Package for orientation of Secondary School Mathematics Teachers of Tamil Nadu, Karnataka, Kerala and Andhra Pradesh

(PAC Programme for Secondary School Mathematics Teachers of Southern Region)

ACADEMIC COORDINATOR

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CONTENTS

FOREWORD

Regional Institute of Education, Mysore conducted PAC programme on "**Development of Package for Orientation of Secondary School Mathematics Teachers of Tamil Nadu, Karnataka, Kerala and Andhra Pradesh**" during the year 2002-2003. Mr B C Basti was the Programme Coordinator. Several resource persons and teachers from southern states developed the material. Before finalisation of the material it was also field tested.

The material includes number of illustrations and examples covering findamental concepts. I express my great appreciation for the task carried out by Mr Basti. It is sincerely hoped that the material will find maximum utilization.

Prof G Ravindra
Principal, RIE,
Mysore

30th March 2003

PREFACE

The PAC programme titled "**Development of a Package for Orientation of Secondary School Mathematics Teachers of Tamil Nadu, Karnataka, Kerala and Andhra Pradesh**" was organized at RIE, Mysore in three phases.

In the first phase from 9th to 12th December 2002, the topics for developing orientation package were identified and also developed by teachers drawn from southern region.

In the second phase from 10th to 14th February 2003, the already developed package was reviewed by local resource persons. Then the Programme Coordinator undertook the field test of the package in several schools of Karnataka.

In the third phase from 17th to 19th March 2003 local resource persons finalized the package.

I thank Principal, RIE, Mysore and Department of Extension Education, RIE, Mysore for their cooperation in this developmental work.

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NUMBER SYSTEM

Introduction: Numbers are mathematical objects. Numeration and number systems originated as and when the necessity of counting and measuring arouse in day-to-day life. In ancient days, different countries used different numerals to represent numbers.

The most commonly used number system is the decimal system with Indo-Arabic notations which was originated in our country and spread all over the world through the Arabs, otherwise called denary system.

The numbers that we are using are in 'base ten' system. In this system, any number can be written with the numerals 0,1,2,3,4,5,7,8 and 9. There are two values for each numeral. One is the actual value and the other is the place value. A digit in a number has place value according to its place. Units' place is represented by 10^0, tenth place is represented by 10^1, hundredth place is represented by 10^2 and so on.

For example, 468 has

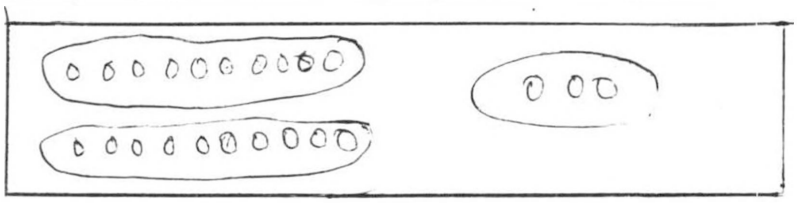
8 in the units' place	=	$8 \times 10^0 = 8$
6 in the tenth place	=	$6 \times 10^1 = 60$
4 in the hundred's place	=	$4 \times 10^2 = 400$

		468

10^2	10^1	10^0
100th Place	10th Place	Place
4	6	8

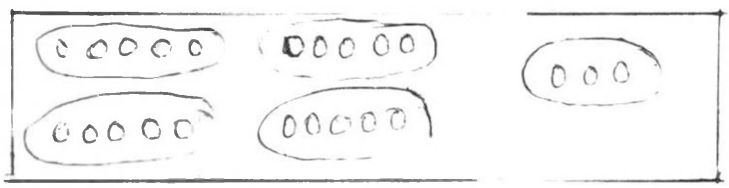
Base five system:- Quenary system.

The aim of teaching systems of numerations in bases other than ten is, to understand the familiar decimal system clearly. Suppose we want to write a numeral in base ten we group them in 10's as shown below by dots. For example number 23 can be grouped in 10^1 as follows.



$2 \times 10^1 + 3 \times 10^0 = 20 + 3 \times 10^0 = 20 + 3 = 23$

If we want to write the same number 23 in base five system we group them in 5's as shown below by dots.



$4 \times 5^1 + 3 \times 5^0$

$= 20 + 3 \times 1$

$= 20 + 3$

$= 23$

As we find the place value in base ten system as 10^0, 10^1, 10^3....., the place value in base five system becomes $5^0, 5^1, 5^2, 5^3$..... Here gouping is done in units or powers of 5.

Eg 1. Express $(234)_5$ in base 10 system.

$$(234)_5 = 2 \times 5^2 + 3 \times 5^1 + 4 \times 5^0$$

$$= 2 \times 25 + 3 \times 5 + 4 \times 1$$

$$= 50 + 15 + 4$$

$$\therefore (234)_5 = 69_{(10)}$$

2. Express $(1203)_5$ in base 10 system

$$(1203)_5 = 1 \times 5^3 + 2 \times 5^2 + 0 \times 5^1 + 3 \times 5^0$$
$$= 1 \times 125 + 2 \times 25 + 0 + 3 \times 1$$
$$= 125 + 50 + 3$$
$$(1203)_5 = 178_{(10)}$$

Now let us examine how to write the corresponding numeral in base five system for a given numeral in base ten system. This can be done by repeated division of the given number by 5 as shown below.

Eg: 1 Express 48 in base 5 system.

$$\begin{array}{r|l} 5 & 48 \\ \hline 5 & 9\text{-}3 \\ \hline & 1\text{-}4 \end{array}$$

$9\text{-}3 \longrightarrow 5^0$

$1\text{-}4 \longrightarrow 5^1$

$\longrightarrow 5^2$

$\therefore 48 = (143)_5$

Eg:2 Express 238 in base 5 system

$$\begin{array}{r|l} 5 & 238 \\ \hline 5 & 47\text{-}3 \\ \hline 5 & 9\text{-}2 \\ \hline & 1\text{-}4 \end{array}$$

$47\text{-}3 \longrightarrow 5^0$

$9\text{-}2 \longrightarrow 5^1$

$1\text{-}4 \longrightarrow 5^2$

$\longrightarrow 5^3$

$\therefore 238 = (1423)_5$

Base two system – Binary System

Modern electronic computing machine uses the two based number system. The advantage is that only two symbols are needed here 0 and 1.

The place value in this system will be , 2^0, 2^1,2^2, 2^3..... Here the grouping is done in units of two's.

Ex: 1 Express $(110)_2$ in base 10 system

$$(110)_2 = 1 \times 2^2 + 1 \times 2^1 + 0 \times 2^0$$
$$= 1 \times 4 + 1 \times 2 + 0 \times 1$$
$$= 4 + 2 + 0$$
$$= 6$$
$$\therefore (110)_2 = 6_{(10)}$$

Ex : 2 Express $(11011)_2$ in base 10 system.

$$(11011)_2 = 1 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 1 \times 2^0$$
$$= 1 \times 16 + 1 \times 8 + 0 + 1 \times 2 + 1 \times 1$$
$$= 16 + 8 + 2 + 1$$
$$(11011)_2 = 27$$

Now let us examine how to write the corresponding numeral in base two system for a given numeral in base ten. This can be done by repeated devision of the number by 2 as shown below.

Eg: 1. Express 9 in base 2 system.

2 | 9
2 | 4-1 → 2^0
2 | 2-0 → 2^1
1-0 → 2^2
→ 2^3

$\therefore 9 = (1001)_2$

2. Express 15 in base 2 system

2 | 15
2 | 7-1 → 2^0
2 | 3-1 → 2^1
1-1 → 2^2
→ 2^3

$\therefore 15 = (1111)_2$

Exercise

1. Express the following decimal numbers in base 5 system.
   58, 162, 248, 369, 526

2. Express the following base 5 numbers in base 10 system.
   $(132)_5$, $(123)_5$, $(104)_5$, $(310)_5$, $(1234)_5$

3. Express the following decimal numbers in binary system.
   7,11,17,21,43

4. Express the following binary system numbers in decimal form
   $(110)_2$, $(1101)_2$, $(1001)_2$, $(11111)_2$, $(101010)_2$

5. Express 47, 59, in base 7 system.

6. Express $(320)_7$ and $(102)_7$ in base 10 system.

Note: Decimal base system, that we use in our daily life is common. But any numeral like 3,4,6,7. Can be taken as base and that particular base system can be developed whenever it is required for practical purposes.

SEQUENCES AND SERIES

Introduction

Sequences

A function whose domain is a set of successive positive integers, for eg. the function defined by

$S(n) = n + 3, \qquad n \in \{1,2,3,....\}$ is

Called a Sequence Function. The elements in the range of such a function arranged in the order S(1), S(2), S(3), S(4),....are said to form a sequence. Thus the sequence can be found out by successively substituting the numbers 1,2,3,....for n.

$S(1) = 1 + 3 = 4, \quad S(2) = 5, \quad S(3) = 6, \quad S(4) = 7.$

and the first four terms are 4, 5, 6 and 7.

Notation : A sequence with general term S(n) is denoted by { S(n) } = S(1), S(2), S(3),.....

The nth term or general term is n + 3.

A sequence is called finite or infinite according as its domain is {1,2,3,....n} for some positive integer or N itself.

Exercise 1

Find the first four terms in a sequence whose general term is $S(n) = \dfrac{n\,(n+1)}{2}$.

$S(1) = \dfrac{1\,(1+1)}{2} = 1$

$S(2) = \dfrac{2\,(2+1)}{2} = 3$

$S(3) = \dfrac{3\,(3+1)}{2} = 6$

$S(4) = \dfrac{4\,(4+1)}{2} = 10$

Ans: 1,3, 6, 10

Example 2

Find the first four terms in a sequence whose general term is $S(n) = (-1)^n 2^n$.

$S(1) = (-1)^1 2^1 = -2$

$S(2) = (-1)^2 2^2 = 4$

$S(3) = (-1)^3 2^3 = -8$

$S(4) = (-1)^4 2^4 = 16$

Ans: -2, 4, -8, 16

Exercise

1. Write down the following sequences whose nth term is given below.

1. $S(n) = n - 5$.
2. $S(n) = 2n - 5$
3. $S(n) = 1 + \frac{1}{n}$
4. $S(n) = \frac{3}{n^2 + 1}$
5. $S(n) = \frac{n^2 - 2}{2}$
6. $S(n) = \frac{n}{2n - 1}$
7. $S(n) = \frac{n(n-1)}{2}$
8. $S(n) = \frac{5}{n(n-1)}$
9. $S(n) = (-1)^n$
10. $S(n) = (-1)^{n+1}$
11. $S(n) = \frac{(-1)^n \quad (n-2)}{n}$
12. $S(n) = (-1)^{n-1} \; 3^{n+1}$

Series

Given a sequence S(1), S(2), S(3),....the expression : S(1) + S(2) + S(3) +..... is called the series associated with the given sequence { S(n) }. The series is

also denoted by $\Sigma S(n)$. It is finite or infinite according as the sequence { S(n) } is finite or infinite.

Eg. 1. $4 + 7 + 10 + \ldots +)3n + 1)$ is a finite series.

2. $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\ldots$ is an infinite series.

Arithmetic Progression

A sequence { S(n) } is called an Arithmetic Progression (A.P.) if the difference between *any two consecutive terms* is constant i.e.
$S(n) - S(n-1) = \text{constant} = d$ (say) *for all n.* d is called the common difference (C.D) of the A.P. A typical (or standard) A.P: Taking $S(1) = a$, a, $(a + d)$, $(a + 2d)$……is a typical A.P. with $a = 1^{st}$ term and d = C.D. of the A.P.

Eg1. Consider the sequence 2,4,6,8,…. Here the nth term is given by $S(n) = 2n$.

$\therefore\ S(n) - S(n-1)$

$= 2n - 2(n-1)$

$= 2n - 2n + 2$

$= 2$

$\therefore\ d = 2$

Eg 2. The sequence 3, 7, 11, 15. Here the nth term is given by $S(n) = 4n - 1$. Hence $d = 4$. How ?

General Term of the progression = a + (n – 1) d

where a is the first term and d is known as the common difference.

Sum of n terms of an Arithmetic Series :

Find the sum of the first n terms of the Arithmetic Progression, a, a + d, a + 2d,……

Let S_n $= [a] + [a + d] + [a + 2d] + \ldots + [a + \overline{n-1}\, d]$

Again S_n $= [a + \overline{n-1}\, d] + [a + \overline{n-2}\, d] + \ldots + [a]$

Adding $2S_n$ $= [2a + \overline{n-1}\, d] + [2a + \overline{n-1}\, d] + \ldots + [2a + \overline{n-1}\, d]$

(Containing n terms)

$$2S_n = n\,[2a + (n-1)\,d]$$

$$\boxed{S_n = \frac{n}{2}\,[2a + (n-1)\,d]} \qquad (2)$$

It can also be written as

$$S_n = \frac{n}{2}\,(a + a + (n-1)\,d) = \frac{n}{2}\,[\,a + S(n)\,] \quad \therefore\ Sn = \frac{n}{2}\,(a + S(n))$$

When S(n) is the last term of the Arithmetic Series.

Example: find the fourteenth term of the arithmetic progression –6, -1, 4,….Find the common difference.

$d = -1 - (-6) = 5$

$S(n) = a + (n-1)\,d$

$S(14) = -6 + (14 - 1)\,5 = 59$

Exercises

1. Find the seventh term in the arithmetic progression 7, 11, 15,….
2. Find the tenth term in the arithmetic progression -3, -12, -21,…..
3. Find the twelfth term in the arithmetic progression 2, $\frac{5}{2}$, 3,….
4. Find the 17th term in the arithmetic progression 3, -2, -7,…
5. Find the 10th term in the arithmetic progression $\frac{3}{4}$, 2, $\frac{13}{4}$.

Miscellaneous Exercises

1. What term in the arithmetic progression 4, 1, -2,…..is –77 ?
2. If the 5th term of an AP is –16 and the 20th term is –46, what is the 12th term?
3. Find $\sum_{i=1}^{12}(4i+1)$.
4. Find $\sum_{i=1}^{21}(3i-2)$
5. Find the sum of all integers n, where $13 < n < 29$.
6. Find the sum of all integral multiples of 7 between 8 and 110.
7. Show that the sum of the first n odd natural numbers is n^2.
8. What is the 12th term in an arithmetic progression in which the second term is x and the third term is y?

Geometric Progression

A sequence { S(n) } in which the ratio between any two consecutive terms is constant is called *Geometric Progression* (G.P.) i.e.

$\frac{S(n)}{S(n-1)} = r$ (a constant say) for all n and so on. r is called the *common ratio* (C.R.) of the G.P. A typical (standard) G.P. : a, ar, ar^2, ...

when a = the 1st term and r = C.R. of the Geometric Progression.

In a G.P. each term except the first is obtained by multiplying the preceding term by a common multiplier is called a Geometric Progression and is defined by the recursive equation s(n + 1) = r. S(n), where r is the common multiplier and r is known as the common ratio.

Eg. 3, 9, 27, 81, ...is a geometric progression in which each term except the first is obtained by multiplying the preceding term by 3.

If the first term is designed by a

then the second term is a . r

the third term is ar . r = ar^2

the fourth term is ar^2 . r = ar^3

and it appears that the nth term will take the form S(n) = ar^{n-1}.

Thus in general the geometric progression will appear as

a, ar, ar^2, ...ar^{n-1},.....

In a G.P. general term or nth term = ar^{n-1}

A series in which the terms are in a G.P. is called a Geometric Series.

Sum of the terms of a Geometric Series

Consider the GP

a, ar, ar^2,.....

Let $S_n = a + ar + ar^2 + \ldots + ar^{n-1}$

$r.S_n = ar + ar^2 + \ldots + ar^{n-1} + ar^n$

$r.S_n - S_n = ar^n - a$

$S_n(r-1) = a(r^n - 1)$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

Sum of n terms = $\frac{a(r^n - 1)}{r - 1}$ or $\frac{a(1 - r^n)}{1 - r}$ when $r \neq 1$. $\therefore$ $S_n = \frac{a(r^n - 1)}{(r - 1)}$

when r = 1, the formula is not used.

Then, $S_n = a + a + a + \ldots$.n terms

$= na$

Sum of n terms of a Geometric Progression when $n \to \infty$ and $|r| < 1$.

$S_\infty = a + ar + ar^2 + \ldots..$

$$= \frac{a(1 - r^n)}{1 - r} \qquad \text{when } |r| < 1 \text{ and } n \to \infty$$

$$= \frac{a(1 - 0)}{1 - r}$$

$$= \frac{a}{1 - r}$$

Example. Find the sum of the first 15 terms of the geometric progression 3125, 625, 125,.....

$$S = \frac{a(1 - r^n)}{1 - r} \qquad a = 3125$$

$$= \frac{3125\left[1 - \left(\frac{1}{5}\right)^{15}\right]}{1 - \frac{1}{5}} \qquad r = \frac{625}{3125} = \frac{1}{5}$$

$$= \frac{5^{15} - 1}{5^9 \times 4}$$

Exercise

1. Find the sum of the following :

$\frac{1}{243}, \frac{1}{81}, \frac{1}{27}, \ldots\ldots$12 terms.

2. $5\sqrt{2}$, 10, $10\sqrt{2}$,.....8 terms.

3. 0.1, 0.01, 0.001,.....10 terms.

4. If 12, x and –2 are the consecutive terms of an AP, what is x ?

5. If $\frac{1}{\sqrt{3}}$, x, $\sqrt{x}$ are three consecutive terms of a GP, what is x?

6. If 7n – 2 is the nth term of an AP, what is the $(n-1)^{th}$ term ?

7. How many numbers are there between 100 and 200 which are exactly divisible by 7?

8. The sum of the first n terms of an AP is $3n^2$. What is the sum of the first n+1 term?

9. The sum of 3 consecutive terms of an AP is 42. If their product is 2520, find the numbers. [Hint: Take a – d, a, a + d be the numbers.

10. The first term of a GP is 4 and the last term is 2916. The sum of all the terms is 4372. Find the number of terms.

11. A man draws a monthly salary of Rs.2500 at present. He is eligible for an increment of Rs.75 every year. What will be his monthly salary after 5 years?

12. How many terms of the AP 6, 10, 14,....add upto 720?

13. The sum of three consecutive terms of an AP is 3. The sum of their squares is 35. Find the term.

14. The first term of a GP is 4 and the last term is 2916. The sum of all the terms is 4372. Find the number of terms.

SETS AND RELATIONS

Introduction

As a language in mathematics, sets are used. Certain collections are called Sets. In daily life, we come across many sorts of collections as for instance, collection of letters in a language, collection of numbers, etc. In each such collection, a given object either belongs to the collection or does not belong to the collection. This property is called the 'well defined'ness of the elements of the collection. Then a collection of well defined objects is called a Set and the objects forming the set are called the elements (or member) of the set.

Set language provides a precise mode to explain mathematical relationships which exist between elements belonging to given sets.

Coming closely to the concept of a set, we have relation between sets or on a given set. The idea of a relation is familiar to us even in non-mathematical situations – as 'brother of', 'longer than', 'older than', 'has the same colour as' and so on.

John Venn (an English mathematician) developed a diagrammatic method of displaying sets by Venn diagrams. Thus the statement 'All children are innocent' states the relation between two sets – the set of children and the set of innocent beings and the relation is one of 'subset' relation.

Sets are useful in deductive logic, in testing the validity of logical conclusions following from certain assumed statements.

1. Concepts

Sets and Relations

A set is a well-defined collection of distinct objects. A football team is a set of players, a class is a set of students and so on.

Representation of Sets

1. **Roster Method (Tabular form)**

In this method, a set is represented by listing all its elements, separating the elements by commas and enclosing them in braces and each element appears once and only once.

Eg.(1) {1,3,5,7,9} is a set of odd integers less than 10.

(2) The set of letters of the word 'SUCCESS'.

2. **Set builder form (Rule form)**

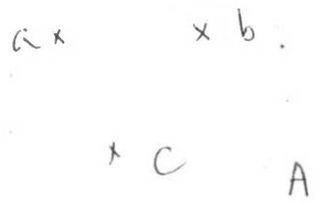
In this method, the members of the sets are represented by stating their characteristic property.

Ex: { x | x is a natural number less than 3 } is a set of natural numbers less than 3.

Notations: $a \in A$ means is an element of A or a belongs to A and $a \notin A$ means a does not belong to A.

3. **Diagrammatic Representation Sets – Euler/ Venn diagram for Sets**

A set is represented by drawing a circle (or any closed curve) and showing all its elements as points inside the circle. Eg. A = { a, b, c }



Venn diagram of set A.

Tips to teachers :

The sets should be well-defined. For example, "the set of all students in your class" is well-defined. "The set of three strongest men in your village" is not well-defined. Ask the students to find collections which are well-defined and collections which are not well-defined.

Types of Sets :

1. Finite Set : A set having finite number of elements is called a finite set.

   Eg. A = { 4,5,6,7} is a finite set. The number of elements in a finite set is called the order of the set denoted by n(A). In the example, n(A) = 4.

2. Singleton Set : A set having only one element is called a singleton set.

   Eg. { x / x is a natural number between 5 and 7}

3. Null set : A set having no elements is a null set denoted by { } or ϕ.

   Eg. { x | x is a natural number between 0 and 1 }

   Note: $n(\phi) = 0$ and ϕ is finite.

Tips to teachers :

Ask the students to find out examples for the above types of sets.

Subset of a Set : A set A is said to be a subset of B if every element of A is also an element of B. Then we write $A \subset B$.

Super Set: If A and B are two sets such that every element of a is also an element of B, then we say that A is a subset of B and B is called the superset of A.

i.e. if $A \subset B$, then $B \supset A$.

Equal Sets : Two sets A and B are said to be equal if they have the same elements. Then we write A = B.

Note : A = B iff $A \subset B$ and $B \subset A$.

Union and Intersection of two Sets :

$A \cup B$ (A union B) is the set consisting of all elements of A together with all elements of B without repeating the elements.

$A \cap B$ (A intersection B) is the set of all elements which are common to both A and B.

Also $x \in A \cap B \Leftrightarrow x \in A$ and $x \in B$.

$x \in A \cup B \Leftrightarrow x \in A$ or $x \in B$

Disjoint Sets : Two sets having no common element are called disjoint Sets. If A and B are disjoint, then

(a) $A \cap B = \phi$

(b) If A and B are finite disjoint sets, then $n(A \cup B) = n(A) + n(B)$.

Result : $\boxed{n(A \cup B) = n(A) + n(B) - n(A \cap B).}$

Example: If A and B are two sets such that $A \cup B$ has 18 elements, A has 8 elements, and B has 15 elements, how many elements does $A \cap B$ have ?

Given, $n(A \cup B) = 18$

$n(A) = 8$

$n(B) = 15$

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

i.e. $18 = 8 + 15 - n(A \cap B)$

$n(A \cap B) = 23 - 18 = 5$

Equivalent Finite Sets : Two finite sets are equivalent if they have the same number of elements.

Eg. The sets $A = \{2,4,6,8\}$ and $B = \{1,3,5,7\}$ are equivalent sets.

More generally, an two sets A and B (finite or infinite) are said to be equivalent if the elements of one set can be paired with the elements of the other set in a one-one way.

Eg. $A = \{1,2,3,4,\ldots\}$

$B = \{2,4,6,8,\ldots\}$

A and B are equivalent (infinite) sets since $n\ (\in A) \leftrightarrow 2n\ (\in b)$ is a one-one correspondence.

Universal Set : In any treatment of sets, it is possible to identify a superset U consisting of all other sets in question as subsets of U. Such a superset U is called the universal set. Its Venn diagram is a rectangle.

Complement of a Set : The complement of a set A is denoted by A^c or A' is the set of all those elements of U which do not belong to a. Venn diagram of A^{-1} is a rectangle.

Difference of two sets

The difference of two sets denoted by A/B or (A – B) is defined as the set of all those elements of the set A which are not in the set B.

i.e. $A / B = \{x / x \in A, x \notin B\}$

Note:

i) If $A \subset B$, then $A - B = \phi$.

ii) If A and B are disjoint finite sets, then $n(A \cup B) = n(A) + n(B)$.

Proof of the Result: If A and B are finite sets, then

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$n(A) = n(A - B) + n(A \cap B)$ (i)

$n(B) = n(B - A) + n(A \cap B)$ (ii)

and $n(A \cup B) = n(A - B) + n(A \cap B) + n(B - A)$ (iii)

$\therefore$ From (i) and (ii)

$n(A) + n(B) = n(A - B) + n(A \cap B) + n(B - A) + n(A \cap B)$

$\therefore n(A) + n(B) = n(A \cup B) + n(A \cap B)$

Hence, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

Properties – Sets and Operations

1. **Associative Laws :**

   a) $(A \cup B) \cup C = A \cup (B \cup C)$
   b) $(A \cap B) \cap C = A \cap (B \cap C)$

2. Commutative Laws :

   a) $A \cup B = B \cup A$
   b) $A \cap B = B \cap A$

3. De Morgan's Laws :

   a) $(A \cup B)^c = A^c \cap B^c$
   b) $(A \cap B)^c = A^c \cup B^c$

4. **Distributive Law :**

a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

5. **Complementation laws :**

a) $(A^c)^c = A$

b) $\phi^c = U$

c) $U^c = \phi$

De Morgan's Laws

These laws state that

$(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$.

Problems :

i) If $A = \{1,2,3,4,5\}$, $B = \{2,4,6,8\}$ find a) $A \cup B$, b) $A \cap B$, c) A/B d) B/A.

a) $A \cup B = \{1,2,3,4,5,6,8\}$

b) $A \cap B = \{2,4\}$

c) $A / B = \{1,3,5\}$

d) $B / A = \{6,8\}$

ii) If $A = \{1,4,9\}$, $B = \{2,4,6\}$, verify that $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

$n(A) = 3$, $n(B) = 3$, $n(A \cap B) = 1$ ($\because$ $A \cap B = \{4\}$).

$A \cup B = \{1,2,4,6,9\}$

$n(A \cup B) = 5$

$n(A) + n(B) - n(A \cap B) = 3 + 3 - 1 = 5 = n(A \cup B)$.

i.e. $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

Exercise :

1. If $A = \{2,4,6,8,10,12\}$, $B = \{1,3,5,7,9,11\}$, find a) $A \cap B$, b) $A \cup B$, c) A/B, d) B / A.

2. If $A = \{-2, -1, 0, 1\}$, $B = \{0,1,2,3\}$, find $n(A)$, $n(B)$, $n(A \cap B)$ and $n(A \cup B)$.

Advanced Exercises for Teachers

1. Prove that $A / B \cup C = A/B \cap A/C$.
2. Prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Relations

Ordered Pairs: Given two sets A and B, an ordered pair (x,y) consists of two elements x and y where $x \in A$ and $y \in B$.

Cartesian Product of Sets: The set of ordered pair (a,b) where $a \in A$, $b \in B$ is called the Cartesian product of two sets A and B. It is denoted by $A \times B$.

Relation : A relation R from a set A to a set B is a subset of $A \times B$. If $(a,b) \in R$, then a is said to be R – related to B and we write as a R b and the relation R from A to B is denoted by $R : A \rightarrow B$.

Example: If $A = \{1,2,3\}$, $B = \{a,b\}$ find $A \times B$.

$A \times B = \{ (1,a), (2,a), (3,a), (1,b), (2,b), (3,b) \}$

Terminology :

a) Given a relation $R : A \rightarrow B$, if$(a,b) \in R$, then b is called the image of a number R.

b) For the relation $R : A \rightarrow B$, A is called the *domain of R*, B is called the *codomain of R*.

c) The set $\{ b \mid (a,b) \in R$ for same $a \in A$ and $b \in B \}$ is called the Range of R.

In the example, A is the domain of R, B is the codomain of R and B is also the range of R.

Note : Range is always a subset of the codomain.

Inverse Relation : Given a relation R from A to B, the relation R^{-1} (which is defined from B to A). $\{ (b,a) / (a,b) \in R \}$ is called the inverse relation of R.

e.g. Given $A = \{1,2,3\}$, $B = \{4,5,6\}$ and R $\{ (1,4), (2,4), (2,6) \}$

$R^{-1} = \{ (4,1), (4,2), (6,2) \}$

Reflexive Relation: A relation R on a set A is said to be reflexive if aRa for all

$a \in A$.

Eg. Let A = {1,2,3} and R]= { (1,1), (2,2), (3,3), (1,2) } is a reflexive relation.

Symmetric Relation: A relation R in a set A is said to be symmetric if aRb $\Rightarrow$ bRa for all a,b $\in$ A. Eg. Let A = {1,2,3}, R = { (1,1), (2,3), (3,2) } is asymmetric relation.

Transitive Relation: A relation R in a set A is said to be transitive if aRb, bRc $\Rightarrow$ aRc for all a,b,c, $\in$ A.

Eg. Let A = {1,2,3}

R = { (1,2), (2,3), (1,3) } is transitive.

Equivalence Relation: A relation which is reflexive, symmetric and transitive is called an equivalence relation.

Example : In the set of straight lines, "is parallel to" is an equivalence relation. Since it is reflexive as any line is parallel to itself. It is symmetric as a || b $\Rightarrow$ b || a. It is transitive as a || b, b || c $\Rightarrow$ a || c.

Exercise: In the set of triangles, find whether 'is congruent to' is an equivalence relation or not.

Worked Examples

1. If A = {2,4,5}, B = {1,2} and U = {1,2,3,4,5,6,7} find $\overline{A}$ and $\overline{B}$.

Solution: $\overline{A}$ = {1,3,6,7}

$\overline{B}$ = {3,4,5,6,7}

2. If U = {0,1,2} and A = {0} state whether the statements are true or false.

i) $\overline{A}$ = U ii) A $\cup$ U = {1,2} iii) $\overline{U}$ = {0

Solution:

i) False. As A = {0}and U = {0,1,2}, $\overline{A}$ = {1,2,} $\neq$ U.

ii) A $\cup$ U = {1,2 } is false as A = {0}, U = {0, 1,2}, A $\cup$ U = {0,1,2}.

iii) False as $\overline{U} = \phi$.

3. If $A = \{2,3,5,7\}$, $B = \{2\}$ find $A \times B$. What is $n(A \times B)$?

Solution:

$A \times B = \{ (2,2), (3,2), (5,2), (7,2) \}$

$n(A \times B) = 4$.

4. Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof: If $x \in A \cap (B \cup C) \Rightarrow x \in A$ and $x \in B \cup C$.

$\Rightarrow$ $x \in A$ and ($x \in B$ or $x \in C$)

$\Rightarrow$ ($x \in A$ and $x \in B$) or ($x \in A$ and $x \in C$)

$\Rightarrow$ ($x \in A\ B$) or ($x \in A \cap C$)

$\Rightarrow$ $x \in (A \cap B) \cup (A \cap C)$

$\therefore A \cap (B \cup C) \Rightarrow (A \cap B) \cup (A \cap C) \Rightarrow A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$

Likewise, it can be shown that $(A \cap b) \cup (A \cap C) \subset A \cap (B \cup C)$. Hence the result.

6. If $A = \{2,3,4,5\}$, $B = \{1,2,3\}$, $C = \{2,4,6\}$ find $A / B \cup C$ and $A/B \cap A/C$. Is $A/B \cup C = A/B \cap A/C$?

$B \cup C = \{1,2,3\} \cup \{2,4,6\} = \{1,2,3,4,6\}$

$A/ B \cup C = \{5\}$ (1)

$A / B = \{ 4,5\}$

$A / C = \{3,5\}$

$A/B \cap A/C = \{5\}$

$\therefore$ $A / B \cup C = A/B \cap A/C$.

7. Prove that the relation 'is parallel to' is an equivalence relation as far as the straight lines are concerned.

i) If a line $l_1 \parallel l_1$ itself. Hence the relation is reflexive.

ii) If a line $l_1 \parallel l_2$, then obviously $l_2 \parallel l_1$. Therefore, the relation is symmetric.

iii) If a line $l_1 \parallel l_2$ and $l_2 \parallel l_3$, then surely $l_1 \parallel l_3$. Therefore, the relation is transitive.

The relation is reflexive, symmetric and transitive. Hence the relation is an equivalence relation.

Exercises (for self-evaluation)

1. Give two examples for : (a) finite set, (b) infinite set, (c) Null set, (d) subset of a set, (e) equivalent set, (f) complement of a set, (g) difference sets, (h) union/intersection of sets, (i) disjoint sets.
2. A group has 20 persons. Among them 15 speak Kannada and 12 speak Telugu. How many speak (a) only Kannada, (b) only Telugu and (c) both Kannada and Telugu?
3. Give an example of a relation which is (a) reflexive but not symmetric, (b) symmetric and transitive but not reflexive, (c) Equivalence relations.

THE LAWS OF EXPONENTS

We know that the result of a repeated addition can be had by multiplication.

Eg. $4 + 4 + 4 + 4 + 4 = 5(4) = 20$

$a + a + a + a + a = 5(a) = 5a$

Likewise the repeated multiplication can be reduced to an exponent form as follows:

$3 \times 3 \times 3 \times 3 \times 3 = 3^5$

$a \times a \times a \times a \times a = a^5$

It may be noticed that in the first case 3 is repeated as a factor 5 times and in the second case 'a' is repeated as a factor 5 times. In all such cases, the number repeated as a factor is called the base, the number which indicates how many times the base is to be used as a factor is called the exponent or index.

This type of expressing products in the form of index or exponents is more convenient and useful in algebra. The very big numbers, like the speed of light, distance between the earth and the moon or sun can be expressed precisely in index form. Even the very small numbers like, diameter of an atom, microseconds of time can be written easily using index form. Hence it is essential to know the laws connected with exponents.

Consider any natural number, say 2 and the expressions lke 2, 2×2, $2 \times 2 \times 2$, etc. can be written as

$2 = 2^1$

$2 \times 2 = 2^2$

$2 \times 2 \times 2 = 2^3$

$2 \times 2 \times \ldots$.m times $= 2^m$

In 2^3, the 3 is called the exponent and the 2 is called the base.

2^3 → Exponent

↓ Base

In 2^3 3 is called as power of 2. In particular in 2^m m is called the index or the exponent and 2 is the base.

Multiplication Property

$$2^3 \times 2^4 = (2 \times 2 \times 2) \times (2 \times 2 \times 2 \times 2)$$
$$= 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$$
$$= 2^7$$

Thus $2^3 \times 2^4 = 2^7$

$$= 2^{3+4}$$

$$x^3 \times x^4 = x^{3+4}$$
$$= x^7$$

In general

$\mathbf{a^m \times a^n = a^{m+n}}$ where m and n are natural numbers.

This property can be extended to problems like the following.

i) $x^3 \times x^2 \times x^5 = x^{3+2+5} = x^{10}$

ii) $y^2 \times y \times y^4 = y^{2+1+4} = y^7$

In general

$$\mathbf{a^m \times a^n \times a^p = a^{m+n+p}}$$

Note: The above property holds true only when the base is same.

Exercise 1.

Simplify

1. $7^4 \times 7^3$
2. $7 \times 7^6 \times 7^4$
3. $x^3 \times x^2 \times x^7$
4. $xy^2 \times x^4y$
5. $x^3y \times x^2y^2 \times xy^3$
6. $3x^2y \times 2xy^2$
7. $\frac{3}{2}xy^3 \times \frac{1}{3}xy^2$
8. $a^{3/2}b^{1/2} \times a^{1/2}b^{1/2}$
9. $\frac{1}{5}x^2y^4 \times \frac{1}{2}xy^2$
10. If $2^n = 64$, find the value of i) 2^{n+2}, ii)_2^{n-3}, iii) 2^{2n+1}, iv) 2^{3n-1}

II. The Division Property

$\mathbf{x^5 \div x^3}$

$$\frac{x^5}{x^3} = \frac{x \times x \times x \times x \times x}{x \times x \times x} = x^2$$

$x^5 \div x^3 = x^{5-3} = x^2.$

In general,

$\mathbf{a^m \div a^n = a^{m-n}}$ $\quad\quad \because$ **m > n.**

$$\frac{x^3}{x^5} = \frac{x \times x \times x}{x \times x \times x \times x \times x} = \frac{1}{x^2}$$

$$x^3 \div x^5 = \frac{1}{x^{5-3}} = \frac{1}{x^2}$$

$\mathbf{a^m \div a^n = \frac{1}{a^{n-m}}}$ $\quad\quad \because$ **m < n**

Note:

i) $a^m \div a^m = a^{m-m} = a^0$

$$\frac{a^m}{a^m} = \frac{1}{1} = 1$$

$\therefore \quad a^0 = 1$

ii) $x^3 \div x^5 = x^{3-5} = x^{-2}$

$$x^3 \div x^5 = \frac{1}{x^{5-3}} = \frac{1}{x^2}$$

$$x^{-2} = \frac{1}{x^2}$$

In general, $\mathbf{a^{-p} = \frac{1}{a^p}}$ **where *p* is a natural number.**

Exercises 2

Simplify

1. $\frac{x^5}{x}$ 2. $\frac{y^{17}}{y^4}$ 3. $\frac{a^3}{a}$ 4. $\frac{c^6}{c^2}$

5. $\dfrac{c^2\ m^3}{cm}$ 6. $\dfrac{14\ y^3}{2y}$ 7. $\dfrac{18\ x^3\ y^2}{3xy^2}$ 8. $\dfrac{-15r^7\ s^5}{-3r^2\ s^2}$

9. $\dfrac{-48x^5y^7}{-3\ x^2\ y}$ 10. $\dfrac{-3\ x^3\ y^2}{xy}$ 11. $\dfrac{(x\ +\ y)^5}{(x+\ y)^2}$

12. $\dfrac{(a-b)^3\ \ (c+d)^{12}}{(a-b)\ \ (c+d)^7}$ 13. $\dfrac{22\ x^8\ y^7}{25\,x^4} \times \dfrac{45\,x^7}{33\ y^3}$

14. $\dfrac{x^8}{y^{12}} \div \dfrac{x^3}{y^{14}}$

II. The power of a power property

$(x^3)^2 = x^3 \times x^3$

$= (x \times x \times x) \times (x \times x \times x)$

$= x^6$

The same result could have been obtained by multiplying the exponents 3 and 2.

i.e. $(x^3)^2 = x^{3 \times 2} = x^6$

Generally,

$$(a^m)^n = a^{mn}$$

Examples :

i) $(a^3)^4 = a^{3 \times 4}\ a^{12}$

ii) $(b^2)^{n+2} = b^{2(n+2)} = b^{2n+4}$

Exercises 3

Simplify

1. $(x^2y)^3$ 2. $(xy^2)^2 \times (x^2y)^3$

3. $\left(\dfrac{a^2b^2}{c}\right)^3$ 4. $(a^2b^3)^6$ 5. $\dfrac{(x^3\ y)^2 \times (xy^2)^3}{(x^2y)^4}$

IV) Power of a Product

Consider the product.

$$(ab)^3 = (ab)\,(ab)\,(ab) = (a\ a\ a)\ (b\ b\ b)$$
$$= a^3b^3$$

In general, $(ab)^m = a^m\, b^m$

Examples:

i) $(a^2b^3)^5 = a^{10}\, b^{15}$

ii) $(-2ab^3)^2 = -2ab^3 \times -2ab^3$
$= 4a^2b^6$

V) . Fractional Index

In a positive fractional index the numerator represents the power and the denominator the root.

For example $x^{1/2} = \sqrt[2]{x} = \sqrt{x}$, $x^{1/3} = \sqrt[3]{x}$, $a^{2/3} = \sqrt[3]{a^2}$.

In general $x^{p/q} = \sqrt[q]{x^p}$

Meaning of $a^{p/q}$ where p and q are any two positive integers.

Consider the product

$a^{p/q}\ .\ a^{p/q}\ .\ a^{p/q}$ .....q factors $= (a^{p/q})^q = a^p$

$\therefore\ a^{p/q} = \sqrt[q]{a^p}$

$\therefore$ $a^{p/q}$ represents the qth root of the pth power of a. Similarly, $a^{p/q} = (\sqrt[q]{a})^p$ represents the pth power of the qth root of a.

Examples :

1. $(16)^{1/2} = \sqrt{16} = 4.$

2. $(27)^{1/3} = \sqrt[3]{27} = 3$

3. $(16)^{3/4} = \sqrt[4]{16^3} = 8$

4. $8^{-4/3} = \frac{1}{8^{4/3}} = \frac{1}{\sqrt[3]{8^4}} = \frac{1}{16}$

Exercises 4

Simplify :

1. $(5\ m^3)^2$
2. $(3a^2b)^2 \times (2ab^2)^3$
3. $\left(\frac{1}{2}\ x^2\ y\right)^3$
4. $\left(\frac{1}{3}\ xy^2\right)^3 \times (3\ xy^2)^3$
5. $(-3xy^2)^3$
6. $(-2xy^2)^2 \times (-3x^2y)^3$

Exercise 5

Simplify

i) $\left(\frac{1}{9}\right)^{\frac{1}{2}} . \left(\frac{1}{27}\right)^{\frac{1}{4}} . (9)^{\frac{1}{6}}$

ii) $\left(\frac{5^{-1}\ .\ 7^2}{5^{-2}\ .\ 7^{-1}}\right)^{\frac{3}{2}}$

iii) $\sqrt[3]{a^4}\ .\ \sqrt[4]{a^2}\ .\ \sqrt[6]{a^2}$

iv) $y^{-51/3}\ .\ \sqrt[4]{y^6}\ .\ \sqrt[6]{y}$

v) $\frac{\sqrt[7]{x^2}\ .\ \sqrt[5]{x^3}}{\sqrt{x^3}\ .\ \sqrt{x^5}}$

Miscellaneous Exercises 6

1. Express the following in the exponential form.

   i) 192 ii) 288 iii) 1296 iv) 729

2. Find the value of each of the following :

   i) 6^3 ii) $(-1)^4$ iii) $(-4)^3$ iv) 6^{-3}

   v) $\frac{1}{2^{-1}}$ vi) $\left(\frac{1}{2}\right)^{-1}$ vii) $a^0\ .\ a^{-2}$ viii) $\frac{a}{a^{-1}}$

ix) $a^3 \div a^2$ x) $\dfrac{1}{b^{-3}}$

3. Simplify : $(x \neq 0,\ m, n, p \in Z)$

i) $x^{m-n} \times x^{n-p} \times x^{p-m}$

ii) $(x^m)^{n-p} \times (x^n)^{p-m} \times (x^p)^{m-n}$

iii) $(x^{m+n})^{m-n} \times (x^{n+p})^{n-p} \times (x^{p+m})^{p-m}$

iv) $\dfrac{1}{1+x^{-m}} + \dfrac{1}{1+x^m}$

4. If $x^m = x^n$, $x \neq 1, x \in R \neq 0$, then $m = n$. Use this rule to solve the following:

i) $2^x = 32$

ii) $5^{3x+1} = 25^{x+2}$

iii) $8^{x+2} = 2^{4x-3}$

iv) $27^{x+1} = 9^{x+3}$

5. Prove that $(x^a)^{b-c} \times (x^b)^{c-a} \times (x^c)^{a-b} = 1$

6. If $x = y^a$, $y = z^b$ and $z = x^c$, prove that $abc = 1$.

7. Simplify m, n being given to be integers.

i) $\dfrac{6^n \times 2^{2n} \times 3^{3n}}{30^n \times 3^{2n} \times 2^n}$ ii) $\dfrac{3^{m+1}}{(3^m)^{m-1}} \div \dfrac{9^{m+1}}{(3^{m-1})^{m+1}}$

iii) $\dfrac{2^{n+1} \times 3^{2n-m} \times 5^{m+n} \times 6^m}{6^n \times 10^{m+2} \times 15^n}$ iv) $\dfrac{2^m \times (2^{m-1})^m \times 2 \times 2^m}{2^{m+1} \times 2^{m-1} \times (2^m)^m}$

8. Prove that

i) $\left(\dfrac{x^p}{x^q}\right)^{p+q} \cdot \left(\dfrac{x^q}{x^r}\right)^{q+r} \cdot \left(\dfrac{x^r}{x^p}\right)^{r+p} = 1$

9. Simplify : $\left(\dfrac{x^q}{x^r}\right)^{q+r-p} \cdot \left(\dfrac{x^r}{x^p}\right)^{r+p-q} \cdot \left(\dfrac{x^p}{x^q}\right)^{p+q-r} = 1$

10. $\left(\dfrac{x^{\frac{2}{3}}}{y^{\frac{3}{2}}} \times \dfrac{y^{\frac{1}{2}}}{x^{\frac{1}{3}}}\right)^2 \div \left(\dfrac{x^{\frac{1}{3}}}{y^{\frac{5}{3}}} \times \dfrac{y^1}{x^{-\frac{2}{3}}}\right)^3$

LOGARITHMS

Introduction

John Napier (1550 –1617) of Scotland designed a simple way of calculations called Napier's bones. John Napier developed a simple computing system, called Napier's bones. In this, there are 10 rods, and on each rod certain numbers are written. We will get the product of two numbers if we arrange the two rods in a particular manner. Hence we can find out the product of two numbers easily. This method is precisely a method of calculations. John Napier and Henry Briggs (1561-1631) constructed ready made logarithmic tables, known as common logarithms.

Integral Power : For any real number 'a', and a positive integer 'n', we define a^n as $a^n = a \times a \times a \times a \times \ldots \times a$ (n factors).

Concepts :

i) $x^m \cdot x^n = x^{m+n}$

ii) $(x^m)^n = x^{mn}$

iii) $\dfrac{x^m}{x^n} = x^{m-n}$

iv) $x^{-m} = \dfrac{1}{x^m}$

v) $x^0 = 1$

Examples:

i) $2^3 \cdot 2^4 = 2^{3+4} = 2^7$

ii) $(2^3)^4 = 2^{3 \times 4} = 2^{12}$

iii) $\dfrac{2^4}{2^3} = 2^{4-3} = 2^1 = 2$

iv) $2^{-4} = \dfrac{1}{2^4} = \dfrac{1}{16}$

v) $2^0 = 1$

Concepts: The nth root of 'a' is denoted by $\sqrt[n]{a}$ or $a^{1/n}$.

Problem : Simplify : $(\sqrt{2}\,\pi)^{1/2}$

$$\left(\sqrt{2}\,\pi\right)^{\frac{1}{2}} = (\sqrt{2})^{\frac{1}{2}} \,.\, \pi^{\frac{1}{2}} = (2^{\frac{1}{2}})^{\frac{1}{2}} \,.\, \pi^{\frac{1}{2}} = 2^{\frac{1}{4}} : \pi^{\frac{1}{2}}$$

Logarithms : Logarithm is just another form of expressing a number in exponential notation. To denote very large or very small numbers, this notation is used.

Eg. $17620 = 1.762 \times 10^4$

$0.0004783 = 4.783 \times 10^{-4}$ etc.

In the equation $2^4 = 16$, the base is 2 and exponent is 4. The same equation can be written as $\log_2 16 = 4$. This can be read as logarithm of 16 to the base 2 is 4. In short, $a = x^m$ and $\log_x a = m$ are equivalent. Here the base must be +ve and not equal to one.

Thus logarithm of a number to a given base is the index or the power to which the base must be raised to equal the given number.

Eg. 1. If $e^x = y$ then $\log_e y = x$

2. If $a^m = n$, then $\log_a n = m$

3. If $b^p = q$, then $\log_b q = p$.

Example

i) Find x if $\log_x 1000 = 3$

$\log_x 1000 = 3 \Leftrightarrow x^3 = 1000$

i.e. $x^3 = 10^3$ $\therefore x = 10$

$\therefore$ $\log_x 1000 = 3$

ii) If $\log_5 a = -2$, find a.

$\log_5 a = -2 \Leftrightarrow 5^{-2} = a$

$$\frac{1}{5^2} = a \qquad \therefore\ a = \frac{1}{25}$$

Laws of Logarithms

1. **Product Rule:** The logarithm of a product of positive numbers is equal to the sum of the logarithm of the numbers.

Proof : We have to prove that $\log_x AB = \log_x A + \log_x B$. Here $x > 0$, $x \neq 1$ and A and B are positive real numbers.

Let $\log_x A = m$ and $\log_x B = n$, then $A = x^m$ and $B = x^n$

$AB = x^m . x^n = x^{m+n}$

Or $\log_x AB = m + n = \log_x A + \log_x B$

$\therefore \log_x AB = \log_x A + \log_x B$

2. Quotient Rule

The logarithm of a quotient of two positive numbers is equal to the logarithm of the numerator minus the logarithm of the denominator.

Proof: We should prove that $\log_x \left(\frac{A}{B}\right) = \log_x A - \log_x B$.

Let $\log_x A = m$ and $\log_x B = n$.

$A = x^m \qquad B = x^n$

$$\frac{A}{B} = \frac{x^m}{x^n} = x^{m-n}$$

$$\therefore \quad \log_x \left(\frac{A}{B}\right) = m - n = \log_x A - \log_x B$$

$$\therefore \quad \log_x \left(\frac{A}{B}\right) = \log_x A - \log_x B.$$

Problems

1. Simplify : $\log_3 24 - \log_3 8$

$$\log_3 24 - \log_3 8 = \log_3 \left(\frac{24}{8}\right) = \log_3 3 = 1$$

2. Simplify : $\log_6 2 + \log_6 3$

$\log_6 2 + \log_6 3 = \log_6 (2 \times 3)$

$= \log_6 6 = 1$

3. Power Rule

The logarithm of a number raised to a power is equal to the power times the logarithm of the number.

Proof: We should prove that $\log_x (A^p) = p \log_x A$.

Let $\log_x A = m$, then $A = x^m$

$\therefore \quad A^p = (x^m)^p = x^{pm}$

$\log_x A^p = pm$

$\therefore \quad \log_x A^P = P. \log_x A.$

Example:

1. Simplify : $\frac{1}{2} \log_3 81$

$\frac{1}{2} \log_3 81 = \log_3 (81)^{1/2}$

$= \log_3 9 = \log_3 3^2 = 2 \log_3 3 = 2 \times 1 = 2$

2. Simplify : $\frac{1}{3} \log_{10} 1000$

$\frac{1}{3} \log_{10} 1000 = \log_{10} (1000)^{1/3}$

$= \log_{10} 10$

$= 1$

4. Change of base rule

This rule states that $\log_x A = \log_y A . \log_x y$

Proof: Let $\log_y A = p$ and $\log_x y = q$.

$A = y^p$ and $y = x^q$

So $A = y^p = (x^q)^p = x^{pq}$

$\therefore \quad \log_x A = pq$

$\therefore \quad \log_x A = \log_y A . \log_x y$

Characteristic and Mantissa : In the logarithmic representation of a number, the whole number part is called the characteristic and the decimal part, the mantissa.

Eg. $\log 16.1 = \log \frac{161}{10}$

$= \log 161 - \log 10$

$= 2.2068 - 1$

$= 1.2068$

Here, the characteristic is 1 and the mantissa is 0.2068.

Common Logarithms: Logarithms with base as 10 are called Common Logarithms.

Note: The characteristic can be negative, but the mantissa is always positive.

Example : If log 2 = 0.3010, what is $\log\left(\frac{1}{2}\right)$?

$$\log\left(\frac{1}{2}\right) = \log 1 - \log 2$$

$$= 0 - (0.3010)$$

$$= -0.3010$$

$$= -1 + 1 - .3010$$

$$= -1 - 0.6990$$

$$= \bar{1}.6990$$

Antilogarithms: For a number N, log is calculated as

$$N \xrightarrow{\textit{log tables}} \log N$$

$$\log N \xrightarrow{\text{Anti log tables}} N$$

Example: Determine the antilog of 2.3207.

Here, characteristic = 2, and mantissa = 0.3207.

From antilog table, 0.3207 corresponds to

$$\begin{array}{r} 2089 \quad + \\ 3 \\ \hline 2092 \end{array}$$

Since the characteristic is 2, there must be three digits behind the decimal : .209.2

$\therefore$ **Antilog of 2.3207 = 209.2**

Calculations using logarithm

Logarithms are useful devices to shorten computation. It must however be remembered that calculations via logarithms will give three figure accuracy.

Example. 1. Evaluate 3.46 × 2.73

Let x = 3.46 × 2.73

log x = log 3.46 × 2.73)

= log 3.46 + log 2.73

= 0.5391 + 0.4362

$= 0.9753$

$\therefore \quad x = \text{antilog } 0.9753 = 9.448$

$\therefore \quad x = 9.448$

2. Evaluate $0.837 \div 0.0028$

Let $x = \dfrac{0.837}{0.0028}$

$\log x = \log \left(\dfrac{0.837}{0.0028} \right)$

$= \log (0.837) - \log (0.0028)$

$= \bar{1}.9227 - \bar{3}.3766$

$= (-1 + 0.9227) - (-3) + 0.3766)$

$= 2.5461$

$x = \text{antilog } (2.5461)$

$= 351.7$

3. Evaluate : $(57.3)^5$

Let $x = (57.3)^5$

$\log x = \log (57.3)^5$

$= 5 \log (57.3)$

$= 5 \times 1.7582$

$= 8.7910$

$\therefore \quad x = \text{antilog } (8.7910)$

$x = 618000000$

4. Evaluate : $\sqrt[3]{(0.1997)^2}$

Let $x = [(0.1997)^2]^{1/3}$

$= (0.1997)^{2/3}$

$\log x = \dfrac{2}{3} \log 0.1997$

$= \dfrac{2}{3} \times \bar{1}.3004$

$\bar{1}.3004 \times \dfrac{2}{3}$

$= \dfrac{\bar{2}.6008}{3}$

$= \dfrac{\bar{3} + 1.6008}{3}$

$= \bar{1}.5336$

$\therefore \quad x = \text{antilog } (\bar{1}.5336)$

$x = 0.3417$

Exercise :

1. Evaluate : $\log_{1/9} 27\sqrt{3}$.
2. Find x if $\log_x 100 = -2$.
3. Simplify :

   (i) $\frac{1}{2} \log_2 64$.

   (ii) $3 \log_a 2 - \log_a 8$

   (iii) $\log_{10} 70 + 2 \log_{10} 5 + \log_{10}\left(\frac{41}{35}\right) - \log_{10}\left(\frac{41}{2}\right)$
4. Show that $\log_b a \,.\, \log_c b \,.\, \log_a c = 1$.
5. Evaluate using logarithms

   (i) 3.456×2.342

   (ii) $86.3 \div 0.427$

   (iii) $(0.0275)^5$

The common logarithm can be conveniently used to solve problems on compound interest. Let 'P' denote the Principal, 'r' the rate of interest percent per annum, 'n' the period in years, and A the amount of P in 'n' years. Then we have

$$A = P\left(1 + \frac{r}{100}\right)^n$$

$$A = P(1 + R)^n \text{ where } R = \frac{r}{100}$$

Taking log on both sides

Log a = log P + n log (1 + R)

This formula involves four quantities, A, P, n and R. Given any three of them, the fourth can be determined.

i.e. $\log P = \log A - n \log(1 + R)$

$$\log(1 + R) = \frac{\log A - \log P}{n}$$

$$n = \frac{\log A - \log P}{\log(1 + R)}$$

Note: If the interest is compounded half yearly. We have $A = P\left(1 + \frac{R}{2}\right)^{2n}$.

Example : Find the compound interest on Rs.10,000/- for 4 years at 5% per annum.

Solution: Here $P = 10{,}000$, $n = 4$, $R = \frac{5}{100} = 0.5$

We have $A = P(1 + R)^n$

$= 10{,}000\,(1 + .05)^4$

$A = 10{,}000\,(1.05)^4$

$\log A = \log 10{,}000 + 4 \log 1.05$

$= 4.0000 + 4(0.0212)$

$= 4.0000 + 0.0848$

$\log A = 4.0848$

$A = 12150$ (Antilog)

Compound Interest = A – P

= Rs.12,150 - Rs.10,000

C.I. = Rs.2,150.

Exercise :

1. Find the compound interest on Rs.5,000/- for 3 years at 5% per annum.
2. Find the compound interest on Rs.10,000 at 10% per annum for 5 years.
3. Find the difference between compound interest and simple interest on Rs.8,000 for 4 years at 5% per annum.
4. What is the present value of Rs.10,000 due in 2 years at 8% per annum compound interest being paid yearly.
5. Mr Mohan borrowed Rs.20,000 from a money lender but he could not repay any amount in a period of 4 years. Accordingly, the money lender demands now Rs.26,500 from him. At what rate percent per annum compound interest did the later lend his money?

QUADRATIC EQUATIONS

Introduction

A quadratic equation is an equation in which the highest degree of the variable is the two. The standard form of a quadratic equation is

$$ax^2 + bx + c = 0$$

where a,b,c are real numbers and $a \neq 0$.

To transform a quadratic equation into the standard form

Example

i) $x(x + 1) - 5 = 0$

$\Rightarrow$ $x^2 + x - 5 = 0$

ii) $x - 4 + \frac{3}{x} = 0$

$\Rightarrow$ $x^2 - 4x + 3 = 0$

iii) $\sqrt{x^2 - 3x} = 2 = 0$

$\Rightarrow$ $x^2 - 3x + 4 = 0$

iv) $x^2 + 7x = 2x + 6$

$\Rightarrow$ $x^2 + 5x - 6$

Exercise 1

Express the following equations in standard form.

i) $x^2 - 9x = 10$

ii) $5x^2 = 125$

iii) $2x^2 = 8x$

iv) $2x^2 + 9x = 2x - 3$

v) $x(x + 3) = 10$

vi) $5(x^2 + 2) = 7(x + 3)$

vii) $x - 5 = \frac{7}{x}$

viii) $\sqrt{2x^2 - 1} = x + 2$

Roots and Solution of a Quadratic Equation

Given a quadratic equation $ax^2 + bx + c = 0$ ($a \neq 0$), a number α is called a root (or solution) of the equation, if $a\alpha^2 + b\alpha + c = 0$. In general, a quadratic equation has two roots. Solving an equation means finding the roots of the equation.

Eg; $2x^2 - 5x + 2 = 0$ has the roots 2 and ½ i.e. { 2, ½ } is the solution of the equation.

Solving a Quadratic Equation by Factorisation

If the left side of a quadratic equation in form can be factored, the solution of the equation depends on the following important principle.

The product of two or more numbers equal to zero if and only if at least one of the factors is equal to zero.

$a \times b = 0$ if and only if $a = 0$ or $b = 0$.

In practice, we apply this principle by equating to zero each linear factor of the left side of the given quadratic equation and solving the resulting linear equation. The following examples will illustrate its application.

i) $x(x - 4) = 5$
$x^2 - 4x - 5 = 0$
$(x - 5)(x + 1) = 0$
$x - 5 = 0$ or $x + 1 = 0$
$x = 5$ or $x = -1$

ii) $2x^2 - 4x = 0$
$2x(x - 2) = 0$
$2x = 0$ or $x - 2 = 0$
$x = 0$ or $x = 2$

iii) $a^2 - 49 = 0$
$(a + 7)(a - 7) = 0$
$a + 7 = 0$ or $a - 7 = 0$
$a = -7$ or $a = 7$

iv) $x - 6 = \sqrt{x}$
Squaring both sides, we get
$(x - 6)^2 = x$
$x^2 - 12x + 36 = x$
$x^2 - 13x + 36 = 0$
$(x - 9)(x - 4) = 0$

$x - 9 = 0$ or $x - 4 = 0$
$x = 9$ or $x = 4$

Exercise 2
Solve by factorizing:

1. $x^2 - 49 = 0$
2. $9x^2 - 64 = 0$
3. $x^2 = 81$
4. $x^2 = \frac{25}{4}$
5. $6x^2 + 2 = 5x^2 + 3$
6. $3x^2 = 4(x^2 - 4)$
7. $x^2 - 2 = 0$
8. $x^2 + 9x + 20 = 0$
9. $3x^2 - 18x = 0$
10. $\frac{8x}{25} = \frac{16}{x}$

Solving Quadratic Equations by the Quadratic Formula

We find the solution as given by the Indian Mathematician Shreedhara in the year 750 A.D. Consider the quadratic equation

$$ax^2 + bx + c = 0, \quad a \neq 0$$

Multiplying by 4a

$$4a^2x^2 + 4abx + 4ac = 0$$

$$(2ax)^2 + 2(2ax)b + 4ac = 0$$

$$(2ax + b)^2 - b^2 + 4ac = 0$$

$$(2ax + b)^2 = b^2 - 4ac$$

Taking the square root

$$\therefore \quad 2ax + b = \pm\sqrt{b^2 - 4ac}$$

$$x = \frac{-b \pm\sqrt{b^2 - 4ac}}{2a}$$

Hence the solution of the general quadratic equation $ax^2 + bx + c = 0$ is

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Putting $\Delta = b^2 - 4ac$

The formula for the solution is

$$x = \frac{-b \pm \sqrt{\Delta}}{2a} \quad \text{when } \Delta = b^2 - 4ac.$$

The quantity $\Delta = b^2 - 4ac$ helps us to distinguish the solutions. So we call $\Delta = b^2 - 4ac$ the *Discriminant* of the quadratic equation.

i) $\Delta = b^2 - 4ac < 0$

Then the quadratic equation has no real solution.

ii) $\Delta = b^2 - 4ac = 0$

Then the quadratic equation has two equal solutions, both equal to $-\frac{b}{2a}$.

iii) $\Delta = b^2 - 4ac > 0$

Then the quadratic equation has two different solutions given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example 1 :

$$x^2 - 13x + 30 = 0$$

For convenience, let us calculate the value of $b^2 - 4ac$ first.

$$b^2 - 4ac = (-13)^2 - 4 \times 1 \times 30$$
$$= 169 - 120$$
$$= 49$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{13 \pm \sqrt{49}}{2}$$

$$x = \frac{13 + 7}{2} \text{ or } \frac{13 - 7}{2}$$

$$x = 10 \text{ or } 3$$

Exercise 3

I. Find the solution of the following equations using formula.

1. $x^2 + 17xc + 60 = 0$
2. $x^2 = 3x + 108$
3. $2x^2 - 11x + 12 = 0$
4. $12x^2 + 4x = 5$
5. $3x^2 = 14x + 5$
6. $5x^2 + 2x = 7$
7. $7x + 2 = 4x^2$
8. $2x^2 = 15x - 27$
9. $7x^2 + 6x + 1 = 0$
10. $x^2 - 8x + 13 = 0$

II. Find the nature of the solution of the following equations. Find the solution, if they exists.

i) $x^2 - 12x + 36 = 0$

ii) $4x^2 + 20x + 25 = 0$

iii) $4x^2 + 49 = 28x$

iv) $x^2 - x + 1 = 0$

v) $x^2 + 25 = 0$

vi) $x^2 - 17x + 14 = 0$

vii) $2x^2 + 15 = 13x$

viii) $4x^2 + 9 = 15x$

Sum and Product of the roots of the Quadratic Equation

We have the elements of the quadratic equation $ax^2 + bx + c = 0$ are

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

The sum of the roots of the solution is

$$= \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2b}{2a}$$

$\therefore$ The sum of the roots $= \dfrac{-b}{a}$

Now let us find how their product is related to a, b and c.

$$\text{Product of the roots} = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)$$

$$= \frac{(-b)^2 - (\sqrt{b^2 - 4ac})^2}{4a^2}$$

$$= \frac{b^2 - (b^2 - 4ac)}{4a^2}$$

$$= \frac{4ac}{4a^2}$$

$\therefore$ Products of the Roots $= \dfrac{c}{a}$

For the quadratic equation $ax^2 + bx + c = 0$.

Sum of the elements of the solution is $= \dfrac{-b}{a}$.

Product of the element of the solution is $= \dfrac{c}{a}$.

Using these formulas, we can find the sum and product of the solution of a quadratic equation without actually solving them.

Example 1 :

Find the sum and product of the solution of the quadratic equation $x^2 + 9x + 20 = 0$. Here $a = 1$, $b = 9$, $c = 20$.

Sum of roots $= \dfrac{-b}{a} = \dfrac{-9}{1} = -9$.

Product of roots $= \dfrac{c}{a} = \dfrac{20}{1} = 20$

Exercise 4

I. For the following equations, find the sum and product of the elements of the solution.

1. $x^2 + 5x + 6 = 0$
2. $2x^2 - 7x - 9 = 0$
3. $4x^2 + 3x = 0$
4. $x^2 - 36 = 0$
5. $3x^2 = 7x + 15$
6. $\frac{x^2}{2} + 4x = 6$
7. $x^2 - 12x + 27 = 0$
8. $x^2 + 15x + 54 = 0$
9. $x^2 + x = 30$
10. $x^2 + 3 = 4x$
11. $x^2 + 3 = 4x$
12. $x^2 = 5x$
13. $x^2 - 7x = 0$
14. $x^2 - 3x - 10 = 0$

We can apply the properties already learnt to solve problems of the type given below.

Example 1 : One root of the equation $3x^2 + kx - 15 = 0$ is -2. Find the value of k and also the other root.

$3x^2 + kx - 15 = 0$

Putting $x = -2$,

$3(-2)^2 + k(-2) - 15 = 0$

$3 \times 4 + k(-2) - 15 = 0$

$-2k - 3 = 0$

$-2k = 3$

$k = \frac{-3}{2}$

Substituting, $\frac{-3}{2}$ for k in the above equation, we get

$$3x^2 - \frac{3}{2}x - 15 = 0.$$

Product of the elements = $\frac{c}{a} = \frac{-15}{3} = -5 \Rightarrow -2 \times 0$. Other root = -5.

$$\therefore \text{ Other root} = \frac{-5}{-2}$$

$$= \frac{5}{2}$$

One element is –2. Second element = $\frac{-5}{-2} = \frac{5}{2}$.

Exercise 5

Against each quadratic equation in column A one root is given in column b. Find the value of k and also the second element.

	A	B
1.	$x^2 - 10x + k = 0$	4
2.	$x^2 - 3x + k = 0$	5
3.	$2x^2 - 7x + k = 0$	-3
4.	$3x^2 + 5x + k = 0$	-2
5.	$x^2 + kx + 22 = 0$	2
6.	$x^2 + kx - 18 = 0$	-3
7.	$x^2 + kx - 15 = 0$	4
8.	$2x^2 + kx + 30 = 0$	-5
9.	$3x^2 + kx - 4 = 0$	-2
10.	$5x^2 + kx - 3 = 0$	-3

Consider the quadratic equation $ax^2 + bx + c = 0$. Dividing the equation by 'a' we get

$$\frac{ax^2}{a} + \frac{bx}{a} + \frac{c}{a} = 0$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

This can be written as

$$x^2 - \left(-\frac{b}{a}\right)x + \frac{c}{a} = 0$$

x^2 – (sum of the element of the solution)x + (product of the element of the solution) = 0.

x^2 – (sum) x + (Pro) = 0

Example 1

Write down the quadratic equation in the general form, if its truth set is {4, -3}.

Sum of the elements = 4 + -3 = 1.

Product of the elements = 3 × -4 = -12

The equation is

x^2 – (sum) x + (prod) = 0

x^2 – (1)x + (-12) = 0

x^2 – x – 12 = 0.

Example 2

The solution set of the quadratic equation is $\{5 + \sqrt{3},\ 5 - \sqrt{3}\}$. Write down the equation in the general form.

Sum of the elements $5 + \sqrt{3} + 5 - \sqrt{3} = 25 - 3 = 22$.

Equation is

x^2 - (sum) x + (pro) = 0

x^2 – (10)x + 22 = 0

x^2 – 10x + 22 = 0

Exercise 6

The solution sets of the quadratic equations are given below. Find the equation in the general form.

1. {2,5}
2. {-4, 5}
3. $\left\{\frac{7}{2}, \frac{-10}{2}\right\}$
4. {-5, -13}
5. $\{\sqrt{7}, -\sqrt{7}\}$
6. $\{6 + \sqrt{5}, 6 - \sqrt{5}\}$
7. $\{1 + \sqrt{3}, 1 - \sqrt{3}\}$
8. $\{2 + \sqrt{2}, 2 - \sqrt{2}\}$

Problem Solving quadratic equation

We have studied the method of solving daily life problems by forming equations based on them. Sometimes when equations are formed we get a quadratic equation. By solving these equations, we can solve such problems. Let us consider a few examples.

Example 1.

Find a number whose square is 2 les than three times the number.

Let n be the number.

$n^2 = 3n - 2$

$n^2 - 3n + 2 = 0$

$(n - 2)(n - 1) = 0$

$n = 2$ or 1

$\therefore$ the numbers are 2 and 1.

Example 2

Find two consecutive integers such that the sum of their squares is 25.

Let n and n + 1 are the consecutive integers.

$n^2 + (n + 1)^2 = 25$

$n^2 + n^2 + 2n + 1 - 25 = 0$

$2n^2 + 2n - 24 = 0$

$n^2 + n - 12 = 0$

$(n + 4)(n - 3) = 0$

$\therefore$ The integers are either -4 and -3 or 4 and 3.

Exercise 7

Solve the following problem by forming quadratic equations based on them.

1. The differences between two numbers is 20. Their product is 480. Find the numbers.
2. Product of two consecutive even numbers is equal to 360. Find the number.
3. The sum of a number and its square is 42. Find the number.
4. Sum of the square of two consecutive counting numbers is equal to 481. Find the numbers.

5. The length of a rectangle is 5 cms more than its breadth. If its area is equal to 176 cm^2 find the length and breadth.

6. In a right triangle, the sum of lengths of shorter sides is equal to 41 cms. If its area is equal to 210 cm^2, find the three sides of a triangle.

7. In a right triangle, the longer side is $x + 5$. The other sides are x and $2x - 5$ respectively. Find the length of the three sides.

8. In the formula $S = ut + \frac{1}{2} gt^2$, $S = 200$, $u = 120$, $g = -32$. Find the value of t.

9. The sum of a number and its reciprocal is $2\frac{4}{15}$. Find the number.

10. In a cricket match, Kapil took one wicket less than twice the number of wickets taken by Sachin. If the product of the number of wickets taken by these two is 15, find the number of wickets taken by each.

ALGEBRAIC EXPRESSIONS – POLYNOMIALS

Introduction

In mathematical problems, often letters (usually of English language) are used to denote numbers. When that is done, the letters used are called *literal numbers*. Thus at any given time, we may denote the population of a country by p or the speed of a car at a given instant by v and so on. Literal numbers take values over a given set of numbers (called the domain). If a mother gives birth to a baby when she is 24 years old, when the baby is x years old, the mother's age is (24 + x) years.

Expressions involving literal number(s) and the operational symbols (+, -, × and ÷) are usually called *algebraic expressions*. The subject of Algebra involves the study of algebraic expressions and algebraic equations. It is said Hindu scholars made extensive use of syncopated notation for unknown quantities and their powers. The Arabic for 'transposition' is 'al-jabr'. Hence the name 'Algebra'.

Algebra is a generalization of Arithmetic (of numbers) with operations on literal numbers and expressions (of these numbers).

Polynomials are special algebraic expressions having their own rules of operations on them.

II Concepts/ Notations/Basic Terminology

a) Constants and Variables

In a given situation, quantities which retain the same value throughout the discussion are called *constants*. In contrast, quantities whose values keep changing (varies) are called *variables*. In algebraic problems, both constant and variables are denoted by letters of English language.

b) Operations on Literal Numbers

i) **Addition :** If a, b are literal numbers, the sum of a and b is defined and denoted by (a + b).

Likewise the sum of a literal number 'a' and a numerical or a number (say) 1 is denoted by a + 1.

ii) **Subtraction :** Given a literal number a, the literal number b such that $a + b = 0$ is called the negative denoted by –a. Accordingly, $a + (-a) = 0$. Given a and b, the number a + (-b) denoted by a – b is called the difference of a and b. Likewise $b - a = b + (-a)$.

iii) **Multiplication:**

a) *of a literal number and a numerical number*

Given a literal number 'a' and the number (a numerical) 2, the product of 2 and a is denoted by 2a (not a2).

As a sum $2a \neq a + a$

Similarly, $3a = 2a + a = a + a + a$, etc.

b) *of two literal numbers a and b*

Product of two literal numbers a and b is defined and denoted by $a \times b$ (or briefly as ab).

c) *of a literal number with itself - powers*

Just as $2 \times 2 = 2^2$

$2 \times 2 \times 2 = 2^3$, etc.

The product of a literal number a with itself in $a \times a$ is denoted by a^2.

Then $a \times a = a^2$ = second power of a (or a - square)

$a^3 = a^2 \times a = a \times a^2 = a \times a \times a$ and so on.

= Third power of a (a – cube)

In a^2, a is called the base, 2 is called the exponent and a^2 is called the second power of a.

More generally, $a \times a \times a \times$ .....n times $= a^n$ when n is any positive integer.

Hence a is the base, n is the exponent or index and a^n is the nth power of a.

The process of getting a^n is called *involution* (or raising a number to a power) and the opposite process (extracting root) is called *Evolution*.

d) *Rules of Operations - Algebra*

1. Addition (+)

a) $a + b = b + a$ (Commutative law for addition)

b) $(a + b) + c = a + (b + c)$ (Associative law for addition)

c) $a + 0 = 0$ (Property of 0).

d) $a + (-a) = 0$ (Property of the negative of a number).

2. Multiplication ($\times$)

a) $ab = ba$ (Commutative law for multiplication)

b) $(ab)c = a(bc)$ (Associative law for multiplication)

c) $a \cdot 1 = a$ (property of unity (=1))

d) $a \cdot \frac{1}{a} = 1$ (when $a \neq 0$) (property of the reciprocal of a number)

3. $a(b + c) = ab + ac$ (Distributive Property)

4. Powers :

a) $a^1 = a$ b) $a^{n+1} = a^n \cdot a$ c) $a^m \cdot a^n = a^{m+n}$

d) $(ab)^n = a^n b^n$ e) $\frac{a^m}{a^n} = a^{m-n}$, if $m > n$.

$\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$ if $m < n$ and $\frac{a^m}{a^n} = 1$, if $m = n$.

e) Monomial / Binomial / Trinomial etc.

An algebraic expression containing

i) a *single* term is called a *monomial*

eg. $4x$, $-6a$, $8a^2$, $3a^2b$, $5abc$

ii) *two terms* is called a *binomial*

Eg. $ax + b$, $x^2 + y^2$, $x^2 - ax$, $a^3 + b^3$

iii) *Three terms is called a Trinomial*

Eg. $a + b + c$, $a^3 + b^2 + c$, $5a + 4b - 2c$

f) Coefficients and Constants in an Algebraic Expression

In a general algebraic expressions, numericals attached to literal numbers (by multiplication) are called the coefficients. Here only numerical coefficients are defined. And the numerical not attached to any literal number is called the constant (term) in the expression.

In $2a^2 + 3ab + 1$, 2 and 3 are coefficients while 1 is the constant term.

In $3x^2yz$, 3, x^2, y, z are all factors. While 3 is a numerical factor, others are literal factors.

g) Like and Unlike Terms

In an algebraic expression, terms having the same literal factors (in same powers) are called like terms and the terms not having the same literal factors are called unlike terms.

Thus in $2x + 3y - 5x + 4y + 2xy$

2x and –5x are like terms,

3y and 4y are like terms,

But 2x and 3y are unlike terms or 2x and 2xy are unlike terms.

Note: Recognition of like terms is an important step in simplifying algebraic expressions.

h) Value of an Algebraic Expression

Given an algebraic expression containing one or more literal numbers, by putting given numerical values for the literal numbers, the value of the algebraic expression is got. The method (of putting the numerical values for literal numbers) is called *substitution.*

Eg. When $x = 0$, $y = 1$, $z = 2$, the value of $(x + 1)\, y^2z = (0 + 1)^2 \,.\, 1^2 \,.\, 2 = 2$

i) Simplification of an Algebraic Expression

An algebraic expression contains

a) numbers (numericals) and literal numbers,

b) operational symbols (+, -, × , ÷ and powers) and

c) brackets – suggesting grouping of terms of different types as ( ) - round, { }- flower, [ ] - square , ____ - bar brackets.

BODMAS RULE

To simplify expressions inside the brackets, the operations are done in the order of the letters of BODMAS.

B → Brackets
O → Of
D → Division (÷)
M → Multiplication (×)
A → Addition (+)
S → Subtraction (-)

Example : Simplify

$$- \left[- \frac{1}{2} \{ 10 + 4 (2a + 1) + 3b \} \right]$$

$$= \frac{1}{2} \{ 10 + 4(2a + 1) + 3b \}$$

$$= 5 + 2 (2a + 1) + 3b$$

$$= 5 + 4a + 2 + 3b$$

$$= 4a + 3b + 7.$$

II Polynomials

a) **Linear Polynomial:** An expression of the form ax + b when a and b are constants and x a variable (called also as an unknown) and $a \neq 0$, is called a *polynomial of degree 1 or a linear polynomial in x.*

Eg. $2x + 3$; $-\frac{1}{2}x + 2$; $\sqrt{2}\,x + \frac{1}{3}$

In ax + b, a is called the coefficient of x and b is the constant term.

b) **Quadratic Polynomial:** An expression of the form $ax^2 + bx + c$ when $a \neq 0$, b, c are constants (Real or complex) and x is a variable is called a *second degree polynomial* or a *quadratic* (expression) *polynomial* in x.

Eg. $3x^2 - 2x + 1$; $\frac{1}{\sqrt{2}}x^2 + 2x - \sqrt{3}$; $x^2 + 4$

c) **Cubic Polynomial:** An expression of the form $ax^3 + bx^2 + cx + d$ when $a \neq 0$, b, c d are constants (Real or complex) and x is a variable is called a *third degree polynomial* or a *cubic polynomial* in x.

The idea can be extended. Accordingly, a polynomial of nth degree in x is an expression of the type : $a_0x^n + a_1x^{n-1} + \ldots + a^n$ when $a_0 \neq 0$, $a_1, a_2, \ldots a_n$ are constants (real or complex) – called the coefficients of various terms of the polynomial, and a_n is called the constant term (term free from x) and x is a positive integer. If $a_0 = 1$, then the polynomial is called a *monic*.

Note: In a polynomial of nth degree,

i) the coefficient of the highest degree (i.e. nth degree) in x is called the leading coefficient and is not zero.

ii) The degrees of terms from the highest decreases.

iii) The polynomial contains only terms with positive exponents

when n = 4, the polynomial is called a biquadratic or quartic and

n = 5, the polynomial is called a Quintic.

iv) A polynomial in x is called a *real polynomial* if the coefficients are all *real numbers.*

v) A polynomial in x is called a complex polynomial if the coefficients are *complex numbers.*

Special Polynomials

i) A polynomial in which all the coefficients are zero is called a Zero Polynomial denoted by 0 (or 0(x)). For a zero polynomial, degree is not defined.

ii) A polynomial containing only the constant term is called a constant polynomial (its degree is zero) if it is not a zero polynomial.

Algebra of Polynomials

1. **Equal polynomials:** Hence forth we denote polynomials by P(x), Q(n), etc.

Let P(x) and Q(x) be two polynomials of the same degree. Then $P(x) \equiv Q(x)$ if and only if the coefficients of like terms in P(x) and Q(x) are equal.

Eg. $ax^2 + bx + c \equiv 3x^2 + 4x + 5$ means a = 3, b = 4, c = 5.

2. **Operations on Polynomials :** Similar to the way we define the operations – Addition (+), Subtraction (-) , Multiplication (×) and Division (÷) on numbers, we define these operations for polynomials also.

a) Addition : While adding polynomials, like terms in Polynomials are added. Then the sum of two polynomials P(x) and Q(x) denoted by P(x) + Q(x) is got.

Eg. If $P(x) = x^2 - 2x + 3$, $Q = 3x^2 + 4x - 1$

Then $P(x) = (x^2 - 2x + 3) + (3x^2 + 4x - 1)$

$= (x^2 + 3x^2) + (-2x + 4x) + (x - 1)$

$= 4x^2 + 2x + 2$

Note: i) Any two polynomials can be added.

ii) The degree of the sum does not exceed the degree of the highest degree among the summands.

b) Subtraction:

i) The negative of a polynomial : Given a polynomial P(x), the polynomial got by changing the signs of each term in P(x) is called the negative of P(x) denoted by –P(x).

Eg. If $P(x) = x^3 - 2x^2 + 3x - 2$

Then $-P(x) = -x^3 + 2x^2 - 3x + 2$

ii) Given two polynomials P(x) and Q(x), the difference of P(x) and Q(x) denoted by $P(x) - Q(x) = P(x) + (-Q(x))$.

Eg. $P(x) = 5x^2 - 2x + 3$, $Q(x) = x^3 - x^2 + 1$

then $P(x) - Q(x) = (5x^2 - 2x + 3) + (-x^3 + x^2 - 1)$

$= -x^3 + 6x^2 - 2x + 2$

c) Multiplication: Given two polynomials P(x) and Q(x), in forming the product of P(x) and Q(x), (i) the terms of P(x) and Q(x) are multiplied in order, (ii) then the like terms are grouped and (iii) lastly, the terms of the product are arranged in decreasing order of degrees of terms.

Eg. 1. Let $P(x) = (2x + 1)$, $Q(x) = (x^2 + 1)$

Then $P(x) . Q(x) = (2x + 1)(x^2 + 1)$

$= 2x(x^2 + 1) + 1(x^2 + 1)$

$= 2x^3 + 2x + x^2 + 1 \quad = 2x^3 + x^2 + 2x + 1$

$\therefore \quad P(x) . Q(x) = 2x^3 + x^2 + 2x + 1$

ii) $P(x) = (x^2 - x + 1)$, $Q(x) = (x^2 - 1)$

$P(x) . Q(x) = (x^2 - x + 1)(x^2 - 1)$

$= x^2 . x^2 + x^2(-1) - x . x^2(-x)(-1) + 1. x^2 + 1(-1)$

$= x^4 - x^2 - x^3 + x + x^2 - 1$

$\therefore P(x) . Q(x) = x^4 - x^3 + x - 1$

Use the Table to determine the sign of the product

×	+	-
+	+	-
-	-	+

Note: **Degree of P(x) . Q(x) = Degree of P(x) + Degree of Q(x)**

d) Division of a polynomial by a linear polynomial – Root of a polynomial equation or a factor of a polynomial.

Given a polynomial P(x) of nth degree, dividing it by a linear polynomial $(x - \alpha)$, α being a number. We denote the quotient by Q(x) and the remainder by R. Then $P(x) = (x - \alpha)\ Q(n) + R$ when Q(n) is the degree $(n - 1)$ and R is a constant).

Remainder Theorem : The remainder got by dividing P(x) by $(x - \alpha)$ is $P(\alpha)$ (i.e. $R = P(\alpha)$).

Proof: $P(x) = (x - \alpha),\ Q(x) + R$.

Putting $x = \alpha$, $P(\alpha) = (\alpha - \alpha)\ Q(\alpha) + R = 0 + R = R$

$\therefore \quad R = P(\alpha)$

Eg. The remainder got by dividing $x^2 - 2x - 2$ by $x - 1$ is $1^2 - 2 . 1 - 2 = 3$.

Since $P(x) = x^2 - 2x - 2$, $x - \alpha = x - 1$, $\therefore \alpha = 1$

$\therefore \quad R = P(\alpha) = 1^2 - 2 . 1 - 2 = -3$.

Root of a Polynomial Equation P(x) = 0

A value α such that $P(\alpha) = 0$ is called a root of $P(x) = 0$ (also called a zero of P(x)).

Theorem : If $x = \alpha$ is a root of $P(x) = 0$, then $(x - \alpha)$ is a factor of P(x).

Proof : By Remainder Theorem, $P(x) = (x - \alpha)\ Q(n) + P(\alpha)$

If α is a root of $P(x) = 0$, then $P(\alpha) = 0$

$\therefore \quad P(x) = (x - \alpha)\ Q(x)$

Hence $(x - \alpha)$ is a factor of P(x).

Note: Corresponding to each root α of $P(x) = 0$, $(x - \alpha)$ is a factor of P(x).

Synthetic Division (Method of Detached Coefficients)

This method is an easy method of obtaining the quotient and the remainder got by dividing a polynomial P(x) by a linear polynomial $(x - \alpha)$.

The method is illustrated by the following example.

Eg. Find the Quotient and the remainder got by dividing $2x^3 + 3x^2 + 4x - 2$ by $x - 1$.

Method

1. Write the (detached) coefficients along with the sign in order from the given polynomial.

   Thus we have 2 3 4 -2

2. Equalise $x - 1$ to zero; get x

   $\therefore \quad x - 1 = 0 \Rightarrow x = 1$

3. Detached Coefficients :

	2	3	4	-2
	0	2.1	5.1	+9.1
Add:	2	5	9	7

4. Noting Q(x) is of degree 2,

   attach x^2, x and 1 with 2, 5, 9 respectively.

   Then $Q(x) = 2x^2 + 5x + 9.1 = 2x^2 + 5x + 9$ and 7 is the remainder.

   Check: $Q(x) \times (n - 1) + r = P(x)$

$$= (2x^2 + 5x + 9)(x - 1) + 7$$

$$= 2x^3 + 5x^2 + 9x$$

$$- 2x^2 - 5x - 9$$

$$+ 7$$

$$2x^3 + 3x^2 + 4x - 2 = P(x).$$

General Method (of Synthetic Division)

Let $P(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n \quad (a_o \neq 0)$ and the divisor be $(x - \alpha)$.

Then, the coefficient in P(x) :		a_o	a_1	a_2	..... a_n
$x - \alpha = 0$ or $n = \alpha$		0	$a_o\alpha$	$b_1\alpha$	..... $b_{n-1}\alpha$
	Add:	a_0	b_1	b_2	b_{n-1} $a_n + \alpha b_{n-1} = R$

The quotient $Q(x) = a_0 x^{n-1} + b_1x^{n-2} + \ldots + b_{n-1}$ and the remainder $= R = a_n + \alpha \cdot b_{n-1}$.

Note: While writing the coefficients in P(x)

a) the coefficients along with the sign must be written.

b) The coefficients must be written in the same order.

c) If a power of x in P(x) is missing, the corresponding coefficient must be taken as zero.

d) At each stage, corresponding entries must be added.

e) The number so got after adding are the coefficients of Q(x) which is a polynomial of degree (n – 1) and

f) The last entry is the remainder R, as shown.

2 (e) If P(x), Q(x) are two polynomials and deg. Q(x) ≤ deg. P(x), then P(x) = Q(x) . S(x) + R(x) when R(x) is called the *Remainder* and S(x) is the quotient . [0 ≤ Deg R(x) < deg Q(x)]

Problems

1. Write the following statements as algebraic expressions using numbers, literal numbers and operations.

		Ans
a)	sum of 6 and x	$6 + x$
b)	3 more than a	$3 + a$
c)	one third y	$\frac{1}{3}y$
d)	y less than 7	$7 - y$
e)	2 less than the quotient of x and y	$\frac{x}{y} - 2$
f)	3 more than twice the product of x and y	$3 + 2xy$

2.

a)	The age of Harish 5 years ago was x. What is his age now?	(x + 5) years.
b)	My salary 15 years ago was Rs.1200 and my yearly Increment is x rupees. My present salary is	(1200 + 15x) Rs.
c)	A box's length, breadth and height are l, b, h respectively. Its volume is .....and total surface area is	Volume = lbh Surface Area = 2(lb + bh + lh)

d) The selling price of an article is S and cost price is C. $P = S - C$
Profit is P = ...

e) a,b,c are the sides of a triangle. Then its perimeter is..... $a + b + c$.

3. a) Add: $-xy^2$, $7x^2y$, $-6x^2z^2$, $-18x^2z^2$, $-5x^2y^2$, $2xy^2$, $6x^2y^2$ and $-9x^2z^2$.

Ans: Grouping like terms

Sum = $(-xy^2 + 2xy^2) + (7x^2y) + (-6x^2z^2 - 9x^2z^2 - 18x^2z^2) + (-5x^2y^2 + 6x^2y^2)$

$= xy^2 + 7x^2y - 33x^2z^2 + x^2y^2$

b) Subtract : $4xy - 2yz + 7xz - 4z + 3x - 2y + 1$ from

$xz + 2xy - 3yz - 4x + 6y + z - 10$

Difference =

$(xz + 2xy - 3yz - 4x + 6y + z - 10) - (4xy - 2yz + 7xz - 4z + 3x - 2y + 1)$

$=(xz - 7xz) + (2xy - 4xy) + (-3yz + 2yz) + (-4x + 3x) + (6y + 2y) + (z + 4z) + (-10 - 1)$

$= -6xz - 2xy - yz - 7x + 8y + 5z - 11$

c) Multiply $(xy + x + y)$ and $(2x - 3y + 1)$

$(xy + x + y)\ (2x - 3y + 1)$

$= xy \cdot 2x + xy(-3y) + xy \cdot 1 + x \cdot 2x + x(-3y) + x \cdot 1 + y \cdot 2x + y(-3y) + y \cdot 1$

$= 2x^2y - 3xy^2 + xy + 2x^2 - 3xy + x + 2xy - 3y^2 + y$

$= 2x^2y - 3xy^2 + 2x^2 - 3y^2 + x + y$

4. Simplify each expression

a) $2a - (b - a) - b + (a - b)$

$= 2a - b + a - b + a - b$

$= 4a - 3b$

b) $m(10m - 9) + 7m - m(3m + 5) - 8$

$= 10m^2 - 9m + 7m - 3m^2 - 5m - 8$

$= 7m^2 - 7m - 8$

c) (i) If $F(x,y,z) = x^2 + y^2 + z^2 + x - y + z + 1$, find $F(1, -1, 0)$.

$F(1, -1, 0) = 1^2 + (-1)^2 + 0^2 + 1 - (-1) + 0 + 1 = 1 + 1 + 1 + 1 + 1 = 5.$

(ii) If $f(x,y) = (x^2 - xy + 2y^2)$

Find f(1,2) and f(2,1)

$f(1,2) = 1^2 - 1 . 2 + 2 . 2^2 = 1 - 2 + 8 = 5$

$f(2,1) = 2^2 - 2 . 1 + 2 . 1^2 = 4 - 2 + 2 = 4$

5. a) Simplify : $85 - [12x - 7(8x - 3) - 2\{ 10x - 5(2 - 4x) \}]$

$= 85 - [12x - 56x + 21 - 2 \{10x - 10 + 20x \}]$

$= 85 - [-44x + 21 - 2 (30x - 10)]$

$= 85 - [-44x + 21 - 60x + 20]$

$= 85 - [-104x + 41]$

$= 44 + 104 x$

b) $(a + b + c) (a^2 + b^2 + c^2 - ab - bc - ca)$

$= a^3 + ab^2 + ac^2 - a^2b - abc - a^2c + a^2b + b^3 + bc^2 - ab^2 - b^2c - abc + a^2c + b^2c + c^3 - abc - bc^2 - ac^2$

$= a^3 + b^3 + c^3 - 3abc.$

6. Given $P(x) = x^3 - x^2 + 3x + 2$, $Q(x) = x^4 - 2x^2 + x - 1$ find (a) $P(x) + Q(x)$, b) $P(x) - Q(x)$ c) $2Q(x) - 3P(x)$.

a) $P(x) + Q(x) = x^3 - x^2 + 3x + 2 + x^4 - 2x^2 + x - 1$

$= x^4 + x^3 - 3x^2 + 4x + 1$ $\therefore$ Degree of $P(x) + Q(x)$ is 4.

b) $P(x) - Q(x) = (x^3 - x^2 + 3x + 2) - (x^4 - 2x^2 + x - 1)$

$= x^3 - x^2 + 3x + 2 - x^4 + 2x^2 - x^1 + 1$

$= -x^4 + x^3 + x^2 + 2x + 3$ Degree = 4

c) $2Q(x) - 3P(x) = 2(x^4 - 2x^2 + x - 1) -3(x^3 - x^2 + 3x + 1)$

$= 2x^4 - 4x^2 + 2x - 2 - 3x^3 + 3x^2 - 9x - 6$

$= 2x^4 - 3x^3 - x^2 - 7x - 8$ Degree = 4

d) $P(x) . Q(x) = (x^3 - x^2 + 3x + 2) (x^4 - 2x^2 + x - 1)$

$= x^7 - 2x^5 + x^4 - x^3$

$- x^6 + 2x^4 - x^3 + x^2$

$+\ 3x^4 - 6x^3 + 3x^2 - 3x$

$+\ 2x^4 - 4x^2 + 2x - 2$

$= x^7 - x^6 - 2x^5 + 8x^4 - 8x^3 - x - 2;$ Degree = 7.

7. Using Remainder theorem, find the remainder on dividing $x^4 + 2x^3 - 3x^2 + x - 5$ by $x + 2$.

$P(x) = x^4 + 2x^3 - 3x^2 + x - 5.$

Remainder on dividing P(x) by $(n - \alpha)$ is $P(\alpha)$. Here $\alpha = -2$.

$\therefore$ The remainder $= P(-2) = (-2)^4 + 2(-2)^3 - 3(-2)^2 + (-2) - 5$

$= 16 - 16 - 12 - 2 - 5 = -19.$

8. a) Find the quotient and the remainder got by dividing $x^3 + x^2 + x + 1$ by $x + 1$.

$x + 1 = 0 \Rightarrow x = -1.$

1	1	1	1
0	-1	0	-1
1	0	1	0

$\therefore$ Quotient: $1 \,.\, x^2 + 0 \,.\, x + 1 = x^2 + 1$

Remainder = 0.

b) Find the quotient and remainder got by dividing $2x^4 - 3x^2 + 4x - 6$ by $x + 3$

$x + 3 = 0 \Rightarrow x = -3.$

2	0	-3	4	-6
0	-6	18	-45	+123
2	-6	15	-41	117

$\therefore$ Quotient : $2x^3 - 6x^2 + 15x - 41$, Remainder = 117.

9. a) Find the quotient and remainder by dividing $x^3 + 1$ by $x^2 + 1$.

$$\begin{array}{r|l} & x \quad = \text{quotient} \\ \hline x^2+1 & x^3 + 1 \\ & x^3 + x \\ \hline \text{Subtract} \rightarrow & \boxed{-x + 1 \quad = \text{Remainder}} \end{array}$$

Then $\dfrac{x^3 + 1}{x^2 + 1} = x + \dfrac{-x+1}{x^2 + 1}$

b) Find S(x) and R(x) such that $(x^4 + 1) = (x^2 + x + 1)\ S(x) + R(x)$ where deg $R(x) \leq 1$.

Dividing $x^4 + 1$ by $x^2 + x + 1$.

$$
\begin{array}{r|l}
 & x^2 - x \\
\hline
x^2 + x + 1 & x^4 + 1 \\
 & x^4 + x^3 + x^2 \\
\hline
\text{Subtract} \rightarrow & -x^3 - x^2 + 1 \\
 & -x^3 - x^2 - x \\
\hline
\text{Subtract} \rightarrow & x + 1
\end{array}
$$

$\therefore$ $S(x) = x^2 - x$

$R(x) = x + 1$

Exercise

1. Write the following statements as algebraic expressions.
   a) The product of 5 less than x and 2 more than x.
   b) The sum of x^2 and $(x + 1)$
   c) Twice x added to half of x^2.
   d) Product of x and $(y + 1)$ added to the quotient of y and $(x + 1)$.
   e) The sum of 3rd power of x, twice 2nd power of x and –2.

2. a) Find the sum of : $3a - 5b + 7c$, $-2a + 10b - 6c$ and $a + 3b - 2c$.
   b) Subtract $x^2 - xy + y^2$ from $-2x^2 - 3xy + 2y^2$
   c) Find the products of $(2a + 1)$, $(a - 2)$ and $(3a + 2)$.

3. Simplify :
   a) $10m^2 - 9m + 7m + m^2 - mn + 4n^2 + 3mn - n^2$
   b) $(x^2 + 3x - 2) - (4x^2 - 2x + 1) + (x^2 + 4)$
   c) Write down the degree of each term in the expressions.
      i) $x^3 + x^2y + 2xyz + xy - x + 1$
      ii) $a^3 + ab^2 - a^2 + bc$

4. a) Find the value of $x^2 + xy + y^2$ when $x = 1, y = 2$.

b) Find the value of $(x^3 + 1)\ (x^2 + y)\ (x + z)$ when $x = 1, y = 2, z = 3$.

c) If $f(a,b,c) = a^3 + b^3 + c^3 - 3abc$, find $f(1, -1, 2)$.

5. Simplify :

a) $a - [2a - 5\ \{\ b + a - 2\ (b - \overline{1 - a}\ )\ \}\ ]$

b) $xy - [yz - zx - \ \{xy - (3y - z)\ x + y(z - 2x)\ \}]$

c) $a(b - c) + b(c - a) + c(a - b)$

d) $(x - y)\ (x^2 + xy + y^2) + (y - z)\ (y^2 + yz + z^2) + (z - x)\ (z^2 + zx + x^2)$

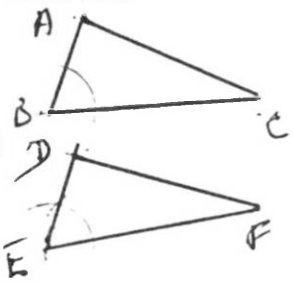
SIMILAR TRIANGLES

Introduction

Congruence and similarity of triangles

The geometrical figures are said to be congruent if they have exactly the same shape and size. Two line segments are congruent if and only if their lengths are equal. Two angles are congruent if and only if their measures are equal. Two angles are congruent if end only if their measures are equal.

Two triangles are congruent if and only if there exists a correspondence between their vertices such that the corresponding sides and the corresponding angles of the two triangles are equal.



Notation: If Δ ABC is congruent to Δ EFG, we write $\Delta ABC \cong \Delta EFG$

Now, we shall consider figures which have the same shape but not necessarily the same size. Design engineers and architects most of ten deal with such figures. Triangles having the same shape are called similar triangles.

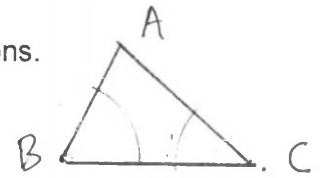
Thales, (about 600 B.C.), who introduced the study of Geometry in Greece, proved an important truth concerning similar triangles. " The ratio of the lengths of any two corresponding sides in similar triangles is always the same irrespective of their actual sizes ". This fact enables us to find the unknown sides of one of the triangles when one of its sides and all the sides of a similar triangle are known. Historians tell us that Thales found the height of a pyramid in Egypt on the basis of the length of the sides of the shadow. By the use of similar triangles it became possible to measure heights and distances which otherwise could not be measured.

Similar polygons

Two polygons are said to be similar to each other if

1) Their corresponding angles are equal and
2) The lengths of their corresponding sides are proportional.

Similar triangles

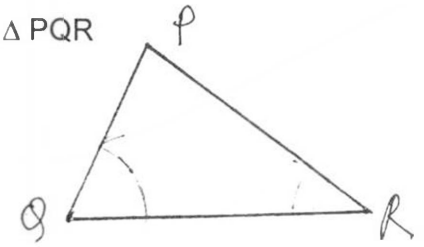


Comparing to the definition of similarity of polygons.

Two triangles are said to be similar if

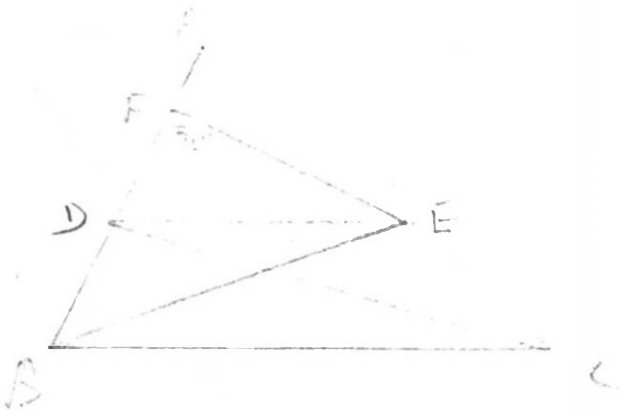
1) Their corresponding angles are equal, and
2) Their corresponding sides are proportional.

If Δ ABC is similar to Δ PQR then it is written as Δ ABC ~ Δ PQR



Thales theorem (Basic proportionality theorem)

In a triangle, a line drawn parallel to one side, to intersect the other sides in distinct points, divides the two sides in the same ratio.



<u>Proof:</u> In Δ ABC, DE || BC and DE intersects AB in D and AC in E. We have to prove that $\frac{AD}{DB} = \frac{AE}{EC}$

Join BE, CD and draw EF $\perp$ BA

$$\frac{area\ of\ \Delta\ ADE}{area\ of\ \Delta BDE} = \frac{\frac{1}{2}ADEF}{\frac{1}{2}DBEF} = \frac{AD}{DB} \text{ ---------(1)}$$

Similarly

$$\frac{area\ of\ \Delta\ ADE}{area\ of\ \Delta CDE} = \frac{AE}{EC} \text{ ----------(2)}$$

Since triangles on the same base (DE) and between the same parallel lines (DE&BC) having the same area, we have,

Area of Δ BDE = area of Δ CDE

∴from (1) and (2) $\frac{AD}{DB} = \frac{AE}{EC}$

Converse:

If a line divides any two sides of a triangle, in the same ratio, the line must be parallel to the third side.

Proof:_ _In Δ ABC, draw a line L intersecting AB in D and AC in E such that

$$\frac{AD}{DB} = \frac{AE}{EC}$$

We have to prove that the L line || BC

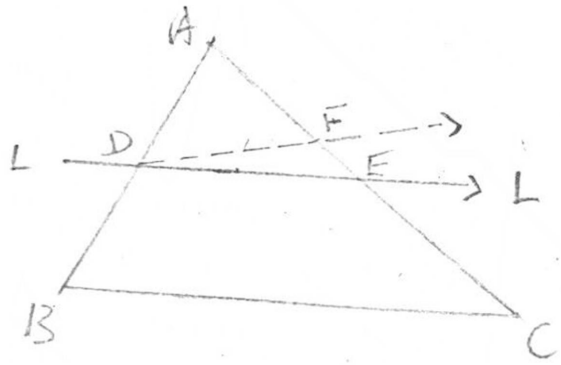
Let us suppose that is L not parallel to BC. Then through D, there must be another line parallel to BC. Let DF || BC

Since DF || BC, by basic proportionality theorem, we get

$$\frac{AD}{DB} = \frac{AF}{FC}$$

But $\frac{AD}{DB} = \frac{AE}{EC}$

∴ $\frac{AF}{FC} = \frac{AE}{EC}$



i.e., $\frac{AF}{FC} + 1 = \frac{AE}{EC} + 1$

$$\frac{AF + FC}{FC} = \frac{AE + EC}{EC}$$

$\therefore \frac{AC}{FC} = \frac{AC}{EC}$

hence FC= EC, but this is impossible unless the points F and e coincide i.e., DF is the line L itself. Hence L || BC.

Example: Determine the point on a line segment which divides it in a given ratio 2:3

Draw a line AC making any acute angle with A.B.

Mark the points A_1, A_2, A_3, A_4 and A_5 at equal distance

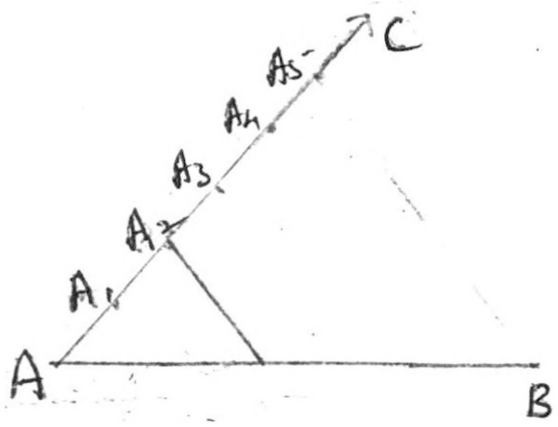
Since the ratio is 2:3

$\therefore AA_1 = A_1A_2 = A_2A_3 = A_3A_4 = A_4A_5$

Join A_5 to B from A_2 draw a line parallel to A_5B to intersect AB in P. P is the required point on AB such that AP:PB = 2:3.

In Δ ABA_5, PA_2 || BA_5. $\therefore$ By basic proportionality theorem, $\frac{AP}{PB} = \frac{AA_2}{A_2A_5} = \frac{2}{3}$

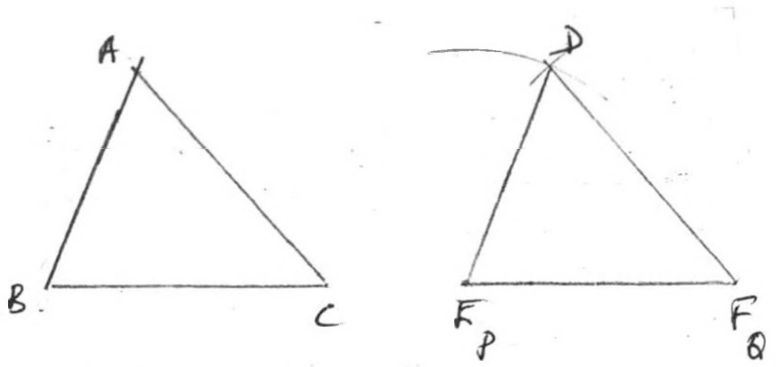
i.e., AP : PB = 2:3

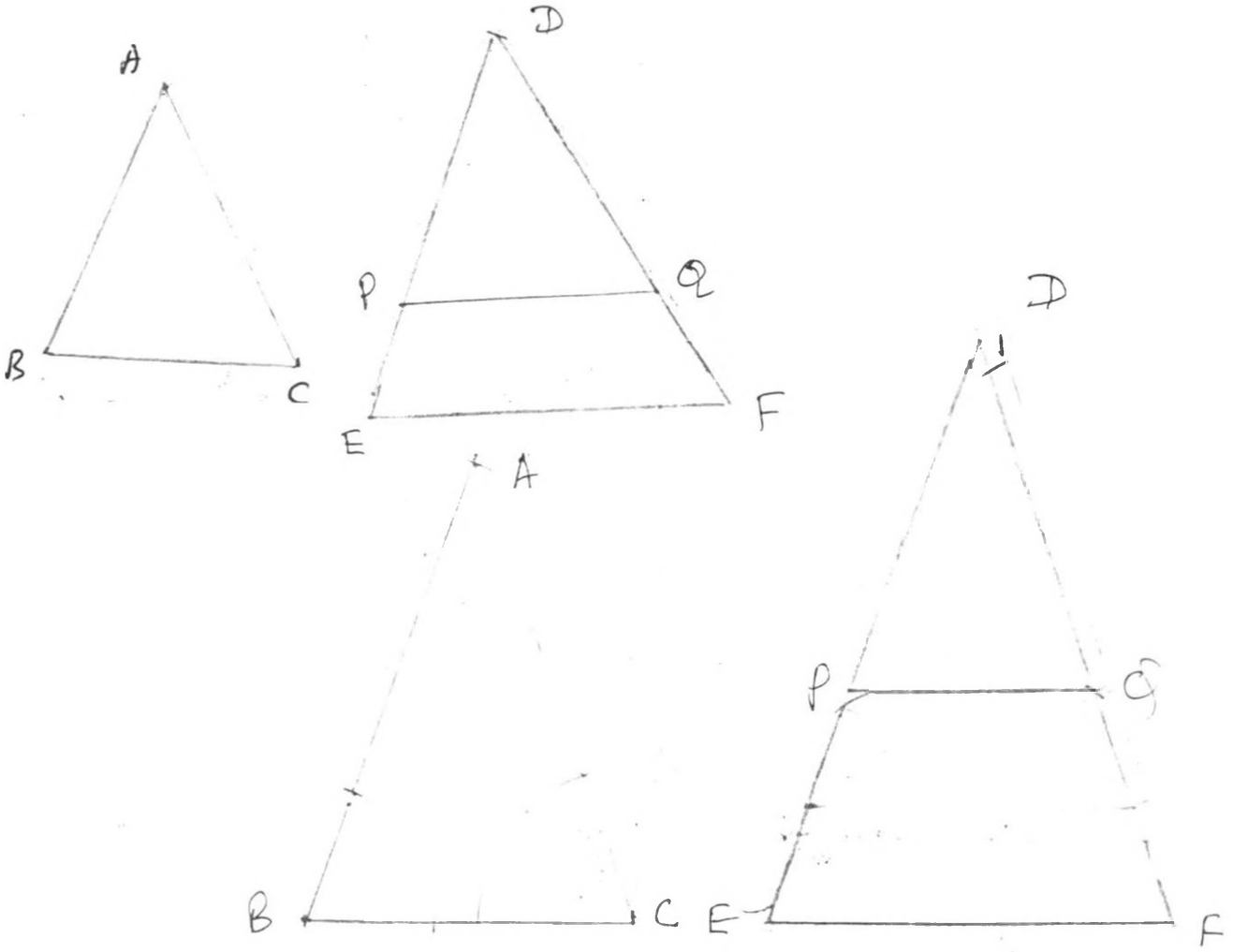


<u>Characteristic property of similar triangles.</u>

If in two triangles, corresponding angles are equal, then the triangles are similar.

Proof:





We make use of the definition of the similarity of two triangles that the Δ les are similar if (1) their corresponding angles are equal and (2) their corresponding sides are proportional. If the corresponding angles are given equal, we must prove that the corresponding sides are proportional. For that purpose, we proceed as follows.

Given two Δ les Δ ABC and Δ DEF such that $\angle A = \angle D$, $\angle B = \angle E$ and $\angle C = \angle F$. We should prove that Δ ABC ~ΔDEF. Mark a point, P on the line DE and Q on the line DF such that AB=DP and AC=DQ and join PQ.

a) AB=DE. Thus P coincides with E. by ASA congruence, Δ ABC$\cong$ Δ DEF. Thus AB = DE, BC =EF and AC=DF.

Thus Q coincides with F.

$$\therefore \frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$$

Since corresponding angles are given equal,

Δ ABC ~Δ DEF

b) AB$\angle$DE, thus P lies in DE

In Δ les ABC and DPQ

AB=DP, AC=DQ, $\angle A = \angle D$

$\therefore \angle ABC \cong \Delta$ DPQ (SAS property)

$\therefore \angle B = \angle DPQ$

$\angle B = \angle E$ $\therefore \angle E = \angle DPQ$

Consequently PQ|| EF

$$\therefore \frac{DP}{DE} = \frac{DQ}{DF} \text{----------(1)}$$

i.e., $\frac{AB}{DE} = \frac{BC}{EF}$ ----------(2)

FROM (1) and (2)

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$$

service corresponding angles are equal.

Δ ABC ~ Δ DEF

(iii) AB > DE . Then P lies on DE produced.

Problem: Show that the diagonals of a parallelogram bisect each other.

Hint:

ABCD is a || gram

AC & BD are diagonals and

Interact at E

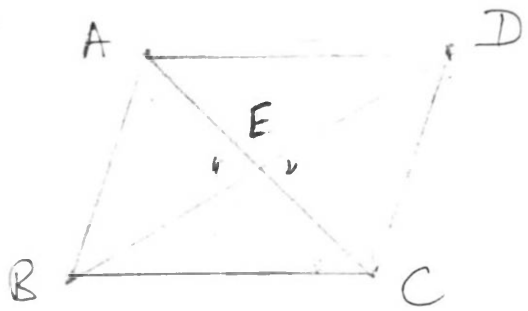
To prove AE =CE, Δ BE =DE

Proof: The two triangles ABE and CED are similar

∵ ABE = CDE (Alternate ∠)

and BAE = DCE (")

AED = CED (Vertically opp ∠les)



$\therefore \frac{AB}{CD} = \frac{AE}{CE} = \frac{BE}{DE}$

AB=CD

$\therefore \frac{AE}{CE} = 1 \quad \frac{BE}{DE} = 1$

∴ AE =CE BE=DE

Characteristic properties of similar triangles

I AAA similarity: If in two triangles, corresponding angle are equal, then the triangles are similar

Corollary: (AA similarity): If two angles of one triangle are respectively equal to two angles of another triangles, then two triangles are similar.

Since the sum of the three angles of any triangles = 180^0, the third angle of the triangles must also be equal.

∴By AAA similarity, the two triangles must be similar.

II SSS similarity : If the corresponding sides of the two triangles are proportional, then they are similar.

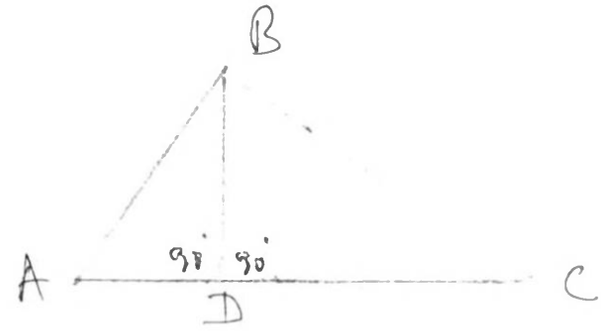
III. SAS Similarity: If in two triangles, one pair of corresponding sides are proportional and the included angles are equal, then the two triangles are similar.

From the above properties we can modify the definition of similarity of triangles as (1) Two triangles are similar if their corresponding angles are equal

OR

(2) Two triangles are similar if their corresponding sides are proportional.

Pythagoras theorem: In a right angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.



Proof: In Δ ADB and Δ ABC,

$\angle A = \angle A$

$\angle ADB = \angle ABC$ (each 90^0)

$\therefore \Delta ADB \sim \Delta ABC$

$\therefore \frac{AD}{AB} = \frac{AB}{AC}$ $\therefore AB^2 = AD.AC$----------(1)

Similarly $BC^2 = DC.AC$ ----------(2)

(1) + (2) $\Rightarrow AB^2 + BC^2 = AD.AC + DC.AC$

$= AC\ (AD + DC)$

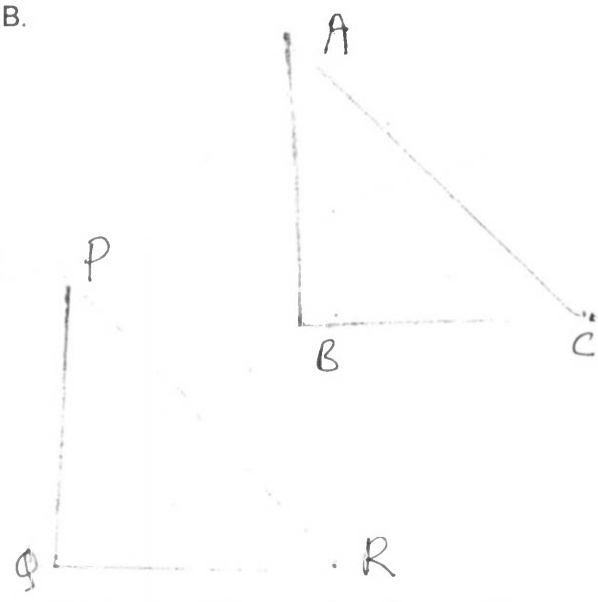
$= AC\ ACC$

$= AC^2$

$\therefore AB^2 + BC^{2+} = AC^2$

<u>Converse</u>: In a triangle, if the square of one side is equal to the sum of the squares of the other two sides, then the angle opposite to the first side is a right angle.

Proof: Δ ABC is such that $AB^2 + BC^2 = AC^2$. We should prove that Δ ABC is a right angled triangle, right angled at B.



In Δ PQR, $\angle Q = 90^0$

$PQ^2 + QR^2 = PR^2$ ----------(1)

Or $AB^2 + BC^2 = PR^2$

But $AB^2 + BC^2 = AC^2$ (given) ----------(2)

From (1) and (2) $PR^2 = AC^2$

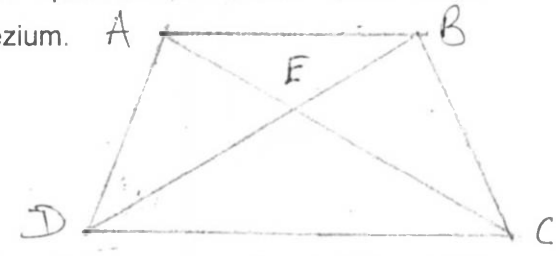
PR=AC

By SSS similarity, Δ ABC $\cong$ Δ PQR

i.e., $\angle B = \angle Q$

but $\angle Q = 90^0$ $\therefore \angle B = 90^0$

Problem: If the diagonals of a quadrilateral divide each other proportionally, prove that it is a trapezium.



In quadrilateral ABCD, given

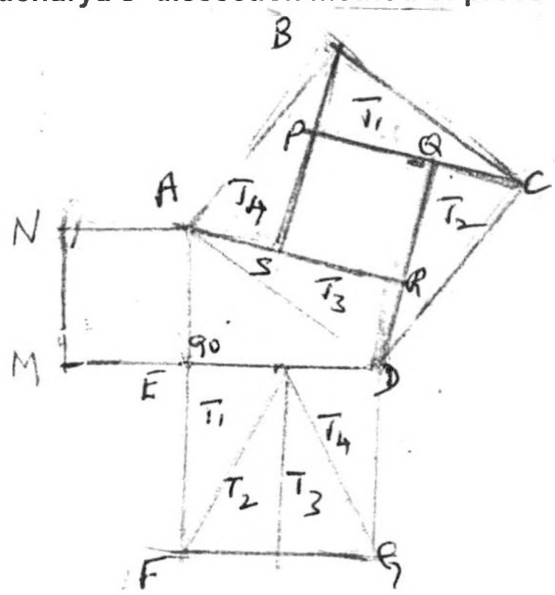
that $\frac{AE}{EC} = \frac{DE}{EB}$

comparing Δ les AED and BEC, AE and EB are proportional, DE and EC are also proportional. The included angles $\angle$AED and $\angle$BEC are equal.

$\therefore$ By SAS property, Δ AED and Δ BEC are similar Δ les.

∴ BC must be proportional to AD since AE and EC, DE and EB are parts of the same line AD and BC are parallel. If the diagonals of a quadrilateral divide each other it is a trapezium.

Bhaskaracharya's dissection method of prove Pythagoras theorem.



Proof: From the figure, the square ABCD can be divided into 4 triangles of equal are and a square, say T_1, T_2 T_3 and T_4. The area of the square AEMN is the same as that of the square PQRS. Arrange the squares DEFG, AEMN and ABCD such that a right angled Δle AED is formed.

Area of the square ABCD = AD^2

" DEFG = DE^2

" AEMN =EA^2

also, area of the square ABCD

= area of the Δles ($T_1 + T_2 + T_3 + T_4$)

+ area of the squares PQRS

= area of DEFG + area of PQRS

i.e., $AD^2 = DE^2 + EA^2$

Exercise

(1) Prove that the area of the equilateral triangle described on the side of a square is half the area of the equilateral triangle described on its diagonal.

(2) The perimeters of two similar Δles are 30 cm., and 20 cm, respectively. If one side of the first Δle is 15 cm., determine the corresponding side of the second triangle.

(3) A vertical stick 12 cm long casts a shadow 8 cm on the ground. At the same time a tower casts the shadow 40m long on the ground. Determine the height of the lower.

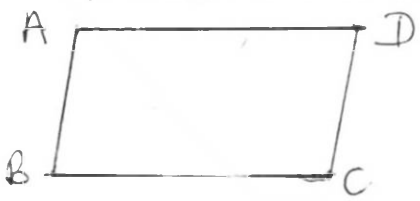
(4) A ladder reaches a window which is 12m above the ground on one side of the street keeping its foot at the same point, the ladder is turned to the other side of the street to reach a window 9 m high. Find the width of the street if the length of the ladder is 15 m.

Hint to solve the exercise:

PARALLELOGRAM

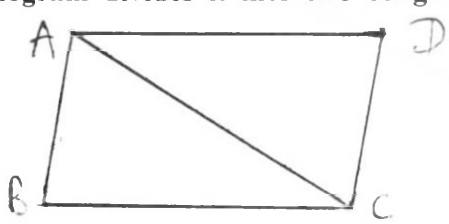
Introduction

We know that a closed figure formed by four line segments is called a quadrilateral. A parallelogram is a particular type of quadrilateral in which opposite sides are parallel to each other. ABCD is a parallelogram in which $\overline{AD} \parallel \overline{BC}$ and $\overline{AB} \parallel \overline{CD}$.



Properties of a Parallelogram

Property 1 : The diagonal of a parallelogram divides it into two congruent triangles.



Given that ABCD is a parallelogram.

Join AC.

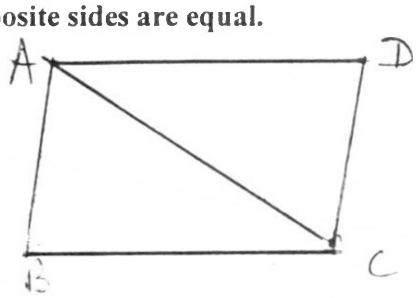
AC is a diagonal.

To prove that AC divides the parallelogram into two congruent triangles i.e. $\Delta ABC \cong \Delta ADC$.

Proof

Statement		Reason	
1.	$\angle CAD = \angle ACB$	1.	Alternate angles
2.	$\angle ACD = \angle BAC$	2.	Alternate angles
3.	$AC = AC$	3.	Identity Property
4.	$\Delta ABC \cong \Delta ADC$	4.	ASA property

Property 2 : In a parallelogram, the opposite sides are equal.



Data: $\overline{AB} \parallel \overline{CD}$ and $\overline{AD} \parallel \overline{BC}$

To prove that $\overline{AB} = \overline{CD}$ and $\overline{AD} = \overline{BC}$.

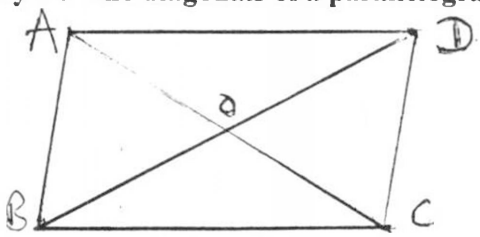
Construction: Join AC.

Proof: By property 1,

$\Delta ABC \cong \Delta CDA$

$\therefore$ AB = CD and BC = DA (corresponding parts of the congruent triangles).

Property 3 : The diagonals of a parallelogram bisect each other.



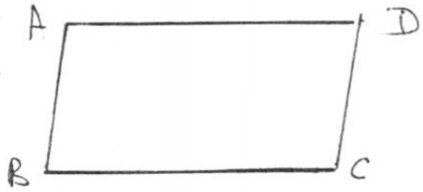
Data: ABCD is a parallelogram. Diagonals $\overline{AC}$ *and* $\overline{BD}$ intersect at O.

To prove : AO = CO and BO = DO.

Proof: Consider the triangles AOB and COD.

	Statement		Reason
1.	AB = CD	1.	Opposite sides of parallelogram are equal.
2.	$\angle$ OBA = $\angle$ODC	2.	Alternate angles
3.	$\angle$OAB = $\angle$OCD	3.	Alternate angles
4.	$\Delta AOB \cong \Delta COD$	4.	ASA property
5.	AO = CO and BO = DO	5.	Corresponding sides of congruent triangles AOB and COD.

Property 4 : In a quadrilateral, if opposite angles are equal, it is a parallelogram.



Data: ABCD is a quadrilateral in which opposite angles

$\angle A = \angle C = x$ and $\angle B = \angle D = y$.

To prove : ABCD is a parallelogram.

Proof:

Statement		Reason	
1.	$\angle A + \angle B + \angle C + \angle D = 360^\circ$	1.	Sum of the angles of a quadrilateral.
2.	$2x + 2y = 360^\circ$ or $x + y = 180^\circ$	2.	Data
3.	$\angle A + \angle B = 180 - \angle A + \angle D = 180^\circ$	3.	Data
4.	AD ‖ BC and AB ‖ CD	4.	Statement (3)
5.	ABCD is a parallelogram.	5.	Statement (4)

Exercise

1. In a quadrilateral if diagonals bisect each other, prove that it is a parallelogram.
2. Prove that a quadrilateral is a parallelogram, if a pair of opposite sides ax parallel and equal.
3. In a parallelogram, if a diagonal bisects one angle, prove that it also bisects the opposite angle.
4. If ABCD is a quadrilateral in which $\overline{AB} \parallel \overline{CD}$ and AD = BC, prove that $\angle A = \angle B$. [Hint: Extend $\overline{AB}$ and draw a line $\overline{CE}$ parallel to $\overline{AD}$].
5. In a parallelogram, show that the angle bisectors of two adjacent angles intersect at right angles.
6. Let ABCD be a parallelogram and let AP, CQ be the perpendiculars from A and C on its diagonal BD. Prove that AP = CQ.
7. AB and CD are two parallel lines and a transversal l intersects AB at X and CD at Y. Prove that the bisectors of the interior angles form a rectangle.
8. Say which of the following statements are true and which are false.

i) In a parallelogram, the diagonals are equal.

ii) In a parallelogram, the diagonals bisect each other.

iii) In a parallelogram, the diagonals intersect at right angles.

iv) In any quadrilateral, if a pair of opposite sides is equal, it is a parallelogram.

v) If all angles of a quadrilateral are equal, it is a parallelogram.

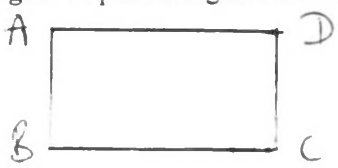
vi) If all sides of a quadrilateral are equal, it is a parallelogram.

vii) If three sides of a quadrilateral are equal, it is a parallelogram.

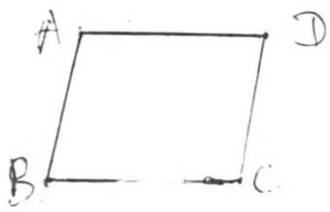
viii) If three angles of a quadrilateral are equal, it is a parallelogram.

Types of Parallelograms

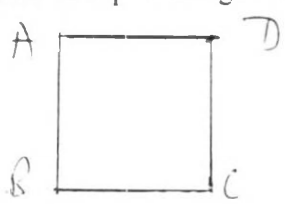
1. **Rectangle:** A parallelogram is called a rectangle if all its angles are equal.



2. **Rhombus** : If all the sides of a parallelogram are equal, then it is called rhombus.

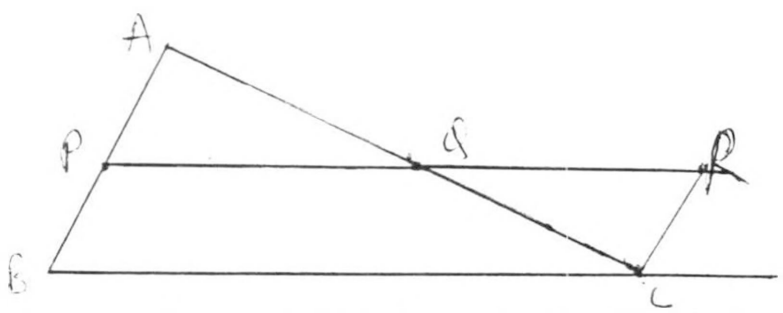


3. **Square :** If all the angles and all the sides of a parallelogram are equal, then it is a square.



Theorems on triangles and parallel lines

Theorem : In a triangle, the line segment joining the midpoints of any two sides is parallel to the third side and is half of it.



Data : ABC is a triangle, in which midpoints P of AB and Q of AC are joined.

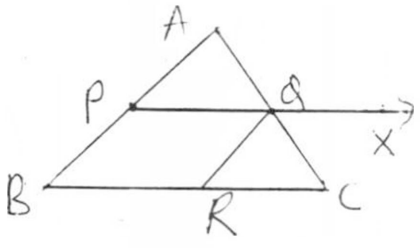
Conclusion : PQ || BC and PQ = $\frac{1}{2}$ BC.

Construction: Extend PQ to R so that PQ = QR. Join CR.

Proof :

Statement		Reason	
1.	AQ = QC	1.	Data
2.	PQ = QR	2.	Construction
3.	$\angle AQP = \angle CQR$	3.	Vertically opposite angles.
4.	$\Delta AQP \cong \Delta CQR$	4.	SAS Property
5.	AP = CR	5.	Corresponding parts
6.	$\angle PAQ = \angle RCQ$	6.	Alternate angles are equal.
7.	AP ‖ CR	7.	Corresponding parts
8.	BP = AP	8.	Data
9.	PBCR is a parallelogram.	9.	Statements (7) and (8)
10.	PR = BC and PR ‖ BC	10.	Statement 9.
11.	$PQ = \frac{1}{2}$ BC	11.	Q is the mid point of AC.

Theorem : The line drawn through the midpoint of one side of a triangle parallel to another side bisects the third side.



Data: ABC is a triangle. Line PX is drawn parallel to BC to intersect the third side. AC at Q.

To prove : AQ = QC.

Construction: Draw a line QR parallel to AB which intersects BC at a point R.

Proof :

Statement		Reason	
1.	PQ ‖ BR	1.	Data
2.	PB ‖ QR	2.	Construction
3.	PQRB is a parallelogram.	3.	Statements (1) and (2).
4.	AP = PB	4.	Data
5.	QR = AP	5.	Data
6.	QR ‖ AB and AB and AC intersect.	6.	Statements (4) and (5).
7.	$\angle RQC = \angle PAQ$	7.	Corresponding angles.
8.	$\angle RCQ = \angle PQA$	8.	Corresponding angles.
9.	$\Delta QRC \cong \Delta APQ$	9.	SAA theorem.
10.	QC = AQ	10.	Corresponding parts.

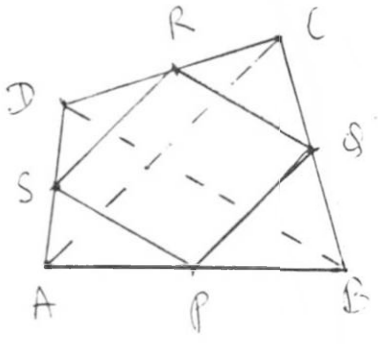
Example 1: The line segments joining the midpoints of the sides of a quadrilateral form a parallelogram.

Data: ABCD is a quadrilateral. P, Q, R S are the mid points of AB, BC, CD and AD.

To Prove: PQRS is a parallelogram.

Construction: Join AC and BD.

Proof:



	Statement		Reason
1.	PQ ∥ AC and $PQ = \frac{1}{2}AC$	1.	Theorem
2.	SR ∥ AC and $SR = \frac{1}{2}AC$	2.	Theorem
3.	PS ∥ BD and $PS = \frac{1}{2}BD$	3.	Theorem
4.	QR ∥ BD and $QR = \frac{1}{2}BD$.	4.	Theorem
5.	PQRS is a parallelogram.	5.	PQ ∥ SR and PS ∥ QR.

Example 2

E, F are respectively, the midpoints of non-parallel sides of a trapezium ABCD.

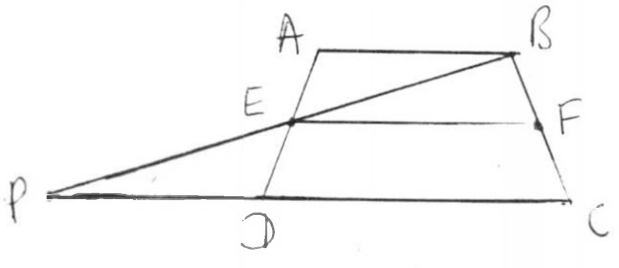
Prove that 1. EF ∥ AB and 2. $EF = \frac{1}{2}(AB + CD)$.

Data: ABCD is a trapezium whose non-parallel sides are AD and BC with midpoints E and F respectively.

Conclusion : 1. EF ∥ AB 2. $EF = \frac{1}{2}(AB + CD)$.

Construction: Join BE and extend it to intersect CD produced at P.

Proof:



	Statement		Reason
1.	In triangles AEB and DEP, AE = ED.	1.	Data
2.	$\angle ABE = \angle EPD$	2.	Alternate angles.
3.	$\angle AEB = \angle DEP$	3.	Vertically opposite angles.
4.	$\Delta AEB \cong \Delta DEP$	4.	SAA property.
5.	BE = PE and AB = DP	5.	Corresponding parts of the triangle.
6.	In triangle BPC, E is the midpoint of BP and F is the midpoint of BC.	6.	Statements (4) and (5).
7.	EF $\parallel$ PC and EF $\parallel$ AB $EF = \frac{1}{2} PC$ $= \frac{1}{2} (PD + DC)$ $= \frac{1}{2} (AB + CD)$	7.	Theorem PC $\parallel$ AB

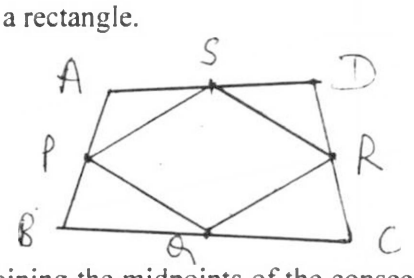
Exercises

1. ABC is a triangle, AD is a median and E is the midpoint of AD. BE is joined and produced to intersect AC in a point F. Prove that $AF = \frac{1}{3} AC$.

2. Show that the quadrilateral formed by joining the midpoints of the consecutive sides of a square is also a square.

3. Show that the quadrilateral formed by joining the midpoints of the consecutive sides of a rectangle is a rhombus.

4. ABCD is a rhombus and P, Q, R, S are the midpoints of AB, BC, CD, DA respectively. Prove that PQRS is a rectangle.

5. 

Prove that the figure formed by joining the midpoints of the consecutive sides of a quadrilateral is a parallelogram.

6. Show that the line segments joining the midpoints of opposite sides of a quadrilateral bisect each other.

7. In a triangle ABC, points M and N on sides AB and AC respectively are taken so that $AM = \frac{1}{4}AB$ and $AN = \frac{1}{2}AC$. Prove that $MN = \frac{1}{4}BC$.

8. In a parallelogram ABCD, E and F are the mid points of sides AB and CD. Prove that the line segments AF and CE bisect the diagonal BD.

9. ABCD is a rhombus and AB is produced to E and F such that AE = AB = BF. Prove that ED and FC are perpendicular to each other.

10. BM and CN are perpendicular to a line passing through the vertex A of a triangle ABC. If L is the midpoint of BC, prove that $\angle M = \angle N$.

CIRCLE

Introduction

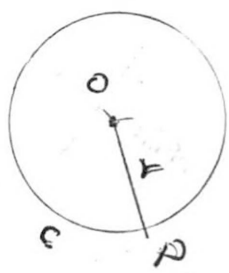
We have seen many objects whose shapes are circular in nature. For example, Coins, Wheels used in machines and Vehicles. We shall try to know about the properties of a circle in this chapter.

Concepts

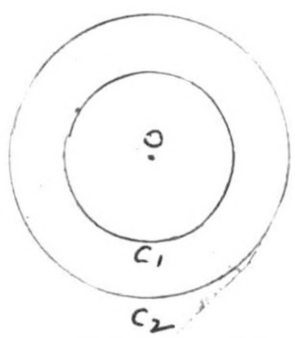
A circle is a set of points which moves in a plane in such a way that its distance from a fixed point is always a positive constant.

The fixed point is called the center and the constant distance is called the radius.

A circle with center o and radius r is denoted by c (o,r).

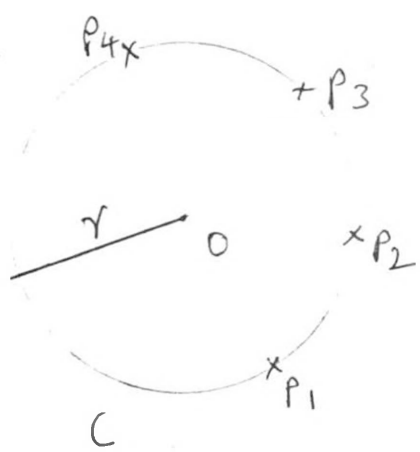


Concentric Circles



Circles having the same center and different radii are called concentric circles.

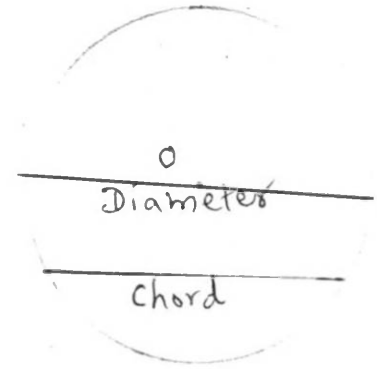
Arc of a circle:



A continuous piece of circle is called an arc. In the above fig. Pieces P_1P_2, P_2P_3, P_3P_4. are all arcs of the circle c(o,r). We denote the area from P_1 to P_2 and is represented as $\overset{\frown}{P_1P_2}$.

Chord of a circle

A line segment joining two points on a circle Is called a chord of a circle. A chord passing through The center of a circle is called a diameter of the circle.



Classification of arcs.

1. **Semi Circle**

If the end points of an arc of a circle are the end points of a diameter, that arc is called a Semi Circle.

2. **Minor arc**

If the length of an arc is less than that of a semicircle, it is called a minor arc of the circle.

3. **Major Arc**

If the length of an arc is greater than that of a semicircle it is a major arc.

Exercise 1

1. Examine the fig. and classify each of the following arcs as minor arc, major arc and semi circle.

(i)	$\overline{AFD}$	(ii)	$\overline{CFD}$
(iii)	$\overline{DBF}$	(iv)	$\overline{ADF}$
(v)	$\overline{CAD}$	(vi)	$\overline{ADC}$
(vii)	$\overline{ADB}$	(viii)	$\overline{ACF}$

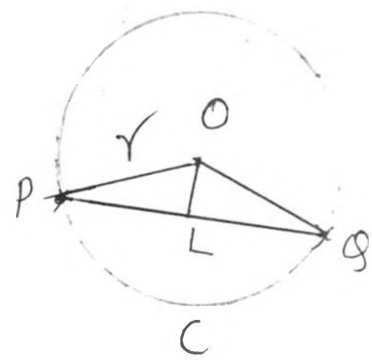
THEOREMS

Theorem 1: The perpendicular from the center of a circle to a chord bisects the chord.

Given

A chord PQ of a circle c(o,r) and $\perp r$ oL to the chord PQ.

To prove LP=LQ



Const.

Draw the line segment OL$\perp'$ to PQ.

Proof:

In two Right triangles OLP and OLQ

OP = OQ = r (radii)

OL = OL

$\angle OLP = \angle OLQ = 90^0$

Δ OLP $\cong$ Δ OLQ

$\therefore$ LP = LQ

Converse of Theorem.1

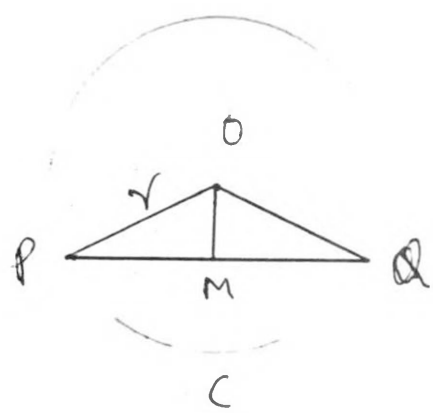
The line joining the center of a circle to the mid-point of a chord is $\perp r$ To the chord.

Given

A chord PQ of a circle c(o,r) with mid point m.

To prove

OM is $\perp r$ to PQ.



Const.

Draw the line segment OP and OQ

Proof

In Δ^{le} OPM and OQM

PM = MQ (m is the mid point of PQ)

OP = OQ = r (radii)

OM = OM.

$\therefore \Delta$ OPM $\cong$ Δ OQM (S.S.S)

$\angle$OMP = $\angle$OMQ

$\angle$OMP + $\angle$OMQ = 180^0

$\angle$OMP = $\angle$OMQ = 90^0

Hence, OM is $\perp r$ to PQ

Note

Perpendicular bisectors of two chords of a circle intersect at its centre.

Theorem 2

If there are two equal chords in a circle, then the chords are at equal distances from the center of the circle.

Given

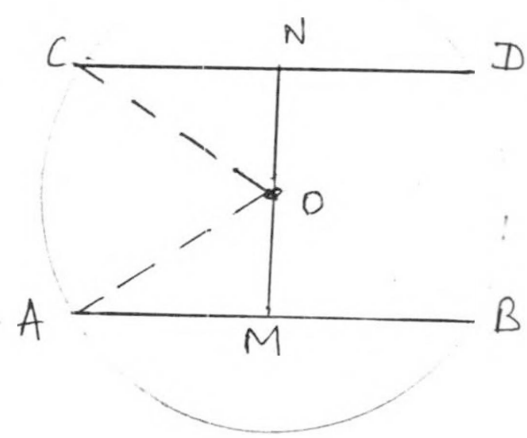
In the circle (o,r) chord AB=Chord CD.

OM $\perp r$ AB

ON $\perp r$ CD

To prove that

OM=ON



Proof:

In Δ OAM and Δ OCN

AM = $\frac{1}{2}$AB Given

CN = $\frac{1}{2}$CD "

$\therefore$ AB = CD

AM = CN

$\angle M = \angle N = 90^0$ (def. Of $\perp r$)

AO = CO (radii)

$\therefore$ Δ OAM = Δ OCN (RHS Theorem)

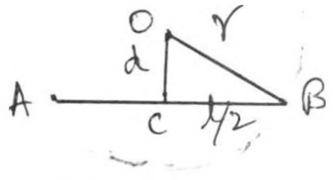
$\therefore$ OM = ON

From the theorem we studied, we come to the conclusion that the radius at a circle, half of a chord and the $\perp r$ from the center to the chord will form a rt. angled $\Delta^{le.}$

In the Rt. Δ^{le} OBC

$OB^2 = OC^2 + CB^2$

$r^2 = d^2 + \left(\frac{L}{2}\right)^2$



using this property, if any two of the measure l, r, and d are known we can find the third measure.

Example 1.

A chord 30 cm long is drawn in a circle 8 cm away from its center.

Calculate the radius of the circle.

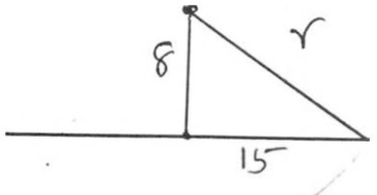
$r^2 = d^2 + \left(\frac{L}{2}\right)^2$

$= 8^2 + 15^2$

$= 64 + 225$

$= 289$

r = 17 cm

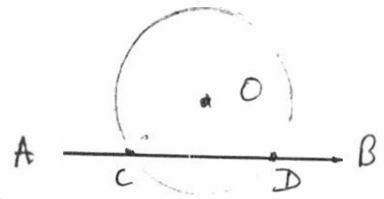


Exercise 2.

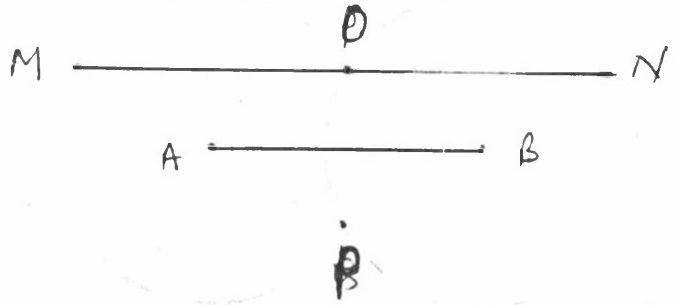
1. A chord 24 cm long is drawn in a circle 5 cm away from its center. Calculate the diameter of the circle.

2. A chord of a circle which is at a distance of 12 cm from its center is 18 cm long. What will be the length of another chord of the same circle which is at a distances of 9 cm from the center of the circle.

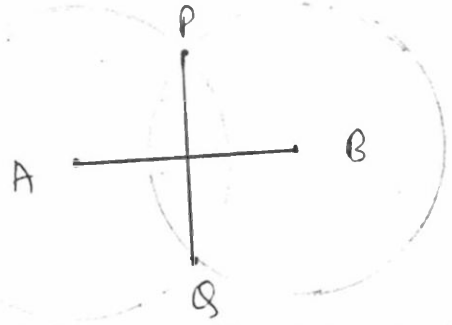
3. In the following fig there are two concentric circles with center O. Chord AB of the longer circle intersects the same circle at C and D. Prove that AC = BD.



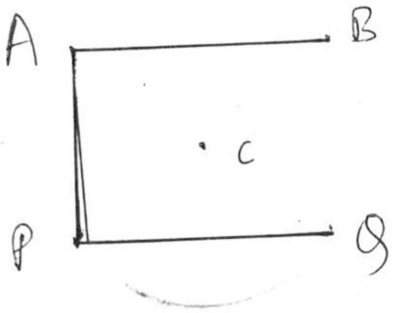
4. In the following figure A and B are centers of two circles which intersect at O and P . If MN || AB. Prove that MN = 2 x AB.



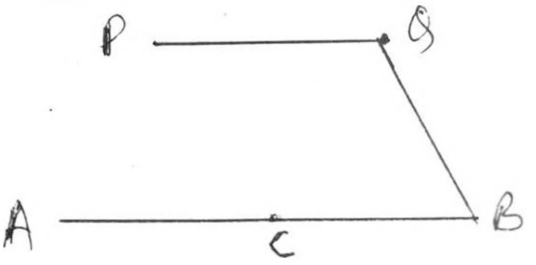
5. In the fig A and B are centers of the circles that intersect at P and Q. Prove that AB is the $\perp r$ bisector of PQ.



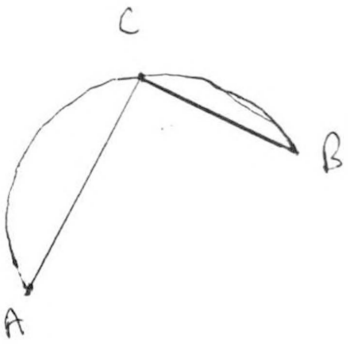
6. In the fig. PQ || AB. C is the center of the circle. The diameter of the circle is 30 cms. If AB=24 cm and PQ= 18 cm. Calculate AP.



7. In fig AB is the diameter of the circle with center C. PQ || AB. AB = 50 cm and PQ = 14 cm. Calculate BQ.



Inscribed Angle of an Arc



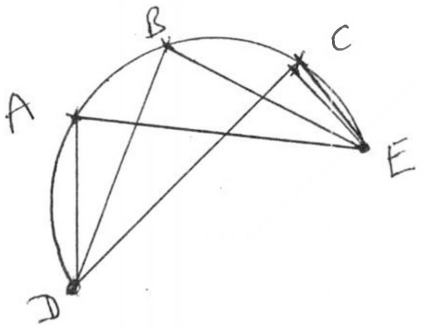
The vertex of the angle is a point of the arc. The vertex is not an end point of the arc. Each arm of the angle contains an end point of the arc.

If an angle and an arc are related in this manner we say that the angle is inscribed in that arc. In the above fig. (∠ACB is an inscribed angle of $\overline{ACB}$.

An angle is an inscribed angle of an arc if.

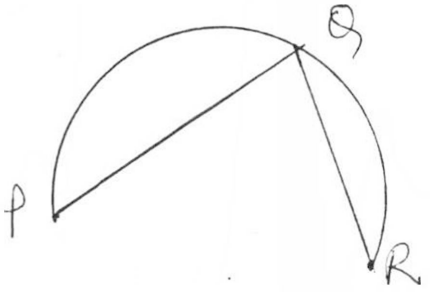
(i) The vertex of the angle is a point of the are other than the end point.

(ii) Each arm of the angle contains an end point of the arc.



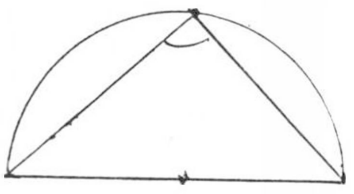
Any number of angles can be inscribed in an arc.

Now we shall see that the nature of angles inscribed in different kinds of arc.



I. Angle inscribed in a major arc is an acute angle.

Now we shall see what the measure an angle inscribed in a semicircle.

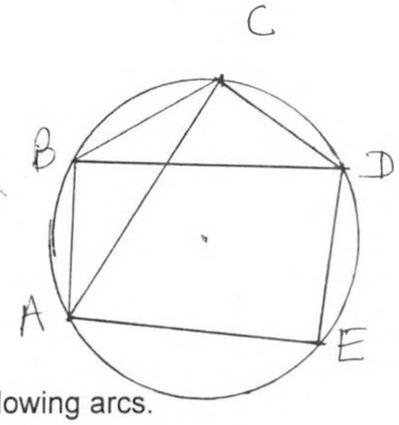


II. Angle inscribed in a semi circle is a right angle.

III. Angle inscribed in a minor arc is an obtuse angle.

Exercise 3

(i) Refer the figure.

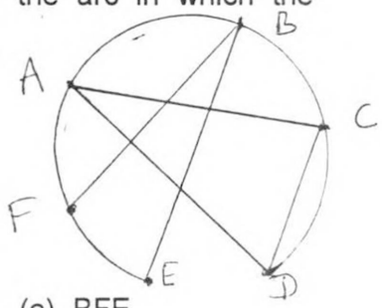


Write the name of an angle inscribed in each of the following arcs.

(i) $\overline{ABC}$ (ii) $\overline{BCD}$ (iii) $\overline{CDE}$ (iv) $\overline{DEA}$

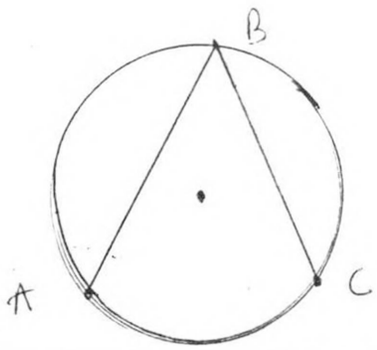
(v) $\overline{ABD}$

2. See the following fig and write the names of the arc in which the following angles are inscribed.



(a) ACD (b) BEF (c) CDA (d) DEF (e) BFE

Intercepted Arc of an Angle



All points of the arc other than the end points are in the interior of the angle thus

I. each arm of the angle contains an end point of the arc.

II. all points of the arc other than the end points are in the interior of the angle. The arc is said to be an intercepted arc of that angle.

Intercepted arc Theorem

The measure of an angle inscribed in an arc of a circle is half the degree measures of its intercepted arc.

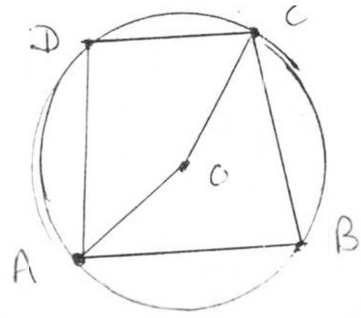
Angles inscribed in the same arc of a circle are of equal measure.

Cyclic Quadrilateral

A quadrilateral is called cyclic if all the four of its vertices lie on the circumference of a circle.

Theorem

The sum of either pair of opposite angles of a cyclic quadrilateral is 180^0.



Given

A cyclic quadrilateral ABCD.

To prove

$A + C = B + D = 180^0$.

Const.

Let o be the center of the circle through A,B,C,D. Join OA and OC.

Proof.

AC subtends $\angle^{le}$ ADC in alternate segment.

$$\angle D = \frac{1}{2}\overline{AC}$$

$$\angle B = \frac{1}{2}\overline{CA}$$

$$\angle D + \angle B + \frac{1}{2}\left(\overline{AC} + \overline{CA}\right)$$

$= \frac{1}{2} x 360^0$

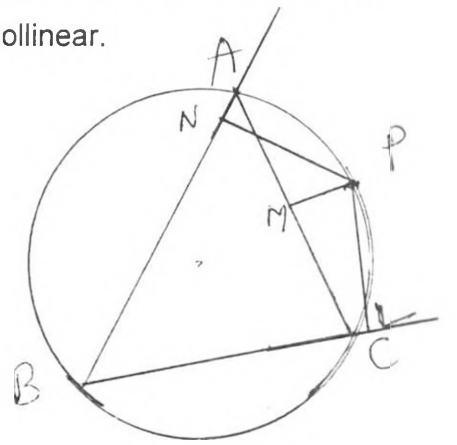
$= 180^0$

Again $A + C = 360 - (\angle B + \angle D)$

$= 180^0$

Example

Let P be a point on circumcircle of Δ ABC and $\perp r$ PL, PM and PN and drawn on the lines through the line segments BC, CA and AB respectively. Prove that the points L, M , N are collinear.



Proof

Join PA and PC

Join NM and ML.

To prove NML is a line segment.

Now

$\angle PMA + \angle ANP = 90^0 + 90^0 = 180^0$

$\therefore$ Points A,M,P,N are concyclic

$\therefore \angle PMN = \angle PAN.$

(Angles of same segment)

Again

$\angle PMC + \angle PLC = 90^0$

$\therefore$ quadrilateral PMLC is cyclic

$\therefore \angle PML + \angle PCL = 180^0$

Also

$\angle PAB + \angle ACB = 180^0$

$\angle PAN + \angle PAB = 180^0$

$\therefore \angle PAN = \angle PCB = \angle PAC$

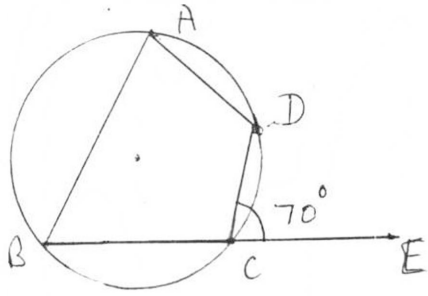
$180^0 = \angle PMC + \angle PCL = \angle PML + \angle PAN$

$= \angle PML + \angle PMN = 180^0$

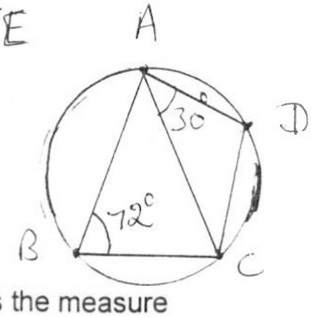
Hence NML is a line Segment.

Exercise – 3

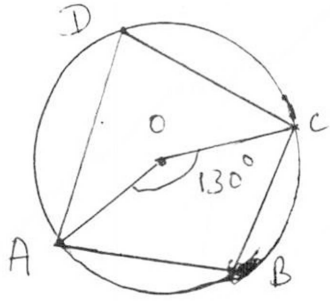
1. ABCD is a cyclic rectangle. Prove that the center of the circle through A,B,C,D is the point of intersection of its diagonals.
2. PQRS is a cyclic quadrilateral if $\angle P = 80^0$. What is the measure of $\angle R$.
3. PQRS is a cyclic quadrilateral $\angle PQR = 120^0$ What is $\angle S$?
4. ABCD is a cyclic quadrilateral. O is the center of the circle in which it is inscribed $\angle A = 80^0$. What is $\angle BAD$.
5. In the following fig. $\angle DCE = 70^0$. What is the measure of $\angle A$?



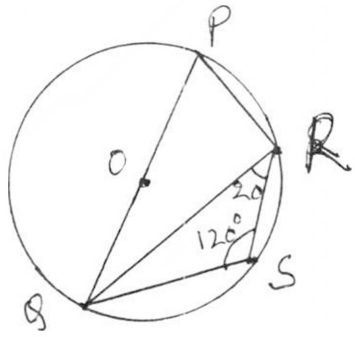
6. In fig. AB=AC
   $\angle ABC = 72^0$, $\angle DAC = 30^0$.
   What is the measure of DCA?



7. In fig. O is the center of the circle $\angle AOC = 130^0$. What is the measure of angle B.

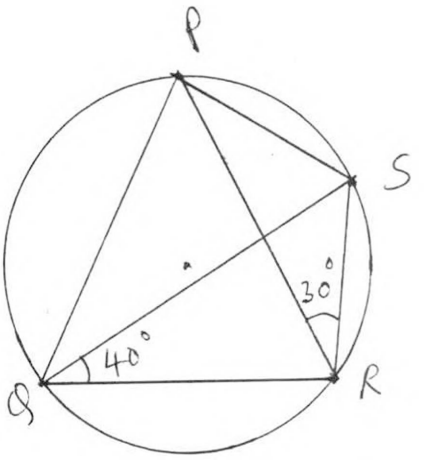


8. In fig O is the center of the circle $\angle S = 120^0$ $\angle QRS = 20^0$. Find the angles of Δ^{le} PQR.



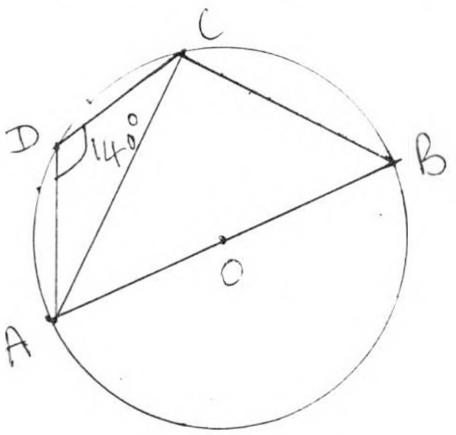
RQS = 40^0, PRS = 30^0. Find PSR?

9. In fig. PQRS is cyclic quadrilateral.



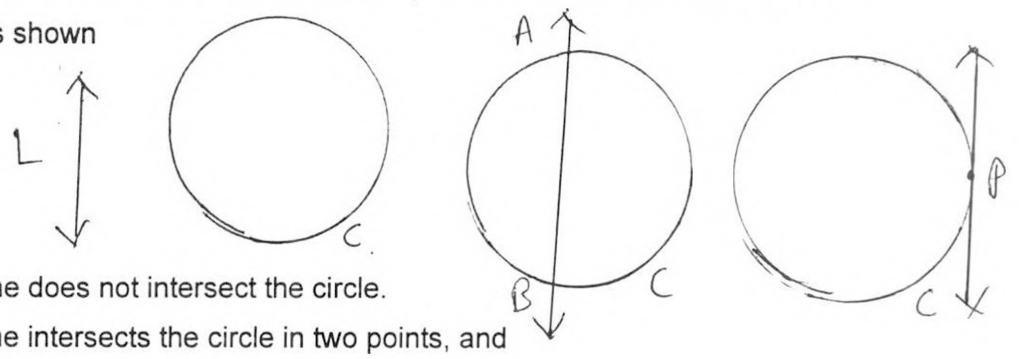
$\angle RQS = 40^0$ $\angle PRS = 30^0$ Find $\angle PSR$.

10. In fig. ABCD is a cyclic quadrilateral whose side AB is a diameter of the circle through A,B,C,C if $\angle ADC = 140^0$ find $\angle BAC$.



Tangent to a Circle

If a circle and a line are drawn on a plane, there will be three different situation as shown



I. The line does not intersect the circle.

II. The line intersects the circle in two points, and

III. The lines intersects the circle in only one point P.

Definition

A line which intersects a circle in two distinct points is called a second of the circle.

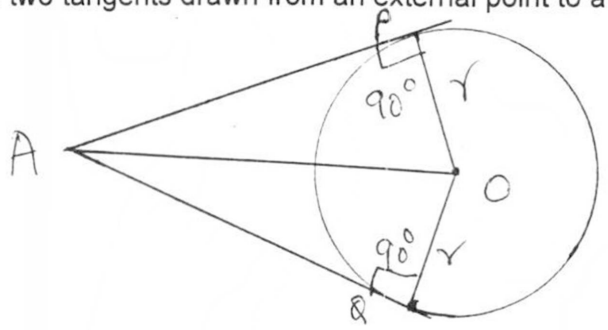
A tangent to a circle is a line that intersects the circle in exactly one point.

Theorem

A tangent to a circle is $\perp r$ to the radius through the point of tangency.

Theorem

The length of the two tangents drawn from an external point to a circle are equal.



Given

AP, AQ are two tangent segments from A to a circle C (o,r)

To prove

AP = AQ.

Const. Draw a line segment OA, OP and OQ.

Proof:

We have

$\angle OPA = \angle OQA = 90$

In two Δ^{les} OPA and OQA.

OP = OQ = r

OA = OA

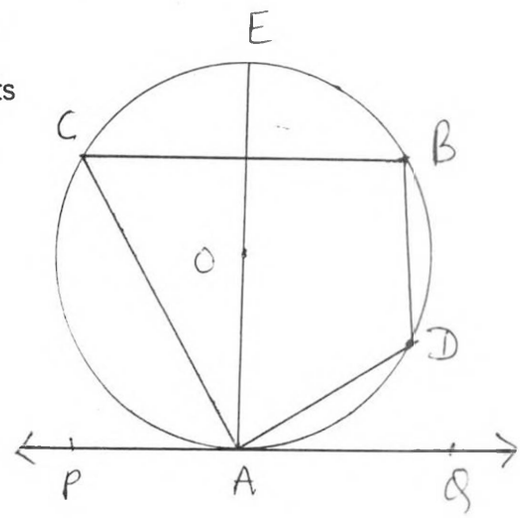
$\therefore \Delta$ OPA $\cong$ Δ OQA (RHS)

$\therefore$ AP = AQ

Theorem

If a chord is drawn through the point of contact of a tangent to a circle, then the angles which this chord makes with the given tangent are equal respectively to the angles formed in the corresponding alternate segments.

Given PQ is a tangent to a circle with point of Contact A. AB is the chord and C, D are points In $\overline{AB}$, $\overline{BA}$ respectively other than A and B.



To prove

$\angle BAQ = \angle ACB$

And $\angle BAP = \angle ADB$

Const.

Draw the diameter AOE and join EB

Proof

$\angle ABE = 90^0$ and

$\angle EAQ = 90^0$ (OA $\perp r$ PQ)

In Δ ABE

$\angle AEB + \angle EAB = 90$

$\angle EAB + \angle BAQ = \angle EAQ = 90^0$

$\therefore \angle AEB = \angle BAQ$

$\therefore \angle AEB = \angle ACB$

$\therefore \angle ACB = \angle BAQ$

In the quadrilateral ADBC is cyclic

$\therefore \angle ACB + \angle BDA = 180^0$

$\angle BAQ + \angle BDA = 180^0$

$\therefore \angle BDA = 180^0 - \angle BAQ = \angle BAP$

or $\angle BAP = \angle ADB$

Exercise –4

1. If Δ ABC is isosceles with AB=AC and c (o,r) is the incircle of the Δ ABC touching BC at L . Prove that the point L bisects BC.
2. $\overline{PQR}$ is a minor arc of the circle with center O. $\angle$PQR =100^0. What is the measure of $\angle$POR.
3. PQRS is a cyclic quadrilateral $\angle$P= 120^0. What is the measure of $\angle$R
4. PQRS is a cyclic quadrilateral show that $\angle P - \angle Q = \angle S - \angle R$.
5. The chord PQ and RS of a circle meet at M. If PR || QS. Show that PM= RM.
6. Two tangent segments BC,BD are drawn to a circle c (o,r) such that DBC = 120^0 Prove that BO = 2BC.
7. AB is a diameter and AC is a chord of a circle such that $\angle$BAC = 30^0. The tangent at C intersects AB produced in D. Prove that BC=BD.
8. Two circles c (o,r) and c(o^1, r^1) intersect in two points A and B and O line on c(0^1,r^1). A tangent CD is drawn to the circle c(o^1r^1) at A. Prove that $\angle$OAC= $\angle$OAB [x join OB andOO^1)
9. With the vertices of a triangle ABC as centers, three circles are drawn each touching the other two externally. If the sides of the Δ^{le} are 4 cms, 6 cm, and 8 cm find the radii of the circle.
10. In a cyclic quadrilateral ABCD. The diagonal CA bisects the angle C. Prove that diagonal BD is parallel to the tangent at A to the circle through A,B,C,D.

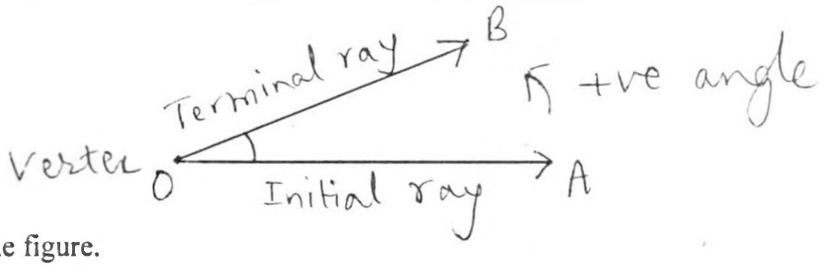
TRIGONOMETRY

Trigonometry is one of the branches of Mathematics. The word Trigonometry is derived from two Greek words – "Trigonon" (meaning Triangle) and "Metron" (meaning a measure) and hence the literal meaning the measurement of a triangle. Thus, trigonometry deals with the measurement of sides and the angles of a triangle and the investigation of various relations which exists among them.

In trigonometry, we have a good deal of combinations of algebra and geometry. There are algebraic symbols, formulas and equations, which makes the subject more interesting and useful for practical applications. To mention a few, it is useful in measuring heights of mountains, summits which cannot be reached, the distance of distant objects etc. Since trigonometric functions are widely used in mathematics, our understanding of the subject will be incomplete without the knowledge of trigonometry.

Basic Concepts

Angle: When a radius vector OA terms about the fixed point O, from its initial position to the terminal position OB, an angle AOB is said to be formed.



$A\hat{O}B$ from the figure.

An angle is said to be a positive angle if it is measured in anti-clockwise direction and it is a negative angle if it is measured in clockwise direction.

Measurement of angles : We have several units of measuring angles. They are 1. Sexagesimal system, 2. Centesimal system and 3. Radian (or circular) measure.

In sexagesimal system, a right angle (angle between the horizontal and a plumb line) is divided into 90 equal parts and each part is called 1 degree.

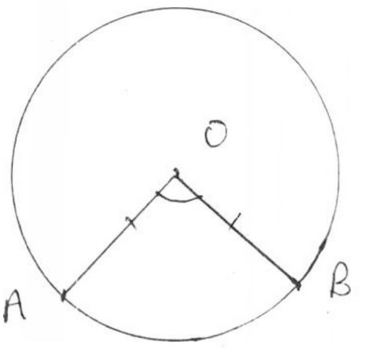
i.e. 1 Right angle = 90°

$1^\circ = 60'$ (minutes)

$1' = 60''$ (seconds)

This system is used in different branches like engineering, astronomy, navigation.

In radian measure, one radian is the unit. Radian is defined as the angle subtended at the center of a circle by an arc whose length is equal to the radius of the circle.



In the figure, OA = OB = arc AB

$\therefore$ $A\hat{O}B = 1$ radian.

Relation between degrees and radians

The circumference of a circle subtends at the center an angle whose measure is 2π units in radians. Also we know that when radius vector completes one revolution it subtends an angle of 360° at the center.

$\therefore$ 2π radians = 360°

or π radians = 180°

$$1 \text{ radian} = \frac{180^\circ}{\pi} = 57^\circ\ 17'\ 44.8'' \quad \text{or } 1^\circ = \frac{\pi}{180} \text{ radians.}$$

Radian measure of some common angles.

Degree	30°	45°	60°	90°	180°	270°	360°
Radians	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π

Exercise 1

1. Express the sexagesimal measures of the angles in radian measure.

   i) 75° ii) 240° iii) 150° iv) $22\frac{1}{2}^\circ$ v) $7\frac{1}{2}^\circ$

2. Express the circular measure of angles in degrees.

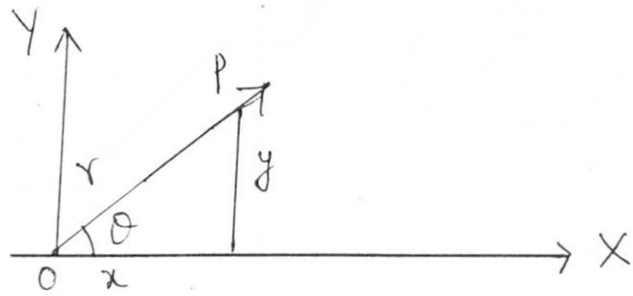
   i) $\frac{\pi}{12}$ ii) $\frac{3\pi}{4}$ iii) $\frac{5\pi}{6}$ iv) $\frac{7\pi}{4}$

3. Angles of a triangle are in the ratio of 1 : 2 : 3. Find the angles both in degrees and in radians.

4. The angles of a quadrilateral are in the ratio of 2 : 3 : 5 : 8. Express it in degrees and in radians.

Trigonometric functions of acute angle

Let a radius vector OA, starting from the position OX and rotating round 'O' trace out an angle $X\hat{O}A = \theta$. Let P be a point on the terminal position of OA. Draw PM $\perp$ r to OX.



Now the three sides OM, MP and OP of the right angled triangle OMP can be arranged two at a time in six ($3P_2$) different ways and hence six ratios can be formed

with them. These six ratios are called trigonometric functions or t-ratios. They are defined as follows:

1. Sine $\theta = \dfrac{opp.\ side}{hypotenuse} = \dfrac{y}{r} = \sin\theta$

2. Cosine $\theta = \dfrac{adj\ \ side}{hypotenuse} = \dfrac{x}{r} = \cos\theta$

3. Tangent $\theta = \dfrac{opp.\ side}{adj\ \ side} = \dfrac{y}{x} = \tan\theta$

4. Cosecant $\theta = \dfrac{hypotenuse}{opp\ side} = \dfrac{r}{y} = \text{cosec}\ \theta$

5. secant $\theta = \dfrac{hypotenuse}{adj\ \ side} = \dfrac{r}{x} = \sec\theta$

6. cotangent $\theta = \dfrac{adj\ \ side}{opp\ side} = \dfrac{x}{y} = \cot\theta$

Relations between Trigonometric Functions

1. Reciprocal Relations

a) $\sin\theta \times \text{cosec}\ \theta = \dfrac{y}{r} \times \dfrac{r}{y} = 1$

$\therefore\ \sin\theta = \dfrac{1}{\text{cosec}\ \theta}$ and $\text{cosec}\ \theta = \dfrac{1}{\sin\theta}$

b) $\cos\theta \times \sec\theta = \dfrac{x}{r} \times \dfrac{r}{x} = 1$

$\therefore\ \cos\theta = \dfrac{1}{\sec\theta}$ and $\sec\theta = \dfrac{1}{\cos\theta}$

c) $\tan\theta \times \cot\theta = \dfrac{y}{x} \times \dfrac{x}{y} = 1$

$\therefore\ \tan\theta = \dfrac{1}{\cot\theta}$ and $\cot\theta = \dfrac{1}{\tan\theta}$

2. Quotient Relations

a) $$\frac{\sin\theta}{\cos\theta} = \frac{y/r}{x/r} = \frac{y}{x} = \tan\theta$$

$$\therefore \tan\theta = \frac{\sin\theta}{\cos\theta}$$

b) $$\frac{\cos\theta}{\sin\theta} = \frac{x/r}{y/r} = \frac{x}{y} = \cot\theta$$

$$\therefore \cot\theta = \frac{\cos\theta}{\sin\theta}$$

3. Square Relations

a) $\sin^2\theta + \cos^2\theta = 1$

$$\sin^2\theta + \cos^2\theta = \frac{y^2}{r^2} + \frac{x^2}{r^2} = \frac{y^2 + x^2}{r^2} = \frac{r^2}{r^2} = 1$$

From this relation, we can obtain,

$\sin^2\theta = 1 - \cos^2\theta$ and $\cos^2\theta = 1 - \sin^2\theta$

b) $\sec^2\theta - \tan^2\theta = 1$

$$\sec^2\theta - \tan^2\theta = \frac{r^2}{x^2} - \frac{y^2}{x^2} = \frac{r^2 - y^2}{x^2} = \frac{x^2}{x^2} = 1$$

From this we can obtain

$\sec^2\theta = 1 + \tan^2\theta$ and $\tan^2\theta = \sec^2\theta - 1$.

c) $\text{cosec}^2\theta - \cot^2\theta = 1$

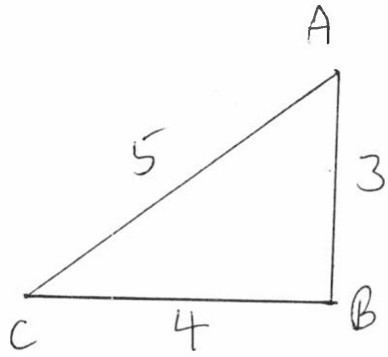
$$\text{cosec}^2\theta - \cot^2\theta = \frac{r^2}{y^2} - \frac{x^2}{y^2} = \frac{r^2 - x^2}{y^2} = \frac{y^2}{y^2} = 1$$

From this we can obtain

$\text{cosec}^2\theta = 1 + \cot^2\theta$ and $\cot^2\theta = \text{cosec}^2\theta - 1$.

The above three inter relations are very important and are called identities. In solving the problems, these identities play very important role.

Examples :



1. If $\sin\theta = \frac{3}{5}$ find $\cos\theta$ and $\tan\theta$.

Given $\sin\theta = \frac{3}{5} = \frac{\textit{opposite side}}{\textit{hypotenuse}}$

From the figure, $BC^2 = AC^2 - AB^2$

$= 5^2 - 3^2 = 25 - 9 = 16$

$BC = 4$

$\therefore \cos\theta = \frac{4}{5}$ and $\tan\theta = \frac{3}{4}$.

2. If $\sec\theta + \tan\theta = 2$, find $\sec\theta$ and $\tan\theta$.

Given $\sec\theta + \tan\theta = 2$ (1)

We have $\sec^2\theta - \tan^2\theta = 1$

$\therefore (\sec\theta + \tan\theta)(\sec\theta - \tan\theta) = 1$

$2(\sec\theta - \tan\theta) = 1$

$\therefore \sec\theta - \tan\theta = \frac{1}{2}$ (2)

Adding (1) and (2), we get

$2\sec\theta = \frac{5}{2}$

$\therefore \sec\theta = \frac{5}{4}$

Subtracting (2) from (1)

$2\tan\theta = \frac{3}{2}$

$\therefore \tan\theta = \frac{3}{4}$

Note: In solving the problems of this type, we can use the figure or the identities as the case may be.

3. Prove the following identities :

i) $(1 - \sin^2 \theta) \sec^2 \theta = 1$

ii) $(\sec^2 \theta - 1) \cot^2 \theta = 1$

iii) $\dfrac{\sin \theta}{1 - \cos \theta} = \dfrac{1 + \cos \theta}{\sin \theta}$

Solution:

(i) LHS $= (1 - \sin^2 \theta) \sec^2 \theta$

$= \cos^2 \theta \times \sec^2 \theta$

$= 1 =$ RHS

(ii) LHS $= (\sec^2 \theta - 1) \cot^2 \theta$

$= \tan^2 \theta \times \cot^2 \theta$

$= 1 =$ RHS

(iii) LHS $= \dfrac{\sin \theta}{1 - \cos \theta} \times \dfrac{1 + \cos \theta}{1 + \cos \theta}$

$= \dfrac{\sin \theta \,.\, (1 + \cos \theta)}{1 - \cos^2 \theta}$

$= \dfrac{\sin \theta \, (1 + \cos \theta)}{\sin^2 \theta}$

$= \dfrac{1 + \cos \theta}{\sin \theta} =$ RHS

4. Prove that : $\sqrt{\dfrac{1 - \sin \theta}{1 + \sin \theta}} = \sec \theta - \tan \theta.$

Proof: LHS $= \sqrt{\dfrac{1 - \sin \theta}{1 + \sin \theta}}$

$= \sqrt{\dfrac{1 - \sin \theta}{1 + \sin \theta} \times \dfrac{1 - \sin \theta}{1 - \sin \theta}}$

$= \sqrt{\dfrac{(1 - \sin \theta)^2}{1 - \sin^2 \theta}}$

$$= \sqrt{\frac{(1 - \sin\theta)^2}{\cos^2\theta}}$$

$$= \frac{1 - \sin\theta}{\cos\theta}$$

$$= \frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta}$$

$$= \sec\theta - \tan\theta = \text{RHS}$$

Exercise

1. If $\tan\theta = \frac{3}{4}$, write down the values of other t-ratios.

2. If $\sin\theta = \frac{8}{17}$, find $\tan\theta + \sec\theta$.

3. If $\cot\theta = \frac{12}{5}$, find the values of $\sec\theta$ and $\sin\theta$.

4. Prove the following identities.

i) $\cos\theta \,.\, \text{cosec}\,\theta = \cot\theta$

ii) $(1 - \sin^2\theta)\ (1 + \tan^2\theta) = 1$

iii) $\frac{\sin\theta}{\text{cosec}\,\theta} + \frac{\cos\theta}{\sec\theta} = 1$

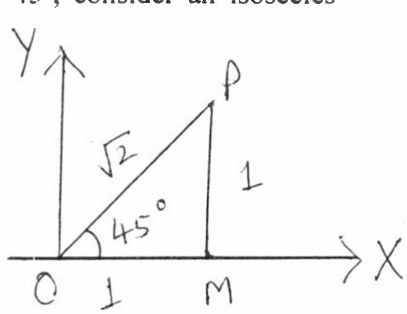
iv) $\tan\theta + \cot\theta = \sec\theta \,.\, \text{cosec}\,\theta$

v) $(\sin\theta + \cos\theta)^2 + (\sin\theta - \cos\theta)^2 = 2$

vi) $(1 + \cot A)^2 + (1 - \cot A)^2 = 2\,\text{cosec}^2 A$.

Trigonometric functions of Standard Angles

a) To find the values of trigonometric functions of 45°, consider an isosceles right angled triangle OMP as shown in the figure.



Let OM = MP = 1 unit.

Then $OP = \sqrt{1+1} = \sqrt{2}$

$\sin 45 = \frac{PM}{OP} = \frac{1}{\sqrt{2}}$ $\quad\therefore \text{cosec } 45 = \sqrt{2}$

$\cos 45 = \dfrac{OM}{OP} = \dfrac{1}{\sqrt{2}}$ $\quad\therefore \sec 45 = \sqrt{2}$

$\tan 45 = \dfrac{PM}{OM} = \dfrac{1}{1} = 1$ $\quad\therefore \cot 45 = 1$

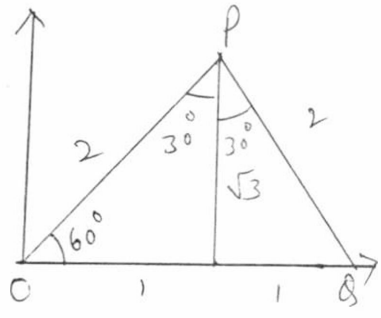
b) To find the values of trigonometric functions of 30° and 60°, consider an equilateral triangle OPQ of side 2 units each. Draw PM $\perp$ OQ.

Then OM = MQ = 1 unit and

$O\hat{P}M = 30^\circ$. In the triangle OPM,

OM = 1, OP = 2

$\therefore$ PM = $\sqrt{2^2 - 1^2} = \sqrt{3}$ as shown in the figure.



i) $\sin 60^\circ = \dfrac{PM}{OP} = \dfrac{\sqrt{3}}{2}$ $\quad\therefore \operatorname{cosec} 60^\circ = \dfrac{2}{\sqrt{3}}$

$\cos 60^\circ = \dfrac{OM}{OP} = \dfrac{1}{2}$ $\quad\therefore \sec 60^\circ = 2$

$\tan 60^\circ = \dfrac{PM}{OM} = \dfrac{\sqrt{3}}{1}$ $\quad\therefore \cot 60^\circ = \dfrac{1}{\sqrt{3}}$

ii) $\sin 30^\circ = \dfrac{OM}{OP} = \dfrac{1}{2}$ $\quad\therefore \operatorname{cosec} 30^\circ = 2$

$\cos 30^\circ = \dfrac{PM}{OP} = \dfrac{\sqrt{3}}{2}$ $\quad\therefore \sec 30^\circ = \dfrac{2}{\sqrt{3}}$

$\tan 30^\circ = \dfrac{OM}{PM} = \dfrac{1}{\sqrt{3}}$ $\quad\therefore \cot 30^\circ = \sqrt{3}$

4. To find the values of t-functions of 0° and 90°.

In the figure as $\theta = 0^\circ$, x = r and y = 0.

$\therefore$ $\sin 0 = \dfrac{y}{r} = \dfrac{0}{r} = 0$ $\quad\therefore \operatorname{cosec} = \infty$

$\cos 0 = \dfrac{x}{r} = \dfrac{r}{r} = 1$ $\quad\therefore \sec 0 = 1$

$\tan 0 = \frac{y}{x} = 0$ $\therefore \cot 0 = \infty$

When $\theta = 90^o$, $x = 0, y = r$.

$\therefore$ $\sin 90 = 1$ $\operatorname{cosec} 90 = 1$

$\cos 90 = 0$ $\sec 0 = \infty$

$\tan 90 = \infty$ $\cot 90 = 0$

Table for T-ratios of standard angles

θ	**0**	**30**	**45**	**60**	**90**
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

Example :

1. Find the value of $\tan^2 60 - 2\tan^2 45$.

$$\tan^2 60 - 2\tan^2 45 = (\sqrt{3})^2 - 2(1)^2$$
$$= 3 - 2$$
$$= 1$$

2. Find the value of $\sin\frac{\pi}{2} \cdot \cos\frac{\pi}{3} + \cos\frac{\pi}{2} \sin\frac{\pi}{3}$

$$\sin\frac{\pi}{2} \cdot \cos\frac{\pi}{3} + \cos\frac{\pi}{2} \cdot \sin\frac{\pi}{3} = 1 \cdot \frac{1}{2} + 0 \cdot \frac{\sqrt{3}}{2}$$

$$= \frac{1}{2} + 0 = \frac{1}{2}$$

3. Determine x if $x \,.\, \sin 30 \sin^2 45 = \dfrac{\cot^2 30 \,.\sec 60 \,.\, \tan 45}{\operatorname{cosec}^2 45 \,.\, \operatorname{cosec} 30}$

$$x \,.\, \sin 30 \sin^2 45 = \frac{\cot^2 30 \,.\sec 60 \,.\, \tan 45}{\operatorname{cosec}^2 45 \,.\, \operatorname{cosec} 30}$$

$$x \times \frac{1}{2} \cdot \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{(\sqrt{3})^2 \,.\, 2 \,.\, 1}{(\sqrt{2})^2 \,.\, 2}$$

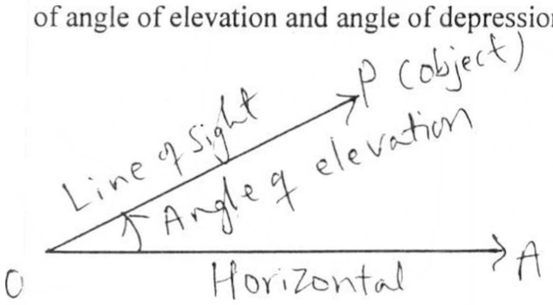
$$x \,.\, \frac{1}{4} = \frac{3 \times 2}{2 \times 2}$$

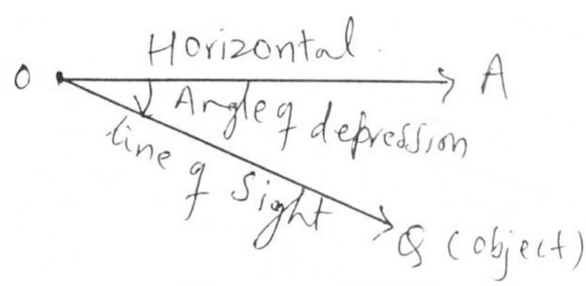
$$\therefore \; x = \frac{3 \times 2 \times 4}{2 \times 2}$$

$$\therefore \; x = 6$$

Heights and Distances

Suppose a person wants to know the width of a river or the height of a mountain, direct measurement is evidently inconvenient or impossible. These can be measured indirectly using trigonometric functions. For this, we require the meaning of angle of elevation and angle of depression.





If we have the line of sight above the horizontal, there is an angle of elevation and if it is below the horizontal, there is an angle of depression.

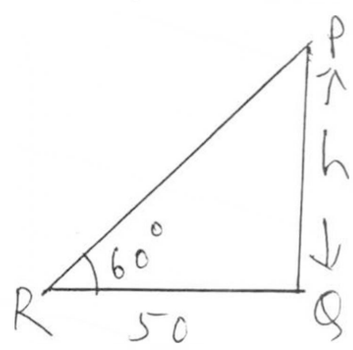
Examples

1. The angle of elevation of the top of a tower as seen from a point at a distance of 50 meters is 60°. Find the height of the tower.

Let PQ = h be the height of the tower.

From the Δ PQR

$$\tan 60^\circ = \frac{PQ}{RQ}$$



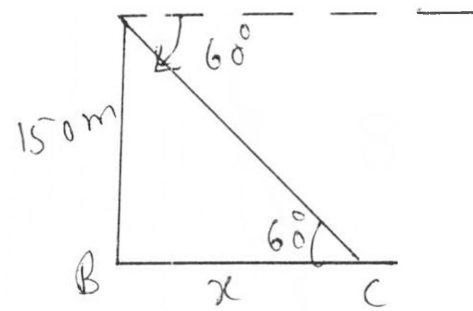
$$\sqrt{3} = \frac{PQ}{50}$$

$\therefore$ PQ = $50\sqrt{3}$ meters.

$\therefore$ Height of the tower is $50\sqrt{3}$ meters.

2. The angle of depression of a ship as seen from the top of a light house of height 150 m is 60°. How far is the ship from the light house ?

Let AB be the light house and C the position of ship. Let the distance of the ship from the light house be 'x'.



From the triangle ABC

$$\tan 60^\circ = \frac{150}{x}$$

$$\sqrt{3} = \frac{150}{x}$$

$$x = \frac{150}{\sqrt{3}} = \frac{150 \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}} = \frac{150 \,.\, \sqrt{3}}{3} = 50\sqrt{3} \text{ meters.}$$

$\therefore$ The distance of the ship from the light house = $50\sqrt{3}$ meters.

Exercises

1. Find the value of each of the following.

a) $\sin 60 \ \cos 30 + \cos 60. \ \sin 30$

b) $2 \sec^2 30 - 3 \cos^2 45 + \tan^2 60$

c) $\dfrac{\sin 30 + \cos 45 - \tan 60}{\cot 30 - \sin 45 - \cos 60}$

2. A ladder leaning against a wall makes an angle of 60° with the ground. The foot of the ladder is 3.5 m away from the wall. Find the length of the ladder.

3. Find the angular elevation of the sun when the shadow of a pole 30 m long is $10\sqrt{3}$ m.

4. During a storm, a tree breaks in such a way that its top touches the ground and makes an angle of 30° with the ground. At what height from the bottom, does the tree break if its original height is 18 m?

STATISTICS

The word 'Statistics' comes from the Latin word "status" which means a political state or a kingdom. The word Statistics has been defined by several statisticians. One of them is as follows: "Statistics is the science which deals with the methods of collecting, classifying, presenting, comparing and interpreting numerical data collected to throw some light on any sphere of enquiry". Thus the various stages in any statistical study are 1. Collection, 2. Classification, 3. Tabulation, 4. Comparison by averages, diagrams, 5. Correlation, 6. Interpretation and forecasting. A scientific treatment of these methods of Statistics is termed as "Science of Statistics".

Basic Concepts

i) ***Collection of Data***

For any statistical enquiry the basic problem is to collect facts and figures relating to a particular phenomenon under study. A statistician conducts the statistical enquiry and collects the data. The data collected in any statistical enquiry are voluminous and crude in form. It should be grouped or classified under appropriate classes or heads systematically.

ii) ***Classification and Tabulation of Data***

Classification is the first step of statistical method in dealing with raw statistical data. It is a process of arranging things in groups or classes according to the common characteristics possessed by them. After classification, the next step is to present the data in some tabular form. Tabulation may be defined as an orderly arrangement of data in rows and columns. This is helpful in analyzing and interpreting collected data.

iii) ***Variables of Observation***

In census and surveys, we make observations on many things. For example, in all India census, the enumerators record the age, sex and place of residence of each person. These are called variables because the results observed are different for different persons.

iv) ***Quantitative and Qualitative Variables***

In All India census, the values of age – variable are numbers. The variables of observation with the numbers as possible values are called quantitative variables. eg. height, weight, income of a worker, etc.

In the census example, the values of the sex variables are not numbers, but the names 'male' and 'female' describing a certain type or quality. Such variables with the names of things, places, attributes, etc. as possible values are called qualitative variables. Eg. Religion, caste, colour of hair, marital status, etc.

v) ***Units of Observation***

The term unit of observation is used to describe what the values of a variable are attached to. In the census example, the units of observation are the persons alive at the time of the census and to each unit of observation, we associate the values of three variables – age, sex and place of residence. Thus different variables of observation may be associated with the same units of observation.

Frequency Tables

The frequency table is one of the important methods to present a raw data in a form suitable for making the information contained in the raw data easily understandable. There are two steps in drawing up a frequency table or raw data. The first step is to use the possible values of a variable of observation to define a number of classes. The second step is to count the number of units of observation for which the values of the variable fall in a given class and to write down the number against the class. The number against each class is called the frequency of the class and the total of all frequencies is called the total frequency. The total frequency is simply the number of all units of observations for which we have recorded the value of the variable of observation. The frequency table breaks up or distributes the total frequency into different classes of the different table defined by means of the values of the variable of observation. For this reason, we use the term Frequency Distribution in place of frequency table.

Construction of a Frequency Table

Range of Raw Data: The difference between the maximum and minimum number occurring in the raw data is called the range of raw data.

Class Intervals and Class Limits: We can condense the data into classes as follows. Decide upon the number of classes into which raw data are to be grouped. Usually it should not be less than 5 or more than 15. The classes must include the minimum as well as the maximum number occurring in the data.

$$\text{The size of the class interval} = \frac{Range}{Desired\ \ No.\ of\ \ Classes}$$

Types of Classes :_ There are two types of classes i.e. Inclusive classes and Exclusive classes.

a) Consider the class 5 – 10, 10 – 15, 15 – 20 etc.

Here in the class, the lower limit is 5 and 10 is the upper limit. Here the upper limit of the class is the lower limit of the next class. Thus there are no gaps. These types of classes are called inclusive classes.

b) Consider the classes 0 – 9, 10 – 19, 20 – 29, etc.

Here in the class, the lower limit is 0 and 9 is the upper limit. Here the upper limit of the class is not the lower limit of the next class. These types of classes are called exclusive classes.

Mid value (or class mark) : The mid point of a class is called its mid value or class mark.

Eg. The class mark of 29.5 – 39.5 is $\frac{29.5 + 39.5}{2} = \frac{69}{2} = 34.5$

Tally Marks: To find the frequency in each class, we use tally marks '|'. Take each number from the data and place a tally mark opposite to the class to which it belongs. The tally marks can be recorded in bunches of 5. After the occurrence of 4 times, the fifth occurrence is recorded by a cross tally (||||) on the first 4 tallies.

Eg. 30 sixteen year old boys were tested to find their pulse rate. The following figures were obtained for the number of beats per minute. Using the class intervals 51-55, 56-60, etc. of equal width, prepare a frequency table.

55	72	70	66	74	70	74	53	57	62
71	58	68	75	79	68	63	59	54	51
61	66	78	73	59	52	66	60	72	56

Least observation = 51.

Greatest observation = 79

The required frequency table is as follows :

No. of Beats/Min	Tally Marks	Frequency
51 – 55	||||	5
56 – 60	|||| |	6
61 – 65	|||	3
66 – 70	|||| ||	7
71 – 75	|||| ||	7
76 – 80	||	2
	Total	30

Cumulative Frequency Table

The cumulative frequency of any class is the total of the frequencies of that class and classes coming before it in the frequency table. The table showing the manner in which cumulative frequencies are distributed is called a cumulative frequency table.

Example: The ages of 50 teachers working in a city are as follows :

Ages	No. of Teachers
20 – 25	2
25 – 30	4
30 – 35	5
35 – 40	10
40 – 45	15
45 – 50	8
50 –55	5
55 – 60	1

Construct a cumulative frequency table.

Ans: The required cumulative frequency distribution table is as follows :

Ages	No. of Teachers	Cumulative Frequency
20 – 25	2	2
25 – 30	4	6
30 – 35	5	11
35 – 40	10	21
40 – 45	15	36
45 – 50	8	44
50 –55	5	49
55 – 60	1	50
	50	

Graphical Representation of Frequency Distribution

The most important methods of graphical representation of frequency distribution are 1. The bar diagram and 2. the Pie diagram.

Bar Diagram: In a bar diagram, the classified data is represented by horizontal and vertical bars, the bars are of the same width even though different classes may have different widths. The height of the bars is proportional to the size of the item (frequency of the class). We could also make the heights of the rectangles proportional to the relative frequencies to get the bar diagram of the relative frequency distribution.

Example: Draw a bar diagram for the following frequency distribution.

Age Group	Village A	Village B
0 – 5	18	126
5 – 10	26	21
10 – 20	25	21
20- 30	29	26
30 – 40	21	15
40- 50	14	8
50 – 60	6	2
60 – 80	4	3

Bar Diagram

Pie Diagram

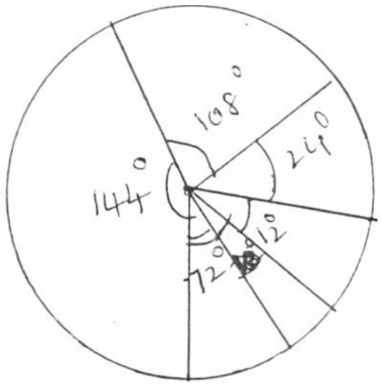
A statistical diagram in which the total allocation is represented by a circle and various items are shown by its sectors, called a pie diagram. Example: Draw a Pie diagram for the following data.

Wages	Frequency
25 – 30	2
30 – 35	9
35 – 40	12
40 – 45	6
45 – 50	1
Total	30

Answer: For each frequency, the corresponding angular measurement is as follows:

For the frequency 2, the corresponding degree measurement is $\frac{2}{30} \times 360 = 24^{\circ}$.

Similarly, we get 108°, 144°, 72° and 12° for the frequencies 9, 12, 6 and 1 respectively.



Measures of Location

In the case of quantitative variables the information contained in the raw data or in the associated frequency table can also be summarized by means of a few numerical values. Such a summary is partly provided by what are called measures of location.

The important measures of location are

1. Arithmetic mean (AM)

I. For raw data : The mean is given by $\bar{x} = \frac{\sum x}{\sum f}$.

Example: Find the A.M. 20, 26, 34, 39, 41, 32, 45, 28, 25, 30

AM = (20 + 26 + 34 + 39 + 41 + 32 + 45 + 28 + 25 + 30) ÷ 10

= 32

II Mean for a frequency distribution

$$\text{AM} = A + \frac{\sum fd \times c}{\sum f}$$

where A = assumed mean, $d = \frac{x - A}{c}$ and c = class interval.

Example: Find the A.M.

Classes	Frequency
0 – 9	32
10 – 19	65
20 – 29	100
30- 39	184
40 – 49	288
50 – 59	167
60 – 69	98
70 – 79	46
80 – 89	20
Total	1000

Answer:

Class Mark (x)		Frequency (f)	$d = \frac{x - A}{c}$	fd
0 – 9	(4.5)	32	-4	-128
10 – 19	(14.5)	65	-3	-195
20 – 29	(24.5)	100	-2	-200
30- 39	(34.5)	184	-1	-184
40 – 49	(44.5)	288	0	
50 – 59	(54.5)	167	1	167
60 – 69	(64.5)	98	2	196
70 – 79	(74.5)	46	3	138
80 – 89	(84.5)	20	4	80
Total		1000		-126

$$A = 44.5 \qquad A.M. = A + \frac{\Sigma fd}{\Sigma f} \times c$$

$$= 44.5 + \frac{-126}{1000} \times 10$$

$$= 44.5 - 1.26$$

$$= 43.24$$

2. Median : Median is the mid value of the given item.

For a raw data

Arrange the given item in the ascending or descending order and find the middle item. If there are two items in the middle, find their average.

Example 1. Find the median.

25, 20, 23, 32, 40, 27, 30, 25, 20, 10, 15, 41

Answer: Arranging the given data in the ascending order, we get

10, 15, 20, 20, 23, <u>25, 25</u>, 27, 30, 32, 40, 41

The middle items are 25, 25

$$\therefore \text{ Median} = \frac{25 + 25}{2} = 25$$

2. Find the median :

2, 30, 12, 25, 20, 8, 10, 4, 15

Answer: Arranging the ascending order

2, 4, 8, 10, <u>12</u>, 15, 20, 25, 30

$\therefore$ Median = 15

For a frequency distribution: The median is calculated by the formula

$$\text{Median} = l + \left(\frac{N}{2} - m\right) \times \frac{c}{f}$$

where l= lower limit of median class

N = Total No. of frequencies
m = cumulative frequency upto the median class
c = class width of median class
f = frequency of median class.

Example: Find the median.

Classes	14.5 – 19.5	19.5 – 24.5	24.5 – 29.5	
Frequencies	53	140	98	
Classes	29.5 – 34.5	34.5 – 39.5	39.5 – 44.5	
Frequencies	32	12	9	
Classes	44.5 - 49.5	49.5 – 54.5	54.5 – 59.5	Above 59.5
Frequencies	5	3	3	2

Answer :

Classes	**Frequency**	**Cumulative Frequency**
14.5 – 19.5	53	53
19.5 – 24.5	140	193
24.5 – 29.5	98	291
29.5 – 34.5	32	323
34.5 – 39.5	12	335
39.5 – 44.5	9	344
44.5 – 49.5	5	349
49.5 – 54.5	3	352
54.5 – 59.5	3	355
Above 59.5	2	357

Median class = the class in which the $\left(\frac{N}{2}\right)$th item lies

i.e. the $\frac{357}{2}$ th item lies

i.e. the $(178.5)^{th}$ item lies IN 19.5 – 24.5.

$$\therefore \text{ Median} = l + \left(\frac{N}{2} - m\right) \times \frac{c}{f}$$

$$= 19.5 + \left(\frac{357}{2} - 53\right) \times \frac{5}{140}$$

$$= 19.5 + (178.5 - 53) \times \frac{5}{140}$$

$$= 19.5 + 4.48$$

$$= 23.98$$

3. **Mode:** Mode is the most repeated item in the given data.

For a raw data: Obtain the most repeated item in the given data.

Problem: Find the mod : 2,48,6,2,9,5,2,7,5,2

The most repeated item = 2 $\therefore$ Mode = 2.

For a frequency distribution: Mode is found using the formula.

$$\text{Mode} = l + \frac{Cf_2}{f_1 + f_2}$$

where

l = lower limit of modal class

c = class interval

f_1 = frequency of the class preceding modal class

f_2 = frequency of the class succeeding modal class.

Problem: Find the mode.

Classes	Frequency
14.5 – 19.5	53
19.5 – 24.5	140
24.5 – 29.5	98
29.5 – 34.5	32
34.5 – 39.5	12
39.5 – 44.5	9
44.5 – 49.5	5
49.5 – 54.5	3
54.5 – 59.5	3
Above 59.5	2

Answer: Modal class = class having the highest frequency = 19.5 – 24.5

$$\text{Mode} = 19.5 + \frac{5 \times 98}{53 + 98}$$

$$= 19.5 + \frac{490}{151}$$

$$= 19.5 + 3.24$$

$$= 22.74$$

Measures of Dispersion

The commonly used measures of dispersion are 1. Standard Deviation, 2. Mean Deviation

1. Standard Deviation

The standard deviation is calculated using the formula

$$\sigma^2 = \frac{h^2}{N}\left[\sum f\, d^2 - \frac{(\sum fd)^2}{N}\right]$$

where h = size of class interval

N = total no. of frequencies

d = deviations

f = frequency σ^2 = Variance, σ = S.D.

Problem : Calculate the S.D.

Marks	Frequency
20 – 30	3
30 – 40	6
40 – 50	13
50 – 60	15
60 – 70	14
70 – 80	5
80 – 90	4

Answer :

Classes	**Frequency f**	**x**	**d =** $\frac{x-A}{c}$	**fd**	**fd²**
20 – 30	3	25	-3	-9	27
30 – 40	6	35	-2	-12	24
40 – 50	13	45	-1	-13	13
50 – 60	15	55	0	0	0
60 – 70	14	65	1	14	14
70 – 80	5	75	2	10	20
80 – 90	4	85	3	12	36
Total	60			2	134

$$(\sigma^2) = \frac{h^2}{N}\left[\sum fd^2 - \frac{(\sum fd)^2}{N}\right]$$

$$= \frac{100}{60}\left[134 - \frac{2^2}{60}\right]$$

$$= \frac{5}{3}\left(134 - \frac{1}{15}\right) = \frac{2009}{9} = 223.22$$

$\therefore$ S.D. = $\sigma = \sqrt{223.22} = 14.01$

Exercises.

1. Construct a frequency table for the following data with a class interval of 30-40, 40-50, etc.:

42	49	37	82	37	75	62	54	79	84
75	63	44	74	44	36	69	54	48	74
39	48	45	61	71	47	38	80	51	31

2. Draw a bar diagram.

Wages :	110 – 115	115 – 120	120 – 125	125 – 130	130 – 135
No. of : Workers	3	4	4	9	8

3. Draw Pie Diagram :

Ages :	25 – 31	31 – 37	37 – 43	43 – 49	49 – 54
No. of: Teachers	10	12	6	2	2

4. Find the A.M.

i) 8, 16, 27, 12, 6, 0, 4, 21, 27, 8

ii)

Classes	Frequency
0 – 9	12
10 – 19	26
20 – 29	55
30 – 39	78
40 – 49	44
50 – 59	30
60 – 69	19

5. Find the Median :

i) 12, 26, 19, 4, 26, 15, 36, 26, 27

ii)

Classes	Frequency
40 – 45	3
45 – 50	5
50 – 55	6
55 – 60	10
60 – 65	12
65 – 70	10
70 – 75	4

6. Find the mode.

i) 0, 26, 11, 27, 12, 11, 30, 10, 8, 11

ii)

Classes	Frequency
15 – 19	25
20 – 24	29
25 – 29	37
30 – 34	56
35 – 39	32
40 – 44	18
45 – 49	11

7. Find the S.D.

Classes	Frequency
10 – 20	3
20 – 30	1
30 – 40	1
40 – 50	8
50 – 60	17
60 – 70	38
70 – 80	9

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