and to articulate logically. Apart from being a specific subject, it should be treated as a concomitant to any subject involving analysis and reasoning.

The National Curriculum Framework for School Education (NCFSE) 2000 document echoes such sentiments as well. Yet, despite this history of exhortations, mathematics education has remained pretty much the same, focussed on narrow aims.

4. Problems in Teaching and Learning of Mathematics

Any analysis of mathematics education in our schools will identify a range of issues as problematic. We structure our understanding of these issues around the following four problems which we deem to be the core areas of concern:

1. A sense of fear and failure regarding mathematics among a majority of children,
2. A curriculum that disappoints both a talented minority as well as the non-participating majority at the same time,
3. Crude methods of assessment that encourage perception of mathematics as mechanical computation, and
4. Lack of teacher preparation and support in the teaching of mathematics.

Each of these can and need to be expanded on, since they concern the curricular framework in essential ways.

4.1 Fear and Failure

If any subject area of study evokes wide emotional comment, it is mathematics. While no one educated in Tamil would profess (or at the least, not without a sense of shame) ignorance of any Tirukkural, it is quite the social norm for anyone to proudly declare that (s)he never could learn mathematics. While these may be adult attitudes, among children (who are compelled to pass mathematics examinations) there is often fear and anxiety. Mathematics anxiety and 'math phobia' are terms that are used in popular literature.[11]

In the Indian context, there is a special dimension to such anxiety. With the universalisation of elementary education made a national priority, and elementary education a legal right, at this historic juncture, a serious attempt must be made to look into every aspect that alienates children in school and contributes towards their non-participation, eventually leading to their dropping out of the system. If any subject taught in school plays a significant role in alienating children and causing them to stop attending school, perhaps mathematics, which inspires so much dread, must take a big part of the blame.

Such fear is closely linked to a sense of failure. By Class III or IV, many children start seeing themselves as unable to cope with the demands made by mathematics. In high school, among children who fail only in one or two subjects in year-end examinations and hence are detained, the maximum numbers fail in mathematics. This statistic pursues us right through to Class X, which is when the Indian state issues a certificate of education to a student. The largest numbers of Board Exam failures also happen in mathematics.

There are many perceptive studies and analyses on what causes fear of mathematics in schools. Central among them is the cumulative nature of mathematics. If you struggle with decimals, then you will struggle with percentages; if you struggle with percentages, then you will struggle with algebra and other mathematics subjects as well. The other principal reason is said to be the predominance of symbolic language. When symbols are manipulated without understanding, after a point, boredom and bewilderment dominate for many children, and dissociation develops.

Failure in mathematics could be read through social indicators as well. Structural problems in Indian education, reflecting structures of social discrimination, by way of class, caste and gender, contribute further to failure (and perceived failure) in mathematics education as well. Prevalent social attitudes which see girls as incapable of mathematics, or which, for centuries, have associated formal computational abilities with the upper castes, deepen such failure by way of creating self-fulfilling expectations.

A special mention must be made of problems created by the *language* used in textbooks, especially at the elementary level. For a vast majority of Indian children, the language of mathematics learnt in school is far removed from their everyday speech, and especially forbidding. This becomes a major force of alienation in its own right.

4.2 Disappointing Curriculum

Any mathematics curriculum that emphasises procedure and knowledge of formulas over understanding is bound to enhance anxiety. The prevalent practice of school mathematics goes further: a silent majority give up early on, remaining content to fail in mathematics, or at best, to see it through, maintaining a minimal level of achievement. For these children, what the curriculum offers is a store of mathematical facts, borrowed temporarily while preparing for tests.

On the other hand, it is widely acknowledged that more than in any other content discipline, mathematics is the subject that also sees great motivation and talent even at an early age in a small number of children[12]. These are children who take to quantisation and algebra easily and carry on with great facility.

What the curriculum offers for such children is also intense disappointment. By not offering conceptual depth, by not challenging them, the curriculum settles for minimal use of their motivation. Learning procedures may be easy for them, but their understanding and capacity for reasoning remain under-exercised.

4.3 Crude Assessment

We talked of fear and failure. While what happens in class may alienate, it never evokes panic, as does the examination. Most of the problems cited above relate to the tyranny of procedure and memorization of formulas in school mathematics, and the central reason for the ascendancy of procedure is the nature of assessment and evaluation. Tests are designed (only) for assessing a student's knowledge of procedure and memory of formulas and facts, and given the criticality of examination performance in school life, concept-learning is replaced by procedural memory. Those children who cannot do such replacement successfully experience panic, and suffer failure.

While mathematics is the major ground for formal problem solving in school, it is also the only arena where children see little room for play in answering questions. Every question in mathematics is seen to have one unique answer, and either you know it or you don't. In Language, Social Studies, or even in Science, you may try and demonstrate partial knowledge, but (as the students see it), there is no scope for doing so in mathematics. Obviously, such a perception is easily coupled to anxiety.

Amazingly, while there has been a great deal of research in mathematics education and some of it has led to changes in pedagogy and curriculum, the area that has seen little change in our schools over a hundred years or more is evaluation procedures in mathematics. It is not accidental that even a quarterly examination in Class VII is not very different in style from a Board examination in Class X, and the same pattern dominates even the end-of chapter exercises given in textbooks. It is always application of some piece of information given in the text to solve a specific problem that tests use of formalism. Such antiquated and crude methods of assessment have to be thoroughly overhauled if any basic change is to be brought about.

4.4 Inadequate Teacher Preparation

More so than any other content discipline, mathematics education relies very heavily on the preparation that the teacher has, in her own understanding of mathematics, of the nature of mathematics, and in her bag of pedagogic techniques. Textbook-centred pedagogy dulls the teacher's own mathematics activity.

At two ends of the spectrum, mathematics teaching poses special problems. At the primary level, most teachers assume that they know all the mathematics needed, and in the absence of any specific pedagogic training, simply try and uncritically reproduce the techniques they experienced in their school days. Often this ends up perpetuating problems across time and space.

At the secondary and higher secondary level, some teachers face a different situation. The syllabi have considerably changed since their school days, and in the absence of systematic and continuing education programmes for teachers, their fundamentals in many concept areas are not strong. This encourages reliance on 'notes' available in the market, offering little breadth or depth for the students.

While inadequate teacher preparation and support acts negatively on all of school mathematics, at the primary stage, its main consequence is this: **mathematics pedagogy rarely resonates with the findings of children's psychology**. At the upper primary stage, when the language of abstractions is formalised in algebra, inadequate teacher preparation reflects as **inability to link formal mathematics with experiential learning**. Later on, it reflects as **incapacity to offer connections within mathematics or across subject areas to applications in the sciences**, thus depriving students of important motivation and appreciation.

4.5 Other Systemic Problems

We wish to briefly mention a few other systemic sources of problems as well. One major problem is that of **compartmentalisation**: there is very little systematic communication between primary school and high school teachers of mathematics, and none at all between high school and college teachers of mathematics. Most school teachers have never even seen, let alone interacted with or consulted, research mathematicians. Those involved in teacher education are again typically outside the realm of college or research mathematics.

Another important problem is that of **curricular acceleration**: a generation ago, calculus was first encountered by a student in college. Another generation earlier, analytical geometry was considered college mathematics. But these are all part of school curriculum now. Such acceleration has naturally meant pruning of some topics: there is far less solid geometry or spehencal geometry now. One reason for the narrowing is that calculus and differential equations are critically important in undergraduate sciences, technology and engineering, and hence it is felt that early introduction of these topics helps students proceeding further on these lines. Whatever the logic, **the shape of mathematics education has become taller and more spindly, rather than broad and rounded.**

While we have mentioned **gender** as a systemic issue, it is worth understanding the problem in some detail. Mathematics tends to be regarded as a 'masculine domain'. This perception is aided by the complete lack of references in textbooks to women mathematicians, the absence of social concerns in the designing of curricula which would enable children questioning received gender ideologies and the absence of reference to women's lives in problems. A study of mathematics textbooks found that in the problem sums, not a single reference was made to women's clothing, although several problems referred to the buying of cloth, etc.[13]

Classroom research also indicates a fairly systematic devaluation of girls as incapable of 'mastering' mathematics, even when they perform reasonably well at verbal as well as cognitive tasks in mathematics. It has been seen that teachers tend to address boys more than girls, which feeds into the construction of the normative mathematics learner as male. Also, when instructional decisions are in teachers' hands, their gendered constructions colour the mathematical learning strategies of girls and boys, with the latter using more invented strategies for problem-solving, which reflects greater conceptual understanding.[14] Studies have shown that teachers tend to attribute boys' mathematical 'success' more to ability, and girls' success more to effort.[15] Classroom discourses also give some indication of how the 'masculinising' of mathematics occurs, and the profound influence of gender ideologies in patterning notions of academic competence in school.[16] With performance in mathematics signifying school 'success', girls are clearly at the losing end.

5. Recommendations

While the litany of problems and challenges magnifies the distance we need to travel to arrive at the vision articulated above, it also offers hope by way of pointing us where we need to go and what steps we may/must take.

We summarise what we believe to be the central directions for action towards our stated vision. We group them again into four central themes:

1. Shifting the focus of mathematics education from achieving 'narrow' goals to 'higher' goals,
2. Engaging every student with a sense of success, while at the same time offering conceptual challenges to the emerging mathematician,
3. Changing modes of assessment to examine students' mathematisation abilities rather than procedural knowledge,
4. Enriching teachers with a variety of mathematical resources.

There is some need for elaboration. How can the advocated shift to 'higher' goals remove fear of mathematics in children? Is it indeed possible to simultaneously address the silent majority and the motivated minority? How indeed can we assess processes rather than knowledge? We briefly address these concerns below.

5.1 Towards the Higher Goals

The shift that we advocate, from 'narrow' goals to 'higher' goals, is best summarized as a shift in focus from mathematical content to mathematical learning environments.

The content areas of mathematics addressed in our schools do offer a solid foundation. While there can be disputes over what gets taught at which grade, and over the level of detail included in a specific theme, there is broad agreement that the content areas (arithmetic, algebra, geometry, mensuration, trigonometry, data analysis) cover essential ground.

What can be levelled as major criticism against our extant curriculum and pedagogy is its failure with regard to mathematical processes. We mean a whole range of processes here: formal problem solving, use of heuristics, estimation and approximation, optimization, use of patterns, visualisation, representation, reasoning and proof, making connections, mathematical communication. Giving importance to these processes constitutes the difference between doing mathematics and swallowing mathematics, between mathematisation of thinking and memorising formulas, between trivial mathematics and important mathematics, between working towards the narrow aims and addressing the higher aims.

In school mathematics, certainly emphasis does need to be attached to factual knowledge, procedural fluency and conceptual understanding. New knowledge is to be constructed from experience and prior knowledge using conceptual elements. However, invariably, emphasis on procedure gains ascendancy at the cost of conceptual understanding as well as construction of knowledge based on experience. This can be seen as a central cause for the fear of mathematics in children.

On the other hand, the emphasis on exploratory problem solving, activities and the processes referred to above constitute learning environments that invite participation, engage children, and offer a sense of success. Transforming our classrooms in this manner, and designing mathematics curricula that enable such a transformation is to be accorded the highest priority.

5.1.1 Processes

It is worth explaining the kind of processes we have referred to and their place in the curricular framework. Admittedly, such processes cut across subject areas, but we wish to insist that they are central to mathematics. This is to be seen in contrast with mathematics being equated to exact but abstruse knowledge with an all-or-nothing character.

Formal problem solving, at least in schools, exists only in the realm of mathematics. But for physics lessons in the secondary stage and after, there are no other situations outside of mathematics where children address themselves to problem solving. Given this, and the fact that this is an important 'life skill' that a school can teach, mathematics education needs to be far more conscious of what tactics it can offer. As it stands, problem solving only amounts to doing exercises that illustrate specific definitions in the text. Worse, textbook problems reduce solutions to knowledge of specific tricks, of no validity outside the lesson where they are located.

On the other hand, many *general tactics* can indeed be taught, progressively during the stages of school. Techniques like abstraction, quantification, analogy, case analysis, reduction to simpler situations, even guess-and-verify, are useful in many problem contexts. Moreover, when children learn a variety of approaches (over time), their toolkit gets richer and they also learn which approach is best when.

This brings us to the use of **heuristics**, or rules of thumb. Unfortunately, mathematics is considered to be 'exact' where one uses 'the appropriate formula'. To find a property of some triangle, it is often useful to first investigate the special case when the triangle is right angled, and then look at the general case afterwards. Such heuristics do not always work, but when they do, they give answers to many other problems as well. Examples of heuristics abound when we apply mathematics in the sciences. Most scientists, engineers and mathematicians use a big bag of heuristics – a fact carefully hidden by our school textbooks.

Scientists regard **estimation of quantities** and **approximating solutions,** when exact ones are not available, to be absolutely essential skills. The physicist Fermi was famous for posing estimation problems based on everyday life and showing how they helped in nuclear physics. Indeed, when a farmer estimates the yield of a particular crop, considerable skills in estimation and approximation are used. School mathematics can play a significant role in developing and honing such useful skills, and it is a pity that this is almost entirely ignored.

Optimisation is never even recognized as a skill in schools. Yet, when we wish to decide on a set of goods to purchase, spending less than a fixed amount, we optimise Rs. 100 can buy us A and B or C, D and E in different quantities, and we decide. Two different routes can take us to the same destination and each has different advantages or disadvantages. Exact solutions to most optimisation problems are hard, but **intelligent choice based on best use of available information** is a mathematical skill that can be taught. Often, the numerical or geometrical facility needed is available at the upper primary stage. Developing a series of such situations and abilities can make school mathematics enjoyable as well as directly useful.

Visualisation and **representation** are again skills unaddressed outside mathematics curriculum, and hence mathematics needs to develop these far more consciously than is done now. Modelling situations using quantities, shapes and forms is the best use of mathematics. Such representations aid visualization and reasoning, clarify essentials, help us discard irrelevant information. Rather sadly, representations are taught as ends in themselves. For example, equations are taught, but the use of an equation to represent the relationship between force and acceleration is not examined. What we need are illustrations that show a multiplicity of representations so that the relative advantages can be understood. For example, a fraction can be written in the form p/q but can also be visualised as a point on the number line; both representations are useful, and appropriate in different contexts. Learning this about fractions is far more useful than arithmetic of fractions.

This also brings us to the need for **making connections**, within mathematics, and between mathematics and other subjects of study. Children learn to draw graphs of functional relationships between data, but fail to think of such a graph when encountering equations in physics or chemistry. That algebra offers a language for succinct substitutable statements in science needs underlining and can serve as motivation for many children. Eugene Wigner once spoke of the unreasonable effectiveness of mathematics in the sciences. Our children need to appreciate the fact that mathematics is an effective instrument in science.

The importance of **systematic reasoning** in mathematics cannot be overemphasized, and is intimately tied to notions of aesthetics and elegance dear to mathematicians. Proof is important, but equating proof with deduction, as done in schools, does violence to the notion. Sometimes, a picture suffices as a proof, a construction proves a claim rigorously. The social notion of proof as a process that convinces a sceptical adversary is important for the practice of mathematics. Therefore, school mathematics should encourage proof as a systematic way of argumentation. The aim should be to develop arguments, evaluate arguments, make and investigate conjectures, and understand that there are various methods of reasoning.

Another important element of process is **mathematical communication**. Precise and unambiguous use of language and rigour in formulation are important characteristics of mathematical treatment, and these constitute values to be imparted by way of mathematics education. The use of jargon in mathematics is deliberate, conscious and stylized. Mathematicians discuss what is appropriate notation since good notation is held to aid thought. As children grow older, they should be taught to appreciate the significance of such conventions and their use. For instance, this means that setting up of equations should get as much coverage as solving them.

In discussing many of these skills and processes, we have repeatedly referred to offering a *multiplicity* of approaches, procedures, solutions. We see this as crucial for liberating school mathematics from the tyranny of the one right answer, found by applying the one algorithm taught. When many ways are available, one can compare them, decide which is appropriate when, and in the process gain insight. And such a multiplicity is available for most mathematical contexts, all through school, starting from the primary stage. For instance, when we wish to divide 102 by 8, we could do long division, or try 10 first, then 15, and decide that the answer lies in between and work at narrowing the gap.

It is important to acknowledge that mathematical competence is situated and shaped by the social situations and the activities in which learning occurs. Hence, school mathematics has to be in close relation to the social worlds of children where they are engaged in mathematical activities as a part of daily life. Open-ended problems, involving multiple approaches and not solely based on arriving at a final, unitary, correct answer are important so that an external source of validation (the teacher, textbooks, guidebooks) is not habitually sought for mathematical claims. The unitary approach acts to disadvantage all learners, but often acts to disadvantage girls in particular.

5.1.2 Mathematics that people use

An emphasis on the processes discussed above also enables children to appreciate the relevance of mathematics to people's lives. In Indian villages, it is commonly seen that people who are not formally educated use many modes of mental mathematics. What may be called *folk algorithms* exist for not only mentally performing number operations, but also for measurement, estimation, understanding of shapes and aesthetics. Appreciating the richness of these methods can enrich the child's perception of mathematics. Many children are immersed in situations where they see and learn the use of these methods, and relating such knowledge to what is formally learnt as mathematics can be inspiring and additionally motivating.

For instance, in Southern India, **kolams** (complex figures drawn on the floor using a white powder, similar to *rangoli* in the north, but ordinarily without colour) are seen in front of houses. A new kolam is created each day and a great variety of kolams are used. Typically women draw kolams, and many even participate in competitions. The grammar of these kolams, the classes of closed curves they use, the symmetries that they exploit - these are matters that mathematics education in schools can address, to the great benefit of students. Similarly, art, architecture and music offer intricate examples that help children appreciate the cultural grounding of mathematics.

5.1.3 Use of technology

Technology can greatly aid the process of mathematical exploration, and clever use of such aids can help engage students. Calculators are typically seen as aiding arithmetical operations; while this is true, calculators are of much greater pedagogic value. Indeed, if one asks whether calculators should be permitted in examinations, the answer is that it is quite unnecessary for examiners to raise questions that necessitate the use of calculators. On the contrary, in a non-threatening atmosphere, children can use calculators to study iteration of many algebraic functions. For instance, starting with an arbitrary large number and repeatedly finding the square root to see how soon the sequence converges to 1, is illuminating. Even phenomena like chaos can be easily comprehended with such iterators.

If ordinary calculators can offer such possibilities, the potential of graphing calculators and computers for mathematical exploration is far higher. However, these are expensive, and in a country where the vast majority of children cannot afford more than one notebook, such use is luxurious. It is here that governmental action, to provide appropriate alternative low-cost technology, may be appropriate. Research in this direction will be greatly beneficial to school education.

It must be understood that there is a spectrum of technology use in mathematics education, and calculators or computers are at one end of the spectrum. While notebooks and blackboards are the other end, use of graph paper, geo boards, abacus, geometry boxes etc. is crucial. Innovations in the design and use of such material must be encouraged so that their use makes school mathematics enjoyable and meaningful.

5.2 Mathematics for All

A systemic goal that needs to be underlined and internalised in the entire system is universal inclusion. This means acknowledging that forms of social discrimination work in the context of mathematics education as well and addressing means for redress. For instance, gendered attitudes which consider mathematics to be unimportant for girls, have to be systematically challenged in school. In India, even caste based discrimination manifests in such terms, and the system cannot afford to treat such attitudes by default.

Inclusion is a fundamental principle. Children with special needs, especially children with physical and mental disabilities, have as much right as every other child to learn mathematics, and their needs (in terms of pedagogy, learning material etc) have to be addressed seriously. The conceptual world of mathematics can bring great joy to these children, and it is our responsibility not to deprive them of such education.

One important implication in taking **Mathematics for all** seriously is that even the *language used in our textbooks* must be sensitive to language uses of all children. This is critical for primary education, and this may be achievable only by a multiplicity of textbooks.

While the emphasised shift towards learning environments is essential for engaging the currently nonparticipating majority in our classrooms, it does not in any way mean dilution of standards. We are not advising here that the mathematics class, rather than boring the majority, ends up boring the already motivated minority. On the other hand, a case can be made that such open problem situations offer greater gradations in challenges, and hence offer more for these few children as well.

It is widely acknowledged that mathematical talent can be detected early, in a way that is not observable in more complex fields such as literature and history. That is, it is possible to present challenging tasks to highly talented youngsters. The history of the task may be ignored; the necessary machinery is minimal; and the manner in which such youngsters express their insights does not require elaboration in order to generate mathematical inquiry.

All this is to say that challenging all children according to their mathematical taste is indeed possible. But this calls for systemic mechanisms, especially in textbooks. In India, few children have access to any mathematical material outside their mathematics textbooks, and hence structuring textbooks to offer such a variety of content is important.

In addition, we also need to consider mechanisms for identification and nurturing of such talent, especially in rural areas, by means of support outside main school hours. Every district needs at least a few centres accessible to children in the district where such mathematical activity is undertaken periodically. Networking such talent is another way of strengthening it.

5.2.1 Assessment

Given that mathematics is a compulsory subject in all school years, all summative evaluation must take into account the concerns of universalization. Since the Board examination for Class X is for a certificate given by the State, implications of certified failure must be considered seriously. Given the reality of the educational scenario, the fact that Class X is a terminal point for many is relevant; applying the same single standard of assessment for these students as well as for rendering eligibility for the higher secondary stage seems indefensible. When we legally bind all children to complete ten years of schooling, the SSLC certificate of passing that the State issues should be seen as a basic requirement rather than a certificate of competence or expertise.

Keeping these considerations in mind, and given the high failure rate in mathematics, we suggest that the Board examinations be restructured. They must ensure that all numerate citizens pass and become eligible for a State certificate. (What constitutes numeracy in a citizen may be a matter of social policy.) Nearly half the content of the examination may be geared towards this.

However, the rest of the examination needs to challenge students far more than it does now, emphasizing competence and expertise rather than memory. Evaluating conceptual understanding rather than fast computational ability in the Board examinations will send a signal of intent to the entire system, and over a period of time, cause a shift in pedagogy as well.

These remarks pertain to all forms of summative examinations at the school level as well. Multiple modes of assessment, rather than the unique test pattern, need to be encouraged. This calls for a great deal of research and a wide variety of assessment models to be created and widely disseminated.

5.3 Teacher Support

The systemic changes that we have advocated require substantial investments of time, energy, and support on the part of teachers. Professional development, affecting the beliefs, attitudes, knowledge, and practices of teachers in the school, is central to achieving this change. In order for the vision described in this paper to become a reality, it is critical that professional development focuses on mathematics specifically. Generic 'teacher training' does not provide the understanding of content, of instructional techniques, and of critical issues in mathematics education that is needed by classroom teachers.

There are many mechanisms that need to be ensured to offer better teacher support and professional development, but the essential and central requirement is that of a large treasury of resource material which teachers can access freely as well as contribute to. Further, networking of teachers so that expertise and experience can be shared is important. In addition, identifying and nurturing resource teachers can greatly help the process. Regional mathematics libraries may be built to act as resource centres.

An important area of concern is the teacher's own perception of what mathematics is, and what constitute the goals of mathematics education. Many of the processes we have outlined above are not considered to be central by most mathematics teachers, mainly because of the way they were taught and a lack of any later training on such processes.

Offering a range of material to teachers that enriches their understanding of the subject, provides insights into the conceptual and historical development of the subject and helps them innovate in their classrooms is the best means of teacher support. For this, providing channels of communication with college teachers and research mathematicians will be of great help. When teachers network among themselves and link up with teachers in universities, their pedagogic competence will be strengthened immensely. Such systematic sharing of experience and expertise can be of great help.

6. Curricular Choices

Acknowledging the existence of choices in curriculum is an important step in the institutionalization of education. Hence, when we speak of shifting the focus from content to learning environments, we are offering criteria by which a curriculum designer may resolve choices. For instance, visualization and geometric reasoning are important processes to be ensured, and this has implications for teaching algebra. Students who 'blindly' manipulate equations without being able to visualize and understand the underlying geometric picture cannot be said to have understood. If this means greater coverage for geometric reasoning (in terms of lessons, pages in textbook), it has to be ensured. Again, if such expansion can only be achieved by reducing other (largely computational) content, such content reduction is implied.

Below, while discussing stage-wise content, we offer many such inclusion /exclusion criteria for the curriculum designer, emphasizing again that the recommendation is not to dilute content, but to give importance to a variety of processes. Moreover, we suggest a principle of postponement: in general, if a theme can be offered with better motivation and applications at a later stage, wait for introducing it at that stage, rather than go for technical preparation without due motivation. Such considerations are critical at the secondary and higher secondary stages where a conscious choice between breadth and depth is called for. Here, a quotation from William Thurston is appropriate:

> *The long-range objectives of mathematics education would be better served if the tall shape of mathematics were de-emphasized, by moving away from a standard sequence to a more diversified curriculum with more topics that start closer to the ground. There have been some trends in this direction, such as courses in finite mathematics and in probability, but there is room for much more.*[17]

6.1 Primary Stage

Any curriculum for primary mathematics must incorporate the progression from the concrete to the abstract and subsequently a need to appreciate the importance of abstraction in mathematics. In the lowest classes, especially, it is important that activities with concrete objects form the first step in the classroom to enable the child to understand the connections between the logical functioning of their everyday lives to that of mathematical thinking.

Mathematical games, puzzles and stories involving number are useful to enable children to make these connections and to build upon their everyday understandings. Games – not to be confused with open-ended play - provide non-didactic feedback to the child, with a minimum amount of teacher intervention[18]. They promote processes of anticipation, planning and strategy.

6.1.1 Mathematics is not just arithmetic

While addressing number and number operations, due place must be given to non-number areas of mathematics. These include shapes, spatial understanding, patterns, measurement and data handling. It is not enough to deal with shapes and their properties as a prelude to geometry in the higher classes. It is important also to build up a vocabulary of relational words which extend the child's understanding of space. The identification of patterns is central to mathematics. Starting with simple patterns of repeating shapes, the child can move on to more complex patterns involving shapes as well as numbers. This lays the base for a mode of thinking that can be called algebraic. A primary curriculum that is rich in such activities can arguably make the transition to algebra easier in the middle grades.[19] Data handling, which forms the base for statistics in the higher classes, is another neglected area of school mathematics and can be introduced right from Class I.

6.1.2 Number and number operations

Children come equipped with a set of intuitive and cultural ideas about number and simple operations at the point of entry into school. These should be used to make linkages and connections to number understanding rather than treating the child as a *tabula rasa*. To learn to think in mathematical ways children need to be logical and to understand logical rules, but they also need to learn conventions needed for the mastery of mathematical techniques such as the use of a base ten system. Activities as basic as counting and understanding numeration systems involve logical understandings for which children need time and practice if they are to attain mastery and then to be able to use them as tools for thinking and for mathematical problem solving[20]. Working with limited quantities and smaller numbers prevents overloading the child's cognitive capacity which can be better used for mastering the logical skills at these early stages.

Operations on natural numbers usually form a major part of primary mathematics syllabi. However, the standard algorithms of addition, subtraction, multiplication and division of whole numbers in the curriculum have tended to occupy a dominant role in these. This tends to happen at the expense of development of number sense and skills of estimation and approximation. The result frequently is that students, when faced with word problems, ask "Should I add or subtract? Should I multiply or divide?" This lack of a conceptual base continues to haunt the child in later classes. All this strongly suggests that operations should be introduced contextually. This should be followed by the development of language and symbolic notation, with the standard algorithms coming at the end rather than the beginning of the treatment.

6.1.3 Fractions and decimals

Fractions and decimals constitute another major problem area. There is some evidence that the introduction of operations on fractions coincides with the beginnings of fear of mathematics. The content in these areas needs careful reconsideration. Everyday contexts in which fractions appear, and in which arithmetical operations need to be done on them, have largely disappeared with the introduction of metric units and decimal currency. At present, the child is presented with a number of contrived situations in which operations have to be performed on fractions. Moreover, these operations have to be done using a set of rules which appear arbitrary (often even to the teacher), and have to be memorized - this at a time when the child is still grappling with the rules for operating on whole numbers. While the importance of fractions in the conceptual structure of mathematics is undeniable, the above considerations seem to suggest that less emphasis on operations with fractions at the primary level is called for.[21]

6.2 Upper Primary Stage

Mathematics is amazingly compressible: one may struggle a lot, work out something, perhaps by trying many methods. But once it is understood, and seen as a whole, it can be filed away, and used as just a step when needed. The insight that goes into this compression is one of the great joys of mathematics. A major goal of the upper primary stage is to introduce the student to this particular pleasure.

The compressed form lends itself to application and use in a variety of contexts. Thus, mathematics at this stage can address many problems from everyday life, and offer tools for addressing them. Indeed, the transition from arithmetic to algebra, at once both challenging and rewarding, is best seen in this light.

6.2.1 Arithmetic and algebra

A consolidation of basic concepts and skills learnt at primary school is necessary from several points of view. For one thing, ensuring numeracy in all children is an important aspect of universalization of elementary education. Secondly, moving from number sense to number patterns, seeing relationships between numbers, and looking for patterns in the relationships bring useful life skills to children. Ideas of prime numbers, odd and even numbers, tests of divisibility etc. offer scope for such exploration.

Algebraic notation, introduced at this stage, is best seen as a compact language, a means of succinct expression. Use of variables, setting up and solving linear equations, identities and factoring are means by which students gain fluency in using the new language.

The use of arithmetic and algebra in solving real problems of importance to daily life can be emphasized. However, engaging children's interest and offering a sense of success in solving such problems is essential.

6.2.2 Shape, space and measures

A variety of regular shapes are introduced to students at this stage: triangles, circles, quadrilaterals, They offer a rich new mathematical experience in at least four ways. Children start looking for such shapes in nature, all around them, and thereby discover many symmetries and acquire a sense of aesthetics. Secondly, they learn how many seemingly irregular shapes can be approximated by regular ones, which becomes an important technique in science. Thirdly, they start comprehending the idea of *space*: for instance, that a circle is a path or boundary which separates the space inside the circle from that outside it. Fourthly, they start associating numbers with shapes, like area, perimeter etc, and this technique of *quantization*, or arithmetization, is of great importance. This also suggests that mensuration is best when integrated with geometry.

An informal introduction to geometry is possible using a range of activities like paper folding and dissection, and exploring ideas of symmetry and transformation. Observing geometrical properties and inferring geometrical truth is the main objective here. Formal proofs can wait for a later stage.

6.2.3 Visual learning

Data handling, representation and visualization are important mathematical skills which can be taught at this stage. They can be of immense use as "life skills". Students can learn to appreciate how railway time tables, directories and calendars organize information compactly.

Data handling should be suitably introduced as tools to understand process, represent and interpret day-to-day data. Use of graphical representations of data can be encouraged. Formal techniques for drawing linear graphs can be taught.

Visual Learning fosters understanding, organization, and imagination. Instead of emphasizing only two-column proofs, students should also be given opportunities to justify their own conclusions with less formal, but nonetheless convincing, arguments. Students' spatial reasoning and visualization skills should be enhanced. The study of geometry should make full use of all available technology. A student when given visual scope to learning remembers pictures, diagrams, flowcharts, formulas, and procedures.

6.3 Secondary Stage

It is at this stage that Mathematics comes to the student as an academic discipline. In a sense, at the elementary stage, mathematics education is (or ought to be) guided more by the logic of children's psychology of learning rather than the logic of mathematics. But at the secondary stage, the student begins to perceive the structure of mathematics. For this, the notions of *argumentation* and **proof** become central to curriculum now.

Mathematical terminology is highly stylised, self-conscious and rigorous. The student begins to feel comfortable and at ease with the characteristics of mathematical communication: carefully defined terms and concepts, the use of symbols to represent them, precisely stated propositions using only terms defined earlier, and proofs justifying propositions. The student appreciates how an edifice is built up, arguments constructed using propositions justified earlier, to prove a theorem, which in turn is used in proving more.

For long, geometry and trigonometry have wisely been regarded as the arena wherein students can learn to appreciate this structure best. In the elementary stage, if students have learnt many shapes and know how to associate quantities and formulas with them, here they start reasoning about these shapes using the defined quantities and formulas.

Algebra, introduced earlier, is developed at some length at this stage. Facility with algebraic manipulation is essential, not only for applications of mathematics, but also internally in mathematics. Proofs in geometry and trigonometry show the usefulness of algebraic machinery. It is important to ensure that students learn to geometrically visualise what they accomplish algebraically.

A substantial part of the secondary mathematics curriculum can be devoted to consolidation. This can be and needs to be done in many ways. Firstly, the student needs to integrate the many techniques of mathematics she has learnt into a problem solving ability. For instance, this implies a need for posing problems to students which involve more than one content area: algebra and trigonometry, geometry and mensuration, and so on. Secondly, mathematics is used in the physical and social sciences, and making the connections explicit can inspire students immensely. Thirdly, mathematical modelling, data analysis and interpretation, taught at this stage, can consolidate a high level of literacy. For instance, consider an environment related project, where the student has to set up a simple linear approximation and model a phenomenon, solve it, visualise the solution, and deduce a property of the modelled system. The consolidated learning from such an activity builds a responsible citizen, who can later intuitively analyse information available in the media and contribute to democratic decision making.

At the secondary stage, a special emphasis on experimentation and exploration may be worthwhile. *Mathematics laboratories* are a recent phenomenon, which hopefully will expand considerably in future[22]. Activities in practical mathematics help students immensely in visualisation. Indeed, Singh, Avtar and Singh offer excellent suggestions for activities at all stages. Periodic systematic evaluation of the impact of such laboratories and activities[23] will help in planning strategies for scaling up these attempts.

6.4 Higher Secondary Stage

Principally, the higher secondary stage is the launching pad from which the student is guided towards career choices, whether they imply university education or otherwise. By this time, the student's interests and aptitude have been largely determined, and mathematics education in these two years can help in sharpening her abilities.

The most difficult curricular choice to be made at this stage relates to that between **breadth and depth**. A case can be made for a broadbased curriculum that offers exposure to a variety of subjects; equally well, we can argue for limiting the number of topics to a few and developing competence in the selected areas. While there are no formulaic answers to this question, we point to the Thurston remark quoted above once again.

Indeed, Thurston is in favour of breadth even as an alternative to remedial material which merely goes over the same material once more, handicapping enthusiasm and spontaneity.

> *Instead, there should be more courses available ... which exploit some of the breadth of mathematics, to permit starting near the ground level, without a lot of repetition of topics that students have already heard.*

When we choose breadth, we not only need to decide which themes to develop, but also how far we want to go in developing those themes. In this regard, we suggest that the decision be dictated by *mathematical considerations*. For instance, introducing projective geometry can be more important for mathematics as a discipline than projectile motion (which can be well studied in physics). Similarly, the length of treatment should be dictated by whether *mathematical objectives* are met. For instance, if the objective of introducing complex numbers is to show that the enriched system allows for solutions to all polynomial equations, the theme should be developed until the student can at least get an idea of how this is possible. If there is no space for such a treatment, it is best that the theme not be introduced; showing operations on complex numbers and representations without any understanding of why such a study is relevant is unhelpful.

Currently, mathematics curriculum at the higher secondary stage tends to be dominated by differential and integral calculus, making for more than half the content in Class XII. Since Board examinations are conducted on Class XII syllabus, this subject acquires tremendous importance among students and teachers. Given the nature of Board examinations as well as other entrance examinations, the manipulative and computational aspects of calculus tend to dominate mathematics at this stage. This is a great pity, since many interesting topics (sets, relations, logic, sequences and series, linear inequalities, combinatorics) introduced to students in Class XI can give them good mathematical insight but these are typically given short shrift. Curriculum designers should address this problem while considering the distribution of content between Classes XI and XII.

In many parts of the world, the desirability of having electives at this stage, offering different aspects of mathematics, has been acknowledged. However, implementation of a system of electives is dauntingly difficult, given the need for a variety of textbooks and more teachers, as well as the centralized nature of examinations. Yet, experimenting with ideas that offer a range of options to students will be worthwhile.

6.5 Mathematics and Mathematicians

At all stages of the curriculum, an element of humanizing the curriculum is essential. The development of mathematics has many interesting stories to be told, and every student's daily life includes many experiences relevant to mathematics. Bringing these stories and accounts into the curriculum is essential for children to see mathematics in perspective. Lives of mathematicians and stories of mathematical insights are not only endearing, they can also be inspiring.

A specific case can be made for highlighting the contribution made by Indian mathematicians. An appreciation of such contributions will help students see the place of mathematics in our culture. Mathematics has been an important part of Indian history and culture, and students can be greatly inspired by understanding the seminal contributions made by Indian mathematicians in early periods of history.

Similarly, contributions by *women mathematicians* from all over the world are worth highlighting. This is important, mainly to break the prevalent myth that mathematics has been an essentially male domain, and also to invite more girls to the mathematical enterprise.

7. Conclusion

In a sense, all these are steps advocated by every mathematics educator over decades. The difference here is in emphasis, in achieving these actions by way of curricular choices. Perhaps the most compelling reason for the vision of mathematics education we have articulated is that our children will be better served by higher expectations, by curricula which go far beyond basic skills and include a variety of mathematical models, and by pedagogy which devotes a greater percentage of instructional time to problem solving and active learning. Many students respond to the current curriculum with boredom and discouragement, develop the perception that success in mathematics depends on some innate ability which they simply do not have, and feel that, in any case, mathematics will never be useful in their lives. Learning environments like the one described in the vision will help students to enjoy and appreciate the value of mathematics, to develop the tools they need for varied educational and career options, and to function effectively as citizens.

Our vision of excellent mathematical education is based on the twin premises that **all students can learn mathematics and that all students need to learn mathematics.** Curricula that assume student failure are bound to fail; we need to develop curricula that assume student success. We are at a historic juncture when we wish to guarantee education for all. It is therefore a historic imperative to offer our children the very highest quality of mathematics education possible.

References

Wheeler, David, "Mathematisation Matters," For the Learning of Mathematics, 3,1; 45 - 47, 1982.

Polya, George, "The goals of Mathematical education", in ComMuniCator, the magazine of the California Mathematics Council, 1969.

"Principles and Standards for School Mathematics", National Council of Teachers of Mathematics, USA, 2000.

"The New Jersey Mathematics standards and curriculum framework", The New Jersey Mathematics Coalition, J.G. Rosenstein (Ed), 1997.

Mathematics Academic Content Standards, California State Board of Education, 2004.

"Mathematics Syllabi", Curriculum planning and development division, Ministry of Education, Singapore, 2001.

"Mathematics Learning Area" statement, Curriculum Corporation, Melbourne, Australia, 2001.

Nemetz, T., "Mathematics education in Hungary." In : I. Morris, S.A. Arora: *Moving into the twenty first century*, UNESCO Series: Studies in mathematics education, No.8., 1991, pp. 105-112.

Howson, G.A., "National Curricula in Mathematics", The Mathematical Association, University of Southampton, 1991, pp. 115-125.

Ferrini-Mundy, J., Burrill, G. and Breaux, G. (Eds), "Mathematics Education Around the World: Bridging Policy and Practice", Report of the 2001 IAS/Park City Mathematics Institute Seminar, Institute for Advanced Study, Princeton, 2001.

Buxton, Laurie. Math Panic. London: Heinemann, 1991.

Krutetskii, V.A., *The Psychology of Mathematical Abilities in School Children,* (edited by J. Kilpatrick and I. Wirszup), University of Chicago Press, Chicago, 1976.

Ahmedabad Women's Action Group, "An assessment of the school textbooks published by Gujarat State School Textbooks under NPE", IAWS conference, Calcutta, December 1990.

Fennemma, E., "Gender and mathematics: What do I know and what do I wish was known?", Fifth annual forum of the National Institute for Science Education, Detroit, May 2000.

Weisbeck, L. "Teachers' thoughts about children during mathematics instruction", PhD dissertation, University of Wisconsin, Madison, 1992.

Manjrekar, N., "Gender in the mathematics curriculum", Seminar on Mathematics and Science Education in School: Teaching practices, Learning Strategies and Curricular issues, Zakir Husain Centre for Educational Studies, JNU, New Delhi, March 2001.

Thurston,William, "Mathematical education", *Notices of the American Mathematical Society*, 37, 844-850, 1990.

Sarangapani, Padma, "A way to explore children's understanding of mathematics", *Issues in primary education*, 2(2), 2000.

Subramaniam, K., "Elementary Mathematics: A Teaching Learning Perspective", *Economic and Political Weekly, Special issue on the Review of Science Studies*: Perspectives on Mathematics, Vol 37, no. 35, 2003.

Nunes, T. and Bryant, P.E., Children Doing Mathematics, Oxford, Blackwell, 1996.

Verma, V.S. and Mukherjee, A., "Fractions – towards freedom from fear", National Seminar on Aspects of Teaching and Learning Mathematics, University of Delhi, January 1999.

Singh, Hukum, Avtar, Ram and Singh V.P., *A Handbook for Designing Mathematics Laboratory in Schools*, NCERT, 2005.

Sarangapani, Padma, and Husain, Shama, "Evaluation of Maths Lab at Samuha-Plan, Deodurg", Report, National Institute of Advanced Studies, Bangalore, April, 2004.

COMPUTER APPLICATIONS IN MATHEMATICS TEACHING
– A RESUME

–

Mathematically eLearning
Environments

Introduction

Mathematically engaging eLearning activities to empower teachers and students to recognise and harness the mathematical power of open-ended computer applications, namely Microsoft Software. The activities have been created in **Microsoft Word, Excel and PowerPoint** and can be made active on all computers with this software installed.

Others Software's can be used:

PageMaker, Photoshop, Micro Media Flash Etc.,

Using ICT in Maths

- The availability of ICT has changed the nature of teaching and learning in
- Maths calculators have become more advanced, allowing users to perform increasingly complex functions.
- A range of portable devices exists which allow pupils to collect data, and manipulate it using spreadsheets and databases. Multimedia software programs focus on specific units of study, bringing dynamic movement, sound and graphics to pupils' learning.
- Applications of ICT to maths, and where used,were the cause of significant changes in maths teaching.

Cont...

- Statutory requirements for the use of ICT in maths are enshrined in the National Curriculum, and include the effective use of calculators, production of charts and graphs, and work with geometrical shapes.
- The ICT contributes to the making of maths teaching easier, for example:
  1.problem solving tasks
  2. Practicing of number skills
  3. exploring patterns and relationships.
- There are many specific forms in which ICT may be used in maths teaching, including calculators, spreadsheets, databases and online, interactive resources.

ICT Integrated with Maths

- To enable learners to value the advantages, disadvantages, potentials and uses of computers
- The idea of integrating the use of computer technologies into all areas of learning is a product of the current age of teaching and learning.
- The learning of mathematics within a computerised environment is no exception.
- There are many ways to manufacture a product using a computer. The strategy with this book / CD has been to fuse multi-skilling using Microsoft applications on computers within a context for learning and working mathematically.

Benefits of ICT Integrated with Maths

- Both students and teachers will be familiar with the interface of Microsoft Applications.
- The emphasis in developing these resources, has been to use and work with the capabilities of mathematically able / productivity software.
- Teachers working mathematically in new ways within these environments.
- It is proposed that regular use of mathematically able software increases general dexterity with the software as a teaching and learning tool and more specifically, a mathematical teaching and learning tool.
- Emphasising ICT use in the curriculum is focused on comprehension and demonstration of mathematics as well as doing computations and solving problems.

Teachers can Maximise the Impact of ICT in Maths Teaching by:

- Using ICT as a tool in working towards learning objectives
- Developing a knowledge of the multimedia software available
- Considering how to provide access to ICT resources for all
- Incorporating the use of portable ICT equipment in teaching.

Why use Microsoft Applications?

- All schools have and use the Microsoft Office applications
- The activities actually integrate the technology into mathematical learning, transforming the software into mathematical learning technologies.
- Schools need NOT expend more money on commercially available software for mathematics learning and teaching. (e.g. Microsoft Applications) to enrich mathematical learning.
- An inclusive approach to using ICT to teach, learn, design and practise mathematics.
- For teachers wishing to polish up their skill base in using Microsoft Office as educational technology tool.

The development of information education

- Considering quick development of ICT, it is very important to lead students towards self-study and active development of their knowledge in this field
- It has been created an electronic educational course for teaching of the subject ICT applications in teaching Mathematics which covers numerical and graphical tools of program MS Excel for solution of mathematical tasks

Project E-learning in Preparation of Mathematics Teachers:

- The determining of mathematics teachers' competence for use of ICT and working-out of system of information education of mathematics teachers
- The projection and realization of courses with using of ICT and educational materials for preparation of future mathematics teachers and also as a part of further education of mathematics teachers
- The exploitation of ICT for modernization of traditional methods and forms of mathematics teaching and learning

LIST OF TOPICS

- Introduction Arithmetic
- Real numbers
- Exponents & Radicals
- Fractions of Numbers, for more clink here
- Multiplication
- Squares
- Pythagorean theorem
- Linear equations
- Linear-inequalities
- Geometric patterns
- Trigonometry
- Compound Angle Formula
- Unit Circle & Trig Equations
- Quadratic Equations
- Introduction to Functions
- Differential Equations

FLASH

A RICH MEDIA IN CREATING SOME SYMBOLS FOR MATHEMATICS TEACHING

GETTING STARTED

Objectives

At the end of this session, you will be able to

- Identify the need for Flash MX
- Open Macromedia Flash MX
- Identify the various parts of the Flash screen
- Identify the different file formats available in Flash

Flash MX, developed by Macromedia, Inc., is the most popular software that is used for developing interactive animations and attractive graphics. This software provides various tools to develop and design complex animations easily. Flash MX also provides various powerful methods to create your original artwork or import artwork from other applications.

Flash MX is no longer just an animation tool, but has grown into a powerful system for creating Web applications.

Applications of Flash MX

Flash MX is a powerful tool used by programmers and designers to develop contents both on and off the Web. It is a user-friendly software with clearly structured functionality. This software is used to,

- Develop interactive animated Websites.
- Create Interactive games.
- Develop animated cartoons.
- Create movie trailers.

Therefore, using the strong multimedia capabilities of Flash, you can create your own home page.

Image Formats

Computers display graphics in two image formats.

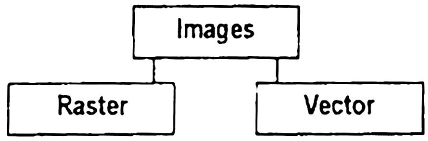


Figure 1: Image formats

Raster Images

Raster images are also called *Bitmap images*. Bitmap images are made up of a combination of coloured dots called pixels. A *pixel* is a combination of three colours Red, Green and Blue as a single colour. These images are resolution dependent. Hence, they lose clarity if displayed at a different resolution. When the bitmap images are highly magnified, the edges will get jagged as pixels. Most of the static images you see on the Web are bitmap images.

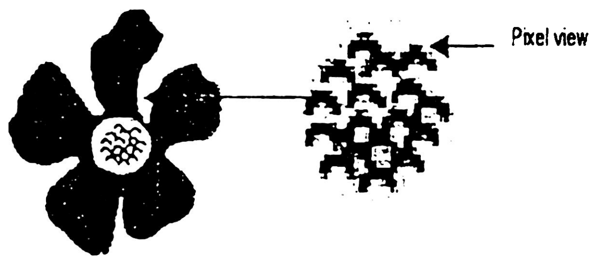


Figure 2: Magnified Bitmap image

In Figure 2, the centre of the flower, created in bitmap format is magnified. When the flower is magnified, the image appears jagged and the pixel view is seen.

> **Note**
> Resolution refers to the number of pixels that can be accommodated within a unit area.

Vector Images

Vector images are made up of lines and curves called *vectors*. These images are resolution independent and hence can be scaled to any size or resolution without losing details or clarity. A vector image can be moved, resized and reshaped without changing its quality of appearance. Vector images occupy very less file size when compared to bitmap images. Flash creates animations in vector format and hence Flash files take less time to load.

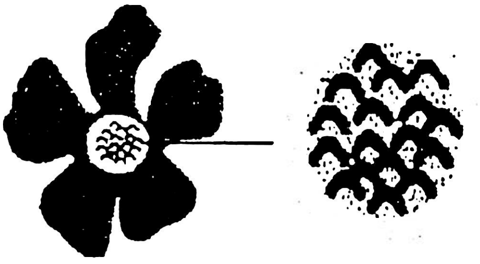


Figure 3: Magnified Vector Image

In Figure 3, the centre of the flower, created in vector format, is magnified. The image retains crisp lines when magnified and the details are not lost.

Animation

Flash is used to create complicated and visually stunning animations easily. Flash animations make the graphics created more attractive by making it move. In Flash, you can make the graphics change shape, move, fade in or out.

Any animation file created using Flash is called a *movie*. Movies can be an integration of a user interface, static objects, movement, sound, and other animated effects.

The two methods of creating animations in Flash are:

- Frame-by-frame animation
- Tweening animation

Frame- by- frame Animation

In this method, the different postures of the required object are created manually to complete the gap between the initial and final position of animation. This type of animation is time consuming but best suited for creating complex animation effects.

Tweening Animation

In this method, only the initial and final positions of the object to be animated are created and Flash creates the intermediate postures to complete the animation. Tweening is easier to create than the frame-by-frame animation.

Exploring Flash MX

To open the Flash MX software under Windows XP,

1. Click the *Start* button.
2. Choose All Programs, Macromedia, Macromedia Flash MX.

The Flash Screen

The Flash screen is displayed (Figure 4) when you open the Macromedia Flash MX software.

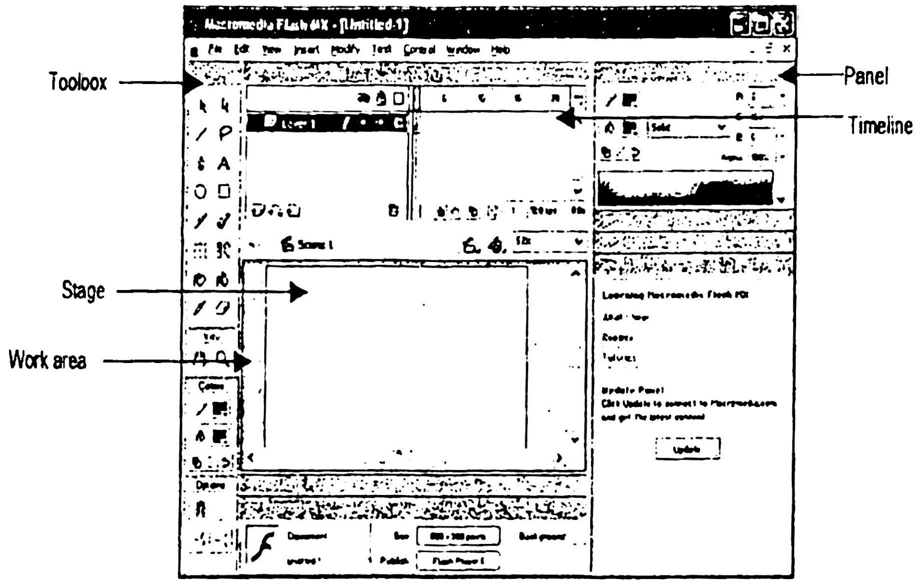


Figure 4: Flash MX screen showing the screen elements

When you open the Flash screen, it displays the following screen elements:

- Stage
- Toolbox
- Panels
- Timeline
- Library
- Property Inspector

The Stage

The white rectangular area in the centre of the screen is called the *Stage*. The Stage is the place where you compose or import the contents for the movie. It is the place where the final movie is displayed. The animated objects created may move across, in and out of the Stage, but only the objects inside the Stage area are visible in the movie. The gray area around the Stage is called *Work area*.

The Toolbox

The Flash Toolbox provides various tools for creating and manipulating the contents on the Stage. The Toolbox is present at the left-hand side of the screen, when you open the Flash

MX screen for the first time. It can be hidden or moved to any place across the Flash screen. You can create a Flash movie using the various drawing and editing tools in the Toolbox.

The Toolbox is divided into four sections namely:

- Tools
- Views
- Colors
- Options

Panels

Panels are the movable sections on the Flash screen. They help to view, organise and change the behaviour of the objects. The panels that are visible on the Flash MX screen by default are:

- Color Mixer
- Color Swatches
- Components
- Answers

You can organise your Flash screen by hiding or moving the panels. To collapse/expand the panel window, click anywhere on the title bar.

Apart from the four panels, Flash MX provides various other panel sets .You can view the panels that you need for a specific task and hide the other panels.

To view/hide a panel,

1. Click the *Window* menu.
2. Choose the required panel name from the menu list displayed.

You can drag the panel and place it anywhere on the screen. Figure 5, displays the Color Swatches panel.

Note	To view/hide all the panels, press the Tab key.

To move the panel,

1. Place the cursor at the dotted portion on the left-hand side of the panel's title bar.
2. Drag the panel, when the mouse pointer changes to a four-headed arrow.

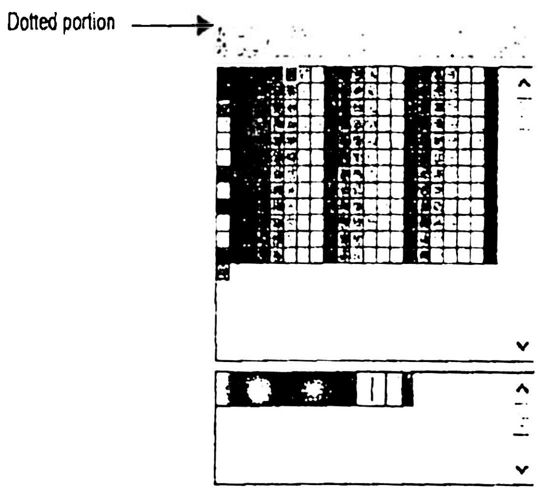


Figure 5 Color Swatches panel

You can reset the panels to the original places and the view the default panels.

To arrange the panels in the original places,

1. Click the *Window* menu.
2. Choose Panel Sets, Default Layout.

To close all the panels, choose *Window* menu and select *Close All Panels*.

The Timeline

The *Timeline* is the most important component of the Flash screen. It is used to control the contents of the movie at different phases of time. By default, the Timeline appears at the top of the Flash screen. A Flash movie creates the illusion of continuous movement by quickly displaying a sequence of still images or postures. The Timeline helps in fixing the time gap between each of the posture. Figure 6, displays the Timeline.

The Timeline contains components like frames and layers. The Timeline divides the movie into frames. *Frames* are essential for creating and arranging the contents of the movie.

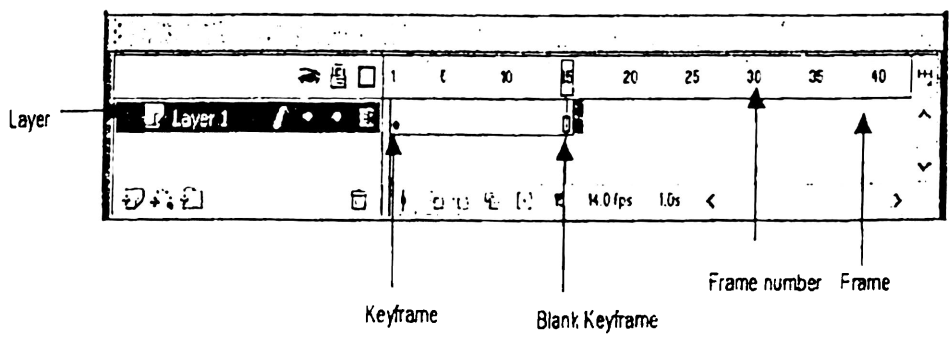


Figure 6 The Timeline

The top portion of the Timeline is marked with numbers, which corresponds to the segments in the next line. Each segment in the second line is called a *frame*.

Keyframe

Keyframes are markers in the Timeline, indicating the start and end point for various animations (refer figure6). It is a special frame in the Timeline that indicates there is something significant in that frame. For example, in the tweening animation of a moving car, the keyframes will contain the initial and final images of the moving car.

Note

```
In frame-by-frame animation, each frame will be a
keyframe.
```

Layers

Layers are transparent holders, which hold the objects in a movie. They are used to create separate objects in the Timeline and hence help in better control and organisation of the movie. Layers are very useful for creating complex animations easily.

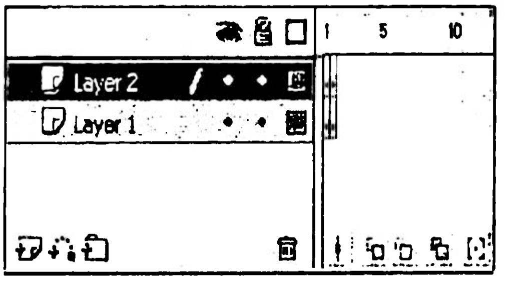


Figure 7: Timeline showing Layers

There can be multiple layers in a movie each containing objects, which act independently. You can draw and edit objects on one layer without affecting objects on another layer.

For example, you can create an animation of a moving ship with an ocean background kept constant throughout the movie. Here, the ship is created and animated separately in one layer and the ocean background in other. When you run the movie, the ship moves independently and the background remains static.

Library in Flash

The *Library* in Flash MX stores and organises various symbols created in a file. It also stores imported files such as video clips, sound clips and bitmaps.

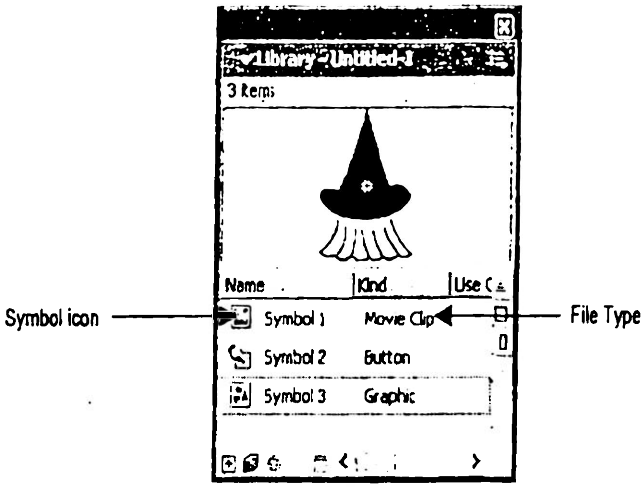


Figure 8 The Library panel

The names of all the symbols in the library are displayed in the Library panel. This panel helps you to view and organise the items according to your comfort. Every item in the library will have an icon next to its name, which indicates the type of the file. Figure 8, displays three items in the Library panel.

> **Note**
>
> To view the Library panel choose *Window, Library* or press F11 key.

Property Inspector

The *Property Inspector* is used to edit the properties of the selected object or tool. It helps in changing the attributes of an object in a document, without accessing the specific panel or menu.

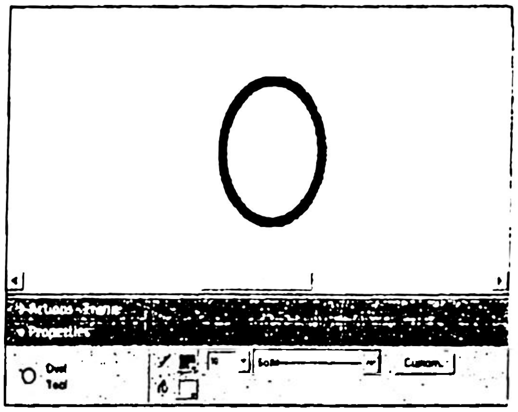


Figure 9 Sample image explaining the Property Inspector

In Figure 9, the Property Inspector displays the properties of the elliptical object in the Stage. Here you can edit the fill colour, line colour, thickness and style easily.

File Formats

The Flash files can be saved in *FLA* or *SWF* formats. The editable version of the movie is stored with the *.fla* file extension. These files contain information that is used to develop and design interactive content.

The *SWF* format or the shock wave file format is the final version or movie that the Flash player displays. Hence, Flash player should be installed to play SWF files. It is the compressed, non-editable form of the FLA document that can be uploaded on Web. The SWF format occupies less memory and can be easily downloaded. Most of the browsers have in-built Flash player and thus helps to play the shock wave file on the Internet.

Opening a Flash File

To open the existing Flash file,

1. Choose *File*, *Open*.
2. Select the file name from the Open dialog box.
3. Click the *Open* button.

To play the Flash movie, press *Ctrl* + *Enter* key. You can also open the file by selecting the file name from the *Recent list* displayed at the end of the *File* menu.

Closing a File and Exiting Flash

To close a Flash file,

- Choose the *File* menu and select the *Close* option (or)
- Click the *Close* button on the top-right corner of that file.

To exit the Flash application,

- Choose the *File* menu and select the *Exit* option.

Tip: To close a FLA file, press the Ctrl + W key combination

Now You Know

- Flash MX is used for developing interactive animations and attractive graphics.
- Flash MX is a powerful tool used to create Web applications.
- Computers display graphics in two formats.
    - Raster images are also called *Bitmap images*. These images are resolution-dependent.
    - Vector images are resolution-independent and do not loose details, when scaled to any size.
- Flash animations make the graphics created, more attractive and interactive by making it move.
- The two methods of creating animations are Frame-by-frame animation and Tweening animation.
- When you open Flash MX, the following screen elements are displayed:
    - Stage - The white rectangular area in the centre of the screen.
    - Toolbox - Provides various tools for creating and manipulating the contents on the Stage.
    - Panels - Helps to view, organise and change the behaviour of the objects.
    - Timeline - Used to control the contents of the movie at different phases of time.
    - Keyframe- Markers in the Timeline indicating the start and end point for various animations.
    - Layers – Transparent holders, which hold the objects in a movie.
    - *Library* - Stores and organises the various symbols created and imported.
    - Property Inspector - Used to edit the properties of the selected object or tool.
- Flash files can be saved in the *FLA* or *SWF* formats.
    - FLA is the editable version of the movie.
    - SWF is the compressed, non-editable format of the movie.
- To play a Flash movie, press Ctrl + Enter keys.
- To close a Flash file, choose *File*, *Close* and to exit Flash choose *File*, *Exit*.

Drawing in Flash MX

2

Objectives

At the end of this session, you will be able to

- View or hide the Toolbox
- Identify the different sections of the Toolbox
- Save a file
- Use the various tools in the Toolbox

Toolbox

The Toolbox contains various tools for drawing and editing the movie content in Flash. The tools in the Toolbox helps to draw, paint, type, select and modify artwork.

By default, the Toolbox is present in the left-hand side of the Flash screen. You can move the Toolbox anywhere around the screen, or hide it.

Dragging or hiding the Toolbox

- To move the Toolbox, click its title bar and drag it.
- To move the Toolbox to its default position, double-click its title bar.
- To view/hide the Toolbox, choose *Window*, *Tools*.

Tip

The shortcut key to view/hide the Toolbox is Ctrl + F2

Figure 10: The Flash MX Toolbox

The Toolbox is divided into four sections (Refer Figure 10):

Tools Section

The *Tools* section has various tools to create and manipulate the visual elements on the Stage. These tools can be used to draw pictures, create text elements, select objects, move objects and edit the existing graphics.

View Section

The *View* section contains tools for zooming and changing the Stage position. You can use the Zoom tools and magnify or reduce the image view on the Stage. The changes in the view and position will affect only the display on the Flash screen and does not have any effect on the final movie

Colors Section

The *Colors* section contains tools for applying colours to the objects. These tools are used to control the fill and stroke colour of the objects. You can also set default colours, switch off the stroke or fill colours and swap the colours using the tools provided in this section.

Note

Filling, places colour inside the selection.
Stroking, adds a stroke of paint over the selected path of an object.

Options Section

The *Options* section displays modifiers that can change the properties of the selected tool. For example, in Figure 10, the Arrow Tool is selected and the Options section displays the various modifiers for the selected tool.

Note

```
All tools do not have modifiers and the Options
section appears blank for such tools.
```

The Arrow Tool

The *Arrow Tool* is used for creating and modifying the graphics on the Stage. This tool is used to select and move objects across the Stage. The Arrow Tool is the default tool selected when you open the Flash MX screen.

To select the Arrow Tool,

- Click the Arrow Tool icon in the Tools section (or)
- Press the alphabet V on the keyboard.

You can also reshape an object by dragging at any point on the line using the Arrow Tool.

The Rectangle Tool

The *Rectangle Tool* is used to draw rectangles with square and rounded comers. This tool can create filled and stroked rectangles.

To draw a rectangle,

1. Click the Rectangle Tool ☐
2. Position the cursor at the required point and drag the mouse diagonally.

Tip

```
You can also select the Rectangle Tool by pressing R
on the keyboard.
```

Consider the following example:

1. Select the Rectangle Tool in the Toolbar.
2. Choose black colour for the outline (Stroke colour) of the rectangle from the *Stroke Color control* in the Colors section.
3. Click the *Fill Color control* in the Colors section and set a light green colour for the fill.
4. The *Property Inspector* displays the properties for the *Rectangle Tool*. Set the borderline thickness to 7 by dragging the Stroke height slider in the Property Inspector.

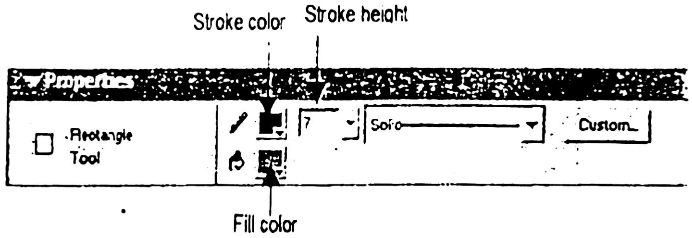
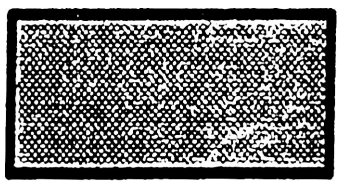


Figure 12: Property Inspector showing Rectangle Tool properties

5. Next, click on the Stage. Hold down the left mouse button and drag to draw the rectangle. The rectangle appears as displayed in Step1.

Step1

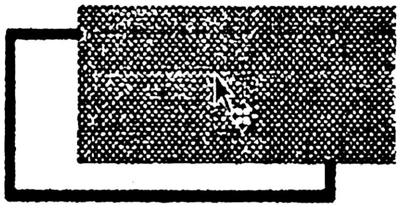
6. Select the *Arrow Tool* by clicking its icon in the Tools section of the Toolbox.
7. Click on the light green portion of the rectangle to select it. The filled portion alone gets selected as shown in Step 2.



Step 2

8. Click the *Fill color* control in the Properties Inspector and change the fill colour to pink.

9. Drag the selected area (filled portion) as shown in Step 3 using the *Arrow Tool*. Now release the mouse button.



Step3

Step 3 displays the filled portion of the rectangle separated from the outline (stroke).

Note

```
To draw a perfect square using the Rectangle Tool,
press the Shift key while dragging.
```

You can also draw a rounded rectangle using the Rectangle Tool. When you select the Rectangle Tool, the *Round Rectangle Radius* modifier gets displayed in the Options section of the Toolbox. This modifier controls the extent to which the corners of the rectangle can be rounded off.

To draw a rounded rectangle,

1. Select the Rectangle Tool.
2. Click the *Round Rectangle Radius* modifier in the Options section of the Toolbox.
3. The Rectangle Settings dialog box is displayed.
4. In the Corner Radius text box, enter the value by which the edges of the rectangle have to be rounded.
5. Click the OK button.

Saving a File

The Flash MX files created can be saved as FLA documents. The movie contents can also be saved as a Flash 5 document.

To save a Flash file,

1. Choose *File*, *Save* option.
2. Type a filename in the File name textbox of the Save As dialog box.

3. Click the *Save* button.

Tip	You can also save the file by clicking the Save icon in the Standard Toolbar.

The Line Tool

The *Line Tool* is used to draw straight and angled lines.

To draw a line,

1. Select the Line Tool from the Toolbox. /
2. Position the cursor at the starting point, drag it till the end point and then release the mouse.

Tip	You can also select the Line Tool by pressing N on the keyboard.

The Property Inspector displays the properties of the *Line Tool* when it is selected. The line colour, thickness and style can be set in the Property Inspector displayed. When you click the Custom button, the Stroke Style dialog box is displayed. This dialog box gives a preview of the line with the selected properties, as shown in Figure 13.

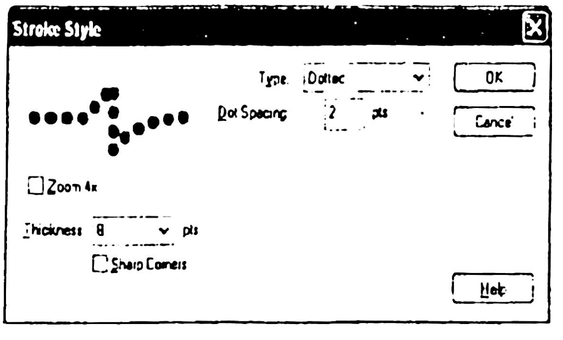


Figure 13: Stroke Style dialog box

Note	To create a horizontal, vertical or a diagonal line using the Line Tool, press Shift key and drag.

The Paint Bucket Tool

The *Paint Bucket Tool* is commonly called *Fill Tool.* It is used to fill empty enclosed shapes and also to change the fill colour of existing shapes.

To fill colour, using the Paint Bucket Tool,

1. Select the *Paint Bucket Tool.*
2. Click the Fill Color control and select the required colour.
3. Click the area that has to be filled.

Tip	You can also select the Paint Bucket Tool by pressing K on the keyboard.

You will not be able to apply fill colour, when the shape drawn is not completely closed. The Paint Bucket Tool is associated with a *Gap Size* modifier, which helps to fill shapes that are not fully enclosed.

To fill the area that is not entirely enclosed,

1. Select the Paint Bucket Tool.
2. Click the Gap Size modifier in the Options section of the Toolbar.
3. Choose the required gap size option.
   - Close Small Gaps
   - Close Medium Gaps
   - Close Large Gaps

If you want to close the gaps manually before filling the area, choose the *Don't Close Gaps* option from the Gap Size Modifier.

The Pencil Tool

The *Pencil Tool* is used to draw freeform lines similar to a normal pencil. You can draw straight or curved lines using the different modes available for the Pencil Tool.

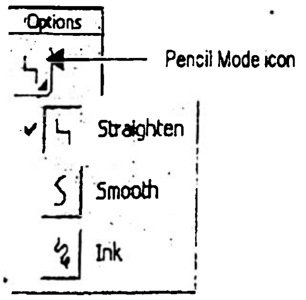
To draw using the Pencil Tool,

1. Select the Pencil Tool.
2. Choose the stroke colour, line thickness and style from the Property Inspector.

3. Click anywhere on the Stage to set the starting point.
4. Drag the cursor till the point to which the line has to be drawn.

> **Tip**
>
> You can also select the Pencil Tool by pressing Y on the keyboard.

There are three modifiers associated with the Pencil Tool. These modifiers control the shape of the freeform line drawn. The *Pencil Mode* icon will be displayed in the Options section of the Toolbar when this tool is selected. Click the small black triangle present at the bottom-right corner of the Pencil Mode icon to view the other modes.



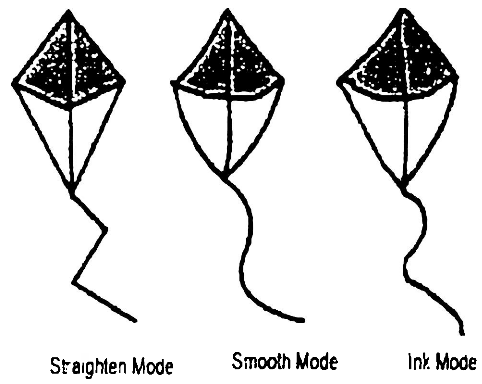


Figure 14 Objects drawn with various Pencil modes

Straighten Mode:

The Straighten mode is used to draw lines and shapes with sharp curves. This mode will flatten out the curves in the lines drawn.

Smooth Mode:

The Smooth mode is used to draw smooth curved shapes and lines. This mode will make the shaky freehand lines neat.

Ink Mode:

In this mode, no modification is applied to the freehand line drawn. This mode is used when you want to view the line exactly the way it is drawn.

In Figure 14, the kites have been drawn using the three modes available for the Pencil Tool. The fill colour is applied using the Paint Bucket Tool. Compare the kites drawn to notice the difference between the three modes.

The Pen Tool

The *Pen Tool* is used to draw smooth flowing curves or straight edged lines precisely.

To draw using the Pen Tool,

1. Select the Pen Tool.
2. Position the pen pointer where you want to begin drawing, and click to set the first anchor point.
3. Click again and set the next anchor point to get a straight line.

To get a smooth curve, keep the left mouse pressed and drag the anchor point. A closed path drawn with a Pen Tool will give a filled shape and an open path will just apply a stroke.

Tip

```
You can also select the Pen Tool by pressing P on the
keyboard.
```

The Oval Tool

The *Oval Tool* is used to draw ellipses and circles.

To draw an ellipse,

1. Select the Oval Tool. O
2. Set the line thickness and border style in the *Stroke Height* and *Stroke Style* drop-down list of the Property Inspector.
3. Drag the mouse pointer over the Stage.

To draw a circle using the Oval Tool, keep the Shift key pressed and drag the mouse pointer over the Stage.

Tip

```
You can also select the Oval Tool by pressing O on the
keyboard.
```

The Brush Tool

The *Brush Tool* is used to paint with the specified colour, without any border. This tool creates artistic effects and enables you to draw objects easily.

To paint using the Brush Tool,

1. Select the Brush Tool.
2. Choose the Brush Size and Brush Style from the Brush Tool's modifiers.
3. Click and drag the mouse cursor to apply the fill on the Stage.

> **Tip**
> You can also select the Brush Tool by pressing B on the keyboard.

The Options section displays four options for the Brush Tool,

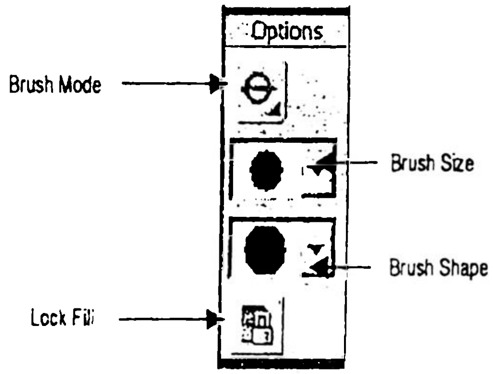


Figure 15: Brush Tool modifiers

Brush Mode Modifier:

Brush Modes in the Options section, controls the mode in which the brush strokes are painted. This option has five modifiers in it.

Paint Normal - Used to paint over any existing object on the screen.

Paint Fills - Used to paint the filled and empty areas without painting the stroke.

Paint Behind - Used to paint the blank areas on the screen without painting over any object.

Paint Selection- Used to paint only the selected areas on the screen.

Paint Inside - Used to fill an empty area and does not affect any filled area. It paints only the fill, where you start painting. This option works smart and paints neatly inside the lines.

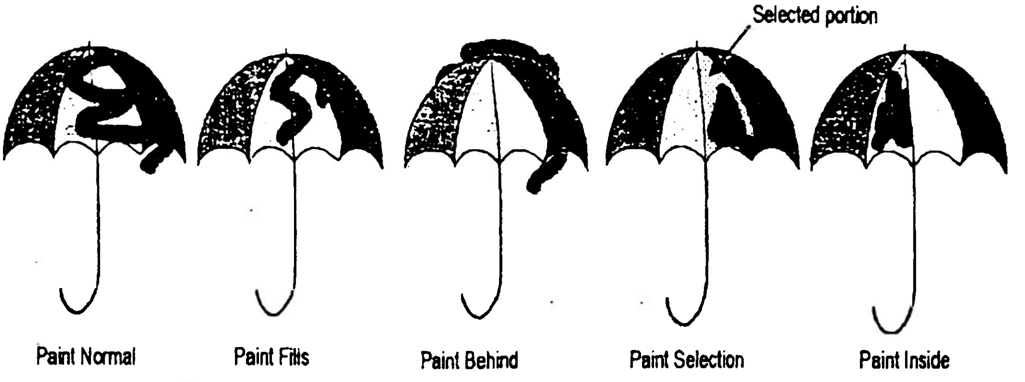


Figure 16: Images explaining Brush Tool modes

Figure 16,explains the various brush modes. You can notice, in image 2 (Paint Fills) the stroke area is not painted. In image 4 (Paint Selection), only the selected area is painted and in image 5 (Paint Inside) the option paints neatly inside the lines.

Brush Size Modifier:

This modifier is used to set the width of the brush. To change the brush width, choose a brush size from the Brush Size drop-down list.

Brush Shape Modifier:

This modifier is used to change the style of the paint fills. To change the brush shape, choose a shape from the Brush Shape drop-down list.

The Zoom Tool

The *Zoom Tool* is used to view an image or the entire Stage at different magnification levels.

To magnify or reduce the view,

1. Click the Zoom Tool.
2. Select the required modifier from the Options section of the Toolbar.
   - Enlarge modifier - To zoom in the view of the Stage.
   - Reduce modifier - To zoom out the view of the Stage.
3. Keep clicking, till the image is zoomed to the required size.

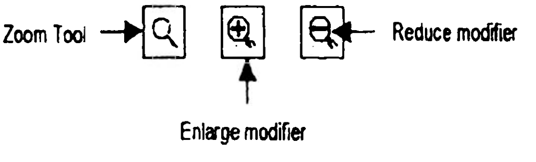


Figure 17: Zoom Tool and its modifiers

You can also select the Zoom Tool by pressing M or Z on the keyboard.

Tip

The Ink Bottle Tool

The *Ink Bottle Tool* is used to apply stroke colour for the objects. This tool works similar to the Paint Bucket Tool. The Paint Bucket Tool is used to change the fill colour, whereas the Ink Bottle Tool is used to change the border colour.

To create a border for a fill,

1. Click the *Ink Bottle Tool.*
2. Set the appropriate colour, height and style for the stroke in the Property Inspector.
3. Click on the object or fill for which you want to create a border.

You can also select the Ink Bottle Tool by pressing S on the keyboard.

Tip

The Text Tool

The *Text Tool* is used to type text on an object or the Stage.

To type text using the Text Tool,

1. Select the Text Tool. A
2. Click on the place where you want to type the text.
3. Type the text in the textbox.
4. Select the text and change its font style, colour and size in the Property Inspector.

You can also select the Text Tool by pressing T on the keyboard.

Tip

The Eyedropper Tool

The *Eyedropper Tool* is used to pick colours and styles from an existing object, for applying the same to the newly created object. You can also pick a bitmap or gradient fill using this tool.

To pick a colour or style using Eyedropper Tool,

1. Select the Eyedropper Tool.
2. Click the stroked or filled area of an existing object to pick its attributes.

The Eyedropper Tool changes to:

- A Paint Bucket Tool when you pick a fill colour from an object.
- An Ink Bottle Tool when you pick a stroke colour from an object.

Tip

You can also select the Eyedropper Tool by pressing I on the keyboard.

Some More Tools

Tool	*Functions*
Eraser Tool (E)	Used to remove the strokes and fills of the underlying object.
Free Transform Tool (Q)	Used to rotate, resize and scale an object.
Fill Transform Tool (F)	Used to transform a gradient Tool.
Subselection Tool (A)	Used to alter specific points of a curve.
Lasso Tool (L)	Used to make a freehand selection of an object.

Table1: Tools and functions

Now You Know

- The Flash Toolbox contains various tools for drawing and editing the movie content.
- The Toolbox is divided into four sections namely, Tools, View, Colors and Options.
- This Arrow Tool is used to select objects for editing and to move objects across the Stage.
- The Rectangle Tool is used to draw rectangles with square and rounded corners.
- The Line Tool is used to draw lines straight and angled lines easily.
- The Paint Bucket Tool is used to fill empty enclosed shapes and change the fill colour of existing shapes.
- The Pencil Tool is used to draw freeform lines. You can also straight or curved lines using the different modes available for this tool.
- The Pen Tool is used to draw smooth flowing curves or straight edged lines precisely.
- The Oval Tool is used to draw ellipses and circles.
- The Brush Tool is used to paint fills with the specified colour without any border.
  - The Brush Mode modifier in the Options section controls the mode in which the brush strokes are painted.
- The Zoom Tool is used to view an image or the entire Stage at different magnification level.
- The Ink Bottle Tool is used to apply stroke colour for the objects.
- The Text Tool is used to type text on an object or the Stage.

THE HISTORY OF SINE AND COSINE FUNCTIONS

The word Sine is originated from the Latin term 'SINUS', which meant 'bay' or 'bosom of garment' . This is a translation of the Arabic word "jaib" meaning the same thing. It is not known how this Arabic term originated. Some believe that it come from the Hindu (Sanskrit) word 'jiva' (the first meaning of which is bowstring; in geometry it meant chord of an arc).But Sine in Hindu terminology is designated by 'ardha-jiva' which means 'half chord'.

The name 'Cosine' appeared only at the beginning of the 17^{th} century as a contraction of the term 'complement sinus' (Sine of the complement), which indicated that the cosine of an angle A is the sine of the complementary angle.

The term 'tangent' and 'secant' (which are translated from the Latin, mean 'contracting and 'cutting') were introduced in 1583 by the German Scholar Finck.

Literal symbolism, which in Algebra come in at the end of the 16^{th} century, was established in Trigonometry only in the middle of 18^{th} century thanks to the efforts of the great Euler (1707 – 1783), who gave trigonometry its modern aspect. The quantities sinx, cosx etc. were regarded by him as functions of a number x, the radian measure of the appropriate angle. Euler assigned to the number x all possible values; positive, negative and even complex numbers also. He also introduced the inverse trigonometric functions.

FIELDS MEDALS IN MATHEMATICS

The Fields Medals are commonly regarded as mathematics' closest analog to the Nobel Prize (Which does not exist in mathematics), and are awarded every four years by the International Mathematical Union to one or more outstanding researchers. **"Fields Medals"** are more properly known by their official name, ***"International medals for outstanding discoveries in mathematics".***

The Field Medals were first proposed at the 1924 International Congress of Mathematicians in Toronto, where a resolution was adopted stating that at each subsequent conference, two gold medals should be awarded to recognize outstanding mathematical achievement. ***Professor J.C. Fields***, a Canadian mathematician who was secretary of the International Congress of Mathematicians held in Toronto, later donated funds establishing the medal which were named in his honor. Consistent with Fields' wish that the awards recognize both existing work and the promise of future achievement, it was agreed to restrict the medals to **mathematicians not over forty** at the year of the Congress. In 1966 it was agreed that, in light of the great expansion of mathematical research, **up to four medals** could be awarded at each Congress.

Each medal carries with it a cash prize of 1500 Canadian dollars. The first two such medals were presented at the Oslo Congress in 1936. After an interruption caused by War, two medals have been presented at each of the Congresses in 1950, 1954, 1958, 1962, 1974 and 2002; four medals at each of the Congresses in 1966, 1970, 1978, 1990, 1994, 1998 and 2006; and three medals at each of the Congresses in 1982 and 1986.

The Fields Medal is made of gold, and shows the head of Archimedes (287-212 BC) together with a quotation attributed to him : *"Transire suum pectus mundoque potiri "* ***("Rise above oneself and grasp the world").*** The reverse side bears the inscription: *"Congregati ex toto orbe mathematici ob scripta insignia tribuere"* ***("the mathematicians assembled here from all over the world pay tribute for outstanding work")***

Why no Nobel prize for Mathematics ?

Nobel prizes were created in the will of the Swedish chemist and inventor of dynamite Alfred Nobel, but Nobel, who was an inventor and industrialist, did not create a prize in mathematics because he was not particularly interested in mathematics or theoretical science. In fact, his will speaks of prizes for those "inventions or discoveries" of greatest practical benefit to mankind. While it is commonly stated that Nobel decided against a Nobel Prize in math because of anger over the romantic attentions of a famous mathematician (often claimed to be Gosta Mittag-Leffler) to a woman in his life, there is no historical evidence to support the story. Furthermore, Nobel was a lifelong bachelor, although he did have a Viennese woman named as his mistress (Lopez-Ortiz).

Note : In the year 1997, film "**GOOD WILL HUNTING**", fictional MIT professor Gerald Lambeau (played by Stallan Skarsgard) is described as having been awarded a Fields medal for his work in combinatorial mathematics.

LIST OF FIELDS MEDAL WINNERS IN MATHEMATICS

Sl.No	YEAR	WINNERS
1)	1936	a) Lars Valerian Ahlfors (Hardvard University) **Sub:** Riemann Surfaces of Inverse Functions b) Jesse Douglas (Massachussetts Institute of Technology) **Sub:** Work on the Plateau problem
2)	1950	a) Laurent Schwartz (University of Nancy) **Sub:** Theory of Distributions b) Atle Selberg (Institute of Advanced Study, Princeton **Sub:** Elementary proof of prime number theorem
3)	1954	a) Kunihiko kodaira (Princeton University) **Sub:** Harmonic integrals & Algebraic varieties b) Jean-Pierre Serre (University of Paris) **Sub:** Cohomology & Sheaf Theory
4)	1958	a) Klaus Friedrich Rot (University of London) **Sub:** Analytic Number Theory b) Rene Thom (University of Strasbourg) **Sub:** Cobordism Theory in Differential Topology
5)	1962	a) Lars V.Hormander (University of Stockholm) **Sub:** Linear Partial Differential Operators b) John Willard Minor (Princeton University) **Sub:** Differential Topology
6)	1966	a) Michael Francis Atiyah (Oxford University) **Sub:** Index Theorem for Elliptic Operators b) Paul Joseph Cohen (Stanford University) **Sub:** Foundations of Mathematics- Forcing c) Alexander Grothendieck (University of Paris) **Sub:** Algebraic Geometry- Schemes d) Stephen Smale (University of California, Berkelery) **Sub:** Dynamic Systems-Structural Stability
7)	1970	a) Alan Baker (Cambridge University)

Sub: Analytic Number Theory- Transcedental Numbers
b) Heisuke Hironaka (Harvard University)
Sub: Algebraic Geometry- Resolution of Singularities
c) Serge P. Novikov (Moscow University)
Sub: Topological Invariance of Pontrjagin class
d) John Griggs Thompson (Cambridge University)
Sub: Finite Simple Groups

8) 1974 a) Enrico Bombieri (University of Pisa)
Sub: Number Theory & Algebraic Geometry
b) David Bryant Mumford (Hardverd University)
Sub: Algebraic Geometry

9) 1978 a) Pierre Rene' Deligne (Institute des Hautes Etudes Scientifiques)
Sub: Weil's Conjecture on Riemann Hypothesis
b) Charles Louis Fefferman (Princeton University)
Sub: Multi Dimensional Complex Analysis
c) Gregori Alexandrovitch Margulis (Moscow University)
Sub: Structure of Lie Groups
d) Daniel G. Quillen (Massachusettes Institute of Technology)
Sub: Serre's Conjecture in Algebraic K-Theory

10) 1982 a) Alain Connes (Institut des Hautes Etudes Scientifiques)
Sub: Operator Algebras & Applications
b) William P. Thurston (Princeton University)
Sub: Low dimensional Manifolds
c) Shing – Tung Yau (Institute for Advanced Study, Princeton)
Sub: Differential Geometry & Partial Differential Equations

11) 1986 a) Simon Donaldson (Oxford University)
Sub: Exotic 4-dimensional Manifolds
b) Gerd Faltings (Princeton University)
Sub: Mordell's Conjecture in Arithmetic Algebraic Geometry
c) Michael Freedman (University of California, San Diego)
Sub: 4-dimensional Poincare conjecture

12) 1990 a) Vladimir Drinfeld (Phys. Inst. Kharkov)
b) Vaughan Jones (University of California, Berkeley)
c) Shigefumi Mori (University of Kyuto?)
d) Edward Witten (Institute for Advanced Study, Princeton)

13) 1994 a) Pierre-Louis Lions (Universite de Paris-Dauphine)
b) Jaen-Christophe Yoccoz (Universite de Paris-Sud, Orsay, France)
c) Jean Bourgain (Institute for Advanced Study, Princeton)
d) Efim Zelmanov (University of Wisconsin)

14)	1998	a) Richard E. Borcherds (Cambridge University) b) W.Timothy Gowers (Cambridge University) c) Maxim Kontsevich (IHES Bures-sur-Yvette) d) Curtis T.McMullen (Hardvard University)
15)	2002	a) Laurent Lafforgue (Institut des Hautes Etudes Scientfiques, Bures-Sur- Yvette, France) b) Vladimir Voevodsky (Institute for Advanced Study Princeton)
16)	2006	a) Andrei Okounkov (Princeton University) b) Grigori Perelman (Russia) (declined award) c) Terence Tao (University of California, Los Angeles) d) Wendelin Werner (Universite de Paris-Sud, Orsay, France)

COGNITIVE DEVELOPMENT IN INFANCY

Piaget's theory of Cognitive Development

In the middle years of the 20^{th} century Jean Piaget developed the most influential theory of child development. He emphasized the properties of **orderly progression and cumulative change.** The child's development proceeds through a series of stages (see table). The stages always occur in the same order, no stage may be missed, and the child never reverts to an earlier stage. Children differ in the age at which they reach each stage. **Development is not a passive process:** children are active agents in their own development. Development is characterized by ***qualitative* changes,** rather than simply growth.

Schemata

For Piaget, development consists of the formation and coordination of schemata : mental ways of understanding the world. Schemata develop from simple behaviours to coordinated sequences of action.

The development of schemata (adaptation) occurs through assimilation, the process of acquiring new information and incorporating it into existing schemata and accommodation, when existing schemata are modified to encompass new formation.

In Piaget's theory, development consists of the formation and coordination of schemata (or schemas) which are mental structures : systems of knowledge, actions and thoughts used to understand the world, and which guide behaviour.

The earliest schemata are behaviours : infants learn that to touch an object they must reach towards it.

As development progresses new schemata are formed, and schemata become coordinated so that sequences of action are possible, such as reaching for an object, grasping it and bringing it to the mouth.

During the sensorimotor stage schemata are limited to such behaviours.

The development of schemata is described as **adaptation,** and takes place through two basic processes.

Assimilation is the process of acquiring new information and incorporating it into existing schemata.

Accommodation occurs when existing schemata are modified to encompass new, discrepant information.

Piaget emphasized orderly progression and cumulative change. During infancy, the child is in the sensorimotor stage, which has six sub-stages: exercises reflexes (birth – 1 month); preliminary circular reactions (1-4 months); secondary circular reactions (4 – 8 months); coordination of secondary reactions (8 – 12 months); tertiary circular reactions (12 – 18 months) and invention of new means (18 – 24 months).

Table 1 : The main stages of Piaget's Theory of Cognitive Development during Infancy

Developmental Stage (approximate age)	Main features
Sensorimotor (birth – 2 years)	A sequence leading to the emergence of symbolic representation : * Reflex actions * Reactions based on infant's own body. * Reactions focused on effects on objects. * Goal-directed coordination of actions. * Greater variety of actions; focus on results * Ability to think of an object in its absence

During infancy, the child is in the **sensorimotor stage** (sometimes written *sensory-motor).* Piaget viewed this as consisting of six sub-stages.

1. **Exercising reflexes** (birth – 1 month). The newborn infant reacts to internal and external stimuli while quickly learning to differentiate stimuli.

2. **Preliminary circular reactions** (1 – 4 months). The infant performs repetitive actions focused on the infant's own body (e.g. thumb sucking)

3. **Secondary circular reactions** (4 – 8 months) Actions that produce changes in the outside world (e.g moving a mobile, or a rattle) are repeated, with the infant closely observing the results.

4. **Coordination of secondary reactions** (8 – 12 months). The infant intentionally coordinates sensory inputs and motor actions, anticipating outcomes, and repeating strategies that have previously been successful.

5. **Tertiary circular reactions** (12 – 18 months). Repeated actions are more varied and the infant 'experiments' with new actions.

6. **Invention of new means** (18 – 24 months). The infant no longer relies on trial and error, being able

to think through problems and mentally find solutions.

Object Permanence

Object permanence is the understanding that objects continue to exist when they are out of sight. This develops through sub-stages 3-5. At sub-stage 4 the infant will search for completely hidden objects, but will show the 'AB error' : searching for a hidden object in a usual hiding place despite having watched it being hidden elsewhere.

The properties of these sub-stages may be illustrated by the development of **object permanence:** the understanding that objects continue to exist when they are out of sight. In the first two sub-stages, infants will look at an object, but will not attempt to look for it if it is hidden from view. Only during sub-stage 3 does the infant start to search for objects, but only to a limited extent, for example continuing eye movements in the direction that an object was moving. At sub-stage 4, the infant will search for completely hidden objects, but will show what has been called the **AB error (or A – not – B effect).** A child at this stage will play a hide-and-seek game, repeatedly retrieving an object hidden in one place (A). If the object is now concealed in a new place (B), while the infant watches, the child will look for it in the original place, A. Piaget's interpretation of this was that the child has not yet developed an understanding that the object has a continued existence when out of sight. At the next stage, some sense of object permanence is achieved. The infant will search for an object, but only where it has been seen to be hidden. If an object is moved from the hiding place without the child seeing, the child will not look elsewhere for it. Only in sub-stage 5 is real object permanence shown. The infant now will search in other places for the 'lost' object.

COGNITIVE DEVELOPMENT IN CHILDHOOD

In the preoperational stage, 2- 7 years, the child cannot perform mental operations: higher order schemata that permit the fuller manipulation of internal representations. The child shows centration: an ability to attend to more than one feature of an object at a time, and egocentrism : an inability to adopt other viewpoints than their own.

Preoperational (2 – 7 years)	*	Can use symbolic representation
	*	Can perform deferred imitation
	*	Cannot place objects in order of size
	*	Cannot at first classify things by colour or shape; subsequently by only one feature
	*	Cannot understand conservation of quantity; only later of number.
	*	Cannot reverse operations.
	*	Egocentrism

Preschool years : Piaget's approach

The child leaves infancy with the ability to use **mental representations;** that is to think about actions and objects in their absence. The child can also imitate actions that were observed earlier **(deferred imitation).** However, Piaget apparently demonstrated that there is a wide variety of cognitive tasks that the child of preschool age cannot perform. These operations were viewed as higher order schemata that permit the fuller manipulation of internal representations. Piaget believed that these operations were not available until around 7 years of age, and he called the period from 2 to 7 years the **preoperational stage**.

Piaget catalogued the types of task that the preoperational child apparently cannot perform. A number of these are examples of a general cognitive approach of **centration** (sometimes called *centering).* This is the apparent inability of the child to attend to more than one feature of an object at a time. One example is a failure of **conservation.** That is, the child is unable to appreciate that the quantity of something remains the same when its shape is changed. For example, a child is shown two glasses of the same shape with the same amount of liquid in each, and will agree that they both have the same amount. The child then watches while the liquid from one glass is poured into a taller, thinner glass. The child will now indicate that the taller vessel contains more liquid. According to Piaget, preschool children also cannot place objects in increasing order of size **(seriation)**; they cannot make **transitive inferences** (i.e. told that A is bigger than B, and that B is bigger than C, cannot see that as a consequence, A is bigger than C); and they cannot **reverse operations** (e.g. watching three different coloured balls being dropped into a tube, they expect them to come out in the same order if the tube is inverted).

Another general feature of the preschooler's thinking is **egocentrism,** which is an inability to adopt other view points than their own. A simple example that, even if they can tell their own left hand from their right, they may reverse left and right on a person standing opposite them. Piaget introduced the 'three mountains task' to demonstrate this. A preschool child seated at one side of a model with three mountains is unable to identify the view that would be seen by a doll seated at a different side of the model.

Middle Childhood and Adolescence

In the stage of concrete operations children show an increased ability to reason logically. However, they cannot handle abstract material. They start to perform tasks showing a weakening of centration and lose their egocentrism. From about 11 to 15 years, a person passes into the stage of formal operations and can handle abstract relationships and hypothetical situations.

Concrete Operations (7 – 11 years)	*	Show logical reasoning.
	*	Limited to real world objects or events of which the child has personal experience.
	*	Can perform hierarchical classifications and **class inclusion** relationships (classifying by more than one feature).
	*	Operations are reversible.
Formal Operations (11 years old)	*	Can form and test hypotheses.
	*	Can reason about abstract ideas and objects and events beyond their experience.
	*	Can think about thought (introspection)

Other Theories of Cognitive Development
Gerome-Bruner's Theory

- **Enactive**
- **Iconic**
- **Symbolic**

Cognitive Development

The literature in the field of cognition and concept learning is voluminous. *Discussion will focus only on what is applicable and relevant to the process of language assessment and intervention in academic subjects.*

The development of language and cognition are intertwined. Although there can be cognition without language as in handicapping conditions such as hearing impairment and retardation, communication and learning is limited (Bloom, 1978; Inhelder and Piaget, 1964; Rice, 1980). A student's use of language reflects his or her level of cognitive functioning (the process of knowing). Carroll (1977) reinforced this notion when he stated "to comprehend language is to comprehend the concepts, propositions, inferences, qualification....and anything else that is expressed in language, either spoken or written". This view has important considerations in teaching moderately and severely retarded students. If the retarded student's language ability approximates his or her cognitive ability and both are significantly below his or her chronological age, then the regular developmental progression would not be applicable. The teacher of the retarded student must present tasks that are functional and appropriate for the student's chronological age but are also at his or her cognitive developmental level. For example, a student with a chronological age of 13 who has a cognitive developmental and language sage of age 4 might be presented with a task of sorting bath and hand towels. This, rather than sorting blocks, would be a functional activity.

The discussion on the analyses of cognitive skills is based on precepts developed by Piaget, who provided a systematic description of the processes by which children acquire knowledge (Schwebel and Raph, 1973). Piaget was not concerned with the specific age at which children acquire skills but rather with the process and sequence in which they are developed. Piaget and his followers believed that the stages of sequential cognitive development are invariant, but that the rate of development varies. They also believed that the earliest stages in the sequence are prerequisite for later stages. Researchers disagree about the invariance and sequential stages but, on the whole, Piaget's stages of cognitive development are widely used as a framework for observing the development of children. These are not discrete stages; rather they form a progression, indicating change.

The following are the Piaget stages of cognitive development:

1. *Sensorimotor stage (birth – 2 years of age).* The child learns about his or her environment through senses and motor responses. Meaningful use of language is a clue that he or she is ready to pass to the next stage.
2. *Preoperational stage (early : 2- 4 years of age; late : 4 – 7 or 8 years of age).* The major characteristic of the early preoperations

stage is that the child develops a symbolic function and language system but is bound to the "here and now". This means that he or she does not have the ability to deal with cause and effect. Progressing to the later preoperational period, the child uses language to form generalisations and develop inferential thinking, becoming less bound to the immediate events in the environment. The major sentence structures and their constituents become fully developed.

3. *Concrete operational stage (7 – 11 or 12 years of age).* The child uses language to develop skills for concrete logical thinking. Operational skills that develop include the ability to take another person's point of view and the ability to conserve, reverse, or manipulate higher levels of classification.
4. *Formal operational stage (12 years of age-adulthood).* Language is used at its highest level to express abstract thoughts and combinations of symbolic logic. The child also develops the ability to hypothesize and evaluate the consequences of his or her actions.

Epstein's (1978) research provides a possible neurobiological support for the Piagetian model of cognitive development. Epstein's notion of brain growth spurts in chronological order supports the Piagetian model.

Continuing research by Epstein suggests that a fifth stage may be distinguish from Piaget's formal operational stage (fourth stage), which would include creative problem solving.

The following concepts and operations are described and analyzed within the context of instructional tasks :

1. Classification (operation)
2. Conservation (operation)
3. Time (concept)
4. Seriation (operation)
5. Space (concept)
6. Casuality (concept)
7. Critical Thinking (operation)

Each section discusses the concept and operation, presents developmental data, vocabulary and implications for instruction.

The cognitive checklists that follow these sections provide developmental information by stage and approximate chronological age. Each checklist serves as a reference to show: (1) the incremental progression within the concept or operational area; (2) whether the student has attained the earlier conceptual requisites for the task; and (3) whether the conceptual requisites for the instructional task are beyond the student's developmental level and may account for his or her failure in the task. The teacher may want to refer to this information when developing hypotheses for intervention.

Classification

The operations of classification are strategies for organization and are present in all academic and non-academic activities. Unless a student has classification skills, he or she will not be able to process and retain all the information received through the senses.

Initially, students must have experiences with the operations of sorting or grouping, combining and ordering familiar objects within his or her environment on the basis of similarities and differences. A student usually starts forming combinations by collecting toy animals in a box, for example, and proceeds further by separating them into sheep, horses, cows, etc. He or she can do this with colour, property, size, or function. The student begins with the likeness of one thing to another and distinguishes it as different from other things. These same objects can be regrouped or reclassified, depending on what properties the student is judging to be the same. "Piaget distinguishes grouping as the principle from which stem classification, seriation, conservation, number and space understanding" (Schewebel and Raph, 1973, p.26). As the student learns sameness, he or she must learn *negatives* to express complementary classes, for example, trucks and things to ride in versus not trucks. For every item that belongs in a class, there are items that do not; it is necessary for the student to comprehend that in putting all red objects together, all objects that are not red must be eliminated. It is often assumed that because a student knows sameness among objects he or she also knows the differences.

There are two interrelated systems of classifications. One is non-numerical, which is used to establish relationships in providing a framework for logical thinking; the other involves numerical experiences, which lead to the idea of sets. As a result of this process, number sequences develop. A set is defined as a "collection of any kind of things belonging together" (Williams and Shuard, 1970, p.30). Sets can be put together into a whole and separated into parts. This is a precursor of *addition, subtraction and comparison.* This activity involves sequencing as well as the important language concepts of *all, some more, less, enough, next, to, before and after.*

Classification requires the ability to manipulate, order, group and transform verbal symbols, both on a receptive and expressive basis. This is a prerequisite for forming new classes, retaining information and transmitting information from one person to another. Classification skills are present in all academic tasks (Gruenewald, 1972).

Development

There are three developmental levels of classification behaviour

1. Perceptual ; Children below 5 years of age tend to select a perceptual or descriptive attribute of shape, colour, size, etc.
2. Functional : Older children most often select attributes based on a functional or relational quality (to cook with, to hear with, happy, sad).
3. Categorical : After 8 years of age, children tend to form groups based on generic or class names (animals, fruits, vehicles)

Inhelder and Piaget (1964) delineated four major stages in classification learning.

Stage 1 : Graphic Collections (about 2 ½ - 5 years of age). The child arranges objects randomly. At this stage, the child groups objects without any specific criterion. He or she can think of only one or two elements at a time.

Stage 2 : Non-graphic Collections (about 4 – 7 years of age). The child sorts objects on a perceptual level, for example, the colour blue. Although the child at this stage may group or make meaningful collections, he or she may not be able to perform class inclusion (all the items belonging together based on a single or multiple criterion).

According to Inhelder and Piaget (1964), true classification demands more than perceptual judgements; it demands mental operations. The child must not only take in information, but must also remake information.

Stage 3 : Class Inclusion. All items belonging together based on a single or multiple criterion are classified. This is the defining characteristic of this stage. According to Kamii and Peper (1969), this is the classification.

An example of class inclusion would be as follows : "Put all the objects that could take you places in the box". The array of objects in front of the child would include the truck, horse, chair, airplane, cow and boat.

Multiple classification refers to the ability to group objects into various subgroups. Objects can be rearanged and regrouped depending on the specific criterion. For examplke, a leather belt can be grouped into (1) clothes, (2) leather goods, or (3) fasteners. Multiple classification demonstrates that classes are not fixed or permanent.

Stage 4 : Period of Formal Operations

Classify inclusion and multiple classification are prerequisites the attainment of formal operations in adolescence (Kamii and Peper, 1969). Concrete operations (classification) structure only the concrete empirical data; formal operations represent reasoning with the structures that achieved in the period of concrete operations. The adolescent begins to reason with categories and tests and verify his or her hypotheses.

Concept Words (Vocabulary)

It is not feasible to list specific vocabulary here. The vocabulary of classification is generated by the requirements of the task. For example :

Perceptual – words, denoting, colour, shape, size, texture, form.

Functional – words denoting use, relationship

Categorical – words denoting class name.

Implications for Instruction

The importance of classification as part of the total language system is based on the observation of many students which had difficulty putting thoughts into words, sequencing ideas or stating reasons for actions. Students may not have sufficient experience in giving precise descriptions of objects, pictures, or events and in giving logical explanations of actions. Classification tasks are intended to develop these skills: however, students may acquire deficiencies in classification skills at an early age because of teachers' misperceptions of the requirements of a task. Teachers often confuse the task of sorting and classifying.

Many early elementary reading and math activities include exercises in classification that are in reality visual and auditory discrimination exercises (sorting). A student may sort objects along various dimensions such as size, colour or shape, usually referred to as attributes. Therefore, a teacher may say to a student, "Put all the blue blocks in this pile and red blocks in that pile". What is happening, however, is that the student is making a visual discrimination based on colour. To determine whether a student is classifying, the teacher must ask questions that will elicit appropriate responses such as, "Show me which ones belong together", or "Show me which ones do not belong together". After the student has grouped the series of objects in any fashion, the important question that must be asked is, "Why do these go together?". Students must be able to demonstrate that they can group according to one or more attributes (i.e. size, shape, colour, etc). Then students must be able to verbalize their decisions for the grouping. They may understand the concept, but until they use language to express it, progression to more abstract tasks will not be possible.

Teachers may also assume incorrectly that because students use categorical terms such as animals, clothing, or fruit, they are able to classify. However, students may not understand the attributes that are used to make these categorical classifications. For example, a student may group sheep, cows and dogs as animals but not be able to give a reason for the grouping, such as four legs, tail, etc.

A teacher may misjudge the ability of students to classify because their responses did not meet the teacher's expectation of the instructional task. For instance, to check whether the students were learning the operation of classifying by using the concepts of big and little, a teache gave the following direction : here are some sponges. Put them into two groups. Her expectation was for the students to demonstrate knowledge of classification by grouping big sponges into one group and little sponges in a second group. This task has been practiced the previous day. The teacher may have assumed that the students understood her direction. One boy squeezed each sponge and put both big and little sponges into each of the two groups. The teacher regarded this response as incorrect because it did not meet her expectation of grouping by big and little sponges. What she had indeed asked for was to sort the sponges into two groups and the student did so, but, the attributes he had selected were that of hard and soft (as demonstrated by his squeezing techniques). The student was classifying at the perceptual level of texture rather than of size.

Evidence indicates that children who are mentally retarded, hearing impaired, learning-disabled, language delayed or otherwise handicapped demonstrate inadequacies in the development of classification skills. Because this development includes all of the concepts discussed in this chapter, the teacher (regular or special education) will benefit from the developmental information presented in each checklist.

Classification skills are embedded in the following academic areas (Kellman and Nyberg, 1980).

Mathematics

Formation of sets

Math operations

Story problems

Social Studies

Outlining

Historical concepts (i.e. major issues of World War II), Geography (class inclusion : home, city, country, state, country, hemisphere).

Language Arts

Reading (grouping of letters to make words, combining small syllables to make larger words).

Reading comprehension (main ideas, part/whole relationships)

Synonyms, antonyms, humour and idioms

Science

Classes of plants and animals

Classes of non-living things

Problem Solving

Creative Thinking

Conservation

Conservation can be defined as "....the ability to realize that certain attributes of an object are constant, even though it changes in appearance" (Pulaski, 1971, p.242). For example, pouring liquid from a tall, thin glass into a short, fat glass may change the appearance but not the volume; or the number of objects in a set remains the same, regardless of how the objects are arranged or combined. Nothing has been added or taken away in either example..

The ability to conserve is basic to the understanding of number, measurement and space. It requires the child to physically and mentally act on an object. Conservation appears during the late preoperational and concrete periods. Inherent in the ability to conserve is the ability of reversibility, which develops when the child is 5 – 8 years of age. The child will be able to conserve when he or she is no longer bound to sensory experiences and can begin to use logic.

Development

The ability to conserve develops at different times with different concepts.

Type	**Approx. age of appearance**
Number	5 – 7
Quantity	5 – 7
Length	7
Area	7 – 8
Time	8 – 9
Weight	9 – 10
Volume	11 – 12

Concept Words (Vocabulary)

Conservation vocabulary includes the following terms ;

Number – more, less, the same as, all, half, whole, few, before, after

Size – tall, short, skinny, fat, wide, narrow, high, low

Area – line them up the same way, some other way, before, after, the same as, more, not as much as.

Length – taller than, higher than, shorter than, the same length as, on top of, under, , alongside of, near, close to, up against.

Volume – empty, full, more, less, too much, too little, all gone.

Weight – heavy, light.

Time – fast, slow, farther, older, younger

Implications for Instruction

Many teachers find that first-grade students have difficulty with addition in a task that requires not only 4 + 1 = 5 but also 1 + --------- = 5. The latter computation requires the ability to conserve and reverse thought. Copeland (1974a) suggested that a student has achieved reversibility when "..... one of two equal sets is rearranged....and he realizes that the number of each set has not changed". Teachers must question whether first-grade children who are presented with a math problem similar to the given example can reverse thought.

The implication of conservation and reversibility also exists in language development. Passive voice, which is a difficult sentence construction for young children, requires the use of reversibility from the active voice, for example, "John hit the ball" (active voice) and "The ball was hit by John" (passive voice). Questions are another example of reversibility (subject/ verb) (i.e. "Is John going home?" as contrasted with "John is going home").

Unless students can conserve, they will have difficulty either adding or subtracting. For instance, a direction might be as follows :
Rename the 10s in 23.
Take one 10 – make 10 ones

Tens	ones	-	tens	ones	
2	3		$\frac{1}{2}$	$\frac{1}{3}$	3

This direction requires the following skills :

1. Understanding equivalency (i.e. one 10 equals 10 ones).
2. Understanding that the quantity concept does not change; 10 is still 10; regardless of how it is displayed, reducing two tens to one and increasing three ones to 13 still does not change the total quantity.
3. Understanding the concept of reversibility, which is inherent in this process.

If the student has difficulty in this task, one of the first things the teacher needs to do is to be certain that the student has one-to-one correspondence. Second, conservation must be established. For instance, give the students 23 sticks and have him or her arrange them first in piles of twos, then in fives and tens, each time counting the total number, regardless of the arrangement. The student must understand that each time, there will be a remainder. Third, have the student count and record the number of tens and the number of ones. Then transfer one pile of tens to the three ones and count the tens and ones. Have the student count the total.

The teacher must be aware that the concept of operations of conservation occur in many other subject areas in addition to match. In social studies, for example, the student must understand that although people may exist in different cultures at different times, they are still people; or in science, changes can occur within a classification system (phylum) without the class being changed.

Time

Temporal concepts can be expressed in two ways : temporal order (sequence) and duration (the interval between two events). A student should understand that the measurement of time is based on the existence of motion that can be time (Copeland, 1974b). For a student to be able to become operational with respect to time, he or she must be able to coordinate order and duration. The ability to order and perceive intervals is not completed until about 9 years of age.

There are many misconceptions about teaching time, which is often done in a mechanical, perceptual manner without developing an understanding of the underlying concepts and skills. This is especially true for calendar time, which is the traditional activity at the beginning of the day for young elementary age students. The utility of this opening activity is questionable because on the whole, it is a rote activity and therefore, not meaningful. Many of the concepts and vocabulary used in this task are not yet within the cognitive developmental level of the student. In some schools, this task is also customarily taught to young children who are hearing impaired or show varying degrees of retardation. Student failure can be incurred if tasks are presented to the student before he or she has the developmental ability to do them.

Development

Temporal behaviour develops in five stages (Elkind and Flavell, 1969).

Stage 1 : Concepts of time based on Order. Time relates to the personal aspects of before and after without capacity of employing calendar or clock concepts. The child does not differentiate time from space or speed. Faster means more time.

Stage 2 : Notion of Velocity or Speed Develops (Kindergarten or First Grade). Speed is a more fundamental notion than time. The child does not have reversibility yet and uses sequence order.

Stage 3 : Ideas of Time, Distance and Speed Coordinated. (6 – 8 years of age). The child answers on a logical rather than perceptual basis. He or she understands the succession of events in time and duration in conservation of age differences (Copeland, 1974 b).

Before a child can be expected to deal logically with telling time in terms of minutes, seconds, days, etc. he must attain the developmental stage of concrete operations (7 – 8 years of age)...At this stage, the ability to do reversible thinking, the ability to deal with seriated relationships is present in the child. At this stage, children also understand transitive relationships, such as "if A = B and B = C, then it follows that A = C"....In this example, B is the middle term that established the relationship between A and AC. A clock is the middle term when used to establish the time relationship between two events. For example, two runners compared themselves to one another through the intermediary of a clock. Bob ran a mile in 5 minutes; George ran it in 6 minutes; Bob took less time than George.

Stage 4 : Speed of movement can be separated from a distance travelled over time. The child understands that the hour hand moving over a short distance is measuring the same time as the minute hand, which moves over a greater distance.

Stage 5 : Historical perspective understood (Adolescence). The child responds to time words before he or she uses them. Temporal concepts at the preschool level are primarily related to the personal aspects of before and after without the capacity of employing calendar and clock concepts.

Temporal Skill	**Years of Age**
Understands the following words :	
Today	2
Tomorrow	2 ½
Yesterday	3
Recognize a special day of the week, such as Sunday.	4
Uses words yesterday and tomorrow correctly.	5
Tells whether it is morning or afternoon	5
Indicates the day of the week	6
Indicates the month	7
Indicates the season	7 – 8
Indicates the year	8
Indicates the day of the month	8 – 9
Is interested in historical time	9 – 10
Estimates duration of a conversation	12
Has the ability to tell which of two events in historical time Occurred earlier or later.	12
Understands BC/AD	12 +
Constructs a time line of historical time	12 +

Concept Words (vocabulary)

The following words are used to express time units ;

Order (succession)	Duration
First, second, third, etc.	Morning
Before	Afternoon
During	Evening
After	Daytime
Since	Night time
While	Old
At the same time	Long time
Next	Today
Last	Tomorrow
One more time	Yesterday
And	Hour
Earlier	Day
Later	Young
A while ago	A little while
Now	Week
Then	Day Month Year Long Short time Long time

Implications for Instruction

If teachers want to ensure successful learning for their students, they need to examine their curricular materials to determine whether the student has the prerequisite understanding or developmental readiness for the task. They can then more easily adapt the materials to the students' cognitive and language level.

Temporal concepts are found in the following academic areas :

Social Studies : Dates, Historical Time.

Language Arts : Time setting of the reading, Selection of events, order of events

Mathematics : Measurement

Science : Motion, Seasons, change over time

The following math problem, for example focuses on measurement.

A turkey is to be cooked 20 minutes for each pound. If a turkey weighing 10 pounds is to be done at 5 pm, what time should it be put in the oven to cook ?

The explicit vocabulary representing the temporal concepts in this example are 5 p.m. and 20 minutes. The underlying temporal concepts and operations (implicit) include:

1. Duration of time – length of cooking time (20 minutes × 10)
2. Minutes to the hour (200 minutes ÷ 60 = 3 1/3 hours)
3. Reversibility (working backward from 5 pm to 1.40 pm)

According to Carpenter et al. (1981) problems involving time duration are difficult. He said, "Fewer than a third of the 9-year olds could find the time8 hours after a given time or find the amount of time between 2 given hours of the day" (p.92). This difficulty in manipulating time generates the following hypotheses.

Hypothesis 1 : Does the student have the concept of hours and minutes within the hour ?

Hypothesis 2 : Does the student have the concept of hours and minutes within the hour ?

Hypothesis 3 : Does the student have the concept of duration as being equivalent to the interval between the points ?

Hypothesis 4 : Does the student have the ability to reverse an operation (i.e. go backwards in time for the given point)?

Teachers must consider the temporal concepts included in the many tasks that students are required.

1. Order events chronologically.
2. Recognise that stories and activities have a beginning and an end.
3. Develop the notion that time involves coordination speeds.
4. Develop the concept of age.
5. Recognise the part/ whole relationship of time.
6. Develop the understanding of before and after, why is necessary to sequence events in a logical order.

Seriation

Another operation that the student needs to organize his or her environment is learning the relationship between objects and putting them in order. This is seriation. As the student acquires the ability to seriate, a profound change in the quality of thinking can be observed (Voyat, 1973). According to Copeland (1974a), young children at the sensorimotor level seem to order objects through a trial-and error procedure. To truly seriate requires the reversibility of thought and transitivity that occurs around 7-8 years of age.

As discussed in the section on conservation, reversibility of thought refers to the ability of the student to perceive order from more than one direction. This operation can be expanded to include reversing order in increasing and decreasing sizes, heights, gradation of colours and textures and qualities.

Problems that kindergarten and first-grade children have with ordinal and cardinal numbers may be based on their lack of this development of reversibility.

Transitivity can be defined as the coordination of a series of relations (Copeland, 1974a). This means that if the first item is related to a second and a second item to a third, then the first item of a series is related to a third. For example, if B is greater than A and C is greater than B, then C is also greater than A. This can be translated into : "If Mary is taller than John and Bill is taller than Mary, then Bill is also taller than John". It follows that Bill is the tallest.

Development

There are four stages in the development of seriation :

Stage 1 : Early Preoperational Period : True seriation is not actually occurring. The child pairs items according to whether they are big or little. He or she cannot use seriation in a constructive sense. By 3 or 4 years of age, the child begins to make arrays of two or three objects when seven objects are presented.

Stage 2 : Late Preoperational period. The child, according to Piaget, has one-to-one correspondence. He or she can seriate 10 items and be able to use different materials. The child still uses a trial and error approach for solving seriation problems.

Stage 3 : (Concrete Operational Period) The child has a strategy for solving the seriation problem that does not depend on trial and error.

The ability to seriate, such as from the smallest to the largest, or to count at the operational level, that is, with true understanding of the inclusion relation involved, develops usually at seven or eight years of age (Copeland, 1974a).

Stage 4 : (Formal Operational Period). The child is able to represent how an ordered group will look before ordering the objects physically. He or she can also understand transitive relationships.

Concept Words (Vocabulary)

Some Vocabulary associated with seriation are :

Number – ordinal (first, second, third, etc.)

Size – big, bigger, biggest

Length – Long, longer, longest

Height – tall, taller, tallest

Space/ time – in front of, behind, before, after

Amount – least, most.

Implications for Instruction

Seriation is included in all subject areas. Teachers may not always be aware that a reading task, such as "Number the pictures according to the sequential episodes and then use as a reference for retelling the story" requirees (1) the understanding of the part/ whole relationships, (2) comprehension of the story, (3) the ability to use complex sentence structure and (4) the ability to seriate. The teacher should question whether the student has the concepts of *before* and *after*, *first, second, third* (ordinal); *one, two, three* (cardinal); and the sequential order of the parts to form a whole (logical ordering).

Other instances of seriation in academic tasks (Kellman and Nyberg, 1980) include ;

Science:

Ordering by attributes of items within classes

Construction of histograms

Social studies :

Showing population densities and typographical maps.

Mathematics :

Counting (if a student cannot order, then it may be impossible for him or her to count with understanding or refer to the position of an object within a set or the number of objects in a set).

Language Arts :

Ordering of events (chronological order in stories)

Ordering letters of the alphabet for dictionary skills and locating books in the library.

Space

Knowledge of space or the world around a person develops in two ways; first, through which Piaget calls perceptual space (what we perceive or what our senses tells us) and second, through representational space (what the mind reconstructs) (Copeland, 1974a).

There are three kinds of spatial relationships :

1. Topology includes relations of proximity (nearness, separation, order sequence) and enclosure (surrounding and continuity). Topology is concerned with an object as a thing in isolation, not as it is related to other objects in space. Changes in shape cannot be coordinated with changes in position. At this point, space lacks organization.

2. Projective space marks the beginning of attempts to locate objects in relation to one another to organise space. It involves the development of perspectives or the ability to view an object from different points of view.

3. Euclidean geometric space is concerned with the relation or coordination of objects in space. At this point, the child is able to distinguish and coordinate different viewpoints.

Development

There are three stages of spatial development.

Stage 1 : (Sensorimotor and Early Preoperational Period). Topological space is the earliest relationship to develop and deals with surroundings, proximity, and order. It consists of the most general and non-metric properties of space.

Stage 2 : (Late Preoperational Period). Projective space develops and deals with spatial relationships on the horizontal and vertical planes. These relationships are first developed in relation to the body image. Concepts related to the self and objects develop in the following order (Kuczaj and Maratsos, 1975) :

1. The child knows the front and back of his or her own body but cannot generalise this information.
2. The child knows the fronts and backs of various objects with fronts, such as cars, telephones and animals.
3. The child can place something in front of or in back of an object with a front and back.
4. The child knows the sides of his own body and the sides of fronted objects (not left and right but sides).
5. The child can generalise notions of front, back and side to new objects that he or she has not seen before.

6. The child can use himself or herself as a reference and place something in front, in back and at the side of non-fronted objects, such as a glass.

Euclidean geometric space appears almost simultaneously with projective space. It involves figures such as line segments, triangles, squares and circles.

Stage 3 : Concrete Operational Period Space for the child becomes a coordinate and objective whole when he is around 9 years of age (Copeland, 1974a).

Concept Words (Vocabulary)

The order of the acquisition of vocabulary expressing spatial terms reflects the movement from topological to projective and Euclidean geometric spatial notions. This vocabulary is constrained by the child's cognitive level.

Topological	Projective	Euclidean Geometry
At	Over	Across
In	Under	Through
Out	Above	Along
On	Below	Toward
Off	In front of	Between
Inside	In back of	
Outside	Behind	
Top	Beside	
Bottom	Next to	
	Right and left (of self	

	5-8 years) Right and left (of person he or she is facing 8-11 years)	

Implications for Instruction

From the time the student enters school, he or she expected to respond to explantions, directions and questions containing spatial words. The teacher should be aware of the type of performance that can be expected by the student at each age. Students at the preschool level should be given activities reflecting the topological notion whereas kindergarten and first-grade children can begin to make the transition to projective and Euclidean geometric activities. Problems may occur with some children in the first grade when they are asked to respond to the direction of printing their name at the top of the page. The direction requires that the student see the top as a horizontal view, whereas they may have been taught the concepts of top and bottom from the vertical view (top of table, top of head, etc). Prepositional concept words such as before, after, in front of and next to require the students have a concept of the self and its body parts. Otherwise he or she may experience difficulty in responding to instruction such as, "Give the book to the child in front of you" or "Put your hands down by your sides".

Other problems in instructional tasks may occur with

Aphabetizing – requires knowledge of before and after.
Number sequences – requires knowledge of before, after, left, right.
Social studies – requires knowledge of across, below, beside, above, next to, along the axis.
Geometry – requires knowledge of through, along, sides, across, between

A familiar example of geometric forms found in first grade workbooks requires the student to judge if the following triangles are congruent or similar.

This task requires the ability of the student to be able to reverse the triangle to be able to understand the shape does not change the size (conserve). Considering that children in the first grade are beginning the transition into projective space, some difficulty in viewing these triangles from different perspectives may occur. It must also be ascertained as to whether the students have developed reversibility and conservation of spatial objects.

The teacher must determine not only whether the student can recognise and name the shape of a square but also whether he or she understands the concepts of square as depicted in other contexts such as signs, boxes, book cases and rooms. The generalisation of a concept takes place gradually over time.

In this context, the following is an example from a third-grade reading task : Mason city, Iowa is a small city near the centre of our country. Find Mason City on the map.

The explicit vocabulary representing spatial concepts in this example are small, centre and near. The implicit concepts included in the request to find the city on the map are top, bottom, left and right. If the student has difficulty with this task, the following hypotheses can be drawn :

Hypotheses 1 : Does the student understand the concept words of small, near and center ?

Hypotheses 2 : Does the student understand the use the spatial concept words implicit in solving the problem ? (These may include top, bottom, left and right).

Hypotheses 3 : Does the student understand and use the implicit operations of coordinating vertical and horizontal axes to find the city on the map ?

If the student does not have the concept(s)to do the task, the teacher must consider whether the reason may be developmental, language competence, or lack of experience or instruction.

Spatial relations are important in the learning process because the ability to perceive and understand the words with these relations underlies reading comprehension, computation and problem solving (Alexander and Nyberg, 1977).

Casuality

Multiple classification and understanding of relationships most likely underlie the major portion of abstract thinking. Piaget found that children below 8 years of age were unable to fully understand casual relationships because of the underlying conceptual and linguistic development (schwebel and Raph, 1973). A young child is likely to consider only one cause or event to the exclusion of others. No single cause can explain a historical, physical or psychological event (Sigel and Hooper, 1968).

A student of 10 or 11 years of age should have acquired the ability to use classes, relations, measurement and numbers in a concrete fashion. Children 8 – 11 years of age are beginning to understand the relationship between physical cause and physical effect, for example, "The sky is very cloudy, it will probably rain".

Sigel and Hooper (1968) discussed three types of casuality : 1. physical, 2. social and 3. psychological and multiplicative. The concept of casuality requires the development of the concepts previously discussed. For years of age generally has acquired a grasp of conservation. He or she can understand that although there may be changes in the physical world, certain things remain constant; likewise, within changes of government, certain institutions remain the same. The student must have acquired the ability of multiple classification to be able to handle the abstractions required to comprehend notions of conflicting ideas and events.

Development

There are four stages in the development of casuality :

Stage 1 : Early Preoperational Period - Children attribute to themselves or others as being the cause of a given event. Explanations are very subjective.

Stage 2 : Late preoperational Period: Children 3 years of age and older are trying to find out why things happen. They seek causes and may assume co-occurring events have a casual relationship. This leads to transductive reasoning, for example, " I had mean thoughts about my brother. He got sick so I made him sick".

Stage 3 : Concrete Operational Period The understanding of casuality that children develop during this period requires that they move from egocentricism and realise that they are not the casual agent. Children must have reversibility so that they know what is cause and what is effect. Children must be able to coordinate a series of events.

Stage 4 : Formal Operational Period. Students comprehend all forms of casuality because they can handle two or more prepositions simultaneously.

Concept Words (vocabulary)

The vocabulary used to express causality includes the words because, if/ then, when, therefore and why. Some of these words may be used in the late preoperational period, but they do not always reflect casual relationships. They are merely used as conjunctions. The meaningful use denoting casual relationships does not appear until higher levels of cognitive development occur.